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Solitons: conservation laws & dressing methods

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Abstract

We review some of the fundamental notions associated to the theory of solitons. More precisely, we focus on the issue of conservation laws via the existence of the Lax pair and also on methods that provide solutions to partial or ordinary differential equations that are associated to discrete or continuous integrable systems. The Riccati equation associated to a given continuous integrable system is also solved and hence suitable conserved quantities are derived. The notion of the Darboux-Bäcklund transformation is introduced and employed in order to obtain soliton solutions for specific examples of integrable equations. The Zakharov-Shabat dressing scheme and the Gelfand-Levitan-Marchenko equation are also introduced. Via this method generic solutions are produced, and integrable hierarchies are explicitly derived. Various discrete and continuous integrable models are employed as examples such as the Toda chain, the discrete non-linear Schrödinger model, the Korteweg-de Vries and non-linear Schrödinger equations as well as the sine-Gordon and Liouville models.

¹This paper is based on a series of lectures presented at Heriot-Watt University by A.D.

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1 Introduction

The main aim of this article is to introduce the key ideas and concepts on the theory of solitons and the associated partial differential equations (PDEs) or ordinary differential equations (ODEs). It is based on a series of lectures presented at Heriot-Watt University as a part of a graduate course, and is primarily addressed to graduate students or researchers who wish to acquire some fundamental knowledge on the theory of solitons.

Solitons are localized traveling wave packets that maintain their shape as they travel and emerge unaltered after interacting with each other. Due to their “particle-like” nature they are of great mathematical significance with a plethora of realistic applications including optical fibres, water wave tsunami formation, Bose-Einstein condensate and chemical reactions among others. Typical examples of equations that exhibit solitonic solutions are the Korteweg-de Vries (KdV), the sinh-Gordon, and the non-linear Schrödinger (NLS) equations. All these are cases of exactly solvable equations known as *integrable* equations and can be described via a unifying scheme under the generic name of Inverse Scattering Transform [1] as will become transparent in these notes. Solitons were first documented by Scott Russell in the 19th century, who observed them along the union canal in Edinburgh (an aqueduct of which is now named after him, a mere 15 minute walk away from the Heriot-Watt campus) while investigating the most efficient design for canal boats. Despite this, it was not until much later that it was suggested as the solution to an equation, such as the shallow wave equation put forward by Korteweg and de Vries near the turn of the 20th century.

As mentioned, solitons are closely related to the idea of integrability, a notion that is associated to the existence of a large number of conserved quantities (i.e. symmetries), therefore a substantial part of this presentation is devoted to the systematic derivation of conserved quantities in integrable systems such as the Toda chain, the modified Korteweg-de Vries equation, the non-linear Schrödinger and sine-Gordon equations. In this context the notion of the Lax pair is introduced [2] together with the idea of the *monodromy* matrix as well as the associated Riccati equation [3], which lead to the derivation of the hierarchy of conserved charges. The main paradigm of a continuum system is the so called AKNS scheme [4]. This offers the main non-relativistic system, and it is naturally associated to the NLS, KdV and mKdV equations. Also, typical examples of relativistic systems such as the sine-Gordon and Liouville models are examined. It is worth noting that our description is exclusively based on the existence of a Lax pair. There is however the algebraic/Hamiltonian description (see for instance some typical textbooks and reviews [3]–[9] and references therein) based on the existence of the so-called classical r -matrix and an associated Poisson algebraic structure [10]. We shall not discuss this formulation in these notes, however we refer the interested reader to [3]–[9].

The other important issue we are addressing in these notes is the derivation of solutions of integrable PDEs/ODEs and the systematic construction of integrable

hierarchies via the dressing process. There is a considerable amount of relevant literature on this well studied subject, however we are only referring to a limited number of relevant sources restricted to the main purposes of this article. The widely used dressing schemes are known under the names: Darboux-Bäcklund dressing transformation, Zakharov-Shabat dressing, Drinfeld-Sokolov method, and they are variations or rather formal descriptions of the inverse scattering transform (see e.g. [11]–[16] as well as [17]–[21] and references therein). The dressing schemes not only provide a sound methodology of producing solutions of integrable PDEs, but also allow the explicit construction of the Lax pairs of integrable hierarchies. Note that in the Hamiltonian framework one can also construct the hierarchy of Lax pairs using a universal expression that requires the existence of the classical (or quantum) r -matrix [10, 22, 23].

We should mention that significant issues, such as Hirota’s bilinear method or Sato’s theory and connections to finite Grassmannians have not been addressed in this presentation. We refer however the interested reader to e.g. [24, 25, 26, 19, 21] and references therein, for pedagogical discussions on these and related issues.

The Korteweg-de Vries Equation

Let us now introduce one of the most important equations in the context of soliton theory that is the Korteweg-de Vries equation. The KdV equation is a partial differential equation (PDE) that theoretically describes solitons, i.e. it admits solitonic solutions, and reads as

$$\frac{\partial u(x, t)}{\partial t} - 6u \frac{\partial u(x, t)}{\partial x} + \frac{\partial^3 u(x, t)}{\partial x^3} = 0. \quad (1.1)$$

Let us derive below the one soliton solution based on rather simple considerations. Later in the text we shall introduce systematic ways of producing solitonic solutions based on the dressing method. Inserting a travelling wave solution $u(x, t) = f(\xi)$, $\xi = x - ct$ the KdV equation becomes:

$$-cf' - 6ff' + f''' = 0$$

where the prime denotes differentiation with respect to ξ , then recalling that $(f^2)' = 2ff'$, this can be integrated to give:

$$-cf - 3f^2 + f'' = A$$

for some arbitrary constant of integration A . Multiplying through by f' , this can be integrated again (adding a further constant of integration B):

$$-\frac{c}{2}f^2 - f^3 + \frac{1}{2}(f')^2 = Af + B.$$

Choosing that f and f' both tend to 0 as $\xi \rightarrow \pm\infty$ (i.e. that the traveling wave solution travels in one local 'lump'), both of the constants A and B will have to be

set to 0, so this becomes:

$$f' = \pm f \sqrt{2f + c} \Rightarrow \pm \int d\xi = \int \frac{1}{f \sqrt{2f + c}} df.$$

To evaluate the right-hand integral, the substitution $f = -\frac{c}{2} \operatorname{sech}^2(\theta)$ is made, from which it follows that $df = c \tanh(\theta) \operatorname{sech}^2(\theta) d\theta$. After this substitution, the integral becomes trivial (after recalling the hyperbolic trigonometric relation $\tanh^2(\theta) = 1 - \operatorname{sech}^2(\theta)$):

$$\begin{aligned} \pm \xi - x_0 &= -\frac{2}{\sqrt{c}} \int d\theta \\ &= -\frac{2}{\sqrt{c}} \theta \end{aligned}$$

for some constant of integration x_0 . Inserting this back into the substitution made earlier finally gives the solitonic solution to the KdV equation

$$f(x - ct) = -\frac{c}{2} \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2} (\pm(x - ct) - x_0) \right). \quad (1.2)$$

The modified KdV equation

We are going to derive yet another significant integrable PDE i.e. the modified KdV (mKdV) equation, starting from the KdV equation. The connection with the time independent Schrödinger equation will also be established.

First, recall the KdV equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

We now introduce the so-called *Miura transformation*: $u = v^2 + \frac{\partial v}{\partial x}$, and substituting this into the KdV equation we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(v^2 + \frac{\partial v}{\partial x} \right) - 6 \left(v^2 + \frac{\partial v}{\partial x} \right) \frac{\partial}{\partial x} \left(v^2 + \frac{\partial v}{\partial x} \right) + \frac{\partial^3}{\partial x^3} \left(v^2 + \frac{\partial v}{\partial x} \right) = \dots \\ &= \left(2v + \frac{\partial}{\partial x} \right) \left(\frac{\partial v}{\partial t} - 6v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} \right) \end{aligned}$$

and consequently v must obey an equation called the modified KdV equation

$$\frac{\partial v}{\partial t} - 6v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0. \quad (1.3)$$

We also introduce the field ψ such that $v = \psi^{-1} \frac{\partial \psi}{\partial x}$, then the Miura transform becomes

$$\begin{aligned} u(x, t) &= \left(\psi^{-1} \frac{\partial \psi}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left(\psi^{-1} \frac{\partial \psi}{\partial x} \right) = \psi^{-1} \frac{\partial^2 \psi}{\partial x^2} \\ \Rightarrow 0 &= \frac{\partial^2 \psi}{\partial x^2} - u\psi. \end{aligned} \quad (1.4)$$

The latter expression has the form of the time independent Schrödinger equation with the potential given by the solution to the original KdV equation, $u(x, t)$, e.g. the one-soliton solution. To generalize this to non-zero spectrum λ of the Schrödinger equation the symmetries of the KdV equation need to be exploited, namely, the solution u is invariant under the transformation $u(x, t) \rightarrow \lambda + u(x + 6t\lambda, t)$, for some real number λ . Then, this transformation introduces a factor $\lambda\psi$:

$$\frac{\partial^2 \psi}{\partial x^2} + (\lambda - u)\psi = 0. \quad (1.5)$$

We shall discuss in what follows the systematic construction and solutions of integrable PDEs and ODEs, as well as the existence of the associated conserved quantities.

2 The Lax Formulation

As already pointed out we are interested in integrable systems, i.e. systems that possess a large number of conserved quantities, can in principle be exactly solved, and usually display solitonic solutions. The issue of conservation laws in relation to such equations will be thoroughly discussed in the subsequent sections.

The idea of integrability is synonymous to the existence of a pair of operators called the *Lax pair* [2], that are associated to a set of linear relations that comprise the so-called auxiliary linear problem. Consider in general a pair of operators (L, M) , the associated spectral problem and the evolution problem respectively:

$$\begin{aligned} L\Psi &= \zeta\Psi \\ \dot{\Psi} &= M\Psi, \end{aligned} \quad (2.1)$$

where the “dot” denotes differentiation with respect to time. Consistency of the two equations above, provided also that $\dot{\zeta} = 0$, yields the *zero curvature condition*:

$$\dot{L} = [M, L], \quad (2.2)$$

where we define the commutator $[X, Y] = XY - YX$. Expression (2.2) provides the equations of motion of the system described by the given Lax pair. These equations are in general integrable PDEs or ODEs depending on the type of system under consideration, i.e. discrete or continuous. In general, the pair (L, M) can be differential operators or $d \times d$ matrices, that depend on some classical fields, solutions of the associated non-linear ODEs or PDEs. We examine below two fundamental examples of such models i.e. the KdV equation and the Toda chain. In the subsequent sections we employ the notion of the Lax pair and the associated auxiliary linear problem (2.1) to construct continuous and discrete integrable models, and also identify the associated non-linear PDEs or ODEs.

- **Differential operators: the KdV equation**

Let us consider the generic pair of differential operators:

$$\begin{aligned} L(x, t) &= -\partial_x^2 + u(x, t) \\ M(x, t) &= a\partial_x^3 + f(x, t)\partial_x + g(x, t) \end{aligned} \quad (2.3)$$

where we use the notation $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_x^n f = \frac{\partial^n f}{\partial x^n}$. Assume also that the pair above satisfies the zero curvature condition. We can then identify the functions f , g as well as the associated non-linear PDE:

$$\begin{aligned} \frac{\partial u}{\partial t} \psi &= \left[\left(a \frac{\partial^3}{\partial x^3} + f \frac{\partial}{\partial x} + g \right) \left(u - \frac{\partial^2}{\partial x^2} \right) - \left(u - \frac{\partial^2}{\partial x^2} \right) \left(a \frac{\partial^3}{\partial x^3} + f \frac{\partial}{\partial x} + g \right) \right] \psi \\ &= \left(a \frac{\partial^3 u}{\partial x^3} + f \frac{\partial u}{\partial x} + \frac{\partial^2 g}{\partial x^2} \right) \psi + \left(3a \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial g}{\partial x} \right) \frac{\partial \psi}{\partial x} \\ &\quad + \left(3a \frac{\partial u}{\partial x} + 2 \frac{\partial f}{\partial x} \right) \frac{\partial^2 \psi}{\partial x^2} \end{aligned}$$

so if we want there to be no terms depending on derivatives of ψ , then this gives us the definitions of $f(x, t)$ and $g(x, t)$, up to some constants of integration:

$$f(x, t) = -\frac{3}{2}au, \quad g(x, t) = -\frac{3}{4}a\partial_x u. \quad (2.4)$$

These can then be inserted back into the equation for the time evolution of u to give the KdV equation (setting $a = -4$):

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}. \quad (2.5)$$

- **$N \times N$ matrices: the Toda chain**

The Lax pair associated to the Toda chain [27] is given by (time dependence is always implied even when it is not explicitly stated)

$$\begin{aligned} L &= \sum_{j=1}^N p_j e_{jj} + \sum_{j=1}^{N-1} e^{(q_{j+1}-q_j)/2} (e_{jj+1} + e_{j+1,j}) \\ M &= \frac{1}{2} \sum_{j=1}^{N-1} e^{(q_{j+1}-q_j)/2} (e_{jj+1} - e_{j+1,j}), \end{aligned} \quad (2.6)$$

where e_{ij} is the $N \times N$ matrix with elements $(e_{ij})_{kl}$. Provided that the Lax pair satisfies the zero curvature condition (2.2) we conclude

$$\begin{aligned} \dot{q}_j &= p_j \\ \dot{p}_j &= e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}}. \end{aligned}$$

Combining the two equations above we obtain the equations of motion for the Toda chain

$$\ddot{q}_j = e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}}. \quad (2.7)$$

A solution of this equation is the discrete soliton expressed as

$$q_j(t) = q_+ + \ln \left(\frac{1 + \frac{\gamma}{1-e^{-2\kappa}} e^{-2\kappa(j+1)+\sigma \sinh(\kappa)t}}{1 + \frac{\gamma}{1-e^{-2\kappa}} e^{-2\kappa j + \sigma \sinh(\kappa)t}} \right) \quad (2.8)$$

where q_+ , κ , σ , and γ are constants, with $\sigma = \pm 2$. This can be verified by inspection.

2.1 Conservation laws

As mentioned integrability is associated to the existence of a large number of conservation laws, usually as many as the degrees of freedom of the system under consideration. Based on the existence of a Lax pair and the relevant zero curvature condition we shall be able in what follows to systematically derive the conserved charges for various examples of integrable models such as the KdV equation, the Toda chain, the discrete NLS model and the AKNS system.

2.1.1 The KdV equation

Recall the Miura transformation, introduced in the previous subsection

$$u = v^2 + \frac{\partial v}{\partial x}, \quad (2.9)$$

and yet another transformation, called the Gardner transformation:

$$v = \frac{1}{2\epsilon} + \epsilon w \quad (2.10)$$

for some ϵ . Combining the two transformations (2.9), (2.10) we conclude

$$u = \frac{1}{4\epsilon^2} + w + \epsilon^2 w^2 + \epsilon \frac{\partial w}{\partial x}. \quad (2.11)$$

The additive constant in the latter expression can be removed by utilizing the symmetry of the KdV equation under the transformation $u(x, t) \rightarrow u(x - 6\lambda t, t) - \lambda$, with $\lambda = \frac{1}{4\epsilon^2}$. Then, inserting this into the KdV equation:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(w + \epsilon^2 w^2 + \epsilon \frac{\partial w}{\partial x} \right) - 6 \left(w + \epsilon^2 w^2 + \epsilon \frac{\partial w}{\partial x} \right) \frac{\partial}{\partial x} \left(w + \epsilon^2 w^2 + \epsilon \frac{\partial w}{\partial x} \right) \\ &\quad + \frac{\partial^3}{\partial x^3} \left(w + \epsilon^2 w^2 + \epsilon \frac{\partial w}{\partial x} \right) \\ &= A + \epsilon \frac{\partial A}{\partial x} + 2\epsilon^2 w A, \end{aligned} \quad (2.12)$$

where we define $A = \frac{\partial w}{\partial t} - 6w \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} - 6\epsilon^2 w^2 \frac{\partial w}{\partial x}$, after some tedious but nevertheless straightforward computations. If $A = 0$, then the original KdV equation is automatically satisfied. Thus, the KdV equation is equivalent to

$$0 = \frac{\partial w}{\partial t} - 6 \left(w + \epsilon^2 w^2 \right) \frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3}. \quad (2.13)$$

In the limit $\epsilon \rightarrow 0$, $w \rightarrow u$, the field w can be expressed as a formal series expansion about $\epsilon \rightarrow 0$:

$$w(x, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n w_n(x, t). \quad (2.14)$$

Choosing $T = w$ and $X = -3w^2 - 2\epsilon^2 w^3 + \frac{\partial^2 w}{\partial x^2}$ it is easy to see that Eq. (2.13) reduces to $\partial_t T + \partial_x X = 0$, and hence we conclude that $\int_{-\infty}^{\infty} w dx$ is a constant of motion, provided that w vanishes at $x \rightarrow \pm\infty$. Then, inserting the asymptotic expansion and considering each power of ϵ in turn, it follows that $\int_{-\infty}^{\infty} w_n dx$ is also a constant of motion for each n . Therefore, there are actually an infinite number of constants of motion for the KdV equation!

As w must agree with u when $\epsilon = 0$, w_0 can be immediately recognized as $w_0 = u$. Then, the remaining w_n can be read by inserting this expansion (2.14) into the transformation (2.11), giving a recursion relation (after removing the $n = 0$ case):

$$0 = w_n + \sum_{\substack{i,j=0 \\ i+j=n-2}}^{\infty} w_i w_j + \frac{\partial w_{n-1}}{\partial x}. \quad (2.15)$$

The first four members of the series expansion read as, $w_0 = u$ and

$$\begin{aligned} w_1 &= -\frac{\partial w_0}{\partial x} = -\frac{\partial u}{\partial x} \\ w_2 &= -w_0^2 - \frac{\partial w_1}{\partial x} = \frac{\partial^2 u}{\partial x^2} - u^2 \\ w_3 &= -2w_0 w_1 - \frac{\partial w_2}{\partial x} = 4u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}. \end{aligned}$$

2.1.2 Conserved quantities from L : The Toda chain

We shall now be working with the general Lax pair (L, M) , where L and M are $N \times N$ matrices that depend on some classical fields (solutions of the associated integrable ODEs/PDEs), as well as some spectral parameter $\lambda \in \mathbb{C}$. For this pair, recall that the zero-curvature condition is $\dot{L} = [M, L]$, while the associated spectral problem is $L\psi = \zeta\psi$. We introduce a function $\mathfrak{g}^{(n)}$, defined to be the trace of the n^{th} power of L , i.e. $\mathfrak{g}^{(n)}(\lambda) = \text{tr}(L^n(\lambda))$. The differential of this with respect to time can be easily calculated, using the properties of the trace (namely, that it pulls through differentiation and that it is cyclic):

$$\begin{aligned} \dot{\mathfrak{g}}^{(n)} &= n \text{tr}(L^{n-1} \dot{L}) \\ &= n \text{tr}(L^{n-1}(LM - ML)) \\ &= n \text{tr}(L^n M) - n \text{tr}(L^{n-1} ML) \\ &= n \text{tr}(L^n M) - n \text{tr}(L^n M) = 0. \end{aligned}$$

So for any value of n , $\mathfrak{g}^{(n)}$ is a conserved quantity.

To see how this can be used to yield the conserved quantities, we look at our typical example of an integrable model with discrete degrees of freedom: the Toda chain (2.6). To find the conserved quantities $t^{(n)}$, we start by taking the trace of L :

$$\mathfrak{g}^{(1)} = \text{tr}(L) = \sum_{j=1}^N p_j, \quad (2.16)$$

which is the total momentum of the system. The second conserved quantity can be similarly found

$$\mathfrak{g}^{(2)} = \text{tr}(L^2) = \sum_{j=1}^N p_j^2 + 2 \sum_{j=1}^{N-1} e^{q_{j+1}-q_j}$$

and after multiplying by a factor of $\frac{1}{2}$, this gives the Hamiltonian for this model

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{j=1}^{N-1} e^{q_{j+1}-q_j}. \quad (2.17)$$

2.2 Semi-discrete Lax-pairs & the auxiliary linear problem

We may now generalize the notion of the Lax pair to describe one dimensional discrete space and continuum time integrable systems. Typical examples of such systems are the Toda chain (which shall be revisited in this context) or the discrete version of the NLS model among others. In this case the Lax pair (L, A) satisfies the semi-discrete auxiliary linear problem, expressed as (see e.g. [3])

$$\begin{aligned} \Psi_{j+1}(t, \lambda) &= L_j(t, \lambda) \Psi_j(t, \lambda) \\ \dot{\Psi}_j(t, \lambda) &= A_j(t, \lambda) \Psi_j(t, \lambda). \end{aligned} \quad (2.18)$$

The Lax pair are $d \times d$ matrices depending on some fields and the spectral parameter. Here we shall only consider examples where L, A are 2×2 matrices.

Differentiating the first of the equations (2.18) with respect to time produces the semi-discrete zero curvature condition

$$\begin{aligned} \dot{\Psi}_{j+1} &= \dot{L}_j \Psi_j + L_j \dot{\Psi}_j \\ A_{j+1} L_j \Psi_j &= \dot{L}_j \Psi_j + L_j A_j \Psi_j \end{aligned}$$

which gives:

$$\dot{L}_j = A_{j+1} L_j - L_j A_j. \quad (2.19)$$

Let us also introduce the discrete space *monodromy matrix*:

$$T_N(\lambda) = L_N(\lambda) \dots L_1(\lambda), \quad (2.20)$$

which is a solution of the first equation of the auxiliary linear problem (2.18). Let us also define the so-called *transfer matrix*: $\mathfrak{t}(\lambda) = \text{tr}(T(\lambda))$, then we can show using

the discrete zero curvature condition that this is constant in time. Indeed, consider the time derivatives of \mathfrak{t} (we suppress the subscript N for brevity):

$$\begin{aligned}
\dot{\mathfrak{t}}(\lambda) &= \text{tr}(\dot{T}(\lambda)) \\
&= \text{tr}\left(\sum_{j=1}^N L_N(\lambda) \dots \dot{L}_j L_{j-1}(\lambda) \dots L_1(\lambda)\right) \\
&= \sum_{j=1}^N \text{tr}\left(L_N(\lambda) \dots L_{j+1}(\lambda) A_{j+1}(\lambda) L_j(\lambda) \dots L_1(\lambda)\right) \\
&\quad - \sum_{j=1}^N \text{tr}\left(L_N(\lambda) \dots L_j(\lambda) A_j(\lambda) L_{j-1}(\lambda) \dots L_1(\lambda)\right) \\
&= \text{tr}\left(A_{N+1}(\lambda) T(\lambda) - T(\lambda) A_1(\lambda)\right) = 0,
\end{aligned}$$

provided that periodic boundary conditions are imposed, i.e. $A_{N+1}(\lambda) = A_1(\lambda)$. The latter expression shows that $\mathfrak{t}(\lambda)$ is the generating function of the integrals of motion: $\mathfrak{t}(\lambda) = \sum_n \frac{t^{(n)}}{\lambda^n}$, where each $t^{(n)}$ is a constant of motion.

2.2.1 The Toda Chain revisited

We revisit the Toda chain based on the so-called dual description [8]. In this frame the Toda model is seen as one dimensional lattice model described by the semi-discrete auxiliary linear problem, with the corresponding Lax pair given by

$$L_j(\lambda) = \begin{pmatrix} \lambda - p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \quad A_j(\lambda) = \begin{pmatrix} \lambda & e^{q_j} \\ -e^{-q_{j-1}} & 0 \end{pmatrix}. \quad (2.21)$$

It is convenient for our purposes here to express the L -operator as $L_j = \lambda D + \tilde{L}_j$, where:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{L}_j = \begin{pmatrix} -p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}.$$

As explained in the preceding subsection to obtain the integrals of motion we need to expand the transfer matrix \mathfrak{t} in powers of $\frac{1}{\lambda}$. Let us first consider the expansion of the monoromy matrix:

$$\begin{aligned}
T(\lambda) &= (\lambda D_N + \tilde{L}_N) \dots (\lambda D_1 + \tilde{L}_1) \\
&= \lambda^N \left(D_N \dots D_1 + \frac{1}{\lambda} \sum_{j=1}^N D_N \dots D_{j+1} \tilde{L}_j D_{j-1} \dots D_1 \right. \\
&\quad \left. + \frac{1}{\lambda^2} \sum_{j>i} D_N \dots D_{j+1} \tilde{L}_j D_{j-1} \dots D_{i+1} \tilde{L}_i D_{i-1} \dots D_1 + \dots \right) \quad (2.22)
\end{aligned}$$

and the transfer matrix reads

$$\mathfrak{t}(\lambda) = \text{tr}(T(\lambda)) = \lambda^N \left(t^{(0)} + \frac{1}{\lambda} t^{(1)} + \frac{1}{\lambda^2} t^{(2)} + \dots \right). \quad (2.23)$$

From the last two expressions (2.22), (2.23) and after some cumbersome, but straightforward calculations we read the first couple of integrals of motion

$$\begin{aligned} t^{(1)} &= -\sum_{j=1}^N p_j \\ t^{(2)} &= \sum_{j>i=1}^N p_j p_i + \sum_{j=1}^N e^{q_{j+1}-q_j}. \end{aligned} \quad (2.24)$$

Notice that $t^{(2)}$ is a *non-local* quantity, i.e. it involves interactions among all the sites of the one dimensional chain. In general, to obtain the *local* conserved quantities i.e. quantities that only involve interactions among neighbor sites one needs to expand: $\ln(t(\lambda)) = \sum_n \frac{I_n}{\lambda^n}$, then one obtains for the first two local integral of motion,

$$\begin{aligned} I_1 &= t^{(1)} \\ I_2 &= t^{(2)} - \frac{(t^{(1)})^2}{2}. \end{aligned} \quad (2.25)$$

The first quantity $I^{(1)}$ is simply the total momentum (2.24), whereas the second charge is the local Toda Hamiltonian with periodic boundary conditions, ($\mathcal{O}_{N+j} = \mathcal{O}_j$, where $\mathcal{O}_j \in \{q_j, p_j\}$):

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{j=1}^N e^{q_j - q_{j-1}}. \quad (2.26)$$

Note that the only difference compared to the expression we obtained via the *dual* description in the previous subsection is that now we consider periodic boundary conditions, whereas in the previous description we dealt with the open case (2.17).

Using the semi-discrete zero curvature condition for the Lax pair (2.19) we recover the familiar equations of motion for the Toda chain

$$\begin{aligned} \dot{q}_j &= p_j \\ \dot{p}_j &= e^{q_{j+1}-q_j} - e^{q_j - q_{j-1}}. \end{aligned}$$

2.2.2 The Discrete Non-Linear Schrödinger model

We have looked a lot at the Toda chain in this section, so let us consider another prototypical integrable model, namely the discrete non-linear Schrödinger model (DNLS) [28, 20]. The Lax pair for this model is given by:

$$\begin{aligned} L_j(\lambda) &= \begin{pmatrix} \lambda + \mathbb{N}_j & x_j \\ X_j & 1 \end{pmatrix} \\ A_j(\lambda) &= \begin{pmatrix} \lambda^2 - x_j X_{j-1} & \lambda x_j - x_j \mathbb{N}_j + x_{j+1} \\ \lambda X_{j-1} - X_{j-1} \mathbb{N}_{j-1} + X_{j-2} & x_j X_{j-1} \end{pmatrix}, \end{aligned} \quad (2.27)$$

where we define $\mathbb{N}_j = 1 + x_j X_j$.

As in the previous example, we are going to derive the conserved quantities of the system by expanding $\ln(t)$ in powers of λ^{-1} , except this time we shall find the first three. Splitting the L matrix up into $L_j = \lambda D + \tilde{L}_j$ (where D is the same as it was in the previous example and \tilde{L}_j can be read off from the above definition of L_j), $T(\lambda)$ takes the same form as it did in that example, as does $\ln(t)$. Therefore, the first integral of motion can be calculated as in the previous section. We keep here terms up to third order in the expansion, then after some lengthy calculations, in the spirit of the previous example we obtain the first few local integrals of motion of the DNLS model

$$\begin{aligned}
I_1 &= \sum_{j=1}^N \mathbb{N}_j \\
I_2 &= \sum_{j=1}^N x_j X_{j-1} - \frac{1}{2} \sum_{j=1}^N \mathbb{N}_j^2 \\
I_3 &= \sum_{j=1}^N x_j X_{j-2} - \sum_{j=1}^N (\mathbb{N}_j + \mathbb{N}_{j-1}) x_j X_{j-1} + \frac{1}{3} \sum_{j=1}^N \mathbb{N}_j^3. \quad (2.28)
\end{aligned}$$

Via the semi-discrete zero-curvature condition (2.19) for the NLS Lax pair (2.27) we also obtain the equations of motion i.e. the ODEs that describe the time evolution of the fields x_i, X_j :

$$\begin{aligned}
\dot{x}_j &= x_j x_{j+1} X_j - x_{j+1} (\mathbb{N}_j + \mathbb{N}_{j+1}) + x_j \mathbb{N}_j^2 + x_j^2 X_{j-1} + x_{j+2} \\
\dot{X}_j &= x_j X_{j-1} X_j + X_{j-1} (\mathbb{N}_j + \mathbb{N}_{j-1}) - X_j \mathbb{N}_j^2 - x_{j+1} X_j^2 - X_{j-2}. \quad (2.29)
\end{aligned}$$

2.3 Continuous systems Lax pair

In the preceding subsection we discussed the class of one-dimensional semi-discrete integrable systems and we derived their time evolution via the discrete zero curvature condition. In the present subsection we focus on continuous space and time integrable systems. Again the key object in our construction is the continuous Lax pair (U, V) , which satisfies the continuous auxiliary linear problem (see e.g. [3])

$$\begin{aligned}
\partial_x \Psi(x, t, \lambda) &= U(x, t, \lambda) \Psi(x, t, \lambda) \\
\partial_t \Psi(x, t, \lambda) &= V(x, t, \lambda) \Psi(x, t, \lambda). \quad (2.30)
\end{aligned}$$

U, V can be in general $d \times d$ matrices depending on some fields and some spectral parameter. Cross-differentiating the two equations above we obtain the continuous zero curvature condition

$$\partial_t U - \partial_x V + [U, V] = 0. \quad (2.31)$$

The continuous analogue of the discrete monodromy matrix (2.20) derived in the previous subsection is given as

$$T(x, y, t, \lambda) = \mathcal{P} \exp \left(\int_y^x U(\xi) d\xi \right), \quad x \geq y \quad (2.32)$$

and it is a solution of the x -part of the continuous auxiliary linear problem. The latter expression is a *path ordered* exponential, which is formally defined as

$$\mathcal{P}\exp\left(\int_y^x U(\xi, t)d\xi\right) = \sum_{n=0}^{\infty} \int_y^x dx_n U(x_n) \int_y^{x_n} dx_{n-1} U(x_{n-1}) \dots \int_y^{x_2} dx_1 U(x_1)$$

$$x \geq x_n \geq x_{n-1} \dots \geq x_1 \geq y. \quad (2.33)$$

The continuous monodromy matrix can be obtained as a suitable continuum limit of the discrete one (2.20) (we refer the interested reader to [29] for a detailed discussion and proof).

We consider that the system runs over some region $[-L, L]$, and define the transfer matrix as in the discrete case, $\mathfrak{t}(t, \lambda) = \text{tr}(T(L, -L, t, \lambda))$. Given that this definition of \mathfrak{t} is the continuous limit of its discrete analogue, the continuous \mathfrak{t} will naturally be an integral of motion, and can be expanded in powers of λ^{-1} to provide an infinite tower of integrals of motion $t^{(n)}$. To obtain the local integrals of motion, as in the discrete case, we need to consider the formal series expansion of $\ln(\mathfrak{t}(\lambda))$.

2.3.1 Riccati equation & conserved quantities: AKNS system

Our main aim now is the derivation of conserved quantities for continuous systems via the solution of an associated Riccati equation. To illustrate the computational details of this process we employ the prototypical AKNS system [4].

We focus on the continuous monodromy matrix, and the fact that it is a solution of the x -part of the auxiliary linear problem (2.30). We use the standard decomposition for the monodromy matrix (see also [3]):

$$T(x, y, t, \lambda) = (\mathbb{I} + W(x, t, \lambda))e^{Z(x, y, t, \lambda)}(\mathbb{I} + W(y, t, \lambda))^{-1}, \quad (2.34)$$

where W is a purely anti-diagonal matrix and Z is a purely diagonal one. Also, the matrix U is split into its diagonal and anti-diagonal parts, $U = U_D + U_A$.

To determine the W and Z matrices, let us insert this expression back into Eq. (2.30):

$$\frac{\partial}{\partial x} \left((\mathbb{I} + W)e^Z \right) = (U_D + U_A)(\mathbb{I} + W)e^Z$$

$$\frac{\partial W}{\partial x} + (\mathbb{I} + W) \frac{\partial Z}{\partial x} = U_D + U_D W + U_A + U_A W.$$

Splitting this into its diagonal and anti-diagonal components gives rise to two equations:

$$\frac{\partial W}{\partial x} + W \frac{\partial Z}{\partial x} = U_D W + U_A$$

$$\frac{\partial Z}{\partial x} = U_D + U_A W, \quad (2.35)$$

and inserting the second equation into the first in place of the $\frac{\partial Z}{\partial x}$ we obtain the fundamental Riccati equation for W

$$\frac{\partial W}{\partial x} + [W, U_D] + WU_AW - U_A = 0. \quad (2.36)$$

Finally, as both W and Z are functions of the spectral parameter λ they can be expanded as (see also [3])

$$\begin{aligned} W(x, t, \lambda) &= \sum_{n=1}^{\infty} \frac{W^{(n)}(x, t)}{\lambda^n}, \\ Z(x, y, t, \lambda) &= \sum_{n=-1}^{\infty} \frac{Z^{(n)}(x, y, t)}{\lambda^n}, \end{aligned} \quad (2.37)$$

then inserting these into our equations (2.35), (2.36) allows the derivation of recursion relations for $W^{(n)}$ and $Z^{(n)}$ depending on the specific form of the U -operator.

- **The AKNS system**

We focus now on the AKNS system [4] for which the U -operator consists of the following diagonal and anti-diagonal parts

$$U_D = \frac{\lambda}{2}\sigma, \quad U_A = \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix}, \quad (2.38)$$

where $\sigma = \text{diag}(1, -1)$. It is worth noting that the AKNS system is a generic system that gives rise to many fundamental integrable models, more precisely:

1. **Non-linear Schrödinger:** $\hat{u} = u^*$
2. **mKdV & sine-Gordon:** $\hat{u} = u$
3. **KdV:** $u = 1$.

Our main aim now is to identify the terms $W^{(n)}$, $Z^{(n)}$ of the series expansion, and hence the conserved quantities. Recalling the generic equations for the diagonal and anti-diagonal parts (2.35), (2.36), and taking into consideration the λ series expansion (2.37) we can derive the following recursion Riccati type relations for the anti-diagonal part:

$$\begin{aligned} W^{(1)}\sigma &= U_A \\ W^{(n+1)}\sigma &= -\frac{\partial W^{(n)}}{\partial x} - \sum_{m=1}^{n-1} W^{(n)}U_AW^{(n-m)}, \end{aligned} \quad (2.39)$$

and for the diagonal part:

$$\frac{\partial Z^{(-1)}}{\partial x} = U_D \quad (2.40)$$

$$\frac{\partial Z^{(n)}}{\partial x} = U_AW^{(n)}. \quad (2.41)$$

It is then straightforward to evaluate the first few terms of the series expansion for the anti-diagonal part via the fundamental recursion relations above,

$$W^{(1)} = \begin{pmatrix} 0 & -\hat{u} \\ u & 0 \end{pmatrix}, \quad W^{(2)} = \begin{pmatrix} 0 & -\partial_x \hat{u} \\ -\partial_x u & 0 \end{pmatrix} \quad (2.42)$$

$$W^{(3)} = \begin{pmatrix} 0 & -\partial_x^2 \hat{u} + u \hat{u}^2 \\ \partial_x^2 u - \hat{u} u^2 & 0 \end{pmatrix} \quad (2.43)$$

and so on. Now that we have at our disposal the $W^{(n)}$, we can use them to calculate the corresponding $Z^{(n)}$ via Eqs. (2.40), (2.41). Integrating over the interval $[-L, L]$ these can be readily read off as

$$Z^{(-1)} = \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} \quad (2.44)$$

$$Z^{(1)} = \begin{pmatrix} \int_{-L}^L u \hat{u} dx & \\ 0 & -\int_{-L}^L u \hat{u} dx \end{pmatrix} \quad (2.45)$$

$$Z^{(2)} = \begin{pmatrix} \int_{-L}^L \hat{u} \partial_x u dx & 0 \\ 0 & -\int_{-L}^L u \partial_x \hat{u} dx \end{pmatrix} \quad (2.46)$$

$$Z^{(3)} = \begin{pmatrix} \int_{-L}^L (\hat{u} \partial_x^2 u - (\hat{u} u)^2) dx & 0 \\ 0 & \int_{-L}^L (-u \partial_x^2 \hat{u} + (\hat{u} u)^2) dx \end{pmatrix}. \quad (2.47)$$

With the knowledge of Z we will be able to derive the integrals of motion by expanding $\ln(\mathbf{t}) = \ln(\text{tr}\{e^Z\})$ in powers of λ . The elements of the exponential of a diagonal matrix are just the exponentials of its elements, i.e. $(e^Z)_{ii} = e^{Z_{ii}}$, so this becomes:

$$\ln(\mathbf{t}) = \ln(e^{Z_{11}} + e^{Z_{22}})$$

We consider one of two limits here, either $\lambda \rightarrow \infty$ or $\lambda \rightarrow -\infty$. In both limits, the leading order term in the exponentials will be $\lambda Z_{ii}^{(-1)}$, but we have already calculated these in Eq. (2.44). Therefore, looking at the $\lambda \rightarrow \infty$ limit, the leading order terms are $e^{\lambda L}$ and $e^{-\lambda L}$ for $e^{Z_{11}}$ and $e^{Z_{22}}$ respectively. Using the limit $\lambda \rightarrow \infty$, $e^{Z_{22}} \rightarrow 0$, then the integrals of motion are just read off from the expansion of

$$\ln(\mathbf{t}) = \lambda Z_{11}^{(-1)} + \lambda^{-1} Z_{11}^{(1)} + \lambda^{-2} Z_{11}^{(2)} + \dots$$

which gives the first three as (ignoring the order λ term, as this is trivially an integral of motion):

$$\begin{aligned} I^{(1)} &= \int_{-L}^L u \hat{u} dx \\ I^{(2)} &= \int_{-L}^L \hat{u} \partial_x u dx \\ I^{(3)} &= \int_{-L}^L (\hat{u} \partial_x^2 u - (\hat{u} u)^2) dx, \end{aligned} \quad (2.48)$$

and are identified with the number of particles, the momentum, and the Hamiltonian of the system respectively.

- **Equivalent description**

We describe in what follows an equivalent, albeit a bit more compact, way of deriving the conserved quantities, based also on the solution of the underlying Riccati equation emerging from the linear auxiliary problem.

The auxiliary function for our example here is expressed as a column two-vector:

$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$. Consider now the x -part of the auxiliary problem (2.30):

$$\begin{aligned} \partial_x \Psi_1 &= \frac{\lambda}{2} \Psi_1 + \hat{u} \Psi_2, \\ \partial_x \Psi_2 &= -\frac{\lambda}{2} \Psi_2 + u \Psi_1, \end{aligned} \quad (2.49)$$

and define the element $\Gamma = \Psi_2 \Psi_1^{-1}$. Then from the latter expressions one arrives at the Riccati equation obeyed by Γ :

$$\partial_x \Gamma = u - \lambda \Gamma - \Gamma \hat{u} \Gamma. \quad (2.50)$$

It is worth noting, in comparison with the previous description based on the decomposition of the monodromy matrix, that Γ is essentially the analogue of the element W_{21} and Ψ_1 is the analogue of $e^{Z_{11}}$, a fact that underlines the equivalence of the two descriptions. Similarly, we could have defined $\hat{\Gamma} = \Psi_1 \Psi_2^{-1}$, then $\hat{\Gamma}$ would be the equivalent of W_{12} and Ψ_2 the equivalent of $e^{Z_{22}}$.

The next important task is to identify the element Γ . This can be achieved by expressing Γ in a formal power series expansion $\Gamma = \sum_k \frac{\Gamma^{(k)}}{\lambda^k}$ and solving the Riccati equation at each order. Then (2.50) reduces to $\Gamma^{(1)} = u$, and

$$\partial_x \Gamma^{(k)} = -\Gamma^{(k+1)} - \sum_{l=1}^{k-1} \Gamma^{(l)} \hat{u} \Gamma^{(k-l)}, \quad k > 0. \quad (2.51)$$

The latter expression is the equivalent of (2.39). Let us report below the first few terms of the expansion

$$\Gamma^{(1)} = u, \quad \Gamma^{(2)} = -\partial_x u, \quad \Gamma^{(3)} = \partial_x^2 u - u \hat{u} u, \quad \dots \quad (2.52)$$

The next natural step is to identify the associated conserved quantities by means of the auxiliary linear problem relations. Let us express the time components of the Lax pair $(U, V^{(n)})$ as $V^{(n)} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}$, (associated to the time t_n) and also recall for both the x and t -parts of the linear problem:

$$\begin{aligned} \partial_x \Psi_1 \Psi_1^{-1} &= \frac{\lambda}{2} + \hat{u} \Gamma, \\ \partial_{t_n} \Psi_1 \Psi_1^{-1} &= \alpha_n + \beta_n \Gamma. \end{aligned} \quad (2.53)$$

After cross differentiating the equations above we conclude

$$\partial_{t_n}(\hat{u}\Gamma) = \partial_x(\alpha_n + \beta_n\Gamma). \quad (2.54)$$

From the latter expression it is clear that the quantities

$$I^{(k)} = \int_{\mathbb{R}} dx \hat{u}(x)\Gamma^{(k)}(x), \quad (2.55)$$

are automatically conserved (see also [3]), where we have assumed vanishing boundary conditions at $\pm\infty$. Substituting the computed values of $\Gamma^{(k)}$ for $k = 1, 2, 3, \dots$ we recover the integrals of motion (2.48), as expected (consider $L \rightarrow \infty$).

3 Darboux-Bäcklund Transformations

After having discussed in detail the systematic construction of conserved quantities via the Lax pair formulation for both discrete and continuous integrable systems we come to the other main objective of this presentation, which is the solution of integrable ODEs/PDEs via general algebraic schemes known as the Darboux-Bäcklund transformation (BT) or the generalized Zakharov-Shabat dressing scheme involving differential and integral operators as dressing transformations.

3.1 Discrete Darboux-BT

Let us first focus on the issue of the discrete Darboux-Bäcklund transformation (BT) in the context of semi-discrete integrable models. Recall the auxiliary linear problem for such systems

$$\begin{aligned} \Psi_{j+1}(t, \lambda) &= L_j(t, \lambda) \Psi_j(t, \lambda), \\ \partial_t \Psi_j(t, \lambda) &= A_j(t, \lambda) \Psi_j(t, \lambda). \end{aligned} \quad (3.1)$$

We introduce the so-called Darboux matrix [17], a $d \times d$ matrix G that transforms the auxiliary function Ψ_j : $\tilde{\Psi}_j = G_j \Psi_j$ and leaves the form of the integrals of motion invariant. The main assumption now is that $\tilde{\Psi}_j$, \tilde{L}_j , \tilde{A}_j also satisfy the auxiliary linear problem, where \tilde{L}_j and \tilde{A}_j comprise the Lax pair associated to this new auxiliary linear problem. Inserting the definition of $\tilde{\Psi}_j$ into the first of these relations:

$$\begin{aligned} G_{j+1} \Psi_{j+1} &= \tilde{L}_j G_j \Psi_j \\ G_{j+1} (L_j \Psi_j) &= \tilde{L}_j G_j \Psi_j, \end{aligned}$$

and inserting the definition into the second relation we conclude

$$\begin{aligned} \partial_t (G_j \Psi_j) &= \tilde{A}_j (G_j \Psi_j) \\ \partial_t G_j \Psi_j + G_j (A_j \Psi_j) &= \tilde{A}_j (G_j \Psi_j). \end{aligned}$$

These then give two expressions relating the original Lax pair (L_j, A_j) with the new Lax pair $(\tilde{L}_j, \tilde{A}_j)$:

$$\begin{aligned} G_{j+1}L_j &= \tilde{L}_jG_j \\ \partial_t G_j &= \tilde{A}_jG_j - G_jA_j. \end{aligned} \quad (3.2)$$

Given a Darboux matrix that satisfies the above construction, and solving these two equations we can actually obtain the Bäcklund transformation that connects solutions between the two systems described by (L_j, A_j) and $(\tilde{L}_j, \tilde{A}_j)$. These can be different solutions of the same PDE (auto-BT) or solutions of different PDEs (hetero-BT), depending on the forms of the two Lax pairs.

3.1.1 The Toda Chain

To illustrate how the BT works we now focus on our typical discrete example of the Toda chain, and derive solutions of the equations of motion via the BT relations. We recall the Lax pair for the model

$$L_j = \begin{pmatrix} \lambda - p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \quad A_j = \begin{pmatrix} \lambda & e^{q_j} \\ -e^{-q_{j-1}} & 0 \end{pmatrix} \quad (3.3)$$

and we chose to consider the following Darboux matrix

$$G_j = \begin{pmatrix} \lambda + \mathbb{N}_j & X_j \\ Y_j & 1 \end{pmatrix}, \quad (3.4)$$

where $\mathbb{N}_j = \Theta + X_jY_j$.

The discrete spatial part of the BT relations produces

$$\begin{aligned} X_j &= e^{q_j}, & Z_{j+1} &= -e^{-\tilde{q}_j} \\ \mathbb{N}_{j+1}q^{q_j} &= -\tilde{p}_jX_j + e^{\tilde{q}_j}, & e^{-\tilde{q}_j}\mathbb{N}_j &= e^{-q_j} + Z_{j+1}p_j, \end{aligned} \quad (3.5)$$

plus two extra compatibility conditions emerging from the diagonal entries, which are omitted for brevity. The t -part of the BT relations provides

$$\dot{X}_j = e^{\tilde{q}_j} - \mathbb{N}_j e^{q_j}, \quad \dot{Z}_j = -e^{\tilde{q}_{j-1}}\mathbb{N}_j + e^{-q_{j-1}}. \quad (3.6)$$

Gathering the information provided by the x and t parts of the BT above we conclude

$$\begin{aligned} \dot{q}_j &= p_j = e^{\tilde{q}_j - q_j} + e^{q_{j+1} - \tilde{q}_j} - \Theta \\ \dot{\tilde{q}}_j &= \tilde{p}_j = e^{\tilde{q}_j - q_j} + e^{q_j - \tilde{q}_{j-1}} - \Theta, \end{aligned} \quad (3.7)$$

and consequently one can show that q_j, \tilde{q}_j satisfy the Toda equations of motion.

We focus our attention on deriving solutions of the Toda equations. To achieve this we consider a trivial solution of the equations, i.e. let $\tilde{p}_j = \tilde{q}_j = 0$ (or equivalently one may consider $q_j = p_j = 0$), then equations (3.7) become

$$0 = e^{-q_j} + e^{q_{j+1}} - \Theta \quad (3.8)$$

$$p_j = e^{-q_j} + e^{q_j} - \Theta. \quad (3.9)$$

We can now readily obtain the stationary solution i.e. consider only the time independent equation (3.8). It is convenient to parametrize as follows (see also [8] and references therein),

$$\Theta = 2\cosh(\xi), \quad e^{q_0} = \frac{\cosh(\xi + \eta)}{\cosh(\xi)}, \quad (3.10)$$

then the solution of the difference equation (3.8) is given as

$$e^{q_j} = \frac{\cosh(\xi + \eta(j+1))}{\cosh(\xi + \eta j)}. \quad (3.11)$$

The second equation (3.9) yields the time dependence of the solution. Compare the solution above with the traveling wave solution introduced earlier in the text.

3.2 Continuous Darboux-BT

In the continuous case, the process is much the same, except that we start from the continuous auxiliary linear problem. We introduce a Darboux matrix G analogously to the discrete case, i.e. $\tilde{\Psi}(x) = G(x)\Psi(x)$, where $\tilde{\Psi}$ is the transformed auxiliary function. As in the semi-discrete case, this allows us to find a pair of continuous relations between the Darboux matrix and the original and “transformed” Lax pairs, (U, V) and (\tilde{U}, \tilde{V}) .

Indeed, after differentiating the transformed auxiliary problem we obtain the fundamental BT relations for the continuous integrable system:

$$\begin{aligned} \partial_x G(\lambda, x, t) &= \tilde{U}(\lambda, x, t)G(\lambda, x, t) - G(\lambda, x, t)U(\lambda, x, t) \\ \partial_t G(\lambda, x, t) &= \tilde{V}(\lambda, x, t)G(\lambda, x, t) - G(\lambda, x, t)V(\lambda, x, t). \end{aligned} \quad (3.12)$$

Solving the pair of relations above allows the derivation of the Darboux matrix and also provides relations between the two different solutions of the associated integrable PDEs. This will become transparent in the various examples discussed below.

3.2.1 The sinh-Gordon Model: auto-BT

A significant physical example is considered in this subsection, i.e. the sinh-Gordon model. This is a relativistic model as opposed to the NLS model, which is associated to a typical reaction-diffusion Hamiltonian. The sinh-Gordon Lax pair reads as (see

e.g. [3]):

$$\begin{aligned} U(\lambda, x, t) &= \frac{1}{2} \begin{pmatrix} -\partial_t w & \sinh(\lambda + w) \\ \sinh(\lambda - w) & \partial_t w \end{pmatrix}, \\ V(\lambda, x, t) &= \frac{1}{2} \begin{pmatrix} -\partial_x w & \cosh(\lambda + w) \\ \cosh(\lambda - w) & \partial_x w \end{pmatrix} \end{aligned} \quad (3.13)$$

where we define $w = \frac{\beta}{2}\phi$, β is the coupling constant of the model, and the sinh-Gordon field ϕ naturally depends on x , t . The zero curvature condition for the Lax pair above provides the sinh-Gordon equation

$$\partial_t^2 \phi - \partial_x^2 \phi = \beta^{-1} \sinh(\beta \phi). \quad (3.14)$$

Note that the sine-Gordon model emerges by simply setting $\phi \rightarrow i\phi$.

We will be using the type I Darboux matrix, given as (there is also a type II matrix, which can be written as the product of two type I matrices)

$$G = \begin{pmatrix} Z & e^{\lambda - \Theta} X^{-1} \\ e^{\lambda - \Theta} X & Z^{-1} \end{pmatrix}. \quad (3.15)$$

Inserting this into the space equation² (3.12) we can immediately read off expressions for X , Z :

$$X = e^{-\frac{1}{2}(w + \tilde{w})}, \quad Z = e^{\frac{1}{2}(w - \tilde{w})} \quad (3.16)$$

and we also obtain four more equations

$$\begin{aligned} 2\partial_x Z &= Z(\partial_t w - \partial_t \tilde{w}) + \frac{e^{-\Theta}}{2}(X^{-1}e^w - e^{-\tilde{w}}X) \\ 2\partial_x X &= \frac{e^{\Theta}}{2}(Ze^{-\tilde{w}} - Z^{-1}e^{-w}) + X\partial_t(w + \tilde{w}) \\ 2\partial_x X^{-1} &= \frac{e^{\Theta}}{2}(Z^{-1}e^{\tilde{w}} - Ze^w) - X^{-1}\partial_t(w - \tilde{w}) \\ 2\partial_x Z^{-1} &= Z^{-1}\partial_t(w - \tilde{w}) - \frac{e^{-\Theta}}{2}e^{\tilde{w}}X^{-1} + \frac{1}{4}Xe^{-w}. \end{aligned} \quad (3.17)$$

Combining now the fundamental sets of relations above we conclude

$$\begin{aligned} \partial_x(w + \tilde{w}) &= -\partial_t(w + \tilde{w}) - \alpha \sinh(w - \tilde{w}) \\ \partial_x(w - \tilde{w}) &= \partial_t(w - \tilde{w}) + \alpha^{-1} \sinh(w + \tilde{w}), \end{aligned} \quad (3.18)$$

where $\alpha = e^{\Theta}$. The equations above provide precisely the Bäcklund transformation of the sinh-Gordon model. It is convenient to express the BT in light cone coordinates: $z = x + t$, $\bar{z} = x - t$,

$$\begin{aligned} \partial_z(w + \tilde{w}) &= \alpha \sinh(\tilde{w} - w) \\ \partial_{\bar{z}}(w - \tilde{w}) &= \alpha^{-1} \sinh(w + \tilde{w}). \end{aligned} \quad (3.19)$$

²Due to the fact that the sinh-Gordon is a relativistic model it suffices to only consider the space (or time) part of (3.12) in order to obtain the BT.

3.2.2 The Liouville Model: hetero-BT

We have seen so far examples of BTs that connect different solutions of the same integrable PDEs. However, one can consider BTs that connect solutions of different integrable PDEs (hetero-BTs). We focus on a particular example of hetero-BT, and we find solutions of the Liouville equations via solutions of the massless free particle equation. Indeed, consider the following Lax pair associated to the Liouville equation

$$\tilde{U}(\lambda) = \frac{1}{2} \begin{pmatrix} -\tilde{\pi} & -2ce^{-\lambda+\tilde{\phi}} \\ -2ce^{\lambda+\tilde{\phi}} & \tilde{\pi} \end{pmatrix}, \quad \tilde{V}(\lambda) = \frac{1}{2} \begin{pmatrix} -\partial_x \tilde{\phi} & 2ce^{-\lambda+\tilde{\phi}} \\ 2ce^{\lambda+\tilde{\phi}} & \partial_x \tilde{\phi} \end{pmatrix}. \quad (3.26)$$

The equations of motion emerging from the zero curvature condition read as ($\tilde{\pi} = \partial_t \tilde{\phi}$)

$$\partial_x^2 \tilde{\phi} - \partial_t^2 \tilde{\phi} - 4c^2 e^{2\tilde{\phi}} = 0. \quad (3.27)$$

The Lax pair for the free massless theory is very simple and is given as

$$U(\lambda) = -\frac{1}{2}\pi \mathbb{I}, \quad V(\lambda) = -\frac{1}{2}\partial_x \phi \mathbb{I}, \quad (3.28)$$

where \mathbb{I} is the 2×2 identity matrix and the corresponding equation of motion is ($\pi = \partial_t \phi$)

$$\partial_x^2 \phi - \partial_t^2 \phi = 0. \quad (3.29)$$

It is clear that the equation of motion remains invariant if we multiply U and V with the same constant matrix.

We choose to consider the following Darboux-matrix:

$$G(\lambda, \Theta) = \begin{pmatrix} A & X e^{-\lambda-\Theta} \\ Z e^{\lambda+\Theta} & B \end{pmatrix}, \quad (3.30)$$

where Θ is an extra free parameter (Bäcklund transformation parameter) and the elements A , B , X , Z are to be determined via (3.12). Indeed, setting:

$$A = X = e^{\frac{1}{2}(\tilde{\phi}-\phi)}, \quad Z = B = e^{\frac{1}{2}(\tilde{\phi}+\phi)}, \quad (3.31)$$

and using light cone coordinates $z = x + t$, $\bar{z} = x - t$ for convenience, we solve (3.12) and obtain:

$$\begin{aligned} \partial_z(\tilde{\phi} - \phi) &= -2ce^{\Theta} e^{(\tilde{\phi}+\phi)}, \\ \partial_{\bar{z}}(\tilde{\phi} + \phi) &= -2ce^{-\Theta} e^{(\tilde{\phi}-\phi)}, \end{aligned} \quad (3.32)$$

which is the celebrated hetero-Bäcklund transformation for the Liouville theory. It is easy to check via (3.32) that the fields $\tilde{\phi}$ and ϕ satisfy the correct equations of motion (3.27) and (3.29) respectively.

One can express solutions of the Liouville theory in terms of solutions of the linear problem associated to the massless free relativistic particle. Indeed, due to the form

of the free particle equation the solutions can be expressed as $\phi(z, \bar{z}) = f(z) + \bar{f}(\bar{z})$, then the BT relations become (we set for simplicity $\Theta = 0$)

$$\begin{aligned} e^{-(\tilde{\phi} + \bar{f} - f)} \partial_z (\tilde{\phi} - f) &= -2ce^{2f}, \\ e^{-(\tilde{\phi} + \bar{f} - f)} \partial_{\bar{z}} (\tilde{\phi} + \bar{f}) &= -2ce^{-2\bar{f}}, \end{aligned} \quad (3.33)$$

which lead to:

$$e^{-(\tilde{\phi} + \bar{f} - f)} = 2c \int^z e^{2f(\xi)} d\xi + 2c \int^{\bar{z}} e^{2\bar{f}(\xi)} d\xi. \quad (3.34)$$

Let $F(z) = \int^z e^{2f(\xi)} d\xi$ and $\bar{F}^{-1}(\bar{z}) = -\int^{\bar{z}} e^{-2\bar{f}(\xi)} d\xi$, then the solution of the Liouville equation can be expressed in terms of F, \bar{F} as

$$\tilde{\phi}(z, \bar{z}) = \frac{1}{2} \ln \left(\frac{\partial_z F \partial_{\bar{z}} \bar{F}}{(1 - F \bar{F})^2} \right). \quad (3.35)$$

This concludes our brief discussion on BTs associated to relativistic integrable models. Next we examine the BT and dressing for the AKNS system.

3.2.3 The AKNS system: BT & dressing

We focus in this subsection on the AKNS (NLS-type) system (see e.g. [31]–[35], [20] for the NLS model and generalizations), we derive the BT relations and also perform the dressing process in order to derive the members of the AKNS hierarchy. Recall that the BT transformation connects two solutions of the same (or distinct) PDEs and is identified provided the existence of given Lax pairs. In the dressing process, on the other hand the form of the so-called “bare” (free of fields) operators are given, and then via the Darboux-dressing process the Lax pairs for the associated hierarchy are identified.

Let us outline what we are going to achieve in this subsection. We first employ a given Lax pair (U, V) , e.g. the NLS one (3.36), and the Darboux matrix to identify the corresponding Bäcklund transformation. We then restrict our attention to the case of trivial solutions $u, \hat{u} = 0$, identify the one-soliton solution, and we also perform the “dressing” process. From this viewpoint the only input is the U -operator, together with the form of the bare operators, then the whole hierarchy of the V -operators can be identified via the dressing process. Note that in all the expressions below x and t dependence is implied, even when it is not explicitly stated.

For the NLS model the Lax pair is given as

$$U(\lambda, x, t) = \begin{pmatrix} \frac{\lambda}{2} & \hat{u} \\ u & -\frac{\lambda}{2} \end{pmatrix}, \quad V(\lambda, x, t) = \begin{pmatrix} \frac{\lambda^2}{2} - \hat{u}u & \lambda\hat{u} + \partial_x \hat{u} \\ \lambda u - \partial_x u & -\frac{\lambda^2}{2} + u\hat{u} \end{pmatrix}, \quad (3.36)$$

where the fields \hat{u}, u depend on x, t . The equations of motion from the zero curvature condition lead to the NLS-type equation:

$$\partial_t u + \partial_x^2 u - 2\hat{u}u^2 = 0. \quad (3.37)$$

Similarly, for \hat{u} ($t \rightarrow -t$). The Darboux matrix is chosen to be of the form

$$G(\lambda, x, t) = \begin{pmatrix} \lambda + \mathcal{A}(x, t) & \mathcal{B}(x, t) \\ \mathcal{C}(x, t) & \lambda + \mathcal{D}(x, t) \end{pmatrix} = \lambda \mathbb{I} + \mathcal{K}. \quad (3.38)$$

This is the fundamental Darboux matrix, we shall see later in the text that the generic Darboux is expressed in a λ -power series. The BT connects two different solutions of the underlying integrable PDEs, i.e the pair U, V are associated to the fields u, \hat{u} , whereas U_0, V_0 are associated to the fields u_0, \hat{u}_0 .

The Darboux transform G is now applied on the auxiliary function Ψ_0

$$\Psi(\lambda, x, t) = G(\lambda, x, t) \Psi_0(\lambda, x, t), \quad (3.39)$$

yielding (see also (4.2)) the two primary Darboux-BT equations

$$\partial_x G = U G - G U_0, \quad \partial_t G = V G - G V_0. \quad (3.40)$$

The x -part of (3.40) leads to the Darboux-BT relations ($\mathcal{D} = -\mathcal{A}$):

$$\begin{aligned} \mathcal{B} &= -(\hat{u} - \hat{u}_0), & \mathcal{C} &= u - u_0, \\ \partial_x \mathcal{A} &= \hat{u} \mathcal{C} - \mathcal{B} u_0, & \partial_x \mathcal{D} &= u \mathcal{B} - \mathcal{C} \hat{u}_0, \\ \partial_x \mathcal{B} &= \hat{u} \mathcal{D} - \mathcal{A} \hat{u}_0, & \partial_x \mathcal{C} &= u \mathcal{A} - \mathcal{D} u_0. \end{aligned} \quad (3.41)$$

For the dressing process described next the trivial solution $u_0 = \hat{u}_0 = 0$ is employed. The t -part of the BT also gives the following:

$$\begin{aligned} \partial_t \mathcal{B} &= \partial_x (\hat{u}_0 + \hat{u}) \mathcal{A} - (u \hat{u} + u_0 \hat{u}_0) \mathcal{B} \\ \partial_t \mathcal{C} &= \partial_x (u + u_0) \mathcal{A} + (u \hat{u} + u_0 \hat{u}_0) \mathcal{B}. \end{aligned} \quad (3.42)$$

We also require that the $\det \mathcal{K}$ is a constant, which yields

$$\mathcal{A} = \sqrt{k^2 + (u - u_0)(\hat{u} - \hat{u}_0)}. \quad (3.43)$$

Gathering all the relations above we end up with the BT for the NLS-type system

$$\begin{aligned} \partial_x (u - u_0) &= (u + u_0) \mathcal{A} \\ \partial_t (u - u_0) &= \partial_x (u + u_0) \mathcal{A} + \mathcal{X} (u - u_0), \end{aligned} \quad (3.44)$$

where we define

$$\mathcal{X} = \frac{1}{2} \left((u - u_0)(\hat{u} - \hat{u}_0) + (u + u_0)(\hat{u} + \hat{u}_0) \right) \quad (3.45)$$

and similarly for \hat{u}, \hat{u}_0

$$\begin{aligned} \partial_x (\hat{u} - \hat{u}_0) &= (\hat{u} + \hat{u}_0) \mathcal{A} \\ \partial_t (\hat{u} - \hat{u}_0) &= -\partial_x (\hat{u} + \hat{u}_0) \mathcal{A} - \mathcal{X} (\hat{u} - \hat{u}_0). \end{aligned} \quad (3.46)$$

From the equations above one may show that both sets u, \hat{u} and u_0, \hat{u}_0 satisfy the NLS-type equations (we leave the explicit proof as an exercise to the interested

reader). Moreover, by considering the trivial solutions $u_0 = \hat{u}_0 = 0$ we can identify the soliton-type solution. Indeed, for $u_0 = \hat{u}_0 = 0$ recall the fundamental relations for \mathcal{A} (3.41), (3.43), which lead to a simple differential equation

$$\partial_x \mathcal{A} = -k^2 + \mathcal{A}^2 \Rightarrow \mathcal{A} = \frac{k(1 + e^{2kx+f(t)})}{1 - e^{2kx+f(t)}}. \quad (3.47)$$

Then from these equations one can easily derive the solution for the fields u , \hat{u} (this too is left as an exercise for the curious reader).

We keep focusing on the case where $u_0 = \hat{u}_0 = 0$ and via the dressing process we are going to identify the “dressed” quantities $V^{(n)}$ of the hierarchy, assuming knowledge of the form of the U -operator as well as the bare operators. Let U_0 , $V_0^{(n)}$ be the “bare” Lax pairs:

$$U_0(\lambda) = \frac{\lambda}{2}\sigma, \quad V_0^{(n)}(\lambda) = \frac{\lambda^n}{2}\sigma, \quad (3.48)$$

$\sigma = \text{diag}(1, -1)$. Also, the “dressed” time components of the Lax pairs can be expressed as formal series expansions

$$V^{(n)}(\lambda, x, t) = \frac{\lambda^n}{2}\sigma + \sum_{k=0}^{n-1} \lambda^k w_k^{(n)}(x, t), \quad (3.49)$$

where the quantities $w_k^{(n)}$ will be identified via the dressing transform.

By solving the x -part of equations (3.40) we obtain

$$\hat{u} = -\mathcal{B}, \quad u = \mathcal{C} \quad \text{and} \quad \partial_x \mathcal{K} = \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} \mathcal{K}. \quad (3.50)$$

From the time part of (3.40) we obtain a set of recursion relations, in exact analogy to the case of dressing via a differential operator, i.e.

$$\begin{aligned} w_{n-1}^{(n)} &= \frac{1}{2}[\mathcal{K}, \sigma], \\ w_{k-1}^{(n)} &= -w_k^{(n)} \mathcal{K}, \quad k \in \{1, 2, \dots, n-1\} \\ \partial_{t_n} \mathcal{K} &= w_0^{(n)} \mathcal{K}. \end{aligned} \quad (3.51)$$

We solve the latter recursion relations, and identify the first few time components of the Lax pairs:

$$\begin{aligned} V^{(0)} &= \frac{1}{2}\sigma, \\ V^{(1)} &= \lambda V^{(0)} + \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix}, \\ V^{(2)} &= \lambda V^{(1)} + \begin{pmatrix} -\hat{u}u & \partial_x \hat{u} \\ -\partial_x u & u\hat{u} \end{pmatrix} \\ V^{(3)} &= \lambda V^{(2)} + \begin{pmatrix} \hat{u} \partial_x u - \partial_x \hat{u} u & -2\hat{u}u\hat{u} + \partial_x^2 \hat{u} \\ -2u\hat{u}u + \partial_x^2 u & u \partial_x \hat{u} - \partial_x u \hat{u} \end{pmatrix}. \\ \dots & \end{aligned} \quad (3.52)$$

In general, the $V^{(n)}$ operator is identified as $V^{(n)} = \lambda V^{(n-1)} + w_0^{(n)}$. Moreover, the recursion relations (3.51) lead to

$$w_k^{(n)} = w_{k-1}^{(n-1)}, \quad w_0^{(n)} = (-1)^{n-1} w_0^{(1)} \mathcal{K}^{n-1}, \quad (3.53)$$

where the latter relations together with the constraints (3.50) suffice to provide $w_0^{(n)}$ at each order³.

We have thus far considered the simplest fundamental Darboux matrix, but nevertheless we were able to fully describe the dressing process. Let us now briefly discuss the generic Darboux expressed as a formal λ -series expansion

$$G(\lambda) = \lambda^m \mathbb{I} + \sum_{k=0}^{m-1} \lambda^k g_k, \quad (3.54)$$

where g_k are 2×2 matrices to be identified. We focus on the fundamental recursion relations arising from the x -part of the Darboux transform (3.40):

$$\begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} = \frac{1}{2} [g_{m-1}, \sigma], \quad (3.55)$$

$$\partial_x g_0 = \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} g_0, \quad \partial_x g_k = \frac{1}{2} [\sigma, g_{k-1}] + \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} g_k. \quad (3.56)$$

The infinite series expansion $m \rightarrow \infty$ is particularly interesting, especially via its connection to a Riccati equation of the type (2.50) (see also for instance [35]).

4 Differential operators

We have so far studied the Darboux-BT transform having chosen Lax pairs as well as Darboux transformations to be $d \times d$ matrices. We now consider Lax pairs being differential operators and the Darboux transform expressed in terms of differential or integral operators.

Let us first recall the main ideas of the generalized Darboux-dressing scheme [12]. Let

$$\mathbb{D}^{(n)} = \mathbb{I} \partial_{t_n} + A^{(n)}, \quad \mathbb{L}^{(n)} = \mathbb{I} \partial_{t_n} + \mathbb{A}^{(n)}, \quad (4.1)$$

where \mathbb{I} is the $d \times d$ unit matrix and $A^{(n)}$, $\mathbb{A}^{(n)}$ are some ‘‘bare’’ (free of fields) and ‘‘dressed’’ operators respectively. In general, they can be $d \times d$ matrices, matrix-differential or matrix-integral operators, and are related via the generic Darboux-dressing transformation

$$\mathcal{G} \mathbb{D}^{(n)} = \mathbb{L}^{(n)} \mathcal{G} \Rightarrow \partial_{t_n} \mathcal{G} = \mathcal{G} A^{(n)} - \mathbb{A}^{(n)} \mathcal{G}. \quad (4.2)$$

³Note that in the special case $\hat{u} = u$ the Lax pair $(U, V^{(3)})$ is the one of the mKdV model, see also [4, 36].

Provided that $[\mathbb{D}^{(n)}, \mathbb{D}^{(m)}] = 0$, the transformation (4.2) leads to the generalized Zakharov-Shabat zero curvature relations, which define the integrable hierarchy

$$\partial_{t_n} \mathbb{A}^{(m)} - \partial_{t_m} \mathbb{A}^{(n)} + [\mathbb{A}^{(n)}, \mathbb{A}^{(m)}] = 0. \quad (4.3)$$

In our setting here we choose to consider $A^{(n)}$, $\mathbb{A}^{(n)}$ as differential operators, whereas the dressing transform \mathcal{G} is chosen to be either a differential or an integral one in the next subsection. We employ our typical example i.e. the KdV equation, as our main paradigm to illustrate the general dressing process.

4.1 Differential transforms: the KdV equation

Before we describe the dressing process as a method of producing solutions as well as constructing the hierarchy we shall first derive the BT for the KdV equation using as the fundamental Darboux-BT transformation the first order differential operator (see also for instance [21, 25]):

$$\mathcal{G} = \partial_x + K. \quad (4.4)$$

Recall that the Lax pair for the KdV equation is given as

$$\mathbb{L}^{(2)} = -\partial_x^2 + u(x), \quad \mathbb{L}^{(3)} = \partial_t - \alpha \partial_x^3 + a_1(x) \partial_x + a_0(x), \quad (4.5)$$

where we define $a_1(x) = \frac{3}{2}\alpha u(x)$, $a_0(x) = \frac{3}{4}\alpha \partial_x u(x)$. The BT relations follow after implementing the transformation:

$$\mathcal{G} (-\partial_x^2 + \hat{u}) = (-\partial_x^2 + u) \mathcal{G} \quad (4.6)$$

$$\mathcal{G} (\partial_t - \alpha \partial_x^3 + \hat{a}_1 \partial_x + \hat{a}_0) = (\partial_t - \alpha \partial_x^3 + a_1 \partial_x + a_0) \mathcal{G}. \quad (4.7)$$

The solution of the latter equation leads to the BT relations for the KdV equation. Indeed, from (4.6) we immediately obtain:

$$\begin{aligned} u - \hat{u} &= 2\partial_x K \\ \partial_x^2 K &= (u - \hat{u})K - \partial_x \hat{u}. \end{aligned} \quad (4.8)$$

It is convenient to introduce the fields w , \hat{w} : $u = \partial_x w$, $\hat{u} = \partial_x \hat{w}$, then (4.8) become

$$\partial_x (w + \hat{w}) = \frac{1}{2}(w - \hat{w})^2 - \frac{\Lambda^2}{2}, \quad (4.9)$$

where $\frac{\Lambda^2}{2}$ is some integration constant. The t -part of the BT (4.7) yields

$$\partial_t K - \alpha \partial_x^2 K + a_1 \partial_x K + (a_0 - \hat{a}_0)K - \partial_x \hat{a}_0 = 0. \quad (4.10)$$

Taking also into account (4.9) we may re-express (4.10) as (set $\alpha = -4$)

$$\partial_t (w - \hat{w}) = 3((\partial_x w)^2 - (\partial_x \hat{w})^2) - \partial_x^3 (w - \hat{w}). \quad (4.11)$$

Equations (4.9), (4.11) provide the Bäcklund transformation for the KdV equation (see e.g. [19] and references therein).

Let us now focus on the special case where $\hat{w} = 0$, and derive the one-soliton solution for the KdV equation. Eq. (4.9) then becomes

$$\partial_x w = \frac{w^2}{2} - \frac{\Lambda^2}{2}, \quad (4.12)$$

which can be easily solved, with the solutions being

$$w(x) = \Lambda \frac{1 + e^{\Lambda(x+f(t))}}{1 - e^{\Lambda(x+f(t))}}. \quad (4.13)$$

Recall $u(x) = \partial_x w(x)$, which yields the one-soliton solution of the KdV equation as can be easily verified.

The time dependence of the solution above can be determined via the t -part of the BT (4.11) ($\hat{w} = 0$):

$$\partial_t w = 3(\partial_x w)^2 - \partial_x^3 w. \quad (4.14)$$

But w satisfies (4.12), which suggests

$$\partial_x^3 w = \partial_x (w \partial_x w) = (\partial_x w)^2 + w^2 \partial_x w. \quad (4.15)$$

Equations (4.14), (4.15) and (4.12) then yield

$$\partial_t w = -\Lambda^2 \partial_x w, \quad (4.16)$$

and hence the function $f(t)$ appearing in (4.13) becomes $f(t) = -\Lambda^2 t$.

We can also construct the two soliton solution via the soliton lattice construction and the use of the permutability theorem as we did for the sinh-Gordon model. Indeed, consider (3.21) for the KdV BT:

$$\begin{aligned} \partial_x (w_i + w_0) &= \frac{1}{2} (w_i - w_0)^2 - \frac{\Lambda_i^2}{2} \\ \partial_x (w_{ij} + w_i) &= \frac{1}{2} (w_{ij} - w_i)^2 - \frac{\Lambda_j^2}{2}, \quad i, j \in \{1, 2\}. \end{aligned} \quad (4.17)$$

We now consider the special case where $w_0 = 0$, and derive the two-soliton solution. The permutability requirement $w_{12} = w_{21} = w$ holds, and hence

$$\begin{aligned} \frac{w_2^2}{2} - \frac{w_1^2}{2} - \frac{\Lambda_2^2}{2} + \frac{\Lambda_1^2}{2} &= \frac{1}{2} (w - w_2)^2 - \frac{1}{2} (w - w_1)^2 + \frac{\Lambda_2^2}{2} - \frac{\Lambda_1^2}{2} \\ \Rightarrow w &= \frac{\Lambda_1^2 - \Lambda_2^2}{w_1 - w_2}, \end{aligned} \quad (4.18)$$

where $w_{1,2}$ are one soliton solutions (4.13) with constants Λ_1, Λ_2 respectively. By taking the derivative of the latter expression with respect to x we obtain the 2-soliton solution for KdV.

Dressing

We briefly describe in what follows the dressing method for the KdV equation utilizing the fundamental Darboux transform introduced above. We introduce the “bare” differential operators:

$$\mathbb{D}^{(2)} = -\partial_x^2, \quad \mathbb{D}^{(3)} = \partial_t - \alpha \partial_x^3. \quad (4.19)$$

and the “dressed” operators $\mathbb{L}^{(n)}$ are given by (4.5), where now u , a_0 and a_1 are to be determined via the dressing process. Indeed, recall the fundamental Darboux-dressing transformation $\mathcal{G} = \partial_x + K$:

$$(\partial_x + K)\mathbb{D}^{(n)} = \mathbb{L}^{(n)}(\partial_x + K), \quad n \in \{2, 3\}. \quad (4.20)$$

This leads to the following set of constraints from equation (4.19) for $n = 2$,

$$\begin{aligned} u &= 2\partial_x K \\ \partial_x^2 K &= uK, \end{aligned} \quad (4.21)$$

which provide a special case of (4.8) ($\hat{u} = 0$), and yield the 1-soliton solution as already discussed.

The t -part of the dressing (4.19) for $n = 3$, leads to

$$\partial_t K(x) - \alpha \partial_x^3 K(x) + a_1(x) \partial_x K(x) + a_0(x) K(x) = 0, \quad (4.22)$$

as well as to contributions proportional to ∂_x^m , $m \in \{1, 2\}$, which essentially determine each one of the factors a_k of the $\mathbb{L}^{(3)}$ operator in terms of the field u

$$\begin{aligned} a_1(x) &= \frac{3\alpha}{2} u(x) \\ a_0(x) &= \frac{3\alpha}{4} \partial_x u(x), \end{aligned} \quad (4.23)$$

where $u(x)$ is now given in (4.21).

Although we have focused on the fundamental transform to describe the dressing we should note that the generic form of the Darboux can be expressed as an M^{th} order differential operator

$$\mathcal{G} = \partial_x^M + \sum_{k=0}^{M-1} g_k(x) \partial_x^k, \quad (4.24)$$

where the functions g_k are to be derived via a set of constraints emerging from the dressing of the Lax pairs (see also e.g. [35]). This is in analogy to the matrix case discussed in the previous section, where the general Darboux is expanded in a formal λ -power series.

4.2 Integral transforms: the KdV equation

Let us introduce the necessary notation in the case where we choose the Darboux transform \mathcal{G} to be an integral operator [12, 13], and derive the Gelfand-Levitan-Marchenko (GLM) equation from the fundamental factorization condition.

Let $\mathcal{G} = \mathbb{I} + \mathcal{K}^+$, and introduce the operators \mathcal{K}^\pm , \mathcal{F} with integral representations (f is the test function):

$$\begin{aligned}\mathcal{F}(f)(x) &= \int_{\mathbb{R}} F(x, y) f(y) dy, \\ \mathcal{K}^\pm(f)(x) &= \int_{\mathbb{R}} K^\pm(x, y) f(y) dy,\end{aligned}\tag{4.25}$$

such that: $K^+(x, y) = 0$, $x > y$, and $K^-(x, y) = 0$, $x < y$. The operators \mathcal{K}^\pm , \mathcal{F} are required to satisfy the factorization condition

$$(\mathbb{I} + \mathcal{K}^+) (\mathbb{I} + \mathcal{F}) = \mathbb{I} + \mathcal{K}^-, \tag{4.26}$$

which leads to the fact that the kernel $K^+(x, y)$ satisfies the Gelfand-Levitan-Marchenko equation and $K^-(x, y)$ obeys an analogous integral equation:

$$\begin{aligned}K^+(x, z) + F(x, z) + \int_x^\infty dy K^+(x, y) F(y, z) &= 0, \quad z > x, \\ K^-(x, z) = F(x, z) + \int_x^\infty dy K^+(x, y) F(y, z), \quad z < x.\end{aligned}\tag{4.27}$$

$F(x, y)$ is the solution of the linear problem, i.e. invariance of the differential operators $\mathbb{D}^{(n)}$ under the action of the operator \mathcal{F} is required:

$$\mathcal{F} \mathbb{D}^{(n)} = \mathbb{D}^{(n)} \mathcal{F}, \quad n \in \{2, 3\}.\tag{4.28}$$

Dependence on time t is implied, but omitted for now for brevity.

4.2.1 Dressing

We keep considering our main example i.e. the KdV equation in order to illustrate the dressing method using an integral operator as the Darboux-dressing transform. The main aim is to identify the functions $u(x)$, $a_1(x)$, $a_0(x)$ appearing in the Lax pair (4.5) via the dressing process as described below (see also [12, 19]).

We first perform the dressing for the time independent part of the Lax pair (i.e.

$\mathbb{D}^{(2)} \rightarrow \mathbb{L}^{(2)}$, (4.5), (4.19):

$$\begin{aligned}
& \int_x^\infty dy K(x, y) (-\partial_y^2) f(y) = \\
& u(x)f(x) - \partial_x^2 \int_x^\infty dy K(x, y) f(y) + \int_x^\infty dy u(x) K(x, y) f(y) \Rightarrow \\
& K(x, x) \partial_x f(x) - \partial_y K(x, y)|_{x=y} f(x) - \int_x^\infty dy \partial_y^2 K(x, y) f(y) = u(x)f(x) + \\
& \partial_x (K(x, x) f(x)) + \partial_x K(x, y)|_{x=y} f(x) + \int_x^\infty dy (-\partial_x^2 + u(x)) K(x, y) f(y).
\end{aligned} \tag{4.29}$$

From the expression above we immediately extract

$$u(x) = -2(\partial_x K(x, y) + \partial_y K(x, y))|_{x=y} \tag{4.30}$$

$$-\partial_x^2 K(x, y) + \partial_y^2 K(x, y) + u(x) K(x, y) = 0. \tag{4.31}$$

We next consider the dressing for the t -part of the Lax pair for generic n^{th} order bare and dressed differential operators ($n \geq 3$) of the form:

$$\mathbb{D}^{(n)} = \partial_{t_n} - \partial_x^n, \quad \mathbb{L}^{(n)} = \partial_{t_n} - \partial_x^n + \sum_{k=1}^{n-1} a_k(x) \partial_x^k. \tag{4.32}$$

Then the basic dressing relation (4.2) is

$$\begin{aligned}
\int_x^\infty dy K(x, y) (\partial_{t_n} - \alpha \partial_y^n) f(y) &= \mathbb{X}_x f(x) + (\partial_{t_n} - \partial_x^n) \int_x^\infty dy K(x, y) f(y) \\
&+ \int_x^\infty dy \mathbb{X}_x K(x, y) f(y),
\end{aligned} \tag{4.33}$$

where we define $\mathbb{X}_x = \sum_{k=0}^{n-1} a_k(x) \partial_x^k$. After repeated integrations by parts, and carefully taking into consideration the boundary terms by iteration, we conclude

$$\partial_{t_n} K(x, y) - \partial_x^n K(x, y) + (-1)^n \partial_y^n K(x, y) + \sum_{k=0}^{n-1} a_k(x) \partial_x^k K(x, y) = 0, \tag{4.34}$$

whereas the use of the boundary terms for each order $\partial_x^m f(x)$ provides the elements a_k . In the special case $n = 3$, the solution of the above constraints leads as expected to: $a_2 = 0$, $a_1 = \frac{3}{2}u$, $a_0 = \frac{3}{4}\partial_x u$, i.e. we recover the KdV Lax pair. The keen reader is encouraged to work out the dressing (4.33), and obtain the generalized relation (4.34) as well as the corresponding boundary contributions (see also e.g. [19, 35]). Note that we have assumed here $\alpha = 1$, which corresponds to a simple re-scaling of the time variable.

4.2.2 Solving the Gelfand-Levitan-Marchenko equation

In this section we are solving the Volterra-type integral GLM equation for given solutions of the linear problem (4.28)

$$K(x, z) + F(x+z) + \int_x^\infty K(x, y) F(y+z) dy = 0. \tag{4.35}$$

We distinguish in what follows two cases corresponding to the “discrete” and “continuous” solutions of the linear problem.

Let us first consider “discrete” solutions of the linear problem (4.28) of the generic form

$$F(x, z, t) = \sum_{n=1}^N b_n e^{-\kappa_n(x+z) + \Lambda_n t}, \quad (4.36)$$

where the general dispersion relation immediately follows from (4.28): $\Lambda_n = -2\alpha\kappa_n^3$. This type of solution for the linear problem will yield the N -soliton solution of the KdV equation. Given the form of the solution of the linear problem (4.36) and the GLM equation we can express the kernel K as

$$K(x, z, t) = \sum_{n=1}^N L_n(x, t) e^{-\kappa_n x}, \quad (4.37)$$

where $L_n(x, t)$ are to be explicitly determined. Given that the kernel K satisfies the GLM equation we obtain

$$\begin{aligned} 0 &= L_n(x) + b_n e^{-\kappa_n x} + b_n \sum_{m=1}^N L_m(x) \int_x^\infty e^{-(\kappa_n + \kappa_m)y} dy \\ &= L_n(x) + b_n e^{-\kappa_n x} + \sum_{m=1}^N \frac{b_n}{\kappa_n + \kappa_m} L_m(x) e^{-(\kappa_n + \kappa_m)x}. \end{aligned} \quad (4.38)$$

The latter equation can be expressed in a compact form as

$$AL = -B \Rightarrow L = -A^{-1}B, \quad (4.39)$$

where L and B are column N -vectors with components L_n and $b_n e^{-\kappa_n x}$ respectively, while A is an $N \times N$ matrix with entries:

$$A_{nm} = \delta_{nm} + \frac{b_n}{\kappa_n + \kappa_m} e^{-(\kappa_n + \kappa_m)x + \Lambda_n t}. \quad (4.40)$$

An interesting observation is in order here. It is easy to check that

$$\partial_x A_{nm} = E_n B_m, \quad (4.41)$$

where $E_n = e^{-\kappa_n x}$. Also, recall that the kernel is expressed as (using also (4.41))

$$K(x, x, t) = \sum_{n=1}^N L_n E_n = \sum_{n,m} A_{nm}^{-1} B_m E_n = \sum_{n,m} A_{nm}^{-1} \partial_x A_{nm}, \quad (4.42)$$

which leads to

$$K(x, x, t) = \text{tr}(A^{-1} \partial_x A) = |A|^{-1} \partial_x |A| = \partial_x (\ln(|A|)). \quad (4.43)$$

We have however derived the field u via dressing in the previous subsection (4.30):

$$u(x) = -2\partial_x^2 \ln(|A|), \quad (4.44)$$

and from the latter expression we readily derive the one soliton solution for the KdV equation

$$u(x) = -4\kappa^2 \operatorname{sech}^2(\kappa x + \Lambda t + x_0), \quad (4.45)$$

where recall $\Lambda = -2\alpha\kappa^3$.

Let us now consider general solutions of the linear problem (4.28) expressed as continuous Fourier transforms

$$F(x, z, t) = \int_{\mathbb{R}} dk b_k e^{\Lambda_k t + ik(x+z)}. \quad (4.46)$$

As in the case of discrete solutions the dispersion relation is cubic:

$$\Lambda_k = -2\alpha ik^3. \quad (4.47)$$

Taking into consideration the solutions of the linear problem as well as the GLM equation we can express the kernel as

$$K(x, z, t) = \int_{\mathbb{R}} dk \mathbb{L}(k, x, t) e^{ikz}. \quad (4.48)$$

Let us also define the quantities:

$$\mathbb{B}(k, x, t) = b_k e^{\Lambda_k t + ikx}, \quad \mathbb{P}(\hat{k}, k, x, t) = \frac{e^{\Lambda_k t + i(k+\hat{k})x}}{i(k+\hat{k})}. \quad (4.49)$$

Provided that the operator $\mathbb{I} - \mathbb{P}$ is invertible i.e. the Fredholm determinant is non-zero ($\det(\mathbb{I} - \mathbb{P}) \neq 0$), then \mathbb{L} is explicitly identified via the integral equation

$$\int d\hat{k} \mathbb{L}(\hat{k}, x, t) \left(\delta(\hat{k}, k) - \mathbb{P}(\hat{k}, k, x, t) \right) = -\mathbb{B}(k, x, t). \quad (4.50)$$

The latter relation provides the formal series expansion for \mathbb{L} i.e. the integral analogue of the matrix relation (4.38) presented in the discrete case previously:

$$\mathbb{L}(k) = -\mathbb{B}(k) - \sum_{m=1}^{\infty} \int_{\mathbb{R}} dk_1 \dots \int_{\mathbb{R}} dk_m \mathbb{B}(k_1) \mathbb{P}(k_1, k_2) \dots \mathbb{P}(k_m, k). \quad (4.51)$$

An interesting observation is in order here. Due to the cubic dispersion relation (4.47) the solutions of the linear problem can be expressed in terms of Airy functions $\operatorname{Ai}(x)$. Indeed, after expressing the coefficients b_k in terms of the initial values of the solutions of the linear problem F_0 at $t = 0$, via an inverse Fourier transform

$$b_k = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi F_0(\xi) e^{-ik\xi}, \quad (4.52)$$

and recalling the definition of the Airy function

$$\operatorname{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} dt \cos\left(\frac{t^3}{3} + xt\right), \quad (4.53)$$

we obtain

$$F(x, z) = \frac{1}{\nu} \int_{\mathbb{R}} d\xi \operatorname{Ai}\left(\frac{x+z-\xi}{\nu}\right) F_0(\xi), \quad (4.54)$$

where we define $\nu = 2(3t)^{\frac{1}{3}}$.

Distinct choices of the initial profile F_0 give rise to different generic solutions, but an obvious choice of initial conditions is for instance, $F_0(\xi) = \delta(\xi)$. Given that F satisfies the linear problem (4.28) ($j = 3$), it is straightforward to see that $\mathbb{F}(\zeta) = \nu F\left(\frac{x+z}{\nu}\right)$ satisfies as expected the Airy equation, (see also [13])

$$\frac{\partial^2 \mathbb{F}(\zeta)}{\partial \zeta^2} - \zeta \mathbb{F}(\zeta) = 0, \quad (4.55)$$

with the parameter ζ defined as $\frac{x+z}{\nu}$, and having also assumed that $\mathbb{F}(\zeta) \rightarrow 0$ when $\zeta \rightarrow \infty$. The kernel can be then expressed as a formal series expansion in terms of Airy functions.

Similar interesting observations can be made for a second order linear differential operator. Indeed, in this case the solutions of the associated linear problem can be expressed in terms of the heat kernel, then via the so-called *Cole-Hopf transformation* another important non-linear PDE is obtained, called the viscous Burgers equation (see for instance [35] and references therein).

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