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Citation for published version:
Rynne, BP 2019, 'Linearized stability implies dynamic stability for equilibria of 1-dimensional, p-Laplacian boundary value problems', *Proceedings of the Royal Society of Edinburgh, Section A: Mathematics*. https://doi.org/10.1017/prm.2018.123

Digital Object Identifier (DOI):
10.1017/prm.2018.123

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Proceedings of the Royal Society of Edinburgh, Section A: Mathematics

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Download date: 01. Jul. 2022

LINEARISED STABILITY IMPLIES DYNAMIC STABILITY FOR EQUILIBRIA OF 1-DIMENSIONAL, p-LAPLACIAN BOUNDARY VALUE PROBLEMS

BRYAN P. RYNNE

Abstract. We consider the parabolic, initial-boundary value problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta_p(v) + f(x,v), && \text{in } (-1,1)\times(0,\infty),\\ v(\pm 1,t) &= 0, && t\in[0,\infty), \\ v &= v_0 \in C^0_0([-1,1]), && \text{in } [-1,1]\times\{0\}, \end{aligned} \tag{1}$$

where Δ_p denotes the p-Laplacian on $(-1,1)$, with $p>1$, and the function $f:[-1,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, and the partial derivative f_v exists and is continuous and bounded on $[-1,1]\times\mathbb{R}$. It will be shown that (under certain additional hypotheses) the 'principle of linearised stability' holds for equilibrium solutions u_0 of (1). That is, the asymptotic stability, or instability, of u_0 is determined by the sign of the principal eigenvalue of a suitable linearisation of the problem (1) at u_0. It is well-known that this principle holds for the semilinear case $p=2$ (Δ_2 is the linear Laplacian), but has not been shown to hold when $p\neq 2$.

We also consider a bifurcation type problem similar to (1), having a line of trivial solutions. We characterise the stability or instability of the trivial solutions, and the bifurcating, non-trivial solutions, and show that there is an 'exchange of stability' at the bifurcation point, analogous to the well-known result when $p=2$.

1. Introduction

We consider the parabolic, initial-boundary value problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta_p(v) + f(x,v), && \text{in } \Omega\times(0,\infty),\\ v(\pm 1,t) &= 0, && t\in[0,\infty), \\ v &= v_0 \in C^0_0(\overline{\Omega}), && \text{in } \overline{\Omega}\times\{0\}, \end{aligned} \tag{1.1}$$

where Ω denotes the interval $(-1,1)$, $p>1$, and Δ_p denotes the p-Laplacian, that is, $\Delta_p(v) = \partial_x(|\partial_x v|^{p-2}\partial_x v)$ on Ω (a precise definition of Δ_p is given in Section 3 below). The function f is assumed to satisfy the following hypothesis:

(H-f) $f:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is continuous, and the partial derivative $f_\xi(x,\xi)$ exists on $\overline{\Omega}\times\mathbb{R}$ and is continuous and bounded.

We are only interested in bounded solutions of (1.1) so the boundedness assumption in (H-f) is not restrictive. Under these hypotheses it is known that for any $v_0\in C^0_0(\overline{\Omega})$ the problem (1.1) has a unique solution $t\to v_{v_0}(t):[0,\infty)\to C^0_0(\overline{\Omega})$, see Theorem 4.2 below (what we mean by a solution of (1.1) will be made precise in Definition 4.1).

We wish to determine the dynamic, asymptotic stability or instability of equilibrium solutions of (1.1) in terms of their 'linearised stability'. Specifically, suppose that u_0 is an equilibrium solution of (1.1). Then we say that u_0 is 'dynamically stable' if, for all v_0 sufficiently close to u_0 (in a suitable sense), $v_{v_0}(t)\to u_0$, in $C^0_0(\overline{\Omega})$, as $t\to\infty$, and u_0 is 'dynamically unstable' if there exists initial conditions v_0 arbitrarily close to u_0 for which this does not hold (this will be made more precise in Theorem 5.1 below). In the semilinear case $p=2$ (Δ_2 is the standard, linear Laplacian) it is well-known that dynamic stability or instability is determined by the location of the spectrum of the linearisation of the operator on the right hand side of (1.1) at u_0; this is the so-called 'principle of linearised stability'. See, for example, [13], which discusses a problem similar to (1.1) (in $N\geqslant 1$ dimensions) with $p=2$. However, this principle has not been shown to hold for the general, quasilinear, p-Laplacian problem (1.1) when $2\neq p>1$. The difficulty in this case is due to the zeros of $\partial_x v$, which cause the term $|\partial_x v|^{p-2}$ in the quasilinear operator $\Delta_p(v)$ to be either zero or ∞. See Remark 3.5 below for some further comments on this.

In this paper we consider the general, quasilinear case $p \neq 2$. We will first define a suitable 'linearisation' operator of (1.1) at u_0 (this in itself is a delicate issue for the p-Laplacian, and will require a considerable amount of preparation). The linearisation operator will be a formally self-adjoint, Sturm-Liouville differential operator on Ω, and we will say that u_0 is 'linearly stable' (respectively, 'linearly unstable') if the principal eigenvalue $\sigma_0(u_0)$ of the linearisation at u_0 is negative (respectively, positive). We then show that, under certain additional conditions, linear stability (respectively, linear instability) implies dynamic stability (respectively, dynamic instability) of u_0. In addition, it will be shown that the rate of convergence to u_0 (or divergence from u_0) is at least exponential, with rate determined by $|\sigma_0(u_0)|$. Thus we obtain a close analogue of the semilinear results in [13]. We give a more detailed discussion and comparison of our results with related results from the literature in Section 5.1 below.

We also show that if $f_\xi < 0$ on a suitable neighbourhood of u_0 then a similar stability result for u_0 can be obtained without using the linearisation operator (and so without some of the associated hypotheses), albeit with a weaker convergence rate. This stability result does not use any linearisation, so cannot be regarded as a 'linearised stability' result. However, since the linearisation is not used in the proof, this result readily extends to the problem (1.1) on a bounded, smooth domain Ω in $\mathbb{R}^N$, with $N \geqslant 1$.

Finally, in Section 6, we consider a bifurcation type problem similar to (1), having a line of trivial solutions. We characterise the stability or instability of the trivial and the bifurcating non-trivial solutions, and we obtain a result on 'exchange of stability' at the bifurcation point, analogous to the well-known result when $p = 2$, see [9, 13].

The overall construction of the linearisation operator, and the proofs, will depend on certain differentiability properties of the inverse p-Laplacian, which are currently only known in 1-dimension. Hence, unfortunately, these results do not seem to extend readily to higher dimensions.

2. Preliminaries

For $q \geqslant 1$, $L^q(\Omega)$ will denote the standard space of real valued functions on Ω whose qth power is integrable, with norm $\|\cdot\|_q$ (throughout, all function spaces will be real); the standard $L^2(\Omega)$ inner product will be denoted by $\langle\cdot,\cdot\rangle$; $W^{1,q}(\Omega)$ will denote the standard Sobolev space of functions on $\overline{\Omega}$ whose first order derivative belongs to $L^q(\Omega)$, with norm $\|\cdot\|_{1,q}$. For $j = 0, 1, \dots,$ $C^j(\overline{\Omega})$ will denote the standard space of j times continuously differentiable functions defined on $\overline{\Omega}$, with the standard sup-norm $|\cdot|_j$. For any $\omega_0 \in C^j(\overline{\Omega})$,

$$B^j_r(\omega_0) := \{\omega \in C^j(\overline{\Omega}) : |\omega - \omega_0|_j < r\}, \qquad r > 0.$$

We also let $C^j_0(\overline{\Omega})$ and $W^{1,q}_0(\Omega)$ denote the sets of functions ω in $C^j(\overline{\Omega})$ and $W^{1,q}(\Omega)$, respectively, satisfying the boundary conditions $\omega(\pm 1) = 0$, and $B^j_{0,r}(\omega_0) := B^j_r(\omega_0) \cap C^j_0(\overline{\Omega})$.

If $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous then, for any $\omega \in C^0(\overline{\Omega})$, we define $h(\omega) \in C^0(\overline{\Omega})$ by

$$h(\omega)(x) := h(x, \omega(x)), \quad x \in \overline{\Omega}.$$

Clearly, the 'Nemitsky' mapping $\omega \to h(\omega) : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is continuous. In particular, we define the function $\phi_p(\xi) := |\xi|^{p-1} \operatorname{sgn} \xi$, $\xi \in \mathbb{R}$, with the corresponding Nemitsky mapping $\phi_p : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$. We also note that it follows from the assumption (H-f) that the Nemitsky mapping associated with the function f in (1.1) is in fact C^1, with Fréchet derivative at $u_0 \in C^0(\overline{\Omega})$ given by

$$Df(u_0)\bar{u} = f_\xi(u_0)\bar{u}, \quad \bar{u} \in C^0(\overline{\Omega}). \tag{2.1}$$

3. The p-Laplacian and its inverse

We define the p-Laplacian operator $\Delta_p : \mathcal{D}(\Delta_p) \to L^1(\Omega)$ by

$$\begin{aligned} \mathcal{D}(\Delta_p) &:= \{u \in C^1_0(\overline{\Omega}) : \phi_p(u') \in W^{1,1}(\Omega)\}, \\ \Delta_p(u) &:= \phi_p(u')', \quad u \in \mathcal{D}(\Delta_p). \end{aligned} \tag{3.1}$$

The operator Δ_p is $(p-1)$-*homogeneous*, in the sense that $\Delta_p(\alpha u) = \phi_p(\alpha)\Delta_p(u)$, for any $\alpha \in \mathbb{R}$ and $u \in \mathcal{D}(\Delta_p)$. The invertibility of Δ_p is well known — see, for example, [5, Theorem 3.1], [14, Theorem 20] ([5] considers periodic boundary conditions, but the proof can readily be modified to deal with Dirichlet boundary conditions). However, for use below we will briefly describe the construction.

We define operators $\mathcal{I} : L^1(\Omega) \to W^{1,1}(\Omega)$ and $T_p : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ by

$$\begin{aligned} \mathcal{I}(\omega)(x) &:= \int_{-1}^{x} \omega, \qquad\qquad \omega \in L^1(\Omega), \\ T_p(\omega) &:= \mathcal{I}(\phi_{p^*+1}(\omega)), \quad \omega \in C^0(\overline{\Omega}), \end{aligned} \tag{3.2}$$

where $p^* := (p-1)^{-1}$, and hence $\phi_{p^*+1}(\phi_p(\xi)) = \xi$, $\xi \in \mathbb{R}$. Clearly, the operators $\mathcal{I}$ and T_p are continuous, and $\mathcal{I}$ is also linear.

Theorem 3.1. *For any $h \in L^1(\Omega)$, the equation $\Delta_p u = h$ has a unique solution $u = u(h) = \Delta_p^{-1}(h) \in \mathcal{D}(\Delta_p)$ given by*

$$\Delta_p^{-1}(h) := T_p\big(\gamma(h) + \mathcal{I}(h)\big), \quad h \in L^1(\Omega), \tag{3.3}$$

where $\gamma(h) \in \mathbb{R}$ is a constant satisfying the equation

$$T_p\big(\gamma(h) + \mathcal{I}(h)\big)(1) = 0. \tag{3.4}$$

Hence, the range $R(\Delta_p) = L^1(\Omega)$, and the operator $\Delta_p^{-1} : L^1(\Omega) \to C_0^1(\overline{\Omega})$ is continuous, and p^*-homogeneous.

3.1. **Differentiability of Δ_p^{-1}.** The following theorem, and Theorem 3.4 below, were proved in [5] for periodic rather than Dirichlet boundary conditions. The proofs in the case of Dirichlet boundary conditions are minor modifications of the proofs in [5] – we will not repeat the proofs here.

Theorem 3.2 ([5, Theorem 3.2])**.** (*A*) *Suppose that $1 < p < 2$. Then $T_p : C^0(\overline{\Omega}) \to C^1(\overline{\Omega})$ is C^1, and for any $g \in C^0(\overline{\Omega})$,*

$$DT_p(g)\bar{g} = p^*\mathcal{I}(|g|^{p^*-1}\bar{g}), \quad \bar{g} \in C^0(\overline{\Omega}). \tag{3.5}$$

(*B*) *Suppose that $p > 2$. Suppose also that $g_0 \in C^1(\overline{\Omega})$ has only simple zeros. Then there exists a neighbourhood U_0 of g_0 in $C^1(\overline{\Omega})$ such that $T_p : U_0 \to W^{1,1}(\Omega)$ is C^1, and for any $g \in U_0$ the derivative $DT_p(g)$ is given by* (3.5)*, with $\bar{g} \in C^1(\overline{\Omega})$.*

We observe that the range space in part (*A*) of Theorem 3.4 is $C^1(\overline{\Omega})$, while in part (*B*) it is $W_0^{1,1}(\Omega)$. This slight loss of regularity in part (*B*) will cause some difficulty below, and a slight extension of Theorem 3.4 (*B*) will be required to deal with this.

For any $\epsilon \in (0, 1]$, let

$$E_\epsilon := [-1, -1 + \epsilon] \cup [1 - \epsilon, 1].$$

For $j = 0, 1$, the space $C^j(E_\epsilon)$ has the obvious meaning, and its norm will be denoted by $|\cdot|_{j,\epsilon}$. By a slight abuse of notation, we also define

$$C_0^j(E_\epsilon) := \{\omega \in C^j(E_\epsilon) : \omega(\pm 1) = 0\},$$

with norm $|\cdot|_{j,\epsilon}$. We now define a 'restriction' operator $P_\epsilon : C_0^1(\overline{\Omega}) \to C_0^1(E_\epsilon)$ by

$$P_\epsilon g := g|_{E_\epsilon}, \quad g \in C_0^1(\overline{\Omega}).$$

Theorem 3.3. *Suppose that $p > 2$, $g_0 \in C^0(\overline{\Omega})$ and $g_0(\pm 1) \neq 0$. Then there exists $\epsilon \in (0, 1]$ such that the mapping $P_\epsilon \circ T_p : B_\epsilon^0(g_0) \to C_0^1(E_\epsilon)$ is C^1.*

Proof. Choose $\epsilon \in (0, 1]$ such that $g \neq 0$ on E_ϵ for any $g \in B_\epsilon^0(g_0)$. The proof of the differentiability of $P_\epsilon \circ T_p$ at any $g \in B_\epsilon^0(g_0)$ then follows the corresponding proof of the differentiability of T_p in Theorem 3.2 (*B*) in [5], but in this case, since g is nonzero on E_ϵ, it can be seen that (using the notation in [5]) $|\Xi|_{1,\epsilon}/|\bar{g}|_0 \to 0$ as $|\bar{g}|_0 \to 0$ instead of $\|\Xi\|_1/|\bar{g}|_1 \to 0$ as $|\bar{g}|_1 \to 0$ (as obtained in [5]). This yields the differentiability of $P_\epsilon \circ T_p$ at $g \in B_\epsilon^0(g_0)$. The continuity of the derivative $D(P_\epsilon \circ T_p)$ on $B_\epsilon^0(g)$ then follows readily from the form of DT_p in (3.5) (again using the fact that g is nonzero on E_ϵ). □

Theorem 3.4 ([5, Theorem 3.4])**.** *For $h \in L^1(\Omega)$, let $u = u(h) := \Delta_p^{-1}(h) \in C_0^1(\overline{\Omega})$.*
(A) Suppose that $1 < p < 2$. Then $\Delta_p^{-1} : L^1(\Omega) \to C_0^1(\overline{\Omega})$ is C^1, and for any $h \in L^1(\Omega)$,

$$D\Delta_p^{-1}(h)\bar{h} = p^*\mathcal{I}\big(|u'|^{2-p}\big(D\gamma(h)\bar{h} + \mathcal{I}(\bar{h})\big)\big), \quad \bar{h} \in L^1(\Omega). \tag{3.6}$$

(B) Suppose that $p > 2$. Suppose also that $h_0 \in C^0(\overline{\Omega})$, $u_0 = u(h_0)$, are such that $u_0'(x) = 0 \implies h_0(x) \neq 0$, for $x \in \overline{\Omega}$. Then there exists a neighbourhood V_0 of h_0 in $C^0(\overline{\Omega})$ such that:

- *(a) the mapping $h \to |u(h)'|^{2-p} : V_0 \to L^1(\Omega)$ is continuous;*
- *(b) the operator $\Delta_p^{-1} : V_0 \to W_0^{1,1}(\Omega)$ is C^1, and for any $h \in V_0$ the derivative $D\Delta_p^{-1}(h)$ is given by (3.6), with $\bar{h} \in C^0(\overline{\Omega})$.*

Remark 3.5. To attempt to clarify the significance of the hypothesis in Theorem 3.4 (B) we observe that u_0 satisfies the differential equation $\phi_p(u_0')' = h_0$, so the hypothesis implies that the function $\phi_p(u_0')$ has only simple zeros. It follows from this that if x_0 is a zero of u_0' then integrating this differential equation from x_0 to x yields $\phi_p(u_0'(x)) \approx h_0(x_0)(x - x_0)$, for x near to x_0, with $h_0(x_0) \neq 0$, and hence $|u_0'(x)|^{2-p} \approx |h_0(x_0)|^\alpha |x - x_0|^\alpha$, where $\alpha := (2-p)/(p-1)$ satisfies $-1 < \alpha < 0$ (since $p > 2$). Thus, $|u_0'|^{2-p} \in L^1(\Omega)$. Since $p > 2$ may be arbitrarily large and u_0' has zeros, this does not seem obvious a priori. Further analysis then yields property (a) in the theorem, which then ensures that the formula (3.6) for the derivative $D\Delta_p^{-1}$ makes sense in this case.

A slight extension of Theorem 3.4 (B) will be required below.

Theorem 3.6. *Suppose that $p > 2$ and $h_0 \in C^0(\overline{\Omega})$ is such that $u(h_0)'(\pm 1) \neq 0$. Then there exists $\epsilon > 0$ and a neighbourhood V_0 of h_0 in $C^0(\overline{\Omega})$ such that the mapping $P_\epsilon \circ \Delta_p^{-1} : V_0 \to C_0^1(E_\epsilon)$ is C^1.*

Proof. For $h \in C^0(\overline{\Omega})$, let $g(h) := \gamma(h) + \mathcal{I}(h) \in C^1(\overline{\Omega})$. Differentiating the function $u(h_0) = \Delta_p^{-1}(h_0)$ with respect to x (using (3.2) and (3.3)), shows that $g(h_0) = \phi_p(u(h_0)')$ and, by hypothesis,

$$g(h_0)(0) = \phi_p\big(u(h_0)'(0)\big) \neq 0.$$

Also, the proof of Theorem 3.4 in [5] shows that the mapping $\gamma : C^0(\overline{\Omega}) \to \mathbb{R}$ is C^1, so the result now follows from the form of Δ_p^{-1} in (3.3) and Theorem 3.3. □

3.2. **The inverse of $D\Delta_p^{-1}(h)$.** For a given $h_0 \in L^1(\Omega)$ the derivative operator $D\Delta_p^{-1}(h_0)$ given by (3.6) (with $u_0 = u_0(h_0)$) is an integral operator. It will be convenient to discuss the inverse of this operator, which will be a Sturm-Liouville-type differential operator on Ω. In particular, this inverse operator will enable us to ascertain, and utilise, the spectral properties of $D\Delta_p^{-1}(h_0)$. To unify the discussion of the cases $1 < p < 2$ and $p > 2$, we define the spaces

$$D_p := \begin{cases} L^1(\Omega), & \text{if } 1 < p < 2, \\ C^0(\overline{\Omega}), & \text{if } p > 2 \end{cases}$$

(these correspond to the domain spaces of $D\Delta_p^{-1}(h_0)$ in Theorem 3.4, in the two cases).

We will suppose that $u_0 \in \mathcal{D}(\Delta_p)$ satisfies the following conditions

$$\begin{aligned} &(a) \text{ if } 1 < p < 2 \text{ then } u_0' \neq 0 \text{ a.e. on } \Omega; \\ &(b) \text{ if } p > 2 \text{ then the hypothesis in Theorem 3.4 } (B) \text{ holds (with } h_0 = \Delta_p(u_0)). \end{aligned} \tag{3.7}$$

Note that if $p > 2$ then the result in (a) of Theorem 3.4 (B) shows that $u_0' \neq 0$ a.e. on Ω, so (3.7) implies that $u_0' \neq 0$ a.e. on Ω for all $2 \neq p > 1$.

Now, let $h_0 = \Delta_p(u_0) \in D_p$, and for any $\bar{h} \in D_p$, let $w := D\Delta_p^{-1}(h_0)\bar{h}$ (as given by (3.6)). By successively differentiating w (with respect to x), using (3.6) and the definition of $\mathcal{I}$ (and (3.7) if $1 < p < 2$), we see that

$$\begin{gathered} w \in W_0^{1,1}(\Omega), \quad |u_0'|^{p-2} w' \in W^{1,1}(\Omega), \\ (p-1)\big(|u_0'|^{p-2} w'\big)' = \bar{h}. \end{gathered} \tag{3.8}$$

In addition, by (3.7), the coefficient function $|u_0'|^{p-2}$ in (3.8) satisfies:

- (a) $|u_0'|^{p-2} > 0$ a.e. on Ω;
- (b) $1/|u_0'|^{p-2} = |u_0'|^{2-p} \in L^1(\Omega)$ (by part (a) of Theorem 3.4 (B) when $p > 2$).

The differential operator in (3.8) is a linear, second order, formally self-adjoint Sturm-Liouville operator, and the properties of the coefficient function $|u_0'|^{2-p}$ in this operator, described in (a), (b), are the standard hypotheses in the L^1 theory of such operators, see [3, Chap. 8]. Hence, we will be able to apply this theory to the above operator.

Definition 3.7. For any $u_0 \in \mathcal{D}(\Delta_p)$ satisfying (3.7) we define a linear operator $\Lambda_{p,u_0} : \mathcal{D}(\Lambda_{p,u_0}) \to L^1(\Omega)$ as follows:

$$\mathcal{D}(\Lambda_{p,u_0}) := \{w \in W_0^{1,1}(\Omega) : |u_0'|^{p-2}w' \in W^{1,1}(\Omega)\}, \tag{3.9}$$

$$\Lambda_{p,u_0} w := (p-1)\big(|u_0'|^{p-2}w'\big)', \quad w \in \mathcal{D}(\Lambda_{p,u_0}). \tag{3.10}$$

It follows immediately from Definition 3.7 that

$$\Lambda_{p,u_0}\, D\Delta_p^{-1}(h_0)\,\bar{h} = \bar{h}, \quad \bar{h} \in D_p, \tag{3.11}$$

$$u_0 \in \mathcal{D}(\Delta_p) \implies u_0 \in \mathcal{D}(\Lambda_{p,u_0}) \ \text{ and } \ \Lambda_{p,u_0} u_0 = (p-1)\Delta_p(u_0), \tag{3.12}$$

$$\langle \Lambda_{p,u_0} w_1, w_2 \rangle = \langle w_1, \Lambda_{p,u_0} w_2 \rangle, \quad w_1,\, w_2 \in \mathcal{D}(\Lambda_{p,u_0}). \tag{3.13}$$

Remark 3.8. In view of the symmetry property (3.13) it would be possible, with some extra work, to formulate the operators $D\Delta_p^{-1}(h_0)$ and Λ_{p,u_0} as, respectively, bounded and unbounded, self-adjoint operators in the Hilbert space $L^2(\Omega)$, but we will only require the properties (3.11)-(3.13) here, so we will not pursue this further.

3.3. **A linearisation operator at equilibria of** (1.1)**.** Naturally, by an *equilibrium* of (1.1) we mean a solution u_0 of the problem

$$\Delta_p(u) + f(u) = 0, \quad u \in \mathcal{D}(\Delta_p), \tag{3.14}$$

which we regard as a constant (in time) solution of (1.1). We now wish to define a 'linearisation' of (3.14) at $u = u_0$. To do this we will suppose that u_0 satisfies the following conditions (recall that $\mathcal{D}(\Delta_p) \subset C_0^1(\overline{\Omega})$):

$$u_0'(\pm 1) \neq 0, \quad u_0' \neq 0 \ \text{ a.e on } \Omega; \tag{3.15}$$

$$\text{if } p > 2 \text{ then } u_0'(x) = 0 \implies f(u_0(x)) \neq 0, \quad x \in \overline{\Omega}. \tag{3.16}$$

It follows from equation (3.14) together with (3.15) that u_0' is C^1 near to ± 1. Also, if $p > 2$ then (3.16) ensures that the hypotheses of Theorem 3.4 hold, with $h_0 = -f(u_0)$. Hence, u_0 satisfies (3.7), so we may define the *linearisation* of (3.14) at $u = u_0$ to be the operator

$$\mathcal{L}_{u_0} w := \Lambda_{p,u_0} w + f_\xi(u_0) w, \quad w \in \mathcal{D}(\Lambda_{p,u_0}).$$

By the remarks in Section 3.2, $\mathcal{L}_{u_0}$ is a Sturm-Liouville operator of the form discussed in [3, Chap. 8], and it has the usual spectral properties of such operators, as described in [3, Theorem 8.4.5]. Hence, we may make the following definition.

Definition 3.9. Given a solution u_0 of (3.14) satisfying the conditions (3.15), (3.16), we let $\sigma_0 = \sigma_0(u_0) \in \mathbb{R}$, $\psi_0 = \psi_0(u_0) \in \mathcal{D}(\Lambda_{p,u_0})$, denote the unique principal eigenvalue and positive, normalised eigenfunction of the problem

$$\mathcal{L}_{u_0} \psi_0 = \sigma_0 \psi_0, \tag{3.17}$$

$$|\psi_0|_0 = 1, \ \ \psi_0 > 0 \text{ in } \Omega \ \text{ and } \ \pm\psi_0'(\pm 1) < 0. \tag{3.18}$$

Remark 3.10. The hypothesis (3.15) ensures that the coefficient function $|u_0'|^{p-2}$ in (3.9) is nonzero at ± 1 and C^1 near to ± 1. Thus, by the form of Λ_{p,u_0} in (3.10) the eigenfunction ψ_0 is C^1 near to ± 1, so the inequalities at ± 1 in (3.18) make sense. Any function in $C_0^0(\overline{\Omega})$ which is C^1 near to ± 1 and satisfies the inequalities in (3.18) will be said to be *strongly positive* on Ω.

The following properties of the eigenvalues of $\mathcal{L}_{u_0}$ are also proved in [3, Theorem 8.4.5]. They are of course standard, but we state them here since they are implicit in the principle of linearised stability (that dynamic stability is determined by the sign of σ_0, and not by any other eigenvalues).

Corollary 3.11. *All the eigenvalues of $\mathcal{L}_{u_0}$ are real, and any eigenvalue $\sigma \neq \sigma_0$ of $\mathcal{L}_{u_0}$ satisfies $\sigma < \sigma_0$.*

4. Solutions of (1.1)

We now discuss, briefly, the existence, uniqueness and various properties of solutions of the problem (1.1).

4.1. Existence and uniqueness of solutions.

To state precisely what we mean by a solution of the problem (1.1) we define the spaces

$$\Sigma_T := C([0,T), C_0^0(\overline{\Omega})) \cap C((0,T), W_0^{1,p}(\Omega)) \cap W_{\rm loc}^{1,2}((0,T), L^2(\Omega)), \quad 0 < T \leqslant \infty.$$

The space $W^{1,2}((0,T), L^2(\Omega))$ is defined in [17, Example 10.2], for $0 < T < \infty$, using the notation $H^1((0,T), L^2(\Omega))$; the space $W_{\rm loc}^{1,2}((0,\infty), L^2(\Omega))$ can be defined by an adaptation of the definition in [17].

Definition 4.1. A *solution* of (1.1) is a function $v \in \Sigma_T$, for some $T > 0$, such that $v(0) = v_0$ and for a.e. $t \in [0,T)$:

(a) the function $v : [0,T) \to L^2(\Omega)$ is differentiable at t;
(b) $v(t) \in \mathcal{D}(\Delta_p)$ and $\Delta_p(v(t)) \in L^2(\Omega)$;
(c) $\dfrac{dv}{dt}(t) = \Delta_p(v(t)) + f(v(t))$ (in the $L^2(\Omega)$ sense).

Thus, we regard a solution v of (1.1) as a time-dependent mapping $t \to v(t) : [0,T) \to C_0^0(\overline{\Omega})$, satisfying (1.1) in the sense described in Definition 4.1. In view of this we will rewrite (1.1) in the form

$$\frac{dv}{dt} = \Delta_p(v) + f(v), \quad v(0) = v_0 \in C_0^0(\overline{\Omega}). \tag{4.1}$$

The following theorem states a basic, known result on the existence and uniqueness of solutions of (4.1).

Theorem 4.2. *For any $v_0 \in C_0^0(\overline{\Omega})$, the problem* (4.1) *has a unique solution $v_{v_0} \in \Sigma_\infty$.*

NB: the solution v_0 in Theorem 4.2 exists on $[0,\infty)$ due to the boundedness assumption in hypothesis (H-f); without this assumption the solution might 'blow-up' in finite time.

Proof. If we suppose that $v_0 \in L^2(\Omega)$, and we replace $C_0^0(\overline{\Omega})$ with $L^2(\Omega)$ in the definition of Σ_∞, then the existence and uniqueness results in Theorem 4.2 are described in [7, Remark 2.2] (under weaker hypotheses), together with details and bibliography, and some further properties of the solutions. The discussion in [7] is based on the results of [2] (in particular, [2, Theorem 3.11]). A similar result is proved in [19, Theorem 2.1], for $p > 2$. If we now suppose that $v_0 \in C_0^0(\overline{\Omega})$ then the result as stated (with $C_0^0(\overline{\Omega})$ in the definition of Σ_∞) follows from these results together with [16, Lemma 2.3], or Theorem 1.3 in Chapter III of [10] (which prove continuity at $t = 0$ in $C_0^0(\overline{\Omega})$). □

4.2. Sub and super-solutions of (4.1) and a comparison theorem.

We now define sub and super-solutions of (4.1). These need not satisfy the Dirichlet boundary conditions at ± 1, so it will be convenient to redefine the operator Δ_p and the solution spaces Σ_T to omit these boundary conditions. Consequently, we define $\tilde{\Delta}_p$ by replacing the space $C_0^1(\overline{\Omega})$ with $C^1(\overline{\Omega})$ in the definition of Δ_p in (3.1), and we define $\tilde{\Sigma}_T$ by replacing the spaces $C_0^0(\overline{\Omega})$, $W_0^{1,p}(\Omega)$ with $C^0(\overline{\Omega})$, $W^{1,p}(\Omega)$, respectively, in the definition of Σ_T. We also use the following notation (see p. 1 of [10])

$$\Omega_T := \Omega \times (0,T), \quad \Gamma_T := \big(\overline{\Omega} \times \{0\}\big) \cup \big(\{\pm 1\} \times [0,T)\big), \quad 0 < T \leqslant \infty;$$

the set Γ_T is called the 'parabolic boundary' of Ω_T.

Definition 4.3. Suppose that, for some $T > 0$, $\tilde{v} \in \tilde{\Sigma}_T$ satisfies (a) and (b) in Definition 4.1, with Δ_p replaced by $\tilde{\Delta}_p$, for a.e. $t \in [0,T)$. Then:

(a) $\tilde{v}$ is a *sub-solution* of (4.1) on Ω_T if

$$\frac{d\tilde{v}}{dt}(t) - \tilde{\Delta}_p(\tilde{v}(t)) - f(\tilde{v}(t)) \leqslant 0 \quad \text{(in the } L^2(\Omega) \text{ sense), a.e. } t \in [0,T); \tag{4.2}$$

(b) $\tilde{v}$ is a *super-solution* of (4.1) on Ω_T if (4.2) holds with the inequality reversed.

Clearly, solutions of (4.1) (including equilibrium solutions) are both sub and super-solutions of (4.1).

We now state a comparison theorem for sub and super-solutions of (4.1). We note that Definition 4.3 could be weakened considerably, and the following comparison theorem could be generalised, but these suffice for our purposes here.

Theorem 4.4. *Suppose that* $\tilde{u}$ *is a sub-solution and* $\tilde{v}$ *is a super-solution of* (4.1) *on* Ω_T*, for some* $0 < T \leqslant \infty$*, and* $\tilde{u} \leqslant \tilde{v}$ *on* Γ_T*. Then* $\tilde{u} \leqslant \tilde{v}$ *on* Ω_T*.*

Proof. The proof is based on the proof of [16, Theorem 2.5], which in turn was based on the proof of [10, Lemma 3.1, Ch. VI]; the equation $v_t = \Delta_p(v)$ was considered in [10], while [16] considered the equation $v_t = \Delta_p(v) + \lambda\phi_p(v)$. For completeness, we will describe the differences between the proof here and in [16], but we will omit most details (including notation) which are identical to those in [16].

We first note that Definition 4.3 is stronger than the definitions of sub and super-solutions used in [10] and [16] ('weak' sub and super-solutions are considered there), so our sub and super-solutions have all the properties used in the proofs in [10] and [16]. In particular, we have $\tilde{u}, \tilde{v} \in C(\Omega_T \cup \Gamma_T)$ automatically from $\tilde{u}, \tilde{v} \in \tilde{\Sigma}_T$.

Now, suppose that $0 < T' < T$. For any $t \in [0, T']$, and $0 < h < T - T'$, a similar calculation to that in the proof of [16, Theorem 2.5] yields

$$\begin{aligned} &\int_\Omega [(\tilde{u} - \tilde{v})_h]_+^2(t) + [\text{plap integral}] \\ &\qquad = \int_\Omega [(\tilde{u} - \tilde{v})_h]_+^2(0) + \int_0^t \int_\Omega [f(\tilde{u}) - f(\tilde{v})]_h [(\tilde{u} - \tilde{v})_h]_+, \end{aligned} \tag{4.3}$$

where we note the following.

(*i*) $(\tilde{u} - \tilde{v})_h$ denotes the 'Steklov average' of $\tilde{u} - \tilde{v}$ (see [10], or [16], for further details) and $[(\tilde{u} - \tilde{v})_h]_+$ denotes the positive part of $(\tilde{u} - \tilde{v})_h$.

(*ii*) The 'plap integral' term in (4.3) comes from integrating by parts various p-Laplacian terms, and is identical to the corresponding term in [16].

(*iii*) Both [10] and [16] assume that $\tilde{u} \leqslant \tilde{v}$ on $\Omega \times \{0\}$ (in their notation, in $\mathbb{R}^N$ with $N \geqslant 1$), but on $\partial\Omega \times [0, T]$ it is assumed in [16] that $\tilde{u} = \tilde{v} = 0$, whereas [10] merely assumes that $\tilde{u} \leqslant \tilde{v}$, which is what we assume here (in 1-dimension). The proof in [16] is very similar to that in [10] and in fact holds with the latter assumption. Thus, the calculations in [16] are valid here.

The only term in (4.3) that differs from the corresponding term in the calculation in [16] is the final term, involving f, so by the arguments in [16],

$$\lim_{h \to 0^+} \int_\Omega [(\tilde{u} - \tilde{v})_h]_+^2(0) = 0, \quad \text{(as in [16], using } \tilde{u}(0) \leqslant \tilde{v}(0))$$

$$[\text{plap integral}] \geqslant 0 \quad \text{(as in [16])}.$$

To estimate the final term in (4.3) we note that, since $|\tilde{u}(\cdot)|_0$, $|\tilde{v}(\cdot)|_0$ are bounded on $[0, T']$, it follows from hypothesis (H-f) that there exists a constant $L_{T'} > 0$ such that

$$|f(\tilde{u}(t)) - f(\tilde{v}(t))| \leqslant L_{T'} |\tilde{u}(t) - \tilde{v}(t)|, \quad \text{on } \overline{\Omega}, \text{ for } t \in [0, T'].$$

Now, letting $h \to 0^+$ in (4.3) and using these results, together with the convergence results in [16], yields

$$\int_\Omega (\tilde{u} - \tilde{v})_+^2(t) \leqslant L_{T'} \int_0^t \int_\Omega (\tilde{u} - \tilde{v})_+^2, \quad t \in [0, T'].$$

It follows from this (together with Gronwall's inequality) that the desired inequality holds on $\Omega_{T'}$, and hence on Ω_T, since $T' < T$ was arbitrary, which completes the proof of Theorem 4.4. □

5. Linearised stability of equilibria implies dynamic stability

We suppose throughout this section that u_0 is a solution of (3.14) which satisfies the conditions (3.15), (3.16), and we let $\sigma_0 = \sigma_0(u_0)$, $\psi_0 = \psi_0(u_0)$, denote the principal eigenvalue and eigenfunction of the linearisation of (3.14) at u_0, as defined in Definition 3.9. The following result shows that the dynamic stability, or instability, of u_0 is determined by the sign of σ_0. In the stable case all solutions with initial values v_0 sufficiently close u_0 converge to u_0, exponentially in time with rate determined by the magnitude of σ_0; in the unstable case some solutions diverge away from u_0, exponentially in time.

Theorem 5.1. *Suppose that u_0 is an equilibrium solution of* (4.1) *satisfying* (3.15) *and* (3.16).

(*a*) *Suppose that $\sigma_0 < 0$. Then for any $\kappa \in (0, |\sigma_0|)$ there exists $\delta_0 > 0$ and $C > 0$ such that*
$$|v_0 - u_0|_0 < \delta_0 \implies |v_{v_0}(t) - u_0|_0 \leqslant Ce^{-\kappa t}, \quad t \geqslant 0. \tag{5.1}$$
That is, u_0 is asymptotically stable.

(*b*) *Suppose that $\sigma_0 > 0$. Then for any $\kappa \in (0, \sigma_0)$ there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, there exists $v_{0,\delta} \in C^1_0(\overline{\Omega})$ such that*
$$|v_{0,\delta} - u_0|_1 < \delta|\psi_0|_1 \quad \text{and} \quad |v_{v_{0,\delta}}(t_\delta) - u_0|_0 \geqslant \delta_0, \quad \text{where} \quad e^{\kappa t_\delta} = 4\delta_0/\delta.$$
That is, u_0 is unstable.

Proof of Theorem 5.1.

The case $1 < p < 2$**.**

We will prove the theorem by constructing suitable sub and super-solutions of (4.1). To do this we first construct solutions (s, η) of the equation
$$\Delta_p(u_0 + s\rho + \eta) + f(u_0 + s\rho + \eta) = s\mathcal{L}_{u_0}\rho + \alpha\eta, \quad (s, \eta) \in \mathbb{R} \times C^1_0(\overline{\Omega}), \tag{5.2}$$
where $\rho \in \mathcal{D}(\Lambda_{p,u_0})$ and $\alpha := \sigma_0 + 1$, so that α is not an eigenvalue of $\mathcal{L}_{u_0}$ (the precise value of α is unimportant, so long as it is not an eigenvalue of $\mathcal{L}_{u_0}$).

Lemma 5.2. *For any $\rho \in \mathcal{D}(\Lambda_{p,u_0})$ there exists $\delta_\rho > 0$ and a C^1 function $\eta_\rho : (-\delta_\rho, \delta_\rho) \to C^1_0(\overline{\Omega})$, such that if $|s| < \delta_\rho$ then:*

(*a*) $\eta_\rho(0) = 0$, $\eta'_\rho(0) = 0$
(*where η'_ρ denotes the derivative, with respect to s, of the mapping $s \to \eta_\rho(s)$*);

(*b*) $u_0 + s\rho + \eta_\rho(s) \in \mathcal{D}(\Delta_p)$;

(*c*) $(s, \eta_\rho(s))$ *satisfies* (5.2).

Proof. We consider the equation
$$F(s, \eta) = 0, \tag{5.3}$$
where $F : \mathbb{R} \times C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is defined by
$$F(s, \eta) := \Delta_p^{-1}\big[f(u_0 + s\rho + \eta) - s\mathcal{L}_{u_0}\rho - \alpha\eta\big] + u_0 + s\rho + \eta, \quad (s, \eta) \in \mathbb{R} \times C^0(\overline{\Omega}).$$
Since $\Delta_p^{-1} : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ and the Nemitsky mapping $f : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ are C^1 (by Theorem 3.4 (*A*) and assumption (*H-f*)), the function F is C^1, and since u_0 satisfies (3.14),
$$F(0, 0) = \Delta_p^{-1}(f(u_0)) + u_0 = 0,$$
$$D_\eta F(0, 0) = I + K_{u_0},$$
where $K_{u_0} : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is defined by
$$K_{u_0}\overline{\eta} := D\Delta_p^{-1}(f(u_0))\big((f_\xi(u_0) - \alpha)\overline{\eta}\big), \quad \overline{\eta} \in C^0(\overline{\Omega}).$$
By Theorem 3.4 (*A*), the operator K_{u_0} is a bounded operator from $C^0(\overline{\Omega})$ into $C^1(\overline{\Omega})$, so by the compactness of the embedding $C^1(\overline{\Omega}) \hookrightarrow C^0(\overline{\Omega})$, K_{u_0} is compact. Hence, $D_\eta F(0, 0)$ is non-singular iff the null space $N(D_\eta F(0, 0)) = \{0\}$. Now, by (3.11) and the fact that α is not an eigenvalue of $\mathcal{L}_{u_0}$,
$$\overline{\eta} \in N(D_\eta F(0, 0)) \implies \mathcal{L}_{u_0}\overline{\eta} = \alpha\overline{\eta} \implies \overline{\eta} = 0,$$
so $D_\eta F(0, 0)$ is non-singular. Thus, by the implicit function theorem, there exists $\delta_\rho > 0$ and a C^1 solution function $\eta_\rho : (-\delta_\rho, \delta_\rho) \to C^0(\overline{\Omega})$ of (5.3) with $\eta_\rho(0) = 0$. In addition:

(*i*) by Theorem 3.4, the differentiability of f and the form of F in (5.3), the function $\eta_\rho : (-\delta_\rho, \delta_\rho) \to C_0^1(\overline{\Omega})$ is also C^1;

(*ii*) by Theorem 3.1 and equation (5.3), for $|s| < \delta_\rho$ we have $u_0 + s\rho + \eta_\rho(s) \in \mathcal{D}(\Delta_p)$ and $(s, \eta_\rho(s))$ satisfies (5.2).

Hence, it only remains to prove that $\eta_\rho'(0) = 0$. Differentiating (5.3) with respect to s, at $s = 0$, yields

$$\begin{aligned} &D\Delta_p^{-1}(f(u_0))\big[f_\xi(u_0)(\rho + \eta_\rho'(0)) - \mathcal{L}_{u_0}\rho - \alpha\eta_\rho'(0)\big] + \rho + \eta_\rho'(0) = 0 \\ &\implies \mathcal{L}_{u_0}(\rho + \eta_\rho'(0)) = \mathcal{L}_{u_0}\rho + \alpha\eta_\rho'(0) \qquad \text{(by (3.11))} \\ &\implies \eta_\rho'(0) = 0 \qquad (\alpha \text{ is not an eigenvalue}) \end{aligned}$$

which completes the proof of Lemma 5.2. □

The next result follows from part (*a*) of Lemma 5.2 and the strong positivity of ψ_0 (recall (3.18)).

Corollary 5.3. *For arbitrarily small $\beta > 0$ there exists $0 < \delta_\beta < \delta_\rho$ such that if $|s| < \delta_\beta$ then*

$$\beta s\psi_0 > |\eta_\rho(s)| \quad \textit{and} \quad \beta\psi_0 > |\eta_\rho'(s)| \quad \textit{on } \Omega. \tag{5.4}$$

Proof of part (a) of Theorem 5.1 (the case $\sigma_0 < 0$).

We now construct the desired sub and super-solutions of (4.1). Since $\sigma_0 < 0$ we know that 0 is not an eigenvalue of $\mathcal{L}_{u_0}$, so by [3, Theorem 8.8.1] there exists $\zeta \in \mathcal{D}(\Lambda_{p,u_0})$ such that $\mathcal{L}_{u_0}\zeta \equiv 1$ on Ω. By the argument in Remark 3.10, ζ is C^1 near to ± 1. We now write $\tau := (\delta, \gamma_1, \gamma_2) \in Q := (0,1)^3$, $|\tau| := \max\{\delta, \gamma_1, \gamma_2\}$, and we define

$$\rho_\tau := \psi_0 - \kappa\gamma_1\zeta, \quad \tau \in Q.$$

A slight extension of the proof of Lemma 5.2 (we use this specific function ρ_τ, and add the variable γ_1 to the function F used in the implicit function theorem argument) shows that there exists $\delta_\rho > 0$ and, for each $\tau \in Q$ with $|\tau| < \delta_\rho$, a function η_{ρ_τ} with the properties described in Lemma 5.2 and Corollary 5.3. Hence, we may define

$$S_\tau^\pm(t) := u_0 \pm \delta e^{-\kappa t}\rho_\tau + \eta_{\rho_\tau}(\pm\delta e^{-\kappa t}) \pm \kappa\gamma_2\delta e^{-\kappa t}, \quad t \geqslant 0, \quad |\tau| < \delta_\rho.$$

That is, in the definition of $S_\tau^\pm$ we set $s = \pm\delta e^{-\kappa t}$, $t \geqslant 0$, in the solutions of (5.2) constructed in Lemma 5.2, and add the term $\pm\kappa\gamma_2\delta e^{-\kappa t}$. Since this latter term is constant with respect to x, it differentiates to zero when we apply the p-Laplacian $\tilde{\Delta}_p$ to $S_\tau^\pm(t)$, so by part (*b*) of Lemma 5.2, $S_\tau^\pm(t) \in \mathcal{D}(\tilde{\Delta}_p)$ and $S_\tau^\pm \in \tilde{\Sigma}_\infty$.

Lemma 5.4. *There exists $\tau \in Q$ such that:*
(*a*) *$\pm(S_\tau^\pm - u_0) \geqslant 0$ on Γ_∞, and there exists $\delta_0 > 0$ such that $\pm\big(S_\tau^\pm(0) - u_0\big) > \delta_0$ on $\overline{\Omega}$;*
(*b*) *S_τ^+ is a super-solution and S_τ^- is a sub-solution of (4.1) on Ω_∞.*

Proof. (*a*) By Corollary 5.3, if $|\tau|$ is sufficiently small then,

$$\begin{aligned} \pm\big(S_\tau^\pm(0) - u_0\big) &= \delta(\psi_0 - \kappa\gamma_1\zeta) + \delta\kappa\gamma_2 \pm \eta_{\rho_\tau}(\pm\delta) > \tfrac{1}{2}\delta\kappa\gamma_2 > 0 \quad \text{on } \overline{\Omega}, \\ \pm S_\tau^\pm(t) &= \kappa\gamma_2\delta e^{-\kappa t} > 0 \quad \text{on } \{\pm 1\}, \text{ for } t \geqslant 0, \end{aligned}$$

so part (*a*) of the lemma holds, with $\delta_0 = \frac{1}{2}\delta\kappa\gamma_2$.

(*b*) For $t \geqslant 0$ and $|\tau| < \delta_\rho$, applying the mean value theorem to f shows that

$$f(S_\tau^\pm(t)) = f(S_\tau^\pm(t) \mp \kappa\gamma_2\delta e^{-\kappa t}) + \kappa\gamma_2\delta e^{-\kappa t}\, R_\tau^\pm(t),$$

where $R_\tau^\pm(t) \in C_0^0(\overline{\Omega})$, and there exists a constant $R_{\max} > 0$, such that $|R_\tau^\pm(t)|_0 \leqslant R_{\max}$. Now, setting $s = \delta e^{-\kappa t}$ in (5.2), shows that

$$\tilde{\Delta}_p(S_\tau^\pm(t)) + f(S_\tau^\pm(t)) = \delta e^{-\kappa t}\mathcal{L}_{u_0}\rho_\tau + \alpha\eta_{\rho_\tau}(\pm\delta e^{-\kappa t}) + \kappa\gamma_2\delta e^{-\kappa t}\, R_\tau^\pm(t).$$

Hence, if δ is sufficiently small then, on $\overline{\Omega}$,

$$\begin{aligned}
&\pm\Big\{ -\frac{dS_\tau^\pm}{dt}(t) + \tilde\Delta_p(S_\tau^\pm(t)) + f(S_\tau^\pm(t))\Big\} \\
&= \kappa\delta e^{-\kappa t}\{\rho_\tau + \eta'_{\rho_\tau}(\pm\delta e^{-\kappa t}) + \gamma_2\} + \delta e^{-\kappa t}\{\mathcal{L}_{u_0}\rho_\tau + \kappa\gamma_2 R_\tau^\pm(t)\} + \alpha\eta_{\rho_\tau}(\pm\delta e^{-\kappa t}) \\
&= \kappa\delta e^{-\kappa t}\{-(-\sigma_0/\kappa - 1)\psi_0 - \gamma_1\zeta + \eta'_{\rho_\tau}(\pm\delta e^{-\kappa t}) + \gamma_2 - \gamma_1 + \gamma_2 R_\tau^\pm(t)\} + \alpha\eta_{\rho_\tau}(\pm\delta e^{-\kappa t}) \\
&\leqslant \kappa\delta e^{-\kappa t}\{-\tfrac12(|\sigma_0|/\kappa - 1)\psi_0 - \gamma_1\zeta + \gamma_2 - \gamma_1 + \gamma_2 R_{\max}\} \qquad \text{(by Corollary 5.3)} \\
&\leqslant \kappa\delta e^{-\kappa t}\{-\tfrac14(|\sigma_0|/\kappa - 1)\psi_0 + \gamma_2 - \gamma_1 + \gamma_2 R_{\max}\} \qquad \text{(if } \gamma_1 \text{ is sufficiently small)} \\
&\leqslant 0 \qquad \text{(if } \gamma_2 \text{ is sufficiently small)}
\end{aligned}$$

so, for any τ satisfying the above criteria, $S_\tau^\pm$ are sub and super-solutions. □

Now, if τ and δ_0 are as in Lemma 5.4 then there exists $C > 0$ such that

$$|v_0 - u_0|_0 < \delta_0 \implies u_0 - Ce^{-\kappa t} \leqslant S_\tau^-(t) \leqslant v_{v_0}(t) \leqslant S_\tau^+(t) \leqslant u_0 + Ce^{-\kappa t}, \quad t \geqslant 0$$

(by Theorem 4.4), that is, (5.1) holds, which proves part (a) of Theorem 5.1, when $1 < p < 2$.

Proof of part (b) *of Theorem 5.1* (*the case* $\sigma_0 > 0$).

This is similar to the proof of part (a), using the above constructions, so we merely sketch the argument. In this case we do not require the constants γ_1, γ_2 or the slight extension of Lemma 5.2 used in the proof of part (a). In fact, in this case we set $\rho := \psi_0$, and we use δ_ρ and η_ρ as given by the basic version of Lemma 5.2 as stated above. Now, for $0 < \delta_0 < \delta_\rho/2$ and $\delta \in (0, \delta_0)$, we let

$$S_\delta^+(t) := u_0 + \tfrac12\delta e^{\kappa t}\psi_0 + \eta_\rho(\delta e^{\kappa t}/2), \quad 0 \leqslant t \leqslant t_\delta, \quad \text{where } \delta e^{\kappa t_\delta} = 4\delta_0,$$

Note that t_δ is chosen so that S_δ^+ is well defined on the time interval $[0, t_\delta]$. By Corollary 5.3 there exists sufficiently small $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$,

$$u_0 \leqslant S_\delta^+(0) < u_0 + \delta \text{ on } \Omega, \text{ and } |S_\delta^+(t_\delta) - u_0|_0 \geqslant \delta_0,$$

and, by a similar calculation to that in the proof of Lemma 5.4, S_δ^+ is now a sub-solution of (4.1) on Ω_{t_δ}. So, setting $v_{0,\delta} = S_\delta^+(0)$, we have $|v_{0,\delta} - u_0|_1 < \delta|\psi_0|_1$ and, by Theorem 4.4, $|v_{v_0}(t_\delta) - u_0|_0 \geqslant \delta_0$, which proves part (b) of Theorem 5.1, when $1 < p < 2$.

The case $p > 2$.

In essence, the proof in this case repeats the above proof for the case $1 < p < 2$, but there is now an additional problem. In this case the construction of the function η_ρ as in Lemma 5.2 relies on Theorem 3.4 (B), and so yields a C^1 function $\eta_\rho : (-\delta_\rho, \delta_\rho) \to W_0^{1,1}(\Omega)$ (instead of $\eta_\rho : (-\delta_\rho, \delta_\rho) \to C_0^1(\overline{\Omega})$ as previously). Although η_ρ still has all the other properties described in Lemma 5.2, this does not imply that Corollary 5.3 holds, which was required in the proof of Lemma 5.4, and hence (5.1). In fact, if the values of η_ρ are not differentiable at $x = \pm1$, then the inequalities in (5.4) may fail near to $x = \pm1$, even for arbitrarily small $s > 0$.

To deal with this problem we note that, since u_0 satisfies (3.15), Theorem 3.6 shows that the values of η_ρ are slightly more regular near to ±1 than simply belonging to $W_0^{1,1}(\Omega)$. In fact, for sufficiently small $\epsilon > 0$, the mapping $P_\epsilon\,\eta_\rho : (-\delta_\rho, \delta_\rho) \to C_0^1(E_\epsilon)$ is also C^1. That is, the values of η_ρ are C^1 functions of x near to ±1, and these C^1 functions are, in turn, C^1 functions of s. This additional regularity near to ±1 ensures that Corollary 5.3 holds. Having obtained this, the rest of the proof when $p > 2$ is as in the previous case with $1 < p < 2$. This finally completes the proof of Theorem 5.1. □

5.1. **Some further remarks.**

(a) Theorem 5.1 shows that linearised stability implies dynamic stability for the p-Laplacian problem (4.1), with an exponential rate of convergence of solutions near to the equilibrium u_0, with exponent given by $\kappa \in (0, |\sigma_0|)$. For the corresponding linear problem the corresponding exponent would be $\kappa = |\sigma_0|$, so the above rate seems to be the best one could expect for a nonlinear problem.

(*b*) For the nonlinear, semilinear ($p = 2$) problem a similar result to Theorem 5.1 is obtained on p. 113 of [13], with a similar exponent $\kappa \in (0, |\sigma_0|)$. In fact, [13] considers a slightly more general linear, elliptic operator than the Laplacian, in a smooth, bounded domain in $\mathbb{R}^N$, with $N \geqslant 1$. The proof in [13] is similar to the proof of Theorem 5.1, using sub and super-solutions of the form $S^{\pm}_H := u_0 \pm \delta e^{-\kappa t}\psi_0$ (for the stability proof). These are similar to, but simpler than, the sub and super-solutions $S^{\pm}_\tau$ used above, since they do not involve the function η_ρ or the parameters γ_1, γ_2. There are two reasons for our use of the more complicated sub and super-solutions $S^{\pm}_\tau$, which we now describe.

(1) When $p \neq 2$ the domain $\mathcal{D}(\Delta_p)$ of the p-Laplacian operator Δ_p is not linear, so functions of the form $S^{\pm}_H = u_0 \pm \delta e^{-\kappa t}\psi_0$ will not, in general, lie in the domain $\mathcal{D}(\tilde{\Delta}_p)$. Hence, in general, we cannot even apply $\tilde{\Delta}_p$ to these functions, so they cannot be sub or super-solutions. It is primarily this feature of the problem that necessitates the construction of the function η_ρ in Lemma 5.2 (in particular, see Lemma 5.2 (*b*)) and the use of η_ρ in defining our sub and super-solutions. Thus, we cannot omit the function η_ρ (or some term which does a similar job) from the above proof.

(2) The convergence result obtained in [13], using the sub and super-solutions $S^{\pm}_H$, essentially shows that

$$u_0 - \delta_0\psi_0 \leqslant v_0 \leqslant u_0 + \delta_0\psi_0 \implies |v_{v_0}(t) - u_0|_0 \leqslant Ce^{-\kappa t}, \quad t \geqslant 0. \tag{5.5}$$

Clearly, the set of initial conditions v_0 which (5.5) shows converge to u_0 contains a ball $B^1_r(u_0)$ in $C^1_0(\overline{\Omega})$, but it does not contain any ball $B^0_r(u_0)$ in $C^0_0(\overline{\Omega})$. Thus (5.5) is weaker than the convergence result in (5.1) above, which proves convergence for all v_0 in the ball $B^0_{\delta_0}(u_0)$ in $C^0_0(\overline{\Omega})$.

It is the use of the terms involving γ_1, γ_2 in the sub and super-solutions $S^{\pm}_\tau$ which yields this improvement. Indeed, if we simply put $(\gamma_1, \gamma_2) = (0, 0)$ (that is, if we simply omitted these parameters from the proof) then the above proof of Theorem 5.1 would yield a weaker result similar to (5.5).

(*c*) Similar linearised stability results have been obtained in [6] for a quasilinear problem (in a smooth, bounded domain in $\mathbb{R}^N$, $N \geqslant 1$) obtained by replacing $\Delta_p(v)$ in (1.1) with the operator $\nabla \cdot (d(v)\nabla v)$, where the coefficient function $d(\cdot)$ is smooth and strictly positive on $\overline{\Omega}$, so this operator is non-degenerate (unlike the p-Laplacian).

The paper [6] uses sub and super-solutions somewhat similar to the functions $S^{\pm}_H$ used in [13], involving a strongly-positive eigenfunction ψ_0. Due to the form of the operator in [6] the construction there does not encounter the difficulty with the nonlinear operator-domain mentioned in the previous remark (it does, however, encounter some difficulties near the boundary $\partial\overline{\Omega}$, due to the fact that ψ_0 is 0 on $\partial\overline{\Omega}$, which necessitates an additional term in the sub and super-solutions to overcome).

As with the result in [13], convergence is obtained for a set of initial conditions v_0 lying between (roughly) functions of the form $S^{\pm}_H$ (as in (5.5)), which contains an open ball $B^1_r(u_0)$ in $C^1_0(\overline{\Omega})$, but does not contain any ball $B^0_r(u_0)$ in $C^0_0(\overline{\Omega})$). Also, the sub and super-solutions constructed in [6] are stationary, so although convergence or divergence is obtained, no estimate of the rate of convergence or divergence is obtained in [6].

(*d*) The paper [7] obtains estimates on the rate of convergence of solutions to equilibria of p-Laplacian problems in terms of Łojasiewicz-Simon gradient inequalities on an energy function related to the problem. The estimates in [7] are difficult to obtain, and the results obtained do not appear to overlap with the linearised stability results obtained here.

(*e*) Linearised stability of various quasilinear p-Laplacian-type problems has been discussed in a multitude of papers, see for example [1, 15], among many others. These papers do not discuss dynamic stability, and do not attempt to prove any principle of linearised stability.

5.2. **The case $f_\xi < 0$.** Given any solution u_0 of (3.14) satisfying (3.15) and (3.16), the principal eigenvalue of the operator Λ_{p,u_0} is < 0 (see [3]). Hence, if we have $f_\xi(u_0) \leqslant 0$ on Ω then, by the form of $\mathcal{L}_{u_0}$, we immediately have $\sigma_0 < 0$, so u_0 is a stable equilibrium solution of (4.1), in the sense of Theorem 5.1. With a slightly stronger inequality on $f_\xi(u_0)$ we can obtain a slightly stronger stability result.

Theorem 5.5. *Suppose that u_0 is an equilibrium solution of* (4.1). *Suppose also that there exists $\delta_0 > 0$ and $\kappa > 0$ such that*

$$f_\xi(x, u_0(x) + \xi) \leqslant -\kappa, \quad x \in \overline{\Omega}, \ |\xi| \leqslant \delta_0. \tag{5.6}$$

Then

$$|v_0 - u_0|_0 \leqslant \delta_0 \implies |v_{v_0}(t) - u_0|_0 \leqslant \delta_0 e^{-\kappa t}, \quad t \geqslant 0. \tag{5.7}$$

Proof. Define

$$S^\pm_{\delta_0}(t) := u_0 \pm \delta_0 e^{-\kappa t}, \quad t \geqslant 0.$$

For all $t \geqslant 0$, it is clear that $S^\pm_{\delta_0}(t) \in \mathcal{D}(\tilde{\Delta}_p)$, with $\tilde{\Delta}_p(S^\pm_{\delta_0}(t)) = \Delta_p(u_0) = -f(u_0)$, and hence $S^\pm_{\delta_0}(t) \in \tilde{\Sigma}_\infty$ and

$$\begin{aligned} \pm\Big\{\frac{dS^\pm_{\delta_0}}{dt}(t) - (\tilde{\Delta}_p(S^\pm_{\delta_0}(t)) + f(S^\pm_{\delta_0}(t)))\Big\} &= -\kappa\delta_0 e^{-\kappa t} + f(u_0) - f(u_0 + \delta_0 e^{-\kappa t}) \\ &\geqslant \delta_0 e^{-\kappa t}(-\kappa + \kappa) = 0 \quad \text{on } \overline{\Omega} \end{aligned}$$

(by applying the mean value theorem to f and using (5.7)), so $S^+_{\delta_0}$ is a super-solution and $S^-_{\delta_0}$ is a sub-solution of (4.1) on Ω_∞. We also have $\pm(S^\pm_{\delta_0}(t) - u_0) > 0$ on Γ_∞, so (5.7) now follows from Theorem 4.4. □

Remark 5.6. (*a*) Theorem 5.5 is slightly stronger than Theorem 5.1 in the sense that (3.15) and (3.16) are not assumed, and the value of δ_0 is given relatively explicitly in Theorem 5.5.

(*b*) On the other hand, the exponent κ in the convergence rate in Theorem 5.5 is given by a bound on the values of f_ξ, not by the principal eigenvalue of the linearisation at u_0, so the convergence rate given by Theorem 5.5 is weaker than that given by Theorem 5.1. In fact, apart from the hypothesis (5.6) on the values of the derivative f_ξ, Theorem 5.5 does not rely on any 'linearisation' of the problem, so is not really a 'linear stability' result.

(*c*) It does not seem possible obtain a similarly simple divergence result, since the functions $S^\pm_{\delta_0}$ used in the proof of Theorem 5.5 have $\pm(S^\pm_{\delta_0}(t) - u_0) > 0$ at $x = \pm 1$, for $t \geqslant 0$, so they cannot be used in the comparison theorem, Theorem 4.4, to 'push' the solutions of (4.1) away from u_0.

We also observe that the proof of Theorem 5.5 only relied on Theorem 4.2 (implicitly, for the basic existence of solutions) and Theorem 4.4; none of the above linearisation machinery was used. However, Theorems 4.2 and 4.4 are in fact valid in $N \geqslant 1$ dimensions (see the cited references for these theorems; in particular, for more details on the formulation of the p-Laplacian in $N \geqslant 1$ dimensions), so Theorem 5.5 is also valid in $N \geqslant 1$ dimensions. We state this in the following theorem, but since it is not a linearised stability result we will not pursue this further.

Theorem 5.7. *Theorem 5.5 holds, verbatim, when Ω is a smooth, bounded domain in $\mathbb{R}^N$, with $N \geqslant 1$.*

6. Bifurcating equilibria and exchange of stability

We now consider the problem

$$\frac{dv}{dt} = \Delta_p(v) + \lambda g(v)\phi_p(v), \quad v(0) = v_0 \in C^0_0(\overline{\Omega}). \tag{6.1}$$

where $\lambda \in \mathbb{R}$, and we suppose that $p > 2$ and

$$g_0 := g(x, 0) > 0, \quad \text{on } \overline{\Omega}. \tag{6.2}$$

The problem (6.1) is of the form (4.1), with $f(v) = \lambda g(v)\phi_p(v)$, and we assume that the functions g, $g_\xi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are continuous, and the boundedness condition (H-f) holds (given (6.2) and the form of ϕ_p, we need $p > 2$ for this to hold), so the previous results apply to (6.1). Specifically, by Theorem 4.2, (6.1) has a solution, which we will denote by $v_{\lambda,v_0} \in \Sigma_\infty$.

The equilibria of (6.1) satisfy

$$\Delta_p(u) + \lambda g(u)\phi_p(u) = 0, \quad \lambda \in \mathbb{R}, \quad u \in \mathcal{D}(\Delta_p), \tag{6.3}$$

and it is clear that (6.3) has a line of *trivial* solutions $(\lambda, 0)$, $\lambda \in \mathbb{R}$, in, say, $\mathbb{R}\times C^0_0(\overline{\Omega})$. Regarding (6.3) as a bifurcation problem, we are interested in the existence of non-trivial solutions of (6.3) bifurcating from this line of trivial solutions, and in the stability, or instability, of both the trivial solutions, and the non-trivial bifurcating solutions, when regarded as equilibria of (6.1).

When $p = 2$ the well-known 'simple bifurcation' results of [8] show that a curve of non-trivial solutions of (6.3) bifurcates from the line of trivial solutions at the eigenvalues of the linear Laplacian. Furthermore, the results of [9, 13] show that there is an 'exchange of stability' between the line of trivial solutions and the curve of non-trivial solutions bifurcating from the principal eigenvalue – see the discussion and bifurcation diagram on p. 114 of [13] (which considers a problem similar to (6.1), with $p = 2$). We will make this more precise below.

When $p > 2$ a 'simple bifurcation' theorem similar to that of [8] was proved in [11], which shows that a curve of non-trivial solutions bifurcates from the line of trivial solutions at the eigenvalues of the p-Laplacian Δ_p (this is described more precisely in Theorem 6.1 below). Here, we are interested in the stability of these solutions and we will obtain an exchange of stability result similar to the $p = 2$ result described in [9, 13]. We will obtain the stability of the bifurcating solutions from the above linearised stability results, but we cannot use this method for the trivial solutions since they do not satisfy (3.15) and (3.16), so we will have to tackle these more directly.

6.1. **Bifurcation results.** We begin by describing the 'simple bifurcation' results for (6.3). By [4] or [16, Lemma 1.1], the nonlinear, p-Laplacian eigenvalue problem

$$-\Delta_p(\Psi) = \lambda g_0 \phi_p(\Psi), \quad \Psi \in \mathcal{D}(\Delta_p),$$

has a unique principal eigenvalue λ_0 (with $\lambda_0 > 0$) and positive, normalised principal eigenfunction $\Psi_0 \in \mathcal{D}(\Delta_p)$, that is,

$$\Delta_p(\Psi_0) + \lambda_0 g_0 \phi_p(\Psi_0) = 0, \quad \Psi_0 > 0 \text{ in } \Omega, \ |\Psi_0|_0 = 1. \tag{6.4}$$

Also, by [4], $\pm\Psi_0'(\pm1) < 0$, so Ψ_0 is strongly positive. The following result shows that a 'simple bifurcation' takes place at $(\lambda, u) = (\lambda_0, 0)$.

Theorem 6.1. *There exists $\delta_0 > 0$ and C^1 functions $\lambda : (-\delta_0, \delta_0) \to \mathbb{R}$, $y : (-\delta_0, \delta_0) \to C^0_0(\overline{\Omega})$, such that for each $s \in (-\delta_0, \delta_0)$:*

- (*a*) $(\lambda(s), u(s)) = (\lambda(s), s\tilde{u}(s)) := (\lambda(s), s(\Psi_0 + y(s)))$, *is a solution of* (6.3);
- (*b*) $\lambda(0) = \lambda_0, \quad y(0) = 0, \quad \langle y(s), \Psi_0\rangle = 0$;
- (*c*) *the function* $y : (-\delta_0, \delta_0) \to C^1_0(\overline{\Omega})$ *is continuous, so* $\tilde{u}(s)$ *is strongly positive.*

In addition, there exists a neighbourhood U_0 of $(\lambda_0, 0)$ in $\mathbb{R}\times C^0_0(\overline{\Omega})$ such that the set of non-trivial solutions of (6.3) *in U_0 is the set $\{(\lambda(s), u(s)) : 0 < |s| < \delta_0\}$.*

Proof. Most of these results were first obtained in [11], for a slightly different problem, and slightly less regularity was obtained (in particular, part (*c*) was not obtained). A slightly improved version was obtained in [12]. The full theorem, for the above problem, was obtained in [18, Theorem 4.1] (parts (*a*) and (*b*)) and [18, Corollary 4.5] (part (*c*)). The results were also extended to the case $1 < p < 2$ in [18], and slightly great regularity was obtained in this case. □

The following theorem gives the 'direction' of the bifurcating curve of solutions in the space $\mathbb{R} \times C^0_0(\overline{\Omega})$. We let $g_{\xi,0}(x) := g_\xi(x, 0)$, $x \in \overline{\Omega}$ (and recall that $\Psi_0 \geqslant 0$).

Theorem 6.2. *The derivative $\lambda'(0)$ is given by*

$$\lambda'(0) = -\lambda_0 \frac{\langle g_{\xi,0}, \Psi_0^{p+1}\rangle}{\langle g_0, \Psi_0^p\rangle}. \tag{6.5}$$

Proof. For $0 < |s| < \delta_0$, substituting $(\lambda, u) = (\lambda(s), s\tilde{u}(s))$ into (6.3) and dividing by $\phi_p(s)$ yields

$$\Delta_p(\tilde{u}(s)) + \lambda(s)\, g(u(s))\, \phi_p(\tilde{u}(s)) \equiv 0, \tag{6.6}$$

and applying the operator Δ_p^{-1} to (6.6), and substituting the form of $\tilde{u}(s)$ in Theorem 6.1 yields

$$\Psi_0 + y(s) \equiv -\Delta_p^{-1}\big(\lambda(s)\, g(u(s))\, \phi_p(\Psi_0 + y(s))\big). \tag{6.7}$$

This formula also makes sense at $s = 0$ where, by Theorem 6.1, it reduces to

$$\Psi_0 = -\Delta_p^{-1}\big(\lambda_0\, g_0\, \phi_p(\Psi_0)\big).$$

Now, Ψ_0 has only simple zeros (at ± 1) so by (6.2) and Theorem 3.4 (B), the operator Δ_p^{-1} is differentiable at $\lambda_0 g_0 \phi_p(\Psi_0)$. Also, by [5, lemma 3.8], the Nemitsky operator $\phi_p : C^0(\overline{\Omega}) \to C^0(\overline{\Omega})$ is differentiable at Ψ_0, with derivative given by

$$D\phi_p(\Psi_0)w = (p-1)|\Psi_0|^{p-2}w, \quad w \in C^0(\overline{\Omega}).$$

Hence, we may differentiate (6.7) with respect to s, at $s = 0$, and then apply the operator Λ_{p,ψ_0} and use (3.11) to obtain

$$\Lambda_{p,\psi_0}\, y'(0) + \lambda'(0)\, g_0\, \phi_p(\Psi_0) + \lambda_0\big\{g_{\xi,0}\, \Psi_0\, \phi_p(\Psi_0) + (p-1)\, g_0\, |\Psi_0|^{p-2}\, y'(0)\big\} = 0.$$

Taking the inner product of this equation with Ψ_0 and using (3.12), (3.13) and (6.4) now yields

$$\begin{aligned}\big\langle y'(0), \Lambda_{p,\psi_0}\, \Psi_0 + \lambda_0\, (p-1)\, g_0\, \phi_p(\Psi_0)\big\rangle + \lambda'(0)\big\langle g_0\, \phi_p(\Psi_0), \Psi_0\big\rangle + \lambda_0\big\langle g_{\xi,0}\, \Psi_0\, \phi_p(\Psi_0), \Psi_0\big\rangle \\ = \lambda'(0)\big\langle g_0\, \phi_p(\Psi_0), \Psi_0\big\rangle + \lambda_0\big\langle g_{\xi,0}\, \Psi_0\, \phi_p(\Psi_0), \Psi_0\big\rangle = 0.\end{aligned}$$

and (6.5) follows immediately from this. □

Remark 6.3. (a) Since the mapping $s \to y(s)$ in Theorem 6.1 is continuous into $C_0^1(\overline{\Omega})$, it follows from the strong positivity of Ψ_0 that, for sufficiently small $|s| \neq 0$, the bifurcating solutions $u(s)$ are either strongly positive or strongly negative, and

$$\operatorname{sgn} u(s) = \operatorname{sgn} s = \lambda'(0) \operatorname{sgn}(\lambda(s) - \lambda_0).$$

(b) A similar formula to (6.5) was derived on p. 40 of [11], for the problem considered there, but there is a slight error in the proof in [11], and the term Ψ_0^{p+2} in the integral in the formula [11] should be Ψ_0^{p+1} (using our notation, with $N = 1$ and $\gamma = 1$ in the formula in [11], due to our differentiability conditions).

6.2. **Stability of the non-trivial bifurcating solutions.** Now that we have a curve of non-trivial solutions of (6.3), we wish to consider the stability of these solutions, and by Theorem 5.1 it suffices to consider their linearised stability.

For $0 < |s| < \delta_0$ the eigenvalue problem for the linearisation of (6.3) at the solution $(\lambda(s), u(s))$ takes the form

$$\Lambda_{p,u(s)}w + \lambda(s)\big\{g_\xi(u(s))\, \phi_p(u(s)) + (p-1)\, g(u(s))\, |u(s)|^{p-2}\big\}w = \sigma_0 w, \tag{6.8}$$

and we want a solution with $w \geqslant 0$ and $|w|_0 = 1$. Dividing (6.8) by $|s|^{p-2}$ and setting $\tilde{\sigma}_0 := \sigma_0/|s|^{p-2}$, yields

$$\Lambda_{p,\tilde{u}(s)}w + \lambda(s)\big\{s\, g_\xi(u(s))\, \phi_p(\tilde{u}(s)) + (p-1)\, g(u(s))\, |\tilde{u}(s)|^{p-2}\big\}w = \tilde{\sigma}_0 w. \tag{6.9}$$

Although we have assumed that $s \neq 0$ in the derivation of (6.9), this problem also makes sense when $s = 0$, and indeed, putting $s = 0$ in (6.9) yields the problem

$$\Lambda_{p,\psi_0}\, w + \lambda_0\, (p-1)\, g_0\, |\Psi_0|^{p-2}\, w = \tilde{\sigma}_0\, w. \tag{6.10}$$

Substituting $\tilde{\sigma}_0 = 0$, $w = \Psi_0$ into (6.10) and using (3.12) and (6.4) now shows that $(\tilde{\sigma}_0, w) = (0, \Psi_0)$ satisfies (6.10), and since $\Psi_0 > 0$ on Ω, we see that $\tilde{\sigma}_0 = 0$ is the unique principal eigenvalue of the problem (6.10), with corresponding positive, normalised eigenfunction Ψ_0.

We can now show that there is a curve of principal eigenvalues and eigenfunctions of (6.9), emanating from $(0, \Psi_0)$.

Lemma 6.4. *There exists $\delta_1 > 0$ and a continuous curve of solutions of (6.9) of the form $s \to (\tilde{\sigma}_0(s), w(s)) : (-\delta_1, \delta_1)) \to \mathbb{R} \times C_0^0(\overline{\Omega})$, with the following properties, for $|s| < \delta_1$,*

$$\begin{aligned} w(s) = \Psi_0 + z(s) \in C_0^0(\overline{\Omega}), \quad \langle z(s), \Psi_0\rangle = 0, \\ \tilde{\sigma}_0(0) = 0, \quad z(0) = 0, \quad w(s) > 0 \quad \text{on } \Omega. \end{aligned} \tag{6.11}$$

Proof. Instead of (6.9), we first consider the perturbed linear eigenvalue problem

$$w + \lambda(s)K(s)\big\{s\, g_\xi(u(s))\, \phi_p(\tilde{u}(s)) + (p-1)\, g(u(s))\, |\tilde{u}(s)|^{p-2}\big\}w = \tilde{\sigma}_0 K(s)w, \tag{6.12}$$

where $K(s) := D\Delta_p^{-1}(\tilde{u}(s))$; it follows from (3.11) that any solution of (6.12) yields a solution of (6.9). Now, the operators in (6.12) are bounded and depend continuously on s, so this problem is of the form considered in [9, Lemma 1.3]. The desired results will now follow from [9, Lemma 1.3] and [9, Remark 1.11] if we can show that when $s = 0$ the problem (6.12) satisfies the 'K-simplicity' condition in [9, Definition 1.2] (see the proof of [9, Corollary 1.13] for a similar argument, with a constant operator K). In the present setting, the operator $K(0) = D\Delta_p^{-1}(\Psi_0) : C_0^0(\overline{\Omega}) \to C_0^0(\overline{\Omega})$ is compact, so condition (i) in [9, Definition 1.2] is immediate, and verifying condition (ii) reduces to showing that the equation

$$z + \lambda_0(p-1)\, K(0)\big(g_0\, |\Psi_0|^{p-2}\, z\big) = K(0)\Psi_0, \quad z \in C_0^0(\overline{\Omega}),$$

has no solution z, and by (3.11) it suffices to show that the equation

$$\Lambda_{p,\psi_0}\, z + \lambda_0(p-1)\, g_0\, |\Psi_0|^{p-2}\, z = \Psi_0, \quad z \in C_0^0(\overline{\Omega}), \tag{6.13}$$

has no solution. Taking the inner product of (6.13) with Ψ_0 and using (3.12), (3.13) and (6.4) now yields

$$\langle \Psi_0, \Psi_0\rangle = (p-1)\big\langle z, \Delta_p(\Psi_0) + \lambda_0\, g_0\, \phi_p(\Psi_0)\big\rangle = 0,$$

which shows that (6.13) cannot have a solution, and so completes the proof. □

Now that we have a curve of principal eigenvalues of the linearisation, we want to know the sign of these eigenvalues.

Theorem 6.5. *The function $\tilde\sigma_0(\cdot)$ is differentiable at $s = 0$, with*

$$\tilde\sigma_0'(0) = \lim_{s\to 0} \frac{\tilde\sigma_0(s)}{s} = \lambda_0 \frac{\langle g_{\xi,0}, \Psi_0^{p+1}\rangle}{\langle \Psi_0, \Psi_0\rangle}. \tag{6.14}$$

Proof. For $0 < |s| < \delta_1$, taking the inner product of (6.9) with $\tilde u(s)$, and using (3.12), (3.13) and (6.6) now yields

$$\begin{aligned}(p-1)\big\{\big\langle w(s), \Delta_p(\tilde u(s)) + \lambda(s)\, g(u(s))\, \phi_p(\tilde u(s))\big\rangle\big\} + s\lambda(s)\big\langle w(s), g_\xi(u(s))\, |\tilde u(s)|^p\big\rangle \\ = s\lambda(s)\, \langle g_\xi(u(s))\, |\tilde u(s)|^p, w(s)\rangle = \tilde\sigma_0(s)\langle \tilde u(s), w(s)\rangle.\end{aligned}$$

Rearranging the latter result, and using the properties of $\tilde\sigma_0(s)$ and $w(s)$ in Lemma 6.4, enables us to evaluate the limit in (6.14) to complete the proof. □

Finally, since $\Psi_0 > 0$, $g_0 > 0$, on Ω, comparing (6.5) with (6.14) yields the following result.

Corollary 6.6. $\operatorname{sgn}\tilde\sigma_0'(0) = -\operatorname{sgn}\lambda'(0) \neq 0$.

In the case $p = 2$, this result was obtained in [9, Theorem 1.16], in an abstract setting, and on p. 114 of [13] for a problem similar to (6.1). It is the basis of the idea of 'exchange of stability', which we will discuss, briefly, in Section 6.4.

6.3. **Stability of the trivial solutions.** We now consider the stability of the trivial solutions of (6.1). As already mentioned, we cannot obtain this from the above linearised stability results, but we will obtain an analogue of the results of Theorem 5.1 for these solutions, albeit with a different convergence rate.

Theorem 6.7. (a) *Suppose that $\lambda < \lambda_0$. Then for any $\kappa \in (0, 1/(p-2))$ there exists $\delta_0 > 0$ and $A > 0$ such that*

$$|v_0|_0 \leqslant \delta_0 \implies |v_{\lambda,v_0}(t)|_0 \leqslant (A+t)^{-\kappa}, \quad t \geqslant 0.$$

That is, the trivial solution $(\lambda, 0)$ of (6.1) is asymptotically stable.

(b) *Suppose that $\lambda > \lambda_0$. Then for any $\kappa \in (0, 1/(p-2))$ there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, there exists $v_{0,\delta} \in C_0^1(\overline{\Omega})$ such that*

$$|v_{0,\delta} - u_0|_1 < \delta|\Psi_0|_1 \quad \text{and} \quad |v_{\lambda,v_0}(t_\delta)|_0 \geqslant \delta_0, \quad \text{where} \quad t_\delta := (\tfrac{1}{2}\delta)^{-1/\kappa} - (\delta_0)^{-1/\kappa}.$$

That is, the trivial solution $(\lambda, 0)$ of (6.1) is unstable.

Proof of part (a). For any $\beta \geqslant 0$, let $I_\beta := (-1-\beta, 1+\beta)$ and define $\bar g_\beta : \overline{I}_\beta \to \mathbb{R}$ by

$$\bar g_\beta(x) := \begin{cases} \sup\{g(x,\xi) : |\xi| \leqslant \beta\}, & |x| \leqslant 1, \\ \bar g_\beta(\pm 1), & \pm x > 1. \end{cases} \tag{6.15}$$

Clearly, $\bar g_\beta \in C(\overline{I}_\beta)$, and $\lim_{\beta\to 0^+} |\bar g_\beta|_{\overline{\Omega}} - g_0|_0 = 0$ (where $\bar g_\beta|_{\overline{\Omega}}$ denotes the restriction of $\bar g_\beta$ to the interval $\overline{\Omega}$), so by (6.2) we may suppose that β is sufficiently small that $\bar g_\beta > 0$ on $\overline{I}_\beta$. We also define the p-Laplacian operator $\Delta_{p,\beta}$ on the interval I_β, in the same manner as we defined Δ_p on $\Omega = I_0$ in (3.1), and we let $\bar\lambda_{0,\beta}$ and $\bar\Psi_{0,\beta} \in \mathcal{D}(\Delta_{p,\beta})$ denote the principal eigenvalue and eigenfunction of the p-Laplacian eigenvalue problem

$$-\Delta_{p,\beta}(\bar\Psi_{0,\beta}) = \bar\lambda_{0,\beta}\, \bar g_\beta\, \phi_p(\bar\Psi_{0,\beta}), \quad \bar\Psi_{0,\beta} > 0 \text{ on } I_\beta. \tag{6.16}$$

It can be shown that $\lim_{\beta\to 0^+} \bar\lambda_{0,\beta} = \lambda_0$ ([16, Lemma 1.1 (e)] proves a similar result for the principal eigenvalues of an increasing sequence of domains in $\mathbb{R}^N$). Hence, since $\lambda < \lambda_0$, we may choose $\beta_0 > 0$ such that

$$\lambda < \bar\lambda_{0,\beta_0}, \tag{6.17}$$

and we define $\Psi_a := \bar\Psi_{0,\beta_0}|_{\overline{\Omega}} \in C^1(\overline{\Omega})$. Clearly, $\Psi_a > 0$ on $\overline{\Omega}$, and we suppose that $|\Psi_a|_0 = 1$ (if not, we rescale it).

Next, we write $\kappa = (1-\nu)/(p-2)$, with $\nu \in (0,1)$. Rearranging this shows that

$$-\kappa - 1 = -\kappa(p-1) - \nu, \tag{6.18}$$

which will be used in the calculation below. Now, for $A > 0$, let

$$S_A^\pm(t) := \pm(A+t)^{-\kappa}\, \Psi_a, \quad t \geqslant 0.$$

Lemma 6.8. *If $A > 0$ is sufficiently large then S_A^- is a sub-solution and S_A^+ is a super-solution of* (6.1) *on Ω_∞, and $\pm S_A^\pm \geqslant 0$ on Γ_∞.*

Proof. By the above constructions, $S_A^\pm \in \tilde\Sigma_\infty$, and if A is sufficiently large then, for all $t \geqslant 0$, $|S_A^\pm(t)|_0 \leqslant \beta_0$. Hence, by (6.15)-(6.17),

$$\begin{aligned}
&\pm\Big\{\frac{dS_A^\pm}{dt}(t) - \big(\tilde\Delta_p(S_A^\pm(t)) + \lambda g(S_A^\pm(t))\phi_p(S_A^\pm(t))\big)\Big\} \\
&\qquad = (A+t)^{-\kappa(p-1)}\big\{-\kappa(A+t)^{-\nu}\Psi_a + \big(\bar\lambda_{0,\beta_0}\,\bar g - \lambda g(S_A^\pm(t))\big)\phi_p(\Psi_a)\big\} \\
&\qquad \geqslant 0 \quad \text{on } \overline{\Omega},
\end{aligned}$$

for sufficiently large A, recalling that $\Psi_a > 0$ on $\overline{\Omega}$. By Definition 4.3, this proves the first result. The second is clear from the construction of $S_A^\pm$. □

By choosing A as in Lemma 6.8 and setting $\delta_0 = A^{-\kappa}\min\{\Psi_a(x) : x \in \overline{\Omega}\}$, part (a) of the theorem now follows from Theorem 4.4.

Proof of part (b). This is similar to the proof of part (a), so we merely sketch the argument. For any $\beta > 0$, define $\underline{g}_\beta \in C^0(\overline{\Omega})$ by

$$\underline{g}_\beta(x) := \inf\{g(x,\xi) : |\xi| \leqslant \beta\}, \quad x \in \overline{\Omega}.$$

Clearly, $\lim_{\beta\to 0^+} |\underline{g}_\beta - g_0|_0 = 0$, so by (6.2) we may suppose that β is sufficiently small that $\underline{g}_\beta > 0$ on $\overline{\Omega}$, and we let $\underline{\lambda}_{0,\beta}$ and $\underline{\Psi}_{0,\beta}$ denote the principal eigenvalue and eigenfunction of the problem

$$-\Delta_p(\underline{\Psi}_{0,\beta}) = \underline{\lambda}_{0,\beta}\, \underline{g}_\beta\, \phi_p(\underline{\Psi}_{0,\beta}), \quad \underline{\Psi}_{0,\beta} > 0 \text{ on } \Omega, \ |\underline{\Psi}_{0,\beta}|_0 = 1.$$

Since $\lambda > \lambda_0$ in this case, we may now choose $\beta_0 > 0$ such that $\lambda > \underline{\lambda}_{0,\beta_0}$ and $|\underline{\Psi}_{0,\beta_0}|_1 < 2|\Psi_0|_1$, and we now define $\Psi_a := \underline{\Psi}_{0,\beta_0}$. Now, for $0 < \delta_0 < \beta_0$ and $\delta \in (0, \delta_0)$, let

$$A_\delta := (\tfrac{1}{2}\delta)^{-1/\kappa}, \quad S_\delta^+(t) := (A_\delta - t)^{-\kappa}\, \Psi_a, \quad 0 \leqslant t < A_\delta, \quad t_\delta := A_\delta - (\delta_0)^{-1/\kappa}.$$

If δ_0 is sufficiently small then S_δ^+ is a sub-solution of (6.1) on Ω_{t_δ}, and $|S_\delta^+(0)|_1 = \frac{1}{2}\delta|\Psi_a|_1 < \delta|\Psi_0|_1$, $|S_\delta^+(t_\delta)|_0 = \delta_0$. Hence, by Theorem 4.4, if we set $v_0 = S_\delta^+(0)$ then $|v_{\lambda,v_0}(t_\delta)|_0 \geqslant \delta_0$, which proves part (b) of Theorem 6.7, and so completes the proof. □

Remark 6.9. The exponent κ in the rates of convergence and divergence in Theorem 6.7 is consistent with the homogeneity properties of the problem (6.1), in the sense that the solutions of the ODE problems $z_t = \pm z^{p-1}$, $z(0) > 0$, have the form $z_\pm(t) = \big((p-2)(A \mp t)\big)^{-1/(p-2)}$, $A > 0$.

6.4. **Exchange of stability.** Corollary 6.6 and Theorem 6.7 now yield exchange of stability results for the trivial and bifurcating, non-trivial solutions (in a neighbourhood of $(\lambda_0, 0)$ in $\mathbb{R} \times C_0^0(\overline{\Omega})$), which can be summarised as follows:

- the trivial solutions $(\lambda, 0)$ are stable when $\lambda < \lambda_0$ and unstable when $\lambda > \lambda_0$;
- the bifurcating solutions (λ, u) are stable when $\lambda > \lambda_0$ and unstable when $\lambda < \lambda_0$.

Thus, when $p > 2$ we obtain a similar bifurcation diagram to that on p. 114 of [13], which considered the case $p = 2$. The idea of exchange of stability of bifurcating solutions is well-known, at least when $p = 2$, so we will not discuss this further.

As a final remark, we note that in physical models the positive solutions are often the ones of interest, as is their stability. By Remark 6.3 and the above exchange of stability we see that this is determined by the sign of $\lambda'(0)$ (where $\lambda'(0)$ is given by (6.5)). Specifically:

- if $\lambda'(0) > 0$ then the bifurcating, positive solutions are 'supercritical' and stable;
- if $\lambda'(0) < 0$ then the bifurcating, positive solutions are 'subcritical' and unstable.

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