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STOCHASTIC NAVIER–STOKES–FOURIER EQUATIONS

DOMINIC BREIT AND EDUARD FEIREISL

ABSTRACT. We study the full Navier–Stokes–Fourier system governing the motion of a general viscous, heat-conducting, and compressible fluid subject to stochastic perturbation. Stochastic effects are implemented through (i) random initial data, (ii) a forcing term in the momentum equation represented by a multiplicative white noise, (iii) random heat source in the internal energy balance. We establish existence of a weak martingale solution under physically grounded structural assumptions. As a byproduct of our theory we can show that stationary martingale solutions only exist in certain trivial cases.

1. INTRODUCTION

There is an abundant amount of literature concerning notably the stochastic perturbations of the Navier–Stokes system in the context of incompressible fluid flows starting with [3]. The existence of weak martingale solutions – these solutions are weak in the analytical sense (derivatives only exist in the sense of distributions) and weak in the probabilistic sense (the underlying probability space is part of the solution) – was first shown in [14] and [27] (see also [11], [16] and [38] for further results). Also, the incompressible stochastic Euler equations has been studied extensively, see [12, 28, 32, 39]. Definitely much less is known if compressibility of the fluid is taken into account. Similarly to [3], first existence results were based on a suitable transformation formula, cf. [20]. It allows to reduce the problem to a random system of PDEs, where the stochastic integral does no longer appear. Existence of solutions in this semi-deterministic setting have been shown in [20] (see also [47] for the 1D case in the Lagrange coordinates and [48] for a rather artificial periodic 2D case). The first “truly” stochastic existence result for the compressible Navier–Stokes system perturbed by a general nonlinear multiplicative noise was obtained by Breit and Hofmanová [9] for periodic boundary conditions (see [45] for the extension to the homogeneous Dirichlet boundary conditions and [37], where the system is studied on the whole space). It is based on the concept of finite energy weak martingale solutions. As in [27] these solutions are weak in the analytical sense and weak in the probabilistic sense. Moreover, the time-evolution of their energy can be controlled in terms of the initial energy. This allows to study some asymptotic properties of the system, see [5], [6] and [37]. All these results are concerned with the barotropic case. The natural next step is to take additionally into account the transfer of heat. The aim of this paper is to develop a consistent mathematical theory of viscous, compressible, and heat-conducting fluid flows driven by stochastic external forces.

The motion of the fluid is described by the standard field variables: the mass density $\varrho = \varrho(t, x)$; the absolute temperature $\vartheta(t, x)$; the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ evaluated at the time t and the spatial position x belonging to the reference physical domain $Q \subset \mathbb{R}^3$. The time evolution of the fluid is governed by the full Navier–Stokes–Fourier system with stochastic forcing:

eq:1

$$(1.1a) \quad d\varrho + \operatorname{div}(\varrho\mathbf{u}) dt = 0,$$

eq:12

$$(1.1b) \quad d(\varrho\mathbf{u}) + [\operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta)] dt = \operatorname{div} \mathbb{S}(\vartheta, \nabla\mathbf{u}) dt + \varrho\mathbf{F}(\varrho, \vartheta, \mathbf{u}) dW,$$

eq:11

eq:13

$$(1.1c) \quad d(\varrho e(\varrho, \vartheta)) + [\operatorname{div}(\varrho e(\varrho, \vartheta)\mathbf{u}) + \operatorname{div} \mathbf{q}(\vartheta, \nabla\vartheta)] dt$$

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$$= [\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p(\varrho, \vartheta) \operatorname{div} \mathbf{u}] dt + \varrho H dt.$$

The field equations (1.1) describe the balance of mass, momentum and internal energy. They must be supplemented with a set of constitutive relations characterizing the material properties of a concrete fluid. In particular, the viscous stress tensor \mathbb{S} , the internal energy flux \mathbf{q} as well as the thermodynamic functions p (pressure) and e (specific internal energy) must be determined in terms of the independent state variables $(\varrho, \vartheta, \mathbf{u})$. For the viscous stress tensor we suppose Newton's rheological law

$$\text{eq:nr} \quad (1.2) \quad \mathbb{S} = \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div} \mathbf{u} \mathbb{I}.$$

The internal energy (heat) flux is determined by Fourier's law

$$\text{eq:f1} \quad (1.3) \quad \mathbf{q} = \mathbf{q}(\vartheta, \nabla \vartheta) = -\kappa(\vartheta) \nabla \vartheta = -\nabla \mathcal{K}(\vartheta), \quad \mathcal{K}(\vartheta) = \int_0^{\vartheta} \kappa(z) dz.$$

The thermodynamic functions p and e are related to the (specific) entropy $s = s(\varrho, \vartheta)$ through Gibbs' equation

$$\text{m97} \quad (1.4) \quad \vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right) \text{ for all } \varrho, \vartheta > 0.$$

Randomness in the time evolution of the system is enforced in three ways: **(i)** the initial state (data) is random; **(ii)** the heat source H is a random variable that may depend on time, **(iii)** the driving process W is a cylindrical Wiener process defined on some probability space, with the diffusion coefficient $\varrho \mathbf{F}$ that may depend on the spatial variable x as well as on the state variables ϱ, ϑ and \mathbf{u} . The precise description of the problem setting will be given in Section 2. Our main result, stated in Theorem 2.1 below, asserts the existence of a *martingale solution* to a suitable weak formulation (1.1)–(1.4) with respect to conservative boundary conditions

$$\text{beq:1} \quad (1.5) \quad \mathbf{u}|_{\partial Q} = 0, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0.$$

We combine the deterministic approach for the Navier–Stokes–Fourier system developed in [21] with the stochastic theory [9] for the barotropic system.

In contrast with the earlier approach proposed in [19], the existence theory in [21] is built up around the Second law of thermodynamics. In view of Gibb's relation (1.4), the internal energy equation (1.1c) can be rewritten in the form of the entropy balance

$$\text{eq:13'} \quad (1.6) \quad d(\varrho s) + \left[\operatorname{div}(\varrho s \mathbf{u}) + \operatorname{div} \left(\frac{\mathbf{q}}{\vartheta} \right) \right] dt = \sigma dt + \varrho \frac{H}{\vartheta} dt$$

with the entropy production rate

$$\text{eq:0202} \quad (1.7) \quad \sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} \right).$$

In view of possible, but in the case of viscous fluids still only hypothetical singularities, it is convenient to relax the equality sign in (1.7) to the inequality

$$\text{eq:0202bis} \quad (1.8) \quad \sigma \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla \mathbf{u} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} \right).$$

Well posedness of the problem is then formally guaranteed by augmenting the system by the total energy balance

$$\text{eq0202bisE} \quad (1.9) \quad d \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right] dx = \int_Q \varrho \mathbf{F} \cdot \mathbf{u} dW + \int_Q \frac{1}{2} \varrho |\mathbf{F}|^2 dt + \int_Q \varrho H dt,$$

cf. [21, Chapter 2]. Note that validity of (1.9) requires the system to be energetically insulated - the total energy flux through the boundary must vanish at any time in accordance with (1.5). In comparison with [21], equality (1.9) contains the contribution of the stochastic driving force here interpreted in Itô's sense.

The main advantage of the entropy formulation is the possibility to deduce a relative energy inequality derived for the deterministic Navier–Stokes–Fourier system in [22], and adapted to

the barotropic *stochastic* Navier–Stokes system in [6]. In particular, we may expect the strong solutions to be stable in the larger class of weak solutions (weak–strong uniqueness). Besides, there are other interesting properties derived for simpler systems in [6] and [22] that are likely to extend to the full thermodynamic framework. Note that a weak formulation based on the internal energy balance in the spirit of [19] apparently does not enjoy these properties unless the weak solution is quite “regular”, see [25]. We remark that a stochastic version of [19] recently appeared in [46].

The total energy balance (1.9) is considered as an integral part of the definition of weak solutions. This excludes any kind of semi-deterministic approach in the spirit of [3] or [20]. Instead, we adapt the multi-layer approximation scheme developed in [21, Chapter 3] combined with a refined stochastic compactness method based on the Jakubowski-Skorokhod representation theorem, cf. [31] (which has been used already before in the construction of martingale solutions to SPDEs, for instance in [11] and [42]). Although a similar idea has been applied to the barotropic Navier–Stokes system in [9], the explicit dependence of the diffusion coefficient \mathbf{F} on the temperature along with the total energy balance appended to the problem give rise to rather challenging new difficulties pertinent to the complete, meaning energetically closed, fluid system. One of the most subtle among them is the necessity to perform the change of probability space via Skorokhod–Jakubowski representation theorem *before* showing compactness of the arguments in the diffusion coefficients \mathbf{F} . This is in sharp contrast with the method developed in [9], where the new probability space emerged in a natural way only at the end of each approximate step of the construction of weak solutions.

Following [21, Chapter 3] we start with the original (internal energy) formulation (1.1) regularized via artificial viscosity (ε -layer), artificial pressure (δ -layer), as well as other stabilizing terms, see Section 3. The so-obtained system is then solved by a Galerkin approximation. Here, the momentum equation is solved in a finite-dimensional subspace whereas we keep continuity and internal energy equation in a continuous framework (m -layer). This has the advantage that the maximum principle applies and both density and temperature remain positive. We follow the approach from [7]: Instead of introducing a stopping-time as in [9] we cut several nonlinearities (R -layer) and gain the existence of a strong solution to the so-obtained system (3.5), see Theorem 3.1. Also differently from [9] we do not apply a fixed-point argument to get a solution to (3.5). Instead we apply a simple time-step by means of a modification of the Cauchy collocation method, see (3.14)–(3.18). The latter one can be solved immediately. Eventually, we pass to the limit in the time-step in Section 3.2. At this stage, it is important to keep the temperature strictly positive in order to divide finally equation (1.1c) obtaining the entropy formulation, cf. Section 4.

The remaining part of the existence proof leans on the entropy formulation, with (1.1c) replaced by (1.6) and (1.9). In Section 4 we perform the limit in the Galerkin approximation scheme (limit $m \rightarrow \infty$) obtaining the artificial viscosity approximation. In Section 5, we get rid of the artificial viscosity and related stabilizing terms (limit $\varepsilon \rightarrow 0$). Finally, in Section 6, we remove the remaining artificial terms recovering a weak solution of the original system (limit $\delta \rightarrow 0$).

2. MATHEMATICAL FRAMEWORK AND THE MAIN RESULT

2.1. Random variables (distributions). Let $Q_T = (0, T) \times Q$. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ be a complete stochastic basis with a Borel probability measure \mathbb{P} and a right-continuous filtration (\mathfrak{F}_t) . For a measurable space (X, \mathcal{A}) an X -valued *random variable* is a measurable mapping $\mathbf{u} : (\Omega, \mathfrak{F}) \rightarrow (X, \mathcal{A})$. We denote by $\sigma(\mathbf{U})$ the smallest σ -field with respect to which \mathbf{U} is measurable, that is

$$\sigma(\mathbf{U}) := \{ \{ \omega \in \Omega; \mathbf{U}(\omega) \in A \}; A \in \mathcal{A} \}.$$

The majority of the random variables we deal with are mappings $\mathbf{U} : \Omega \rightarrow L^1(Q_T, R^M)$. Such an object is not a stochastic process in the classical sense as it is only defined a.e. in time. Due to this the concept of *random distributions* has been introduced in [7][Chap. 2.2] to which we refer for more details. The approach based on random distributions is necessary, in particular,

to handle the time dependent random variables as the velocity \mathbf{u} and the temperature ϑ that do not possess, in general, well defined instantaneous values at *any* time t . The interested reader may consult [21, Section 1.2] for a detailed discussion of the problem of instantaneous values of weak solutions to the deterministic version of the Navier–Stokes–Fourier system.

RDD1 **Definition 2.1.** *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space. A mapping*

$$\mathbf{U} : \Omega \rightarrow (C_c^\infty(Q_T; R^M))'$$

is called random distribution if $\langle \mathbf{U}, \varphi \rangle : \Omega \rightarrow \mathbb{R}$ is a measurable function for any $\varphi \in C_c^\infty(Q_T; R^M)$.

In order to introduce a concept of *progressive measurability* we consider the σ -field of all progressively measurable sets in $\Omega \times [0, T]$ associated to the filtration $(\mathfrak{F}_t)_{t \geq 0}$. To be more precise, $A \subset \Omega \times [0, T]$ belongs to the progressively measurable σ -field provided the stochastic process $(\omega, t) \mapsto \mathbf{1}_A(\omega, t)$ is (\mathfrak{F}_t) -progressively measurable. We denote by $L_{\text{prog}}^1(\Omega \times [0, T])$ the Lebesgue space of functions that are measurable with respect to the σ -field of (\mathfrak{F}_t) -progressively measurable sets in $\Omega \times [0, T]$ and we denote by μ_{prog} the measure $\mathbb{P} \otimes \mathfrak{L}_{[0, T]}$ restricted to the progressively measurable σ -field.

RDD3 **Definition 2.2.** *Let \mathbf{U} be a random distribution in the sense of Definition 2.1. We say that \mathbf{U} is adapted to (\mathfrak{F}_t) if $\langle \mathbf{U}, \varphi \rangle$ is (\mathfrak{F}_t) -measurable for any $\varphi \in C_c^\infty(Q_t)$.*

rrr5 **Remark 2.1.**

- *For random distributions adaptedness and progressive measurability is the same (as it is for continuous stochastic processes).*
- *If X is a separable Hilbert space we can equivalently define the space of progressively measurable random variables as the $L^1(\Omega \times [0, T]; X)$ -closure of the (\mathfrak{F}_t) -progressively measurable stochastic processes belonging to $L^1(\Omega; C^1([0, T]; X))$.*

The above concept is convenient when dealing with general distributions. It coincides with the standard concept of progressive measurability as long as the distribution defines a stochastic process, see [7, Chapter 2, Lemma 2.2.18]. Also, if a random distribution is (\mathfrak{F}_t) -adapted, there is a modification which is (\mathfrak{F}_t) -progressively measurable, cf. [7, Chapter 2, Lemma 2.2.18], as in the classical situation.

It is convenient to consider a random variable $\mathbf{U} : \Omega \rightarrow L^1(Q_T; R^M)$ as a mapping ranging in a larger space, i.e. $\mathbf{U} : \Omega \rightarrow W^{-k, 2}(Q_T; R^M)$, where the latter is a separable Hilbert space; whence Polish. Note that, by virtue of the standard embedding $W_0^{k, 2} \hookrightarrow C$ as soon as $k > \frac{5}{2}$, we have $L^1(Q_T) \hookrightarrow W^{-k, 2}(Q_T)$. Accordingly, any such \mathbf{U} may be viewed as a Borel random variable on the Polish space $W^{-k, 2}(Q_T; R^M)$.

2.1.1. *Initial data.* In the context of martingale solutions, the initial state of the system is prescribed in terms of a law Λ - a Borel measure defined on a suitable function space. We consider $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ to describe the initial state. Accordingly, we consider Λ defined on $L^1(Q) \times L^1(Q) \times L^1(Q; R^3)$ satisfying

$$\begin{aligned} \text{Ida} \quad (2.1) \quad & \Lambda \left\{ \varrho_0 \geq 0, \vartheta_0 > 0, 0 < \underline{\varrho} < (\varrho_0)_Q \equiv \frac{1}{|Q|} \int_Q \varrho_0 \, dx < \bar{\varrho} \right\} = 1, \\ & \int_{L^1 \times L^1 \times L^1} \left\| \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) + \varrho_0 |s(\varrho_0, \vartheta_0)| \right\|_{L^1(Q)}^r \, d\Lambda \leq c(r), \text{ for all } r \geq 1. \end{aligned}$$

Here $\underline{\varrho}, \bar{\varrho}$ are two deterministic constants.

rrr1 **Remark 2.2.** *The second stipulation in (2.1) anticipates that the solutions will belong to the class of finite total energy and entropy. As we shall see below, this follows from (1.9) and (1.6) respectively. We provide more comments on (2.1) in Section 2.2.*

2.1.2. *Mechanical bulk force.* The process W is a cylindrical Wiener process on a separable Hilbert space \mathfrak{U} , that is, $W(t) = \sum_{k \geq 1} \beta_k(t) e_k$ with $(\beta_k)_{k \geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathfrak{F}_t)_{t \geq 0}$. Here $(e_k)_{k \geq 1}$ denotes a complete orthonormal system in \mathfrak{U} . In addition, we introduce an auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 = \left\{ v = \sum_{k \geq 1} \alpha_k e_k; \sum_{k \geq 1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}, \quad v = \sum_{k \geq 1} \alpha_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of W are \mathbb{P} -a.s. in $C([0, T]; \mathfrak{U}_0)$ (see [15]).

Choosing $\mathfrak{U} = \ell^2$ we may identify the diffusion coefficients $(\mathbf{F}e_k)_{k \geq 1}$ with a sequence of real functions $(\mathbf{F}_k)_{k \geq 1}$,

$$\varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) dW = \sum_{k=1}^{\infty} \varrho \mathbf{F}_k(x, \varrho, \vartheta, \mathbf{u}) d\beta_k.$$

We suppose that \mathbf{F}_k are smooth in their arguments, specifically,

$$\mathbf{F}_k \in C^1(\overline{Q} \times [0, \infty)^2 \times R^3; R^3),$$

where

$$\boxed{\text{P-1}} \quad (2.2) \quad \|\mathbf{F}_k(\cdot, \cdot, \cdot, 0)\|_{L^\infty} + \|\nabla_{x, \varrho, s, \mathbf{u}} \mathbf{F}_k\|_{L^\infty} \leq f_k, \quad \sum_{k=1}^{\infty} f_k^2 < \infty.$$

Let us remark that (2.2) implies that \mathbf{F} is bounded in ϱ and ϑ but may grow linearly in \mathbf{u} . We easily deduce from (2.2) the following bound

$$\|\varrho \mathbf{F}_k(\varrho, \vartheta, \mathbf{u})\|_{W^{-k, 2}(Q; R^3)} \lesssim \|\varrho \mathbf{F}_k(\varrho, \vartheta, \mathbf{u})\|_{L^1(Q; R^3)} \lesssim f_k (\|\varrho\|_{L^1(Q)} + \|\varrho \mathbf{u}\|_{L^1(Q; R^3)})$$

whenever $k > \frac{3}{2}$. Accordingly, the stochastic integral

$$\int_0^\tau \varrho \mathbf{F} dW = \sum_{k=1}^{\infty} \int_0^\tau \varrho \mathbf{F}_k(\varrho, \vartheta, \mathbf{u}) d\beta_k$$

can be identified with an element of the Banach space space $C([0, T]; W^{-k, 2}(Q))$,

$$\begin{aligned} & \int_Q \left(\int_0^\tau \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) dW \cdot \varphi \right) dx \\ &= \sum_{k=1}^{\infty} \int_0^\tau \left(\int_Q \varrho \mathbf{F}_k(x, \varrho, \vartheta, \mathbf{u}) \cdot \varphi dx \right) d\beta_k, \quad \varphi \in W^{k, 2}(Q; R^3), \quad k > \frac{3}{2}. \end{aligned}$$

2.1.3. *Heat source.* Similarly to [21, Chapter 3, Section 3.1], the heat source H may depend on both t and x and can also be random, specifically, $H \in C([0, T] \times \overline{Q})$ \mathbb{P} -a.s. In addition, we suppose

$$\boxed{\text{ID3}} \quad (2.3) \quad 0 \leq H \leq \overline{H}, \quad H \text{ } (\mathfrak{F}_t) \text{ - progressively measurable,}$$

where \overline{H} is a deterministic constant.

A heat source appears in numerous real-world applications (see e.g. [21]); therefore we find it important to include it in the existence theory. From the mathematical point of view, however, its presence in the system can be accommodated rather easily at the same level of difficulty as the random initial data.

H

2.2. Structural and constitutive assumptions. Besides Gibbs' equation (1.4), we impose several restrictions on the specific shape of the thermodynamic functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$ and $s = s(\varrho, \vartheta)$. They are borrowed from [21, Chapter 1], to which we refer for the physical background and the relevant discussion.

We consider the pressure p in the form

$$\boxed{\text{m98}} \quad (2.4) \quad p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + \frac{a}{3}\vartheta^4, \quad a > 0, \quad p_M(\varrho, \vartheta) = \vartheta^{5/2}P\left(\frac{\varrho}{\vartheta^{3/2}}\right),$$

$$\boxed{\text{mp8a}} \quad (2.5) \quad e(\varrho, \vartheta) = e_M(\varrho, \vartheta) + a\frac{\vartheta^4}{\varrho}, \quad e_M(\varrho, \vartheta) = \frac{3}{2}\frac{p_M(\varrho, \vartheta)}{\varrho} = \frac{3}{2}\frac{\vartheta^{5/2}}{\varrho}P\left(\frac{\varrho}{\vartheta^{3/2}}\right),$$

$$\boxed{\text{md8!}} \quad (2.6) \quad s(\varrho, \vartheta) = s_M(\varrho, \vartheta) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \quad s_M(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right),$$

$$\boxed{\text{md8}} \quad (2.7) \quad S = S(Z), \quad S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} < 0,$$

where

$$\boxed{\text{m103-}} \quad (2.8) \quad P \in C^1[0, \infty) \cap C^2(0, \infty), \quad P(0) = 0, \quad P'(Z) > 0, \quad \text{for all } Z \geq 0,$$

$$\boxed{\text{md7}} \quad (2.9) \quad 0 < \frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} < c, \quad \text{for all } Z > 0,$$

and

$$\boxed{\text{md1}} \quad (2.10) \quad \lim_{Z \rightarrow \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0.$$

The viscosity coefficients μ, η are continuously differentiable functions of the absolute temperature ϑ , more precisely $\mu, \lambda \in C^1[0, \infty)$, satisfying

$$\boxed{\text{m105}} \quad (2.11) \quad 0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta) \leq \bar{\mu}(1 + \vartheta),$$

$$\boxed{*m105*} \quad (2.12) \quad \sup_{\vartheta \in [0, \infty)} (|\mu'(\vartheta)| + |\lambda'(\vartheta)|) \leq \bar{m},$$

$$\boxed{\text{m106}} \quad (2.13) \quad 0 \leq \lambda(\vartheta) \leq \bar{\lambda}(1 + \vartheta).$$

The heat conductivity coefficient $\kappa \in C^1[0, \infty)$ satisfies

$$\boxed{\text{m108}} \quad (2.14) \quad 0 < \underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^3).$$

Remark 2.3. *These structural hypotheses imply, in particular, that solutions of finite total energy belong to certain Lebesgue spaces of integrable functions. Indeed it follows from (2.8) that the (molecular) energy e_M is always non-negative function, in particular,*

$$\varrho e(\varrho, \vartheta) = \varrho e_M(\varrho, \vartheta) + a\vartheta^4 \geq a\vartheta^4.$$

Similarly, hypothesis (2.9) implies that

$$Z \mapsto \frac{P(Z)}{Z^{5/3}}$$

is a non-increasing function of Z . In particular, (2.10) makes sense and we easily deduce from (2.5) that

$$\varrho e(\varrho, \vartheta) \geq \frac{3}{2}p_M(\varrho, \vartheta) \geq c\varrho^{5/3}.$$

Thus, boundedness of the total energy

$$\int_Q \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] dx$$

enforces boundedness of the density ϱ in the Lebesgue space $L^{5/3}(Q)$ as well as boundedness of the absolute temperature ϑ in the Lebesgue space $L^4(Q)$, cf. (2.1).

subsec:solution

2.3. Martingale solutions. The solutions to (1.1) will be weak in both probabilistic and PDE sense. From the point of view of the theory of PDEs, we follow the approach of [21] and consider so-called finite energy weak martingale solutions satisfying the momentum balance (1.1b), the equation of continuity (1.1a), together with the entropy balance (1.6), (1.8), and the total energy balance (1.9). Solutions will satisfy these relations in the weak PDE sense, meaning all derivatives are understood in the sense of distributions.

From the probabilistic point of view, two concepts of solution are typically considered in the theory of stochastic evolution equations, namely, pathwise (or strong) solutions and martingale (or weak) solutions. In the former notion the underlying probability space as well as the driving process is fixed in advance while in the latter case these stochastic elements become part of the solution of the problem. Clearly, existence of a pathwise solution is stronger and implies existence of a martingale solution. In the present work we are only able to establish existence of a martingale solution to (1.1). Due to the classical Yamada-Watanabe-type argument (see, e.g., [33], [41] and [44]), existence of a pathwise solution would then follow if pathwise uniqueness held true (for these concepts in the context of SPDEs we refer to [41]). However, uniqueness for the Navier–Stokes–Fourier equations for compressible fluids is an open problem even in the deterministic setting. In hand with this issue goes the fact that the initial state is determined only by the measure Λ introduced in (2.1).

Remark 2.1. *The notion weak martingale solutions often refers in literature to solutions satisfying some form of energy balance. We follow the tradition of our previous papers and call them finite energy weak martingale solutions in order to emphasise the importance of the energy inequality.*

Let us summarize the above in the following definition.

def:sol

Definition 2.1 (Martingale solution). *Let $Q \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Let Λ be a Borel probability measure on $L^1(Q) \times L^1(Q) \times L^1(Q; \mathbb{R}^3)$ enjoying the properties (2.1). Let $H \in C([0, T] \times Q)$ \mathbb{P} -a.s. be given satisfying (2.3).*

Then

$$((\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P}), \varrho, \vartheta, \mathbf{u}, W)$$

is called (weak) martingale solution to problem (1.1), (1.5) with the initial data Λ provided the following holds.

- (a) $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
- (b) W is an (\mathfrak{F}_t) -cylindrical Wiener process;
- (c) the random variables

$$\varrho \in L^1(Q_T), \vartheta \in L^1(Q_T), \mathbf{u} \in L^2(0, T; W_0^{1,2}(Q; \mathbb{R}^3))$$

are (\mathfrak{F}_t) -progressively measurable, $\varrho \geq 0$, $\vartheta > 0$ \mathbb{P} -a.s.;

- (d) the density $\varrho \in C_w([0, T]; L^{5/3}(Q))$,
the temperature $\vartheta \in L^\infty(0, T; L^4(Q)) \cap L^2(0, T; W^{1,2}(Q))$ \mathbb{P} -a.s.,
the velocity $\mathbf{u} \in L^2(0, T; W_0^{1,2}(Q; \mathbb{R}^3))$ \mathbb{P} -a.s.,
and the momentum $\varrho \mathbf{u} \in C_w([0, T]; L^{5/4}(Q; \mathbb{R}^3))$ \mathbb{P} -a.s. satisfy

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\varrho(t, \cdot)\|_{L^{5/3}(Q)}^{5/3} \right] < \infty,$$

$$\mathbb{E} \left[\operatorname{ess\,sup}_{t \in [0, T]} \|\vartheta(t, \cdot)\|_{L^4(Q)}^4 \right] < \infty, \quad \mathbb{E} \left[\int_0^T \|\vartheta\|_{W^{1,2}(\mathbb{T}^3)}^2 dt \right] < \infty,$$

$$\mathbb{E} \left[\int_0^T \|\mathbf{u}\|_{W_0^{1,2}(Q; \mathbb{R}^3)}^2 dt \right] < \infty, \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\varrho \mathbf{u}(t, \cdot)\|_{L^{5/4}(Q; \mathbb{R}^3)}^{5/4} \right];$$

- (e) There is an \mathfrak{F}_0 -measurable random variable $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ with $\Lambda = \mathbb{P} \circ (\varrho_0, \vartheta_0, \mathbf{u}_0)^{-1}$;

(f) *the equation of continuity*

$$\boxed{\text{wWS237final}} \quad (2.15) \quad \int_0^T \int_Q [\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi] \, dx \, dt = - \int_Q \varrho_0 \psi(0) \, dx;$$

holds for all $\psi \in C_c^\infty([0, T] \times R^3)$ \mathbb{P} -a.s.;

(g) *the momentum equation*

$$\boxed{\text{wWS238final}} \quad (2.16) \quad \begin{aligned} & \int_0^T \partial_t \psi \int_Q \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \, dt \\ & + \int_0^T \psi \int_Q \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \, dx \, dt - \int_0^T \psi \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \, dx \, dt \\ & + \int_0^T \psi \int_Q p(\varrho, \vartheta) \operatorname{div} \boldsymbol{\varphi} \, dx \, dt + \int_0^t \psi \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \boldsymbol{\varphi} \, dx \, dW \\ & = - \int_Q \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, dx; \end{aligned}$$

holds for all $\psi \in C_c^\infty[0, T]$, $\boldsymbol{\varphi} \in C_c^\infty(Q; R^3)$ \mathbb{P} -a.s.

(h) *the entropy balance*

$$\boxed{\text{m217*final}} \quad (2.17) \quad \begin{aligned} - \int_{\mathbb{T}^3} \varrho_0 s(\varrho_0, \vartheta_0) \psi \, dx & \geq \int_0^T \int_Q [s(\varrho, \vartheta) \partial_t \psi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla \psi] \, dx \, dt \\ & + \int_0^T \int_Q \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] \psi \, dx \, dt \\ & + \int_0^T \int_Q \frac{\kappa(\vartheta) \nabla \vartheta}{\vartheta} \cdot \nabla \psi \, dx \, dt + \int_0^T \int_Q \frac{\varrho}{\vartheta} H \psi \, dx \, dt \end{aligned}$$

holds for all $\psi \in C_c^\infty([0, T] \times R^3)$, $\psi \geq 0$ \mathbb{P} -a.s.;

(i) *the total energy balance*

$$\boxed{\text{EI20final}} \quad (2.18) \quad \begin{aligned} - \int_0^T \partial_t \psi \left(\int_Q \mathcal{E}(\varrho, \vartheta, \mathbf{u}) \, dx \right) \, dt & = \psi(0) \int_Q \mathcal{E}(\varrho_0, \vartheta_0, \mathbf{u}_0) \, dx + \int_0^T \psi \int_Q \varrho H \, dx \, dt \\ & + \int_0^T \psi \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW \, dx + \frac{1}{2} \int_0^T \psi \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) dt \end{aligned}$$

holds for any $\psi \in C_c^\infty([0, T])$ \mathbb{P} -a.s. Here, we abbreviated

$$\mathcal{E}(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta).$$

NRR1

Remark 2.4. *The random variables ϱ , $\varrho \mathbf{u}$ are (\mathfrak{F}_t) -progressively measurable and weakly continuous, in particular, the real valued processes*

$$t \mapsto \int_Q \varrho \psi \, dx, \quad t \mapsto \int_Q \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \quad \text{are } (\mathfrak{F}_t)\text{-adapted for any smooth } \psi, \boldsymbol{\varphi} \text{ } \mathbb{P}\text{-a.s.}$$

The entropy $\varrho s(\varrho, \vartheta)$ is “weakly càdlàg”, meaning

$$t \mapsto \int_Q \varrho s(\varrho, \vartheta) \psi \, dx \quad \text{is càdlàg for any smooth } \psi \text{ } \mathbb{P}\text{-a.s.,}$$

cf. the discussion in [21, Chapter 1]. The random variables $(\varrho, \vartheta, \mathbf{u})$ being (\mathfrak{F}_t) -progressively measurable in the sense of random distributions described above. By [7, Chapter 2, Lemma 2.2.17] also the compositions $\varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u})$ and $\varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u}$ are (\mathfrak{F}_t) -progressively measurable random distribution such that the real-valued stochastic processes $\psi \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \boldsymbol{\varphi} \, dx$ and $\psi \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx$ are (\mathfrak{F}_t) -progressively measurable in the classical sense. Consequently, all stochastic integrals are well defined.

NR1 **Remark 2.5.** *The energy equality (2.18), together with the entropy inequality (2.17), forms a natural counterpart of the energy inequality obtained in [6] for the barotropic case. As in the deterministic case, cf. [21], equality holds in (2.18).*

rem:helmholtz **Remark 2.6.** *Let*

$$H_{\bar{\vartheta}}(\varrho, \vartheta) = \varrho(e(\varrho, \vartheta) - \bar{\vartheta}s(\varrho, \vartheta)), \text{ with a constant } \bar{\vartheta} \geq 0,$$

be the ballistic free energy (Helmholtz function), see [21, Sec. 2.2.3]. This quantity enjoys a remarkable coercivity property, namely,

Bcoerc (2.19)
$$H_{\bar{\vartheta}}(\varrho, \vartheta) \geq c_1 (\varrho e(\varrho, \vartheta) + \bar{\vartheta} \varrho |s(\varrho, \vartheta)|) - c_2(\varrho + 1), \quad c_1 > 0, \quad c_2 > 0,$$

see [21, Proposition 3.2].

Combining (2.17) and (2.18) we obtain for any $\bar{\vartheta} \geq 0$

EI2final (2.20)
$$\begin{aligned} & - \int_0^T \partial_t \psi \left(\int_{\mathbb{T}^3} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\bar{\vartheta}}(\varrho, \vartheta) \right] dx \right) dt + \int_0^T \psi \int_{\mathbb{T}^3} \frac{\bar{\vartheta}}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 \right] dx \\ & \leq \psi(0) \int_{\mathbb{T}^3} \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\bar{\vartheta}}(\varrho_0, \vartheta_0) \right] dx + \int_0^T \psi \int_{\mathbb{T}^3} \varrho \left(1 - \frac{\bar{\vartheta}}{\vartheta} \right) H \, dx \, dt \\ & \quad + \int_0^T \psi \int_{\mathbb{T}^3} \varrho \mathbb{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dW \, dx + \frac{1}{2} \int_0^T \psi \left(\int_{\mathbb{T}^3} \sum_{k \geq 1} \varrho |\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) dt \end{aligned}$$

for any deterministic smooth test function $\psi \geq 0$, $\psi(T) = 0$ \mathbb{P} -a.s. Thus, in view of (2.19), the moment estimates in (d) are a consequence of this and the constitutive hypotheses (2.4–2.10). Note that the right-most term in (2.19) is controlled as the total mass

$$\int_Q \varrho \, dx$$

is a constant of motion.

Now we are ready to state the existence theorem which is the main result of this paper.

thm:main **Theorem 2.1.** *Let $Q \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Suppose that the structural assumptions (2.2)–(2.14) are satisfied. Let Λ be a Borel probability measure on $L^1(Q) \times L^1(Q) \times L^1(Q; \mathbb{R}^3)$ satisfying (2.1). Then there exists a martingale solution to problem (1.1), (1.5) in the sense of Definition 2.1 with the initial law Λ .*

rrr3 **Remark 2.7.** *In view of Remark (2.3), condition 2.1 implies, in particular, that*

$$\varrho_0 \in L^{5/3}(Q), \quad \vartheta_0 \in L^4(Q) \quad \mathbb{P} - \text{a.s.}$$

The rest of the paper is devoted to the proof of Theorem 2.1. Concluding remarks are presented in Section 7.

3. BASIC APPROXIMATE PROBLEM

sec:galerkin

The solutions will be constructed by means of a multilevel approximation scheme. In order to simplify presentation, we carry over the proof in the absence of the heat source H . The necessary modifications to accommodate this quantity in the proof are left to the reader and can be found in [21, Chapter 3].

The standard approach to non-linear partial differential equations starts with a finite dimensional approximation of Galerkin type. Unfortunately, this can be applied only to the momentum equation (1.1b) since we need the density ϱ and temperature ϑ to be positive at the first level of approximation. Positivity of both results from a maximum principle, where the latter is usually incompatible with a Galerkin type approximation. It seems therefore more convenient to apply the artificial viscosity method adding diffusive terms to both (1.1b) and (1.1a). In order to prove Theorem 2.1 we will regularize several quantities in the system (1.1). Adopting the approximation scheme developed in [21] we add an artificial viscosity to the continuity and momentum equations and regularize the pressure. For technical reasons, however, we have to

regularize several further quantities as well.

Following [21], we introduce

$$\begin{aligned}
& p_\delta(\varrho, \vartheta) = p(\varrho, \vartheta) + \delta(\varrho^2 + \varrho^\beta), \\
& e_{M,\delta}(\varrho, \vartheta) = e_M(\varrho, \vartheta) + \delta\vartheta, \quad e_\delta(\varrho, \vartheta) = e(\varrho, \vartheta) + \delta\vartheta, \\
\boxed{\text{neu}} \quad (3.1) \quad & s_{M,\delta}(\vartheta, \varrho) = s_M(\vartheta, \varrho) + \delta \log \vartheta, \quad s_\delta(\varrho, \vartheta) = s(\varrho, \vartheta) + \delta \log(\vartheta), \\
& \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta\left(\vartheta^\beta + \frac{1}{\vartheta}\right), \quad \mathcal{K}_\delta(\vartheta) = \int_0^\vartheta \kappa_\delta(z) \, dz.
\end{aligned}$$

Let Δ_0 be the Laplace operator defined on the domain Q , with the homogeneous Dirichlet boundary conditions. Let $\{w_n\}_{n \geq 1}$ be the orthonormal system of the associated eigenfunctions. A suitable platform for the Galerkin method is the following family of finite-dimensional spaces,

$$H_m = \left(\text{span} \left\{ w_n \mid n \leq m \right\} \right)^3, \quad m = 1, 2, \dots$$

endowed with the Hilbert structure of the Lebesgue space $L^2(Q; \mathbb{R}^3)$. Let

$$\Pi_m : L^2(Q; \mathbb{R}^3) \rightarrow H_m$$

be the associated L^2 -orthogonal projection. As ∂Q is regular, we have $\mathcal{D}(\Delta_0) = W^{2,2}(Q; \mathbb{R}^3) \cap W_0^{1,2}(Q; \mathbb{R}^3) \hookrightarrow C(\overline{Q}; \mathbb{R}^3)$. Accordingly,

$$\boxed{\text{wW2a}} \quad (3.2) \quad \|P_m[\mathbf{f}]\|_{L^\infty(Q; \mathbb{R}^3)} \lesssim \|P_m[\mathbf{f}]\|_{W^{2,2}(Q; \mathbb{R}^3)} \lesssim \|P_m[\mathbf{f}]\|_{\mathcal{D}(\Delta_0)} \lesssim \|\mathbf{f}\|_{W^{2,2}(Q; \mathbb{R}^3)},$$

where the associated embedding constants are *independent of m* .

Moreover, as H_m is finite-dimensional, all norms are equivalent on H_m for any fixed m - a property that will be frequently used at the first level of approximation.

Next we introduce a cut-off function

$$\chi \in C^\infty(\mathbb{R}), \quad \chi(z) = \begin{cases} 1 & \text{for } z \leq 0, \\ \chi'(z) \leq 0 & \text{for } 0 < z < 1, \\ \chi(z) = 0 & \text{for } z \geq 1, \end{cases}$$

together with the operators

$$[\mathbf{v}]_R = \chi(\|\mathbf{v}\|_{H_m} - R)\mathbf{v}, \quad \text{defined for } \mathbf{v} \in H_m.$$

Finally, we regularize the diffusion coefficients replacing \mathbf{F} by \mathbf{F}_ε ,

$$\mathbf{F}_\varepsilon = (\mathbf{F}_{k,\varepsilon})_{k \geq 1}, \quad \mathbf{F}_{k,\varepsilon}(x, \varrho, \vartheta, \mathbf{u}) = \chi\left(\frac{\varepsilon}{\varrho} - 1\right) \chi\left(|\mathbf{u}| - \frac{1}{\varepsilon}\right) \mathbf{F}_k(x, \varrho, \vartheta, \mathbf{u}).$$

Consequently, in view of (2.2),

$$\boxed{\text{P-1new}} \quad (3.3) \quad \|\mathbf{F}_{k,\varepsilon}\|_{L^\infty} + \|\nabla_{x,\varrho,s,\mathbf{u}} \mathbf{F}_k\|_{L^\infty} \leq f_{k,\varepsilon}, \quad \sum_{k=1}^{\infty} f_{k,\varepsilon}^2 < \infty.$$

In addition, we regularize $\mathbf{F}_{k,\varepsilon}(\cdot, \varrho, \vartheta, \mathbf{u})$ in a “non-local” way, specifically, we introduce $\mathbf{F}_{\varepsilon,\xi}$,

$$\boxed{\text{P-1New}} \quad (3.4) \quad \mathbf{F}_{k,\varepsilon,\xi} = \omega_\xi * [h_\xi \mathbf{F}_{k,\varepsilon}(\cdot, \varrho, \vartheta, \mathbf{u})], \quad h_\xi \in C_c^\infty(Q),$$

where $(\omega_\xi(x))_{\xi > 0}$ is a family of regularizing kernels.

The basic *approximate problem* reads:

$$\boxed{\text{wW}} \quad \boxed{\text{wW3}} \quad (3.5a) \quad d\varrho + \text{div}(\varrho[\mathbf{u}]_R) \, dt = \varepsilon \Delta \varrho \, dt,$$

$$\begin{aligned}
\boxed{\text{wW4}} \quad (3.5b) \quad & d\Pi_m[\varrho \mathbf{u}] + \Pi_m[\text{div}(\varrho[\mathbf{u}]_R \otimes \mathbf{u})] \, dt + \Pi_m \left[\chi(\|\mathbf{u}\|_{H_m} - R) \nabla (p(\varrho, \vartheta) + \delta(\varrho^2 + \varrho^\beta)) \right] \, dt \\
& = \Pi_m \left[\varepsilon \Delta(\varrho \mathbf{u}) + \chi(\|\mathbf{u}\|_{H_m} - R) \text{div} \mathbb{S}(\nabla \mathbf{u}) + \frac{1}{m} \mathbf{u} \right] \, dt \\
& + \Pi_m \left[\varrho \Pi_m \left[\mathbf{F}_{\varepsilon,\xi}(\varrho, \vartheta, \mathbf{u}) \right] \right] \, dW,
\end{aligned}$$

$$\boxed{\text{m119}} \quad (3.5c) \quad d(\varrho e_\delta(\varrho, \vartheta)) + [\text{div}(\varrho e_\delta(\varrho, \vartheta)[\mathbf{u}]_R) - \text{div}(\kappa_\delta(\vartheta) \nabla \vartheta)] \, dt$$

$$\begin{aligned}
&= \left[\chi(\|\mathbf{u}\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p(\varrho, \vartheta) \operatorname{div}[\mathbf{u}]_R \right] dt \\
&+ \left[\varepsilon \delta(\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \delta \frac{1}{\vartheta^2} - \varepsilon \vartheta^5 + \varepsilon \varrho |\nabla \mathbf{u}|^2 \right] dt,
\end{aligned}$$

to be solved in the space-time cylinder $[0, T] \times Q$, with the Neumann boundary conditions

$$(3.6) \quad \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0,$$

and prescribed initial data

$$(3.7) \quad \begin{aligned} \varrho(0, \cdot) &= \varrho_0 \in C^{2+\nu}(\overline{Q}), \quad \varrho_0 > 0, \quad \vartheta(0, \cdot) = \vartheta_0, \quad \vartheta_0 \in W^{1,2}(Q) \cap C(\overline{Q}), \quad \vartheta_0 > 0, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0 \in H_m. \end{aligned}$$

In (3.5a) and (3.5b) we recognize the artificial ‘‘viscosity’’ terms $\varepsilon \Delta \varrho$, $\varepsilon \Delta(\varrho \mathbf{u})$, the pressure regularization $\delta(\varrho^\beta + \varrho^2)$ as well as the cut-off operators applied to the velocity in the convective terms and other quantities to preserve the total energy balance. Note that equations (3.5a) and (3.5c) are deterministic, meaning they can be solved pathwise, while (3.5b) involves stochastic integration. It is worth noting that (3.5c) expresses the balance of the internal energy while the target problem is formulated in terms of the entropy. In the following we give a precise definition of solutions to the approximate problem.

WD1 **Definition 3.1.** *Let Λ be a Borel probability measure on $C^{2+\nu}(\overline{Q}) \times W^{1,2} \cap C(\overline{Q}) \times H_m$. Then*

$$((\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P}), \varrho, \vartheta, \mathbf{u}, W)$$

is called a martingale solution to problem (3.5a)–(3.5c), (3.6), with the initial data Λ , provided the following holds.

- (a) $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$ is a stochastic basis with a complete right-continuous filtration;
- (b) W is an (\mathfrak{F}_t) -cylindrical Wiener process;
- (c) the functions $\varrho, \vartheta, \mathbf{u}$ belong to the class:

$$\begin{aligned}
&\varrho \in C([0, T]; C^{2+\nu}(\overline{Q})), \quad \varrho > 0 \text{ } \mathbb{P}\text{-a.s.}, \\
&\vartheta \in C([0, T]; W^{1,2}), \quad 0 < \underline{\vartheta} < \vartheta < \overline{\vartheta}, \quad \partial_t \vartheta, \quad K_\delta(\vartheta) \in L^2((0, T) \times Q) \text{ } \mathbb{P}\text{-a.s.}, \\
&\mathbf{u} \in C([0, T], H_m) \text{ } \mathbb{P}\text{-a.s.},
\end{aligned}$$

$\varrho(t, \cdot), \vartheta(t, \cdot), \mathbf{u}(t, \cdot)$ are (\mathfrak{F}_t) -adapted for any $t \geq 0$;

- (d) the approximate equation of continuity

$$(3.8) \quad \partial_t \varrho + \operatorname{div}(\varrho[\mathbf{u}]_R) = \varepsilon \Delta \varrho, \quad \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0 \text{ holds in } (0, T) \times Q, \text{ } \mathbb{P}\text{-a.s.};$$

- (e) the approximate equation of internal energy

$$\begin{aligned}
&\partial_t(\varrho e_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho e_\delta(\varrho, \vartheta)[\mathbf{u}]_R) - \Delta K_\delta(\vartheta) \\
&= \chi(\|\mathbf{u}\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p(\varrho, \vartheta) \operatorname{div}[\mathbf{u}]_R \\
&+ \varepsilon \delta(\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \delta \frac{1}{\vartheta^2} - \varepsilon \vartheta^5 + \varepsilon \varrho |\nabla \mathbf{u}|^2, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0,
\end{aligned}$$

holds a.a. in $(0, T) \times Q$ \mathbb{P} -a.s.;

- (f) the approximate momentum equation

$$\begin{aligned}
\langle \varrho \mathbf{u}(t), \varphi \rangle &= \langle (\varrho \mathbf{u})_0, \varphi \rangle + \int_0^t \langle \varrho[\mathbf{u}]_R \otimes \mathbf{u}, \nabla \varphi \rangle ds - \int_0^t \langle \chi(\|\mathbf{u}\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}), \nabla \varphi \rangle ds \\
&- \varepsilon \int_0^t \langle \varrho \mathbf{u}, \Delta \varphi \rangle ds + \int_0^t \chi(\|\mathbf{u}\|_{H_m} - R) \langle p_\delta(\vartheta, \varrho), \operatorname{div} \varphi \rangle ds \\
&- \frac{1}{m} \int_0^t \langle \mathbf{u}, \varphi \rangle ds + \int_0^t \langle \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] dW, \varphi \rangle,
\end{aligned}$$

holds \mathbb{P} -a.s. for all $\varphi \in H_m$ and all $t \in [0, T]$;

- (g) we have

$$(3.11) \quad (\varrho \mathbf{u})_0 = \varrho(0) \mathbf{u}(0) \text{ } \mathbb{P}\text{-a.s.}, \quad \mathbb{P}[\varrho(0, \cdot), \vartheta(0, \cdot), \mathbf{u}(0, \cdot)]^{-1} = \Lambda[\varrho_0, \vartheta_0, \mathbf{u}_0].$$

Our main goal in this section is to prove the existence of martingale solutions to the approximate problem (3.5a)–(3.5c), (3.6).

wWP1 **Theorem 3.1.** *Let $\beta > 6$. Let Λ be a Borel probability measure on $C^{2+\nu}(\bar{Q}) \times W^{1,2} \cap C(\bar{Q}) \times H_m$ such that*

$$\Lambda \left\{ 0 < \underline{\varrho} \leq \varrho_0, \|\varrho_0\|_{C_x^{2+\nu}} \leq \bar{\varrho}, \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0 \right\} = 1, \quad \Lambda \left\{ 0 < \underline{\vartheta} \leq \vartheta_0 \leq \bar{\vartheta}, \|\vartheta_0\|_{W_x^{1,2}} \leq \bar{\vartheta} \right\} = 1,$$

$$\int_{C^{2+\nu} \times W^{1,2} \cap C \times H_m} \|\mathbf{u}_0\|_{H_m}^r d\Lambda \leq \bar{u}$$

for some positive deterministic constants $\underline{\varrho}$, $\bar{\varrho}$, $\underline{\vartheta}$, $\bar{\vartheta}$ and some $r > 2$.

Then the approximate problem (3.5a)–(3.5c), (3.6) admits a martingale solution in the sense of Definition 3.1. The solution satisfies

wWS110a (3.12)
$$\sup_{t \in [0, T]} \left(\|\varrho(t)\|_{C_x^{2+\nu}} + \|\partial_t \varrho(t)\|_{C_x^\nu} + \|\varrho^{-1}(t)\|_{C_x^0} \right) \leq c \quad \mathbb{P}\text{-a.s.},$$

$$\sup_{t \in [0, T]} \|\vartheta^{-1}\|_{L_x^\infty} + \sup_{t \in [0, T]} \|\vartheta\|_{W_x^{1,2} \cap L_x^\infty} + \int_0^T \left(\|\partial_t \vartheta\|_{L_x^2}^2 + \|\Delta_x \mathcal{K}_\delta(\vartheta)\|_{L_x^2}^2 \right) dt \leq c \quad \mathbb{P}\text{-a.s.}$$

wWS113a (3.13)
$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\mathbf{u}(t, \cdot)\|_{H_m}^r \right] \leq c \left(1 + \mathbb{E} [\|\mathbf{u}_0\|_{H_m}^r] \right),$$

where $c = (m, R, T, \underline{\varrho}, \bar{\varrho}, \underline{\vartheta}, \bar{\vartheta}, \bar{u})$.

In order to prove Theorem 3.1 we adopt the following strategy: The Galerkin projection applied in (3.5b) reduces the problem to a variant of ordinary stochastic differential equation, where the other unknown quantities ϱ , ϑ are uniquely determined by the deterministic equations (3.5a) and (3.5c) in terms of \mathbf{u} and the data. Accordingly, problem (3.5), (3.6) can be solved by means of a simple *iteration scheme*. This is the objective of Section 3.1. In addition, the approximate solutions satisfy the associated energy balance equation yielding the uniform bounds necessary to carry out the asymptotic limits $m \rightarrow \infty$, $R \rightarrow 0$, and $\xi \rightarrow 0$.

wWS1S1

3.1. Iteration scheme. To begin, we fix the initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ satisfying (3.11). The existence of such data along with a suitable probability space follows from the standard Skorokhod representation theorem. Solutions to problem (3.5), (3.6) will be constructed by means of a modification of the Cauchy collocation method. Fixing a time step $h > 0$ we set

wWS14 (3.14)
$$\varrho(t, \cdot) = \varrho_0, \quad \vartheta(t, \cdot) = \vartheta_0, \quad \mathbf{u}(t, \cdot) = \mathbf{u}_0, \quad \text{for } t \leq 0,$$

and define recursively, for $t \in [nh, (n+1)h)$

wWS15 (3.15)
$$\partial_t \varrho + \operatorname{div}(\varrho[\mathbf{u}(nh, \cdot)]_R) = \varepsilon \Delta \varrho, \quad \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0, \quad \varrho(nh, \cdot) = \varrho(nh-, \cdot),$$

$$\partial_t(\varrho e_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho e_\delta(\varrho, \vartheta)[\mathbf{u}(nh, \cdot)]_R) - \Delta \mathcal{K}_\delta(\vartheta)$$

wWS15b (3.16)
$$= \chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}(nh, \cdot)) : \nabla \mathbf{u}(nh, \cdot) - p(\varrho, \vartheta) \operatorname{div}[\mathbf{u}(nh, \cdot)]_R$$

(3.17)
$$+ \varepsilon \delta (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \delta \frac{1}{\varrho^2} - \varepsilon \vartheta^5 + \varepsilon \varrho |\nabla \mathbf{u}(nh, \cdot)|^2,$$

$$\nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0, \quad \vartheta(nh, \cdot) = \vartheta(nh-, \cdot).$$

Note that system (3.15), (3.16) is uncoupled as the former equation is independent of ϑ . Finally, for ϱ and ϑ given through (3.15), (3.16), we solve

wWS16 (3.18)
$$\begin{aligned} & d\Pi_m(\varrho \mathbf{u}) + \Pi_m \left[\operatorname{div} \left(\varrho[\mathbf{u}(nh, \cdot)]_R \otimes \mathbf{u}(nh, \cdot) \right) \right] dt \\ & + \Pi_m \left[\chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \nabla (p(\varrho, \vartheta) + \delta(\varrho^2 + \varrho^\beta)) \right] dt \\ & = \Pi_m \left[\varepsilon \Delta(\varrho \mathbf{u}(nh, \cdot)) + \chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \operatorname{div} \mathbb{S}(\nabla \mathbf{u}(nh, \cdot)) + \frac{1}{m} \mathbf{u}(nh, \cdot) \right] dt \\ & + \Pi_m [\varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}(nh, \cdot)]] dW, \quad t \in [nh, (n+1)h), \quad \mathbf{u}(nh, \cdot) = \mathbf{u}(nh-). \end{aligned}$$

To proceed it is convenient to rewrite (3.18) in terms of $\mathbf{d}\mathbf{u}$ (eventually, in (3.21) below, we obtain an equivalent formulation in terms of a simple iteration). To this end, we write

$$d\Pi_m(\varrho\mathbf{u}) = \Pi_m(d\varrho\mathbf{u}) + \Pi_m(\varrho d\mathbf{u}) = \Pi_m(\partial_t \varrho\mathbf{u}) dt + \Pi_m(\varrho d\mathbf{u}).$$

Next, we introduce a linear mapping $\mathcal{M}[\varrho]$,

$$\mathcal{M}[\varrho] : H_m \rightarrow H_m, \quad \mathcal{M}[\varrho](\mathbf{v}) = \Pi_m(\varrho\mathbf{v}),$$

or, equivalently,

$$\int_Q \mathcal{M}[\varrho]\mathbf{v} \cdot \boldsymbol{\varphi} \, dx \equiv \int_Q \varrho\mathbf{v} \cdot \boldsymbol{\varphi} \, dx \text{ for all } \boldsymbol{\varphi} \in H_m.$$

The operator \mathcal{M} has been introduced in [23, Section 2.2], where one can also find the following properties. We have that $\mathcal{M}[\rho]$ is invertible we have

$$\boxed{\text{eqM-1}} \quad (3.19) \quad \left\| \mathcal{M}^{-1}[\rho] \right\|_{\mathcal{L}(X_n^*, X_n)} \leq \left(\inf_{x \in \mathbb{T}^3} \rho \right)^{-1}$$

as long as ρ is bounded below away from zero, and, clearly,

$$\mathcal{M}^{-1}[\rho](\Pi_m[\rho\mathbf{v}]) = \mathbf{v} \text{ for any } \mathbf{v} \in H_m.$$

Let us finally mention that

$$\boxed{\text{eqM-1b}} \quad (3.20) \quad \left\| \mathcal{M}^{-1}[\rho] - \mathcal{M}^{-1}[\underline{\rho}] \right\|_{\mathcal{L}(X_n^*, X_n)} \leq c(m, \underline{\rho}) \|\rho^1 - \rho^2\|_{L_x^1},$$

provided both ρ^1 and ρ^2 are bounded from below by some positive constant $\underline{\rho}$. Accordingly, relation (3.18) can be written in the form

$$\begin{aligned} \mathbf{u}(t) - \mathbf{u}(nh-) + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[\operatorname{div} \left(\varrho[\mathbf{u}(nh, \cdot)]_R \otimes \mathbf{u}(nh, \cdot) \right) \right] dt \\ + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[\chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \nabla (p(\varrho, \vartheta) + \delta(\varrho + \varrho^\beta)) \right] dt \\ \boxed{\text{wWS17}} \quad (3.21) \quad = \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[\chi(\|\mathbf{u}(nh, \cdot)\|_{H_m} - R) \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}(nh, \cdot)) \right] dt \\ + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m \left[\varepsilon \Delta(\varrho\mathbf{u}(nh, \cdot)) + \frac{1}{m} \mathbf{u}(nh, \cdot) \right] dt \\ + \mathcal{M}^{-1}[\varrho(t)] \int_{nh}^t \Pi_m [\mathbf{F}_{\varepsilon, \xi}(nh, \cdot)] dW, \quad nh < t < (n+1)h. \end{aligned}$$

The iteration scheme (3.15)–(3.18) provides a unique solution for any initial data (3.14). Indeed, as $\mathbf{u}(nh) \in H_m$ is a smooth function, equation (3.15) admits a unique solution for any initial data $\varrho(nh)$. Moreover, as a direct consequence of the parabolic maximum principle, ϱ remains positive as long as the initial datum $\varrho(nh)$ is positive. We may therefore infer that (3.14–3.18) give rise to uniquely determined functions $\varrho, \vartheta, \mathbf{u}$. In fact, we find a solution ϱ such that

$$\varrho \in C([0, T]; C^{2+\nu}(\overline{Q})), \quad \varrho > 0, \quad \mathbb{P}\text{-a.s.}$$

by applying standard results (see, for instance, [36, Theorem 5.1.21]) pathwise. For equation (3.16) we obtain a solution ϑ belonging \mathbb{P} -a.s. to the class

$$\boxed{\text{m125b}} \quad (3.22) \quad Y = \left\{ \begin{array}{l} \partial_t \vartheta \in L^2((0, T) \times Q), \quad \Delta_x \mathcal{K}_\delta(\vartheta) \in L^2((0, T) \times Q), \\ \vartheta \in L^\infty(0, T; W^{1,2}(Q) \cap L^\infty(Q)), \quad \frac{1}{\vartheta} \in L^\infty((0, T) \times Q), \end{array} \right\}$$

by applying [21, Lemma 3.4] pathwise. Finally, knowing ϱ and ϑ we can find the velocity

$$\mathbf{u} \in C([0, T]; X_m) \quad \mathbb{P}\text{-a.s.}$$

solving (3.18) recursively. Note that our construction implies that ϱ, ϑ and \mathbf{u} are (\mathfrak{F}_t) -adapted and continuous in the time variable \mathbb{P} -a.s.

wWS1S2

3.2. The limit for vanishing time step. Our next goal is to let $h \rightarrow 0$ in (3.14–3.18) to obtain a solution of the approximate problem (3.5), (3.6). This step leans essentially on suitable uniform bounds independent of h . To simplify notation, we shall write

$$[v]_h = v(nh, \cdot), \quad [v]_{h,R}(t, \cdot) = [v(nh, \cdot)]_R \text{ for } t \in [nh, (n+1)h], \quad n = 0, 1, \dots$$

As all norms are equivalent on the finite-dimensional space H_m and ∂Q is smooth, we get

$$\|[\mathbf{u}]_{h,R}\|_{C^l(\bar{Q}; R^3)} \leq c(l, m, R) \text{ uniformly for } h > 0, \text{ and } t \in [0, T] \text{ at least for } l \leq 2.$$

Consequently, the approximate equation of continuity (3.15) admits a unique regular solution, the smoothness of which is determined by the initial data. In particular, the solution ϱ belongs to the class

wWS18

$$(3.23) \quad \varrho \in C([0, T]; C^{2+\nu}(\bar{Q})), \quad \partial_t \varrho \in L^\infty([0, T]; C^\nu(\bar{Q}))$$

as soon as $\varrho_0 \in C^{2+\nu}(\bar{Q})$, $\nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0$ for some $\nu > 0$. In addition, the standard parabolic maximum principle yields

wWS19

$$(3.24) \quad 0 < \underline{r}(T, m, R) \min_{\bar{Q}} \varrho_0 \leq \varrho(t, \cdot) \leq \bar{r}(T, m, R) \max_{\bar{Q}} \varrho_0 \text{ for all } t \in [0, T].$$

Note that the regularized velocity $[\mathbf{u}]_{h,R}$ is only piece-wise continuous; whence the same is true for $\partial_t \varrho$ and therefore we do not expect $\partial_t \varrho \in C([0, T]; C^\nu(\bar{Q}))$. Also note carefully that, thanks to the hypotheses imposed on the initial law Λ , ϱ is bounded in the aforementioned spaces only in terms of the initial datum ϱ_0 , meaning, no probabilistic averaging has been applied. In particular, we may infer that \mathbb{P} -a.s.

wWS110

$$(3.25) \quad \operatorname{ess\,sup}_{t \in [0, T]} \left(\|\varrho(t, \cdot)\|_{C^{2+\nu}(\bar{Q})} + \|\partial_t \varrho(t, \cdot)\|_{C^\nu(\bar{Q})} + \|\varrho^{-1}(t, \cdot)\|_{C(\bar{Q})} \right) \lesssim c$$

with $c = c(m, R, T, \underline{\varrho}, \bar{\varrho})$, whenever

wWS111

$$(3.26) \quad 0 < \underline{\varrho} \leq \varrho_0, \quad \|\varrho_0\|_{C^{2+\nu}(\bar{Q})} \leq \bar{\varrho} \text{ } \mathbb{P}\text{-a.s.}$$

for certain *deterministic* constants $\underline{\varrho}, \bar{\varrho}$.

As far as the internal energy equation (3.16) we recall that

wWS111b

$$(3.27) \quad 0 < \underline{\vartheta} \leq \vartheta_0 \leq \bar{\vartheta}, \quad \|\vartheta_0\|_{W^{1,2}(Q)} \leq \bar{\vartheta} \text{ } \mathbb{P}\text{-a.s.},$$

where $\underline{\vartheta}, \bar{\vartheta}$ are deterministic constants. From [21, Lemma 3.3, Corollary 3.2] we obtain the following estimate for the solutions to (3.16)

$$\begin{aligned} & \sup_{t \in (0, T)} \|\vartheta\|_{W^{1,2}(Q)}^2 + \int_0^T \left(\|\partial_t \vartheta\|_{L^2(Q)}^2 + \|\Delta_x \mathcal{K}_\delta(\vartheta)\|_{L^2(Q)}^2 \right) dt \\ & \leq C \left(\|\varrho\|_{W^{1,\infty}((0, T) \times Q)}, \|[\mathbf{u}]_{h,R}\|_{C([0, T]; X_n)}, \left(\inf_{(0, T) \times Q} \varrho \right)^{-1}, \|\vartheta_0\|_{W^{1,2}(Q)} \right), \end{aligned}$$

and

$$0 < \underline{\vartheta} \leq \vartheta \leq \bar{\vartheta},$$

where $\underline{\vartheta}, \bar{\vartheta}$ depend only on $\underline{\varrho}, \bar{\varrho}, \|\varrho\|_{W^{1,\infty}((0, T) \times Q)}, \|[\mathbf{u}]_{h,R}\|_{C([0, T]; X_n)}$.

Thus we may infer that

m135

$$(3.28) \quad \begin{aligned} & \operatorname{ess\,sup}_{t \in (0, T)} \|\vartheta^{-1}\|_{L^\infty(Q)} + \operatorname{ess\,sup}_{t \in (0, T)} \|\vartheta\|_{W^{1,2} \cap L^\infty(Q)}^2 \\ & + \int_0^T \left(\|\partial_t \vartheta\|_{L^2(Q)}^2 + \|\Delta_x \mathcal{K}_\delta(\vartheta)\|_{L^2(Q)}^2 \right) dt \lesssim c(m, R, \underline{\varrho}, \bar{\varrho}, \underline{\vartheta}, \bar{\vartheta}) \end{aligned}$$

uniformly in h , where, similarly to (3.25), the bound is deterministic. This immediately implies the bounds

m135A

$$(3.29) \quad \|\vartheta\|_{C^{\nu(\alpha)}([0, T]; W^{\alpha,2}(Q))} \leq c(m, R, \underline{\varrho}, \bar{\varrho}, \underline{\vartheta}, \bar{\vartheta}), \quad \nu(\alpha) > 0 \text{ for any } 0 \leq \alpha < 1,$$

using the interpolation

$$W^{1,2}(0, T; L^2(Q)) \cap L^2(0, T; W^{2,2}(Q)) \hookrightarrow W^{\gamma,2}(0, T; W^{(1-\gamma)2,2}(Q))$$

for $\gamma > \frac{1}{2}$, cf. Amann [1, Thm. 3.1], together with Sobolev's embedding for Bochner spaces. Now, we are ready to estimate the velocity. In order to do so, we will systematically use the fact that all norms are equivalent on the finite dimensional space H_m . It follows from (3.18) and the equivalence of norms on H_m that

$$\begin{aligned} \int_Q \varrho \mathbf{u}(\tau, \cdot) \cdot \boldsymbol{\varphi} \, dx &\lesssim \|\mathbf{u}_0\|_{H_m} + \int_0^\tau \sup_{0 \leq s \leq t} \|\mathbf{u}\|_{H_m} \, dt + T \\ &\quad + \left\| \int_0^\tau \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}]_h \, dx \, dW \right\|_{H_m} \end{aligned} \quad \text{for any } \boldsymbol{\varphi} \in H_m, \|\boldsymbol{\varphi}\|_{H_m} \leq 1,$$

whenever $0 \leq \tau \leq T$. Consequently, taking the supremum over $\boldsymbol{\varphi}$, we obtain

$$\|\Pi_m[\varrho \mathbf{u}](\tau, \cdot)\|_{H_m} \lesssim \|\mathbf{u}_0\|_{H_m} + \int_0^\tau \sup_{0 \leq s \leq t} \|\mathbf{u}\|_{H_m} \, dt + T + \left\| \int_0^\tau \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}]_h \, dx \, dW \right\|_{H_m},$$

or,

$$\begin{aligned} &\|\Pi_m[\varrho \mathbf{u}](\tau, \cdot)\|_{H_m}^r \\ &\lesssim c(T, k) \left[\|\mathbf{u}_0\|_{H_m}^r + \int_0^\tau \sup_{0 \leq s \leq t} \|\mathbf{u}\|_{H_m}^r \, dt + T + \left\| \int_0^\tau \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}]_h \, dx \, dW \right\|_{H_m}^r \right] \end{aligned}$$

for $0 \leq \tau \leq T$, and for any $r \geq 1$. Next, we pass to expectations and apply Burkholder-Davis-Gundy inequality to control the last integral obtaining

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq \tau} \|\Pi_m[\varrho \mathbf{u}](t, \cdot)\|_{H_m}^r \right] \lesssim \mathbb{E} [\|\mathbf{u}_0\|_{H_m}^r] + \int_0^\tau \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\mathbf{u}\|_{H_m}^r \right] \, dt + T \\ &\quad + \mathbb{E} \left[\int_0^\tau \sum_{k=1}^\infty \left| \int_Q \varrho \Pi_m[\mathbf{F}_{k, \varepsilon, \xi}]_h \, dx \right|^2 \, dt \right]^{r/2} \\ \boxed{\text{wWS112}} \quad (3.30) \quad &\lesssim \mathbb{E} [\|\mathbf{u}_0\|_{H_m}^r] + \int_0^\tau \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\mathbf{u}\|_{H_m}^r \right] \, dt + T + \mathbb{E} \left[\int_0^\tau \sum_{k=1}^\infty \|\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}]\|_{L^\infty(Q)}^2 \, dt \right]^{r/2} \end{aligned}$$

where we have used the uniform bounds for the density obtained in (3.25). Finally, we use (3.2) to bound the last integral,

$$\|\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}]\|_{L^\infty}^2 \lesssim \|\mathbf{F}_{k, \varepsilon, \xi}\|_{W_x^{2,2}} \lesssim c(\xi) \|\mathbf{F}_{k, \varepsilon, \xi}\|_{L^\infty} \lesssim f_{k, \varepsilon} c(\xi).$$

Seeing that

$$\mathbf{u} = \mathcal{M}^{-1}[\varrho][\Pi_m[\varrho \mathbf{u}]]$$

we may use again the bounds (3.25), (3.26) to conclude that

$$\|\Pi_m[\varrho \mathbf{u}]\|_{H_m} \lesssim \|\mathbf{u}\|_{H_m} \lesssim \|\Pi_m[\varrho \mathbf{u}]\|_{H_m}$$

where the constants in \lesssim depend only on $\varrho, \bar{\varrho}$. Consequently, a direct application of Gronwall's lemma gives rise to the estimate

$$\boxed{\text{wWS113}} \quad (3.31) \quad \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\Pi_m[\varrho \mathbf{u}](\tau, \cdot)\|_{H_m}^r \right] + \mathbb{E} \left[\sup_{\tau \in [0, T]} \|\mathbf{u}(\tau, \cdot)\|_{H_m}^r \right] \lesssim c(r, T) \mathbb{E} [1 + \|\mathbf{u}_0\|_{H_m}^r], \quad r \geq 1.$$

In addition to the uniform bound (3.31) we will need compactness of the approximate velocities in the space $C([0, T]; H_m)$. Moreover, we have to control the difference

$$(\mathbf{u} - [\mathbf{u}]_h)$$

uniformly in time. To this end, estimates on the modulus of continuity of \mathbf{u} are needed. Evoking (3.18) we obtain

$$\begin{aligned} & \int_Q [\varrho \mathbf{u}(\tau_1, \cdot) - \varrho \mathbf{u}(\tau_2, \cdot)] \cdot \boldsymbol{\varphi} \, dx \\ &= \int_{\tau_1}^{\tau_2} \int_Q \left(\varrho[\mathbf{u}]_{h,R} \otimes [\mathbf{u}]_h \right) : \nabla \boldsymbol{\varphi} \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_Q \chi(\|[\mathbf{u}]_h\|_{H_m} - R) p_\delta(\varrho, \vartheta) \operatorname{div} \boldsymbol{\varphi} \, dx \, dt \\ &+ \int_{\tau_1}^{\tau_2} \int_Q \varepsilon \varrho[\mathbf{u}]_h \cdot \Delta \boldsymbol{\varphi} \, dx \, dt - \int_{\tau_1}^{\tau_2} \int_Q \frac{1}{m} [\mathbf{u}]_h \cdot \boldsymbol{\varphi} \, dx \, dt \\ &- \int_{\tau_1}^{\tau_2} \int_Q \chi(\|[\mathbf{u}]_h\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla[\mathbf{u}]_h) : \nabla \boldsymbol{\varphi} \, dx \, dt \\ &+ \int_{\tau_1}^{\tau_2} \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}]_h \cdot \boldsymbol{\varphi} \, dx \, dW \text{ for any } \boldsymbol{\varphi} \in H_m, 0 \leq \tau_1 < \tau_2. \end{aligned}$$

With the bound (3.31) at hand, we may repeat the arguments leading to (3.30) to obtain

$$\mathbb{E} \left[\|\Pi_m[\varrho \mathbf{u}(\tau_1) - \varrho \mathbf{u}(\tau_2)]\|_{H_m}^r \right] \leq c(r, T) |\tau_1 - \tau_2|^{r/2} \mathbb{E} \left[(\|\mathbf{u}_0\|_{H_m}^r + 1) \right], \quad r \geq 1$$

whenever $0 \leq \tau_1 < \tau_2 \leq T$, $|\tau_1 - \tau_2| \leq 1$. Thus we may apply Kolmogorov continuity criterion to conclude that $\Pi_m[\varrho \mathbf{u}]$ has \mathbb{P} -a.s. β -Hölder continuous trajectories for all $\beta \in (0, \frac{1}{2} - \frac{1}{r})$.

$$\mathbb{E} \left[\|\Pi_m[\varrho \mathbf{u}]\|_{C^\beta([0, T]; H_m)}^r \right] \leq c(r, T) \mathbb{E} \left[\|\mathbf{u}_0\|_{H_m}^r + 1 \right], \quad r > 2.$$

Recalling the relation

$$\mathbf{u} = \mathcal{M}^{-1}[\varrho] \Pi_m[\varrho \mathbf{u}],$$

boundedness of ϱ from (3.25) and (3.19) we may infer that

$$\boxed{\text{wWS114}} \quad (3.32) \quad \mathbb{E} \left[\|\mathbf{u}\|_{C^\beta([0, T]; H_m)}^r \right] \lesssim \mathbb{E} \left[\|\mathbf{u}_0\|_{H_m}^r + 1 \right]$$

uniformly in h whenever $r > 2$ and $\beta \in (0, \frac{1}{2} - \frac{1}{r})$ with a constant independent of h .

With the estimates (3.25), (3.28), and (3.32) at hand, we are ready to perform the limit $h \rightarrow 0$ in the approximate scheme (3.14–3.18). Consider the joint law of the basic state variables $(\varrho, \vartheta, \mathbf{u}, W)$ ranging in the *pathsace*

$$\begin{aligned} \mathfrak{X} &\equiv C^\iota([0, T]; C^{2+\iota}(Q)) \times [C^\iota([0, T]; L^2(Q)) \cap L^2(0, T; W^{1,2}(Q))] \\ &\times C^\iota([0, T]; H_m) \times C([0, T]; \mathfrak{U}_0), \end{aligned}$$

$\iota \in (0, \bar{\nu})$, where $\bar{\nu} > 0$ is the minimum of the Hölder exponents in (3.25), (3.29), (3.32), and (3.33). Let $[\varrho_h, \vartheta_h, \mathbf{u}_h, W]$ be the (unique) approximate solution issuing from the iteration scheme (3.14–3.18), with the initial data being \mathfrak{F}_0 measurable and satisfying (3.26), (3.27) as well as

$$\boxed{\text{wWS115}} \quad (3.33) \quad \mathbb{E} \left[\|\mathbf{u}_0\|_{H_m}^r \right] \leq \bar{u} \text{ for some } r > 2.$$

Let $\mathcal{L}[\varrho_h, \vartheta_h, \mathbf{u}_h, W]$ denote the joint law of $[\varrho_h, \vartheta_h, \mathbf{u}_h, W]$ on \mathfrak{X} , whereas $\mathcal{L}[\varrho_h]$, $\mathcal{L}[\vartheta_h]$, $\mathcal{L}[\mathbf{u}_h]$ and $\mathcal{L}[W]$ denote the corresponding marginals on, respectively.

In view of the bounds (3.23), (3.24), (3.28), (3.29) and (3.31), (3.32), we conclude that $\mathcal{L}[\varrho_h, \vartheta_h, \mathbf{u}_h, W]$ is *tight* on the *Polish* space \mathfrak{X} . We may therefore apply Skorokhod's representation theorem to obtain the following.

Proposition 3.1. *There exists a complete probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$ with \mathfrak{X} -valued Borel measurable random variables $(\tilde{\varrho}_h, \tilde{\vartheta}_h, \tilde{\mathbf{u}}_h, \tilde{W}_h)$, $h \in (0, 1)$, and $(\varrho, \vartheta, \mathbf{u}, \tilde{W})$ such that (up to a subsequence)*

- (a) *the law of $(\tilde{\varrho}_h, \tilde{\vartheta}_h, \tilde{\mathbf{u}}_h, \tilde{W}_h)$ on \mathfrak{X} is given by $\mathcal{L}[\varrho_h, \vartheta_h, \mathbf{u}_h, W]$, $h \in (0, 1)$,*
- (b) *the law of $(\varrho, \vartheta, \mathbf{u}, \tilde{W})$ on \mathfrak{X} is a Radon measure,*

prop:skorokhod

(c) $(\tilde{\varrho}_h, \tilde{\vartheta}_h, \tilde{\mathbf{u}}_h, \tilde{W}_h)$ converges $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}, \tilde{W})$ in the topology of \mathfrak{X} , i.e.

$$\begin{aligned} \tilde{\varrho}_h &\rightarrow \tilde{\varrho} \text{ in } C^\nu([0, T]; C^{2+\nu}(\bar{Q})) \text{ } \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\vartheta}_h &\rightarrow \tilde{\vartheta} \text{ in } C^\nu([0, T]; W^{1,2}(Q)) \cap L^2(0, T; W^{1,2}(Q)) \text{ } \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{\mathbf{u}}_h &\rightarrow \tilde{\mathbf{u}} \text{ in } C^\nu([0, T]; H_m) \text{ } \tilde{\mathbb{P}}\text{-a.s.}, \\ \tilde{W}_h &\rightarrow \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \text{ } \tilde{\mathbb{P}}\text{-a.s.} \end{aligned} \quad (3.34)$$

Since the trajectories of $\tilde{\varrho}$, $\tilde{\vartheta}$ and $\tilde{\mathbf{u}}$ are $\tilde{\mathbb{P}}$ -a.s. continuous, progressive measurability with respect to their canonical filtrations follows from adaptivity of the approximate sequence. Consequently, they are progressively measurable with respect to the canonical filtration generated by $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}, \tilde{W}]$, namely,

$$\tilde{\mathfrak{F}}_t := \sigma(\sigma_t[\tilde{\varrho}] \cup \sigma_t[\tilde{\vartheta}] \cup \sigma_t[\tilde{\mathbf{u}}] \cup \cup_{k=1}^\infty \sigma_t[\tilde{W}_k]), \quad t \in [0, T].$$

Moreover, it is standard to show that \tilde{W} is a cylindrical Wiener process with respect to $(\tilde{\mathfrak{F}}_t)_{t \geq 0}$. Now, we show that $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ solves the approximate continuity equation.

lem:cont1

Lemma 3.1. *The process $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ satisfies (3.8) in $(0, T) \times Q$, $\tilde{\mathbb{P}}$ -a.s.*

Proof. As a consequence of the equality of laws from Proposition 3.1 and Theorem 4.2, we see that the approximate continuity equation (3.15) is satisfied on the new probability space by $[\tilde{\varrho}_h, \tilde{\mathbf{u}}_h]$. Moreover, the uniform bounds (3.25), (3.32) hold true also for $[\tilde{\varrho}_h, \tilde{\mathbf{u}}_h]$. Hence by Proposition 3.1 and Vitali's convergence theorem we may pass to the limit in (3.15) and deduce that $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ is a weak solution to the approximate continuity equation (3.5a). Furthermore, the bounds (3.25), (3.32) are also valid for the limit process $[\tilde{\varrho}, \tilde{\mathbf{u}}]$. Consequently, (3.5a) is satisfied a.e. in $(0, T) \times \mathbb{T}^3$, $\tilde{\mathbb{P}}$ -a.s. Finally, using parabolic regularity theory, we conclude that (3.8) is satisfied in the classical sense. \square

As the next step, we are now going to show that the quantity $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}, \tilde{W}]$ solves the approximate momentum equation.

prop:m1

Lemma 3.2. *The process $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}, \tilde{W}]$ satisfies (3.10) for all $\varphi \in H_m$ and $t \in [0, T]$, $\tilde{\mathbb{P}}$ -a.s.*

Proof. Modifying slightly the proof, the result of Theorem 4.2 remains valid in the current setting. Hence as a consequence of the equality of laws from Proposition 3.1, the approximate momentum equation (3.18) is satisfied on the new probability space by $[\tilde{\varrho}_h, \tilde{\vartheta}_h, \tilde{\mathbf{u}}_h, \tilde{W}_h]$. It is enough to pass to the limit with respect to h .

We observe that

$$\|[\tilde{\mathbf{u}}_h]_h(t) - \tilde{\mathbf{u}}_h(t)\|_{H_m} \lesssim h^\nu \|\tilde{\mathbf{u}}_h\|_{C_t^1 H_m}$$

and similarly

$$\begin{aligned} \|[\tilde{\varrho}_h]_h(t) - \tilde{\varrho}_h(t)\|_{C_x^{2+\nu}} &\lesssim h^\nu \|\tilde{\varrho}_h\|_{C_t^1 C_x^{2+\nu}}, \\ \|[\tilde{\vartheta}_h]_h(t) - \tilde{\vartheta}_h(t)\|_{L_x^2} &\lesssim h^\nu \|\tilde{\vartheta}_h\|_{C_t^1 L_x^2}. \end{aligned}$$

Now, with the convergences (3.34), the bounds (3.25), (3.32) and the assumption (2.2) at hand we may pass to the limit in the approximate momentum equation (3.18). The only term which needs an explanation is the stochastic integral. By the uniform convergence of $\tilde{\varrho}_h$ and $\tilde{\mathbf{u}}_h$ (recall Proposition 3.1), the continuity of the coefficients $\mathbf{F}_{k,\varepsilon,\xi}$ and the continuity of Π_m , it is easy to see that $\tilde{\mathbb{P}}$ -a.s.

eq:1707a

$$\begin{aligned} \Pi_m \left[[\tilde{\varrho}_h]_h \Pi_m [\mathbf{F}_{k,\varepsilon,\xi}([\tilde{\varrho}_h]_h, [\tilde{\vartheta}_h]_h), [\tilde{\mathbf{u}}_h]_h] \right] \\ \rightarrow \Pi_m \left[\varrho \Pi_m [\mathbf{F}_{k,\varepsilon,\xi}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})] \right] \text{ in } L^q((0, T) \times \mathbb{T}^3) \end{aligned} \quad (3.35)$$

for all $k \in \mathbb{N}$ and all $q < \infty$. On the other hand, we have

$$\tilde{\mathbb{E}} \int_0^T \left\| \Pi_m \left[[\tilde{\varrho}_h]_h \Pi_m [\mathbf{F}_{\varepsilon,\xi}([\tilde{\varrho}_h]_h, [\tilde{\vartheta}_h]_h, [\tilde{\mathbf{u}}_h]_h)] \right] \right\|_{L_2(\mathfrak{U}; L_x^2)}^2 dt$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \tilde{\mathbb{E}} \int_0^T \left\| \left[[\tilde{\varrho}_h]_h \Pi_m [\mathbf{F}_{k,\varepsilon,\xi}([\tilde{\varrho}_h]_h, [\tilde{\vartheta}_h], [\tilde{\mathbf{u}}_h]_h)] \right] \right\|_{L_x^2}^2 dt \\
&\leq \|\tilde{\varrho}_h\|_{L_{\omega,t,x}^\infty}^2 \sum_{k=1}^{\infty} \tilde{\mathbb{E}} \int_0^T \left\| \Pi_m [\mathbf{F}_{k,\varepsilon,\xi}([\tilde{\varrho}_h]_h, \tilde{\vartheta}_h, [\tilde{\mathbf{u}}_h]_h)] \right\|_{L_x^2}^2 dt \\
&\leq \|\tilde{\varrho}_h\|_{L_{\omega,t,x}^\infty}^2 \sum_{k=1}^{\infty} \tilde{\mathbb{E}} \int_0^T \left\| \mathbf{F}_{k,\varepsilon,\xi}([\tilde{\varrho}_h]_h, [\tilde{\vartheta}_h]_h, [\tilde{\mathbf{u}}_h]_h) \right\|_{L_x^2}^2 dt \lesssim \|\tilde{\varrho}_h\|_{L_{\omega,t,x}^\infty}^2 \sum_{k=1}^{\infty} f_{k,\varepsilon}^2 \lesssim c
\end{aligned}$$

using (3.2), (3.3) as well as (3.25). Consequently, we can strengthen (3.35) to

$$\begin{aligned}
&\Pi_m \left[[\tilde{\varrho}_h]_h \Pi_m [\mathbf{F}_{\varepsilon,\xi}([\tilde{\varrho}_h]_h, [\tilde{\vartheta}_h], [\tilde{\mathbf{u}}_h]_h)] \right] \\
&\rightarrow \Pi_m \left[\varrho \Pi_m [\mathbf{F}_{\varepsilon,\xi}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})] \right] \text{ in } L^2(0, T; L_2(\mathfrak{U}; L^2(\mathbb{T}^3)))
\end{aligned} \tag{3.36}$$

$\tilde{\mathbb{P}}$ -a.s. Combining this with the convergence of \tilde{W}_h from Proposition 3.1 we may apply Lemma 4.1 to pass to the limit in the stochastic integral and hence complete the proof. \square

Next, we show:

prop:s1b **Lemma 3.3.** *The process $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ satisfies (3.9) a.a. in $(0, T) \times Q$ $\tilde{\mathbb{P}}$ -a.s.*

Proof. As a consequence of the equality of laws from Proposition 3.1 and Theorem 4.2, we see that the approximate internal energy balance (3.16) is satisfied on the new probability space by $[\tilde{\varrho}_h, \tilde{\vartheta}_h, \tilde{\mathbf{u}}_h]$. Moreover, the uniform bounds (3.28) hold true also for $[\tilde{\varrho}_h]$. Using Proposition 3.1 and we may pass to the limit in (3.16) and deduce that $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$ is a weak solution to the approximate internal energy equation (3.9). Furthermore, the limit process $\tilde{\vartheta}$ also belongs to the class (3.22). Consequently, (3.9) is satisfied a.e. in $(0, T) \times Q$, $\tilde{\mathbb{P}}$ -a.s. \square

Finally, as $\tilde{\vartheta}_h$ obeys the (deterministic) bounds (3.28), (3.29), the limit $\tilde{\vartheta}$ belongs to the same class. In particular, the limit temperature $\tilde{\vartheta}$ enjoys the regularity claimed in Theorem 3.1.

The proof of Theorem 3.1 is hereby complete.

wWS1S3

3.3. Energy balance. We show that any solution of the approximate problem (3.5a–3.5c) satisfies a variant of the energy balance equation. To this end, we take the scalar product of (3.5b) with \mathbf{u} and integrate the resulting expression by parts. We apply Itô's formula to the scalar product

$$\int_Q \Pi_m(\varrho \mathbf{u}) \cdot \mathbf{u} \, dx = \int_Q \varrho |\mathbf{u}|^2 \, dx.$$

As

$$\int_Q \mathcal{M}^{-1}[\varrho] \Pi_m[\mathbf{v}] \cdot \Pi_m[\varrho \mathbf{u}] \, dx = \int_Q \varrho \mathcal{M}^{-1}[\varrho] \Pi_m[\mathbf{v}] \cdot \mathbf{u} \, dx = \int_Q \mathbf{v} \cdot \mathbf{u} \, dx,$$

we deduce from (3.5b) that

$$\begin{aligned}
&d \int_Q \frac{1}{2} \varrho |\mathbf{u}|^2 \, dx \\
&= - \int_Q \left[\operatorname{div}(\varrho [\mathbf{u}]_R \otimes \mathbf{u}) + \chi(\|\mathbf{u}\|_{H_m} - R) \nabla p_\delta(\varrho, \vartheta) \right] \cdot \mathbf{u} \, dx \, dt \\
&\quad + \int_Q \left[\varepsilon \Delta(\varrho \mathbf{u}) + \chi(\|\mathbf{u}\|_{H_m} - R) \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) + \frac{1}{m} \mathbf{u} \right] \cdot \mathbf{u} \, dx \, dt - \frac{1}{2} \int_Q |\mathbf{u}|^2 d\varrho \, dx \\
&\quad + \frac{1}{2} \sum_{k \geq 0} \int_Q \varrho |\Pi_m[\mathbf{F}_{k,\varepsilon,\xi}]|^2 \, dx \, dt + \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon,\xi}] \cdot \mathbf{u} \, dx \, dW.
\end{aligned} \tag{3.37}$$

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Furthermore, equation (3.5a) tells us that

$$\frac{1}{2} \int_Q |\mathbf{u}|^2 d\varrho \, dx = \frac{1}{2} \int_Q \varepsilon |\mathbf{u}|^2 \Delta \varrho \, dx \, dt - \frac{1}{2} \int_Q \operatorname{div}(\varrho [\mathbf{u}]_R) |\mathbf{u}|^2 \, dx \, dt,$$

while

$$\int_Q \operatorname{div}(\varrho[\mathbf{u}]_R \otimes \mathbf{u}) \cdot \mathbf{u} \, dx = -\frac{1}{2} \int_Q \varrho[\mathbf{u}]_R \cdot \nabla |\mathbf{u}|^2 \, dx = \frac{1}{2} \int_Q \operatorname{div}(\varrho[\mathbf{u}]_R) |\mathbf{u}|^2 \, dx,$$

and

$$\varepsilon \int_Q \Delta(\varrho \mathbf{u}) \cdot \mathbf{u} \, dx = -\varepsilon \int_Q \varrho |\nabla \mathbf{u}|^2 \, dx + \frac{1}{2} \int_Q \varepsilon |\mathbf{u}|^2 \Delta \varrho \, dx.$$

Consequently, relation (3.37) reduces to

$$\begin{aligned} \text{d} \int_Q \frac{1}{2} \varrho |\mathbf{u}|^2 \, dx + \chi(\|\mathbf{u}\|_{H_m} - R) \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, dx \, dt + \frac{1}{m} \int_Q |\mathbf{u}|^2 \, dx \, dt + \varepsilon \int_Q \varrho |\nabla \mathbf{u}|^2 \, dx \, dt \\ \text{(3.38)} \quad = \int_Q \chi(\|\mathbf{u}\|_{H_m} - R) p_\delta(\varrho, \vartheta) \operatorname{div} \mathbf{u} \, dx \, dt \\ + \frac{1}{2} \int_Q \varrho |\Pi_m[\mathbf{F}_{\varepsilon, \xi}]|^2 \, dx \, dt + \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}] \cdot \mathbf{u} \, dx \, dW. \end{aligned}$$

Seeing that

$$\chi(\|\mathbf{u}\|_{H_m} - R) p_\delta(\varrho, \vartheta) \operatorname{div} \mathbf{u} = p_\delta(\varrho, \vartheta) \operatorname{div}[\mathbf{u}]_R$$

we rewrite the energy balance (3.38) as

$$\begin{aligned} \text{d} \int_Q \frac{1}{2} \varrho |\mathbf{u}|^2 \, dx = -\chi(\|\mathbf{u}\|_{H_m} - R) \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, dx \, dt + \int_Q p_\delta(\varrho, \vartheta) \operatorname{div}[\mathbf{u}]_R \, dx \, dt \\ \text{(3.39)} \quad - \varepsilon \int_Q \varrho |\nabla \mathbf{u}|^2 \, dx \, dt - \frac{1}{m} \int_Q |\mathbf{u}|^2 \, dx \, dt + \sum_{k \geq 0} \frac{1}{2} \int_Q \varrho |\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}]|^2 \, dx \, dt \\ + \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}] \cdot \mathbf{u} \, dx \, dW. \end{aligned}$$

For the first term on the right-hand side we use the approximate internal energy equation to see

$$\begin{aligned} -\varepsilon \int_Q \varrho |\nabla \mathbf{u}|^2 \, dx - \chi(\|\mathbf{u}\|_{H_m} - R) \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, dx + \int_Q p(\varrho, \vartheta) \operatorname{div}[\mathbf{u}]_R \, dx \\ = \int_Q \left[-\partial_t(\varrho e_\delta(\varrho, \vartheta)) - \varepsilon \delta(\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \delta \frac{1}{\vartheta^2} - \varepsilon \vartheta^5 \right] \, dx. \end{aligned}$$

Next, multiplying the equation of continuity on $b'(\varrho)$ we deduce a renormalized equation

$$db(\varrho) + \operatorname{div}(b(\varrho)[\mathbf{u}]_R) + (b'(\varrho)\varrho - b(\varrho)) \operatorname{div}[\mathbf{u}]_R \, dt = \operatorname{div}(b'(\varrho)\nabla \varrho) \, dt - b''(\varrho) |\nabla \varrho|^2 \, dt$$

for any twice continuously differentiable function b . Inserting this into (3.39) we can write the energy balance in its final form

$$\begin{aligned} \text{d} \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta-1} + \varrho^2 \right) \right] \, dx \\ \text{(3.40)} \quad + \frac{1}{m} \int_Q |\mathbf{u}|^2 \, dx \, dt + \varepsilon \int_Q \vartheta^5 \, dx \, dt \\ = \int_Q \delta \frac{1}{\vartheta^2} \, dx \, dt + \frac{1}{2} \sum_{k \geq 0} \int_Q \varrho |\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}]|^2 \, dx \, dt + \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}] \cdot \mathbf{u} \, dx \, dW. \end{aligned}$$

We have shown the following version of the energy balance for the approximate martingale solutions.

WWP2 **Proposition 3.1.** *Under the hypotheses of Theorem 3.1, let $(\varrho, \vartheta, \mathbf{u}, W)$ be a martingale solution of the approximate problem (3.5a)–(3.5c).*

Then the following total energy equation

$$\begin{aligned}
& - \int_0^T \partial_t \psi \left(\int_Q \mathcal{E}_\delta(\varrho, \vartheta, \mathbf{u}) \, dx \right) dt + \int_0^T \psi \int_Q \left(\varepsilon \vartheta^5 + \frac{1}{m} |\mathbf{u}|^2 \right) dt \\
& = \psi(0) \int_{\mathbb{T}^3} \mathcal{E}_\delta(\varrho_0, \vartheta_0, \mathbf{u}_0) \, dx + \int_0^T \int_Q \frac{\delta}{\vartheta^2} \psi \, dx \, dt \\
& + \frac{1}{2} \int_0^T \psi \left(\int_Q \sum_{k \geq 1} \varrho |\Pi_m[\mathbf{F}_{k,\varepsilon,\xi}(\varrho, \vartheta, \mathbf{u})]|^2 \, dx \right) dt \\
& + \int_0^T \psi \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon,\xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dW \, dx
\end{aligned} \tag{3.41}$$

holds true for any deterministic test function $\psi \in C_c^\infty[0, T]$, \mathbb{P} -a.s. Here, we abbreviated

$$\mathcal{E}_\delta(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta - 1} + \varrho^2 \right).$$

wWR8

Remark 3.1. Consistently with the weak formulation of the field equations in Definition 3.1, we have rewritten (3.40) in the form of a variational equality with a deterministic test function ψ .

3.4. Entropy balance. As equations (3.8) and (3.9) are satisfied in the strong sense and ϑ is strictly positive, we may divide (3.9) by ϑ obtaining the (regularized) entropy balance equation

$$\begin{aligned}
& \partial_t(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta)[\mathbf{u}]_R) - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] \\
& = \frac{1}{\vartheta} \left[\chi(\|\mathbf{u}\|_{H^m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] \\
& + \frac{\varepsilon \delta}{\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{\Delta_x \varrho}{\vartheta} \left(\vartheta s_\delta(\varrho, \vartheta) - e_\delta(\varrho, \vartheta) - \frac{p(\varrho, \vartheta)}{\varrho} \right) - \varepsilon \vartheta^4 + \frac{\varepsilon}{\vartheta} \varrho |\nabla \mathbf{u}|^2
\end{aligned} \tag{3.42}$$

satisfied a.a. in $(0, T) \times Q$, together with the boundary conditions $\nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0$.

EGA

4. GALERKIN APPROXIMATION

Our goal is to perform several steps: (i) letting $R \rightarrow \infty$ in the approximate system (3.5a)–(3.5c), (ii) letting $m \rightarrow \infty$ in the resulting limit, (iii) letting the parameter $\xi \rightarrow 0$. The technique in these three steps is rather similar and is based on the uniform bounds enforced by the data. In the following we amply use the Korn–Poincaré inequality:

$$\|\mathbf{v}\|_{W^{1,2}(Q, \mathbb{R}^3)}^2 \leq c \int_Q \frac{1}{\vartheta} \mathbb{S}(\vartheta, \nabla \mathbf{v}) : \nabla \mathbf{v} \, dx \text{ for all } \mathbf{v} \in W^{1,2}(Q; \mathbb{R}^3), \mathbf{v}|_{\partial Q} = 0, \tag{4.1}$$

cf. also hypothesis (2.11).

4.1. Uniform bounds. We start by introducing the *ballistic free energy*,

$$H_\Theta(\varrho, \vartheta) = \varrho(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)), \quad H_{\delta, \Theta}(\varrho, \vartheta) = \varrho(e_\delta(\varrho, \vartheta) - \Theta s_\delta(\varrho, \vartheta)),$$

cf. [21, Chapter 2, Section 2.2.3]. Combining the total energy balance (3.41), with the entropy equation (3.42) we get the *total dissipation balance*

$$\begin{aligned}
& \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \frac{\delta}{\beta - 1} \varrho^\beta + \delta \varrho^2 \right] (\tau, \cdot) \, dx + \Theta \int_0^\tau \int_{\mathbb{T}^3} \sigma_{R, m, \varepsilon, \delta} \, dx \\
& + \int_0^\tau \int_Q \left(\varepsilon \vartheta^5 + \frac{1}{m} |\mathbf{u}|^2 \right) \, dt \\
& = \int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\delta, \Theta}(\varrho_0, \vartheta_0) + \frac{\delta}{\beta - 1} \varrho_0^\beta + \delta \varrho_0^2 \right] \, dx \\
& + \varepsilon \int_0^\tau \int_Q \frac{\Theta}{\vartheta^2} \left(e_{M, \delta}(\varrho, \vartheta) + \varrho \frac{\partial e_M}{\partial \varrho}(\varrho, \vartheta) \right) \nabla \varrho \cdot \nabla \vartheta \, dx \, dt \\
& + \int_0^\tau \int_Q \left(\frac{\delta}{\vartheta^2} + \varepsilon \Theta \vartheta^4 \right) \, dx \, dt + \int_0^\tau \int_Q \varrho \Pi_m [\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dW \, dx \\
& + \frac{1}{2} \int_0^\tau \left(\int_Q \sum_{k \geq 1} \varrho |\Pi_m [\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})]|^2 \, dx \right) \, dt
\end{aligned} \tag{4.2}$$

for any $0 \leq \tau \leq T$, and any positive constant $\Theta > 0$, \mathbb{P} -a.s., where

$$\begin{aligned}
\sigma_{R, m, \varepsilon, \delta} &= \frac{1}{\vartheta} \left[\chi(\|\mathbf{u}\|_{H_m} - R) \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 + \frac{\delta}{2} \left(\varrho^{\beta-1} + \frac{1}{\vartheta^2} \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] \\
&+ \frac{\varepsilon \delta}{2\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) \frac{|\nabla \varrho|^2}{\varrho \vartheta} + \frac{\varrho}{\vartheta} |\nabla \mathbf{u}|^2.
\end{aligned}$$

Keeping $\varepsilon > 0$, $\delta > 0$, $\xi > 0$ fixed we derive bounds independent of the parameters R and m . As the projections Π_m are bounded by (3.2), we get

$$\begin{aligned}
& \sum_{k \geq 0} \int_Q \varrho |\Pi_m [\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})]|^2 \, dx \lesssim \sum_{k \geq 0} \|\varrho\|_{L_x^1} \|\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})\|_{W_x^{2,2}}^2 \\
& \lesssim c(\xi) \sum_{k \geq 0} \|\varrho\|_{L_x^1} \|\mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u})\|_{L_x^\infty}^2 \lesssim c(\varepsilon, \xi, \bar{\varrho})
\end{aligned} \tag{4.3}$$

using also

$$\|\varrho\|_{L_x^1} = \|\varrho_0\|_{L_x^1} \leq \bar{\varrho}. \tag{4.4}$$

Next, by means of the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
& \mathbb{E} \left[\sup_{0 \leq t \leq \tau} \left| \int_0^t \int_Q \varrho \Pi_m [\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \, dW \right|^r \right] \\
& \lesssim \mathbb{E} \left[\int_0^\tau \sum_{k \geq 0} \left| \int_Q \varrho \Pi_m [\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \right|^2 \right]^{r/2}, \quad r \geq 1.
\end{aligned} \tag{4.5}$$

Furthermore, using once more (3.2), we deduce

$$\begin{aligned}
& \left| \int_Q \varrho \Pi_m [\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \right|^2 \lesssim \left\| \sqrt{\varrho} \right\|_{L_x^2} \left\| \sqrt{\varrho} \mathbf{u} \right\|_{L_x^2} \left\| \Pi_m [\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \right\|_{L_x^\infty} \right|^2 \\
& \lesssim c(\xi, \bar{\varrho}) \left\| \sqrt{\varrho} \mathbf{u} \right\|_{L_x^2}^2 \left\| \mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u}) \right\|_{L_x^\infty}^2 \leq c(\xi, \bar{\varrho}) f_{k, \varepsilon}^2 \left\| \sqrt{\varrho} \mathbf{u} \right\|_{L_x^2}^2.
\end{aligned}$$

Next, we observe that the term δ/ϑ^2 on the right-hand side of (4.2) is dominated by its counterpart δ/ϑ^3 in the entropy production term $\sigma_{R, m, \varepsilon, \delta}$. Analogously, the quantity $\varepsilon \Theta \vartheta^4$ on the right hand side is “absorbed” by the term $\varepsilon \vartheta^5$ at the left hand side of (4.2).

Consequently, it remains to handle the quantity

$$\varepsilon \int_Q \frac{1}{\vartheta^2} \left(e_M(\varrho, \vartheta) + \varrho \frac{\partial e_M(\varrho, \vartheta)}{\partial \varrho} \right) \nabla \varrho \cdot \nabla \vartheta \, dx$$

appearing on the right-hand side of (4.2). To this end, we first use hypothesis (2.6), together with (2.9) and (2.10), in order to obtain

$$\left| \frac{1}{\vartheta^2} \left(e_M(\varrho, \vartheta) + \varrho \frac{\partial e_M(\varrho, \vartheta)}{\partial \varrho} \right) \nabla \varrho \cdot \nabla \vartheta \right| \leq c \left(\frac{\varrho^{\frac{2}{3}} + \vartheta}{\vartheta^2} \right) |\nabla \varrho| |\nabla \vartheta|,$$

where, furthermore,

$$\frac{|\nabla \varrho| |\nabla \vartheta|}{\vartheta} \leq \kappa \frac{|\nabla \varrho|^2}{\vartheta} + c(\kappa) \frac{|\nabla \vartheta|^2}{\vartheta} \text{ for any } \kappa > 0,$$

and, similarly,

$$\frac{\varrho^{\frac{2}{3}} |\nabla \varrho| |\nabla \vartheta|}{\vartheta^2} \leq \kappa \frac{\varrho^{\frac{4}{3}} |\nabla \varrho|^2}{\vartheta} + c(\kappa) \frac{|\nabla \vartheta|^2}{\vartheta^3}.$$

Thus we infer that

$$\begin{aligned} \text{m140} \quad (4.6) \quad & \varepsilon \int_Q \frac{1}{\vartheta^2} \left| e_M(\varrho, \vartheta) + \varrho \frac{\partial e_M(\varrho, \vartheta)}{\partial \varrho} \right| |\nabla \varrho| |\nabla \vartheta| \, dx \\ & \leq \frac{1}{2} \int_Q \left[\delta (\vartheta^{\beta-2} + \frac{1}{\vartheta^3}) |\nabla \vartheta|^2 + \frac{\varepsilon \delta}{\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 \right] \, dx \end{aligned}$$

provided $\varepsilon = \varepsilon(\delta) > 0$ is small enough. Consequently, passing to expectations in (4.2) we may apply Gronwall's inequality to conclude that

$$\begin{aligned} \text{wWS27} \quad (4.7) \quad & \mathbb{E} \left[\sup_{0 \leq \tau \leq T} \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \frac{\delta}{\beta-1} \varrho^\beta + \delta \varrho^2 \right] \, dx + \Theta \int_0^T \int_Q \sigma_{R, m, \varepsilon, \delta} \, dx + \varepsilon \int_0^T \int_Q \vartheta^5 \, dx \, dt \right]^r \\ & \leq c(T, \xi) \mathbb{E} \left[\int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\delta, \Theta}(\varrho_0, \vartheta_0) + \frac{\delta}{\beta-1} \varrho_0^\beta + \delta \varrho_0^2 \right] \, dx \right]^r \end{aligned}$$

for any $r \geq 1$.

Taking into account the properties of the function $H_{\delta, \bar{\vartheta}}$, see [21, Section 2.2.3, (2.49), (2.50)], we obtain the following bounds depending only on the initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ determined in terms of their law Λ , and the parameter ξ :

$$\begin{aligned} \text{m142*} \quad (4.8) \quad & \mathbb{E} \left[\left| \sup_{t \in (0, T)} \int_Q \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta-1} + \varrho^2 \right) \right) \, dx \right|^r \right] \leq c(r, \xi, \Lambda), \\ & \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{T}^3} \chi(\|\mathbf{u}\|_{H_m} - R) \int_{\mathbb{T}^3} \frac{1}{\vartheta} [\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}] \, dx \, dt \right|^r \right] \leq c(r, \xi, \Lambda), \\ & \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{T}^3} \frac{1}{\vartheta} \left(\frac{\kappa(\vartheta)}{\vartheta} + \delta (\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 \, dx \, dt \right|^r \right] \leq c(r, \xi, \Lambda), \\ & \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{T}^3} \left(\varepsilon \delta \frac{1}{\vartheta^3} + \varepsilon \vartheta^5 + \frac{1}{m} |\mathbf{u}|^2 + \varepsilon \frac{\varrho}{\vartheta} |\nabla \mathbf{u}|^2 \right) \, dx \, dt \right|^r \right] \leq c(r, \xi, \Lambda), \\ & \varepsilon \delta \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{T}^3} \frac{1}{\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 \, dx \, dt \right|^r \right] \leq c(r, \xi, \Lambda), \\ & \varepsilon \mathbb{E} \left[\left| \int_0^T \int_{\mathbb{T}^3} \frac{\bar{\vartheta}}{\varrho \vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) |\nabla \varrho|^2 \, dx \, dt \right|^r \right] \leq c(r, \xi, \Lambda). \end{aligned}$$

As all norms are equivalent on the finite-dimensional space H_m and ∂Q is regular,, we deduce from (4.8)₄ that

$$\text{wWS28} \quad (4.9) \quad \mathbb{E} \left[\left| \int_0^T \|\mathbf{u}\|_{C^2(\bar{Q})}^2 \, dt \right|^r \right] \leq c(m, \xi, \varepsilon, \Lambda).$$

Finally, we recall the coercivity properties of the function $H_{\delta,\Theta}$, see [21, Chapter 3, Proposition 3.2]:

$$\boxed{\text{Nhe1m}} \quad (4.10) \quad H_{\delta,\Theta}(\varrho, \vartheta) \geq \frac{1}{4} (\varrho e_\delta(\varrho, \vartheta) + \Theta \varrho |s(\varrho, \vartheta)|) - \left| (\varrho - \bar{\varrho}) \frac{\partial H_{\delta,2\Theta}}{\partial \varrho}(\bar{\varrho}, 2\Theta) + H_{\delta,2\Theta}(\bar{\varrho}, 2\Theta) \right|$$

for any positive $\varrho, \vartheta, \bar{\varrho}, \Theta$. Consequently, (4.8)₁ gives rise to

$$\boxed{\text{Nm142*}} \quad (4.11) \quad \mathbb{E} \left[\left[\sup_{t \in (0,T)} \int_{\mathbb{T}^3} (\vartheta^4 + \delta \varrho |\log(\vartheta)|) \, dx \right]^r \right] \leq c(\xi, \Lambda).$$

$\boxed{\text{wWS2S2}}$

4.2. **Limit** $R \rightarrow \infty$. Keeping m fixed we consider the rather restrictive hypothesis on the initial distribution of the data imposed in Theorem 3.1:

$$\begin{aligned} \Lambda = \Lambda_R, \Lambda_R \left\{ 0 < \underline{\varrho}_R \leq \varrho_0, \|\varrho_0\|_{C_x^{2+\nu}} \leq \bar{\varrho}_R, \nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0 \right\} &= 1 \\ \Lambda_R \left\{ 0 < \underline{\vartheta}_R \leq \vartheta_0, \|\vartheta_0\|_{W_x^{1,2} \cap C_x} \leq \bar{\vartheta}_R \right\} &= 1 \\ \int_{C_x^{2+\nu} \times W_x^{1,2} \times H_m} \|\mathbf{u}_0\|_{H_m}^r \, d\Lambda_R &\leq \bar{u}_R \text{ for any } r \geq 1. \end{aligned}$$

As the uniform bounds will be lost in the limit $R \rightarrow \infty$ we suppose that

$$\boxed{\text{nGL1}} \quad (4.12) \quad \Lambda_R \rightarrow \Lambda \text{ weakly-} (*) \text{ in } \mathcal{M}^+(C^{2+\nu} \times W^{1,2} \times H_m),$$

where

$$\begin{aligned} &\Lambda_R \left\{ 0 < \underline{\varrho} \leq \int_Q \varrho_0 \, dx \leq \bar{\varrho}, \nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0, \varrho_0, \vartheta_0 > 0 \right\} \\ &= \Lambda \left\{ 0 < \underline{\varrho} \leq \int_Q \varrho_0 \, dx \leq \bar{\varrho}, \nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0, \varrho_0, \vartheta_0 > 0 \right\} = 1, \\ &\int_{C^{2+\nu} \times W^{1,2} \times H_m} \left[\|\varrho_0\|_{C_x^{2+\nu}} + \|\vartheta_0\|_{W_x^{1,2} \cap C_x} + \|\mathbf{u}_0\|_{H_m} \right]^r \, d\Lambda_R \leq c(r) \text{ uniformly in } R, \end{aligned}$$

By virtue of Theorem 3.1, the approximate problem (3.5), (3.6) admits a martingale solution $(\varrho_R, \vartheta_R, \mathbf{u}_R)$ with the initial law Λ_R for any fixed $R > 0$. Our first goal is to justify the limit $R \rightarrow \infty$. The strategy is similar to Section 3.2; we establish compactness of the phase variables and use a variant of Skorokhod representation theorem, cf. Theorem 4.1 below.

4.2.1. *Compactness.* We start recalling the standard parabolic maximal regularity estimates (see e.g. [30] or [34]) applied to (3.5a):

$$\boxed{\text{wWS220}} \quad (4.13) \quad \begin{aligned} &\|\partial_t \varrho\|_{L^p(0,T;L^q(Q))} + \|\varrho\|_{L^p(0,T;W^{2,q}(Q))} \\ &\lesssim \|\operatorname{div}(\varrho[\mathbf{u}]_R)\|_{L^p(0,T;L^q(Q))} + \|\varrho_0\|_{C^{2+\nu}(Q)}, \\ &\|\varrho\|_{L^p(0,T;W^{1,q}(Q))} \lesssim \|\varrho[\mathbf{u}]_R\|_{L^p(0,T;L^q(Q))} + \|\varrho_0\|_{C^{2+\nu}(Q)}, \end{aligned}$$

for $1 < p, q < \infty$. In (4.13), the regularity of the initial data can be considerably weakened. However, such generality is not needed here. Now observe that (4.8), (4.9) give rise to

$$\mathbb{E} \left[\|\varrho[\mathbf{u}]_R\|_{L^1(0,T;L^3(Q))}^r \right] \leq c(r, m, R).$$

It is worth noting that this estimate is independent of R as long as

$$\boxed{\text{newEST}} \quad (4.14) \quad \mathbb{E} \left[\left[\int_0^T \|\mathbf{u}\|_{W^{1,2}(Q)}^2 \, dt \right]^r \right] \lesssim 1.$$

As we shall see below, estimate (4.14) remain valid at any stage of approximation.

Now observe that (4.8)₁ together with $\beta \geq 6$, (4.14), combined with the standard Sobolev

embedding $W^{1,2}(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$, gives rise to

$$\begin{aligned} \mathbb{E} \left[\left\| \varrho \mathbf{u} \right\|_{L_t^1 L_x^3}^r \right] &\leq \mathbb{E} \left[\left\| \int_0^T \|\varrho\|_{L_x^6} \|\mathbf{u}\|_{L_x^6} dt \right\|^r \right] \\ &\lesssim \mathbb{E} \left[\left\| \sup_{t \in [0, T]} \|\varrho\|_{L_x^6} \int_0^T \|\mathbf{u}\|_{L_x^6} dt \right\|^r \right] \\ &\lesssim \mathbb{E} \left[\left\| \sup_{t \in [0, T]} \|\varrho\|_{L_x^6}^2 \right\|^r \right] + \mathbb{E} \left[\left\| \int_0^T \|\mathbf{u}\|_{W_x^{1,2}}^2 dt \right\|^r \right] \\ &\lesssim c(r). \end{aligned}$$

Now, (4.8)₁ yields

$$\mathbb{E} \left[\left\| \sup_{t \in (0, T)} \int_Q |\varrho \mathbf{u}|^{\frac{2\beta}{\beta+1}} dx \right\|^r \right] \leq c(r, \xi, \Lambda).$$

Using the interpolation

$$\left(L^\infty(0, T; L^{\frac{2\beta}{\beta+1}}(Q)); L^1(0, T; L^3(Q)) \right)_\theta = L^p(0, T; L^p(Q))$$

for an appropriate $\theta \in (\frac{1}{2}, 1)$ we obtain

$$\mathbb{E} \left[\left\| \varrho \mathbf{u} \right\|_{L^p(0, T; L^p(Q))}^r \right] \lesssim c(r) \text{ for a certain } p > 2.$$

This, plugged in the right-hand of (4.13), implies

$$\boxed{\text{wWS221}} \quad (4.15) \quad \mathbb{E} \left[\left\| \varrho \right\|_{L^p(0, T; W^{1, p}(Q))}^r \right] \lesssim c(r) \text{ for some } p > 2.$$

Finally, the estimates (4.13) and (4.15) can be used again in (4.13) to conclude that

$$\boxed{\text{wWS222}} \quad (4.16) \quad \mathbb{E} \left[\left\| \|\partial_t \varrho\|_{L^p(0, T; L^p(Q))} + \|\varrho\|_{L^p(0, T; W^{2, p}(Q))} \right\|^r \right] \lesssim c(r)$$

for some $p > 1$, where $c(r)$ depends also on the initial data, in particular on $\mathbb{E} \|\varrho_0\|_{C_x^{2+\nu}}^r$.

Now, following the arguments introduced in Section 3.2, we show compactness of $\varrho \mathbf{u}$ with respect to the time variable. Similarly to Section 3.2 and in view of the bounds established above, it is enough to check the time continuity of the stochastic integral, namely

$$\mathbb{E} \left[\left\| \int_{\tau_1}^{\tau_2} \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] dW \right\|_{W_x^{-k, 2}}^r \right], \quad k > \frac{N}{2}.$$

Applying again the Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned} &\mathbb{E} \left[\left\| \int_{\tau_1}^{\tau_2} \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] dW \right\|_{W_x^{-k, 2}}^r \right] \\ &\lesssim |\tau_1 - \tau_2|^{r/2} \mathbb{E} \left[\sum_{k \geq 1} \sup_{0 \leq t \leq T} \left(\|\varrho \Pi_m[\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})]\|_{L_x^1} \right)^2 \right]^{\frac{r}{2}} \text{ for any } r \geq 1. \end{aligned}$$

Next, by Hölder's inequality,

$$\begin{aligned} \|\varrho \Pi_m[\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})]\|_{L_x^1} &\leq \|\varrho\|_{L_x^1} \|\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}(\varrho \vartheta, \mathbf{u})]\|_{L_x^\infty} \\ &\leq c(\bar{\varrho}) \|\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})\|_{W_x^{2, 2}} \lesssim c(\bar{\varrho}, \xi) f_{k, \varepsilon} \end{aligned}$$

using also (3.3) and (4.4). Consequently, we may use the bounds (4.8) and apply the Kolmogorov continuity criterion to obtain

$$\boxed{\text{wWS224}} \quad (4.17) \quad \mathbb{E} \left[\left\| \varrho \mathbf{u} \right\|_{C^s([0, T]; W^{-k, 2}(Q))}^r \right] \lesssim c(r) \text{ for a certain } 0 < s(r) < \frac{1}{2}$$

provided $r \geq 2$, where $c(r)$ behaves like (4.16). Apparently, estimates (4.16), (4.17) imply strong (pointwise) compactness of ϱ and $\varrho \mathbf{u}$ necessary for passing to the limit in the nonlinear

terms. As we shall see below, the same property of the temperature will follow from equation (3.42).

s:AL1

4.2.2. *Asymptotic limit.* Suppose now that m is fixed. Given a family of approximate solutions $(\varrho_R, \vartheta_R, \mathbf{u}_R)_{R>0}$ we let $R \rightarrow \infty$. Unfortunately, the available estimates are considerably weaker than those obtained in Section 3.2, making the choice of the appropriate *path space* more delicate. In particular, we have to use the *weak topologies* that are in general not Polish. This is usually overcome by Jakubowski’s variant of the Skorokhod representation theorem, see [31]. In the context of stochastic Navier–Stokes equations it has firstly been used in [11]. Here and hereafter we systematically follow the approach proposed in [7] and consider bounded sequences in Banach spaces together with their norms as a new random variables, applying the standard Skorokhod theorem. The key result is the following theorem ([7, Chapters 2.7 and 2.8]):¹

RDT4

Theorem 4.1. *Let $(\mathbf{U}_{0,n})_{n \geq 1}$ be a sequence of random variables in a Polish space Y_0 , $(\mathbf{U}_n)_{n \geq 1}$ a sequence of random variables in $L^1(Q_T; R^M)$, and $(W_n)_{n \geq 1}$ a sequence of cylindrical Wiener processes defined on a complete probability space $\{\Omega, \mathfrak{B}, \mathbb{P}\}$. Suppose that the family of laws of $(\mathbf{U}_{0,n})_{n \geq 1}$ is tight in Y_0 . In addition, suppose that for any $\varepsilon > 0$, there exists $M > 0$ such that*

$$\begin{aligned} \mathbb{P} \{ \|\mathbf{U}_n\|_{L^q(Q_T; R^M)} > M \} &< \varepsilon \text{ for some } q \geq 1; \\ \mathbb{P} \{ [\|\mathbf{U}_n\|] > M \} &< \varepsilon \end{aligned}$$

uniformly for $n = 1, 2, \dots$, where $[\cdot] : W^{-m,2}(Q_T) \rightarrow [0, \infty]$, $m > \frac{N+1}{2}$ is a Borel measurable function.

Then there exist subsequences of random variables $(\tilde{\mathbf{U}}_{0,n(j)})_{j \geq 1}$ in Y_0 , $(\tilde{\mathbf{U}}_{n(j)})_{j \geq 1}$, $\tilde{\mathbf{U}}_{n(j)} \in L^1(Q_T; R^M)$ and cylindrical Wiener processes $\tilde{W}_{n(j)}$ on the standard probability space $\{[0, 1], \mathfrak{B}[0, 1], \mathfrak{L}\}$ enjoying the following properties \mathfrak{L} -a.s.:

$$\begin{aligned} [\mathbf{U}_{0,n(j)}, \mathbf{U}_{n(j)}, W_{n(j)}] &\sim [\tilde{\mathbf{U}}_{0,n(j)}, \tilde{\mathbf{U}}_{n(j)}, \tilde{W}_{n(j)}] \quad (\text{equivalence in law}); \\ \tilde{\mathbf{U}}_{n(j)} &\rightarrow \mathbf{U} \text{ in } W^{-m,2}(Q_T; R^M), \quad g(\tilde{\mathbf{U}}_{n(j)}) \rightarrow^* \overline{g(\mathbf{U})} \text{ in } L^\infty(Q_T) \text{ for any } g \in C_c(R^M); \\ \tilde{\mathbf{U}}_{0,n(j)} &\rightarrow \mathbf{U}_0 \text{ in } Y_0, \quad \tilde{W}_{n(j)} \rightarrow W \text{ in } C([0, T]; \mathfrak{L}_0); \\ &\sup_{j \geq 1} [\|\tilde{\mathbf{U}}_{n(j)}\|] < \infty. \end{aligned}$$

If, in addition $q > 1$, then \mathfrak{L} -a.s.

$$\tilde{\mathbf{U}}_{n(j)} \rightarrow \mathbf{U} \text{ weakly in } L^q(Q_T; R^M), \quad f(\tilde{\mathbf{U}}_{n(j)}) \rightarrow \overline{f(\mathbf{U})} \text{ weakly in } L^r(Q_T)$$

for any $f \in C(R^M)$ such that

$$|f(\mathbf{v})| \leq c(1 + |\mathbf{v}|^s), \quad 1 \leq s < q, \quad r = \frac{q}{s} > 1.$$

Here $\overline{f(\mathbf{U})}$ denotes the weak limit of $f(\tilde{\mathbf{U}}_{n(j)})$.

Nrcom1

Remark 4.1. Here and hereafter the symbol $\overline{f(\mathbf{U})}$ denotes a weak L^1 -limit of a sequence $(f(\mathbf{U}_n))_{n \geq 1}$. The existence of such a limit for any f with appropriate growth implies the existence of a Young measure $\{\nu_{t,x}\}_{(t,x) \in Q_T}$ associated to the sequence $(f(\mathbf{U}_n))_{n \geq 1}$. In particular,

$$\langle \nu_{t,x}; f \rangle = \overline{f(\mathbf{U})}(t,x) \text{ for a.a. } (t,x) \in Q_T, \quad f \in C_c(R^M),$$

cf. Pedregal [43].

¹The first part of Thm. 4.1 can be found on pages 55 and 56 (with a special choice of X and Y); the second part (concerning the Young measures) is stated in Thm. 2.8.1.

Nrcom

Remark 4.2. Note that $L^1(Q_T) \hookrightarrow W^{-m,2}(Q_T)$ as soon as $m > (N+1)/2$. In applications, \mathbf{U} is a vector of random variables (unknowns) like $(\varrho, \vartheta, \mathbf{u}, \nabla \vartheta, \nabla \mathbf{u})$ and $[[\cdot]]$ represents the sum of available uniform bounds, like

$$[[(\varrho, \vartheta, \mathbf{u}, \nabla \vartheta, \nabla \mathbf{u})]] = \|\varrho\|_{L^\infty(0,T;L^\beta)} + \|\nabla \vartheta\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2} \dots$$

In addition, we need an abstract result on changing law in a system of stochastic PDE's. Specifically, evoking the situation considered in [7, Chapter 2.9], we consider an abstract stochastic PDE

$$D(\mathbf{U})(\tau) - D_0 + \int_0^\tau \operatorname{div} \mathbf{F}(\mathbf{U}) \, dt = \int_0^\tau \mathbf{G}(\mathbf{U}) \, dW,$$

or, in the weak form,

$$(4.18) \quad \int_0^T [\partial_t \psi \langle D(\mathbf{U}), \varphi \rangle + \psi \langle \mathbf{F}(\mathbf{U}), \nabla \varphi \rangle] \, dt + \int_0^T \psi \langle \mathbf{G}(\mathbf{U}), \varphi \rangle \, dW + \psi(0) \langle D_0, \varphi \rangle = 0$$

RD8

for any $\varphi \in C_c^\infty(Q)$, $\psi \in C_c^\infty[0, T]$, where

$$D = D(x, \mathbf{U}), \quad \mathbf{F} = \mathbf{F}(x, \mathbf{U}), \quad \mathbf{G}_k = \mathbf{G}_k(x, \mathbf{U}), \quad x \in Q, \quad \mathbf{U} \in \mathbb{R}^M,$$

are nonlinear superposition operators given by Carathéodory functions. We report the following result [7, Chapter 2, Theorem 2.9.1]:

RDT3

Theorem 4.2. Let $\mathbf{U} : \Omega \rightarrow L^1([0, T]; L^1(Q))$, $D_0 : \Omega \rightarrow L^1(Q)$ be a random variables such that

$$D(\mathbf{U}), \mathbf{F}(\mathbf{U}), \mathbf{G}_k(\mathbf{U}) \in L^1([0, T]; L^1(Q)) \quad \mathbb{P}\text{-a.s.},$$

$$\int_0^T \sum_{k=1}^{\infty} |\langle \mathbf{G}_k(\mathbf{U}), \varphi \rangle|^2 \, dt < \infty \quad \mathbb{P}\text{-a.s. for any } \varphi \in C_c^\infty(Q).$$

Let $W = \sum_{k \geq 1} W_k e_k$ be a cylindrical Wiener process on \mathfrak{U} . Suppose that the filtration

$$\mathfrak{F}_t = \sigma(\sigma_t[\mathbf{U}] \cup \cup_{k=1}^{\infty} \sigma_t[W_k]), \quad t \geq 0,$$

is non-anticipative with respect to W . Let $\tilde{\mathbf{U}}, \tilde{D}_0$ be another pair of random variables and \tilde{W} another stochastic process such their joint laws coincide, namely,

$$[D_0, \mathbf{U}, W] \sim [\tilde{D}_0, \tilde{\mathbf{U}}, \tilde{W}].$$

Then \tilde{W} is an \mathfrak{U} -cylindrical Wiener process with $\tilde{W} = \sum_{k \geq 1} \tilde{W}_k e_k$ for a sequence $(\tilde{W}_k)_{k \geq 1}$ of mutually independent real-valued standard Wiener processes, the filtration

$$\tilde{\mathfrak{F}}_t = \sigma(\sigma_t[\tilde{\mathbf{U}}] \cup \cup_{k=1}^{\infty} \sigma_t[\tilde{W}_k]), \quad t \geq 0,$$

is non-anticipative with respect to \tilde{W} , and

$$\left[\int_0^T [\partial_t \psi \langle D(\mathbf{U}), \varphi \rangle + \psi \langle \mathbf{F}(\mathbf{U}), \nabla \varphi \rangle] \, dt + \int_0^T \psi \langle \mathbf{G}(\mathbf{U}), \varphi \rangle \, dW + \psi(0) \langle D_0, \varphi \rangle \right] \sim \left[\int_0^T [\partial_t \psi \langle D(\tilde{\mathbf{U}}), \varphi \rangle + \psi \langle \mathbf{F}(\tilde{\mathbf{U}}), \nabla \varphi \rangle] \, dt + \int_0^T \psi \langle \mathbf{G}(\tilde{\mathbf{U}}), \varphi \rangle \, d\tilde{W} + \psi(0) \langle \tilde{D}_0, \varphi \rangle \right] \quad (\text{equivalence in law in } \mathbb{R})$$

for any deterministic $\psi \in C_c^\infty([0, T])$, $\varphi \in C^\infty(\mathbb{T}^3)$.

We apply Theorems 4.1, 4.2 to the sequences $(\mathbf{U}_{0,R})$, $(\mathbf{U}_R)_{R>0}$,

$$\mathbf{U}_{0,R} = [\varrho_{0,R}, \vartheta_{0,R}, \mathbf{u}_{0,R}], \quad \mathbf{U}_R = [\varrho_R, \vartheta_R, \mathbf{u}_R, \nabla \mathbf{u}_r, \nabla \varrho_R, \nabla \vartheta_R],$$

with the associated Wiener processes W_R , the existence of which is guaranteed by Theorem 3.1. Here, the initial data $(\varrho_{0,R}, \vartheta_{0,R}, \mathbf{u}_{0,R})$ are considered in the space $Y_0 \in C^{2+\nu} \times W^{1,2} \cap C \times H_m$. The functional $[[\cdot]]$ is taken as the sum of all norm appearing in the estimates (4.8), (4.9), (4.11), and (4.17) together with the norm of the initial data, specifically,

$$\begin{aligned} [[\mathbf{U}_R]] &= \sup_{t \in (0,T)} \int_Q \left(\frac{1}{2} \varrho_R |\mathbf{u}_R|^2 + H_{\delta, \Theta}(\varrho_R, \vartheta_R) + \delta \left(\frac{\varrho_R^\beta}{\beta - 1} + \varrho_R^2 \right) \right) dx + \dots \\ &+ \|\varrho_R \mathbf{u}_R\|_{C^s([0,T]; W^{-k,2}(Q))}. \end{aligned}$$

In accordance with Theorems 4.1 and 4.2, we obtain a new family of random variables $(\tilde{\varrho}_R, \tilde{\vartheta}_R, \tilde{\mathbf{u}}_R)$, together with the Wiener processes \tilde{W}_R , defined on the standard standard probability space $([0, 1], \mathfrak{B}[0, 1], \mathfrak{L})$, with the associated right-continuous complete filtration $(\tilde{\mathfrak{F}}_t^R)_{t \geq 0}$ such that:

- $\left[([0, 1], \mathfrak{B}[0, 1], (\tilde{\mathfrak{F}}_t^M)_{t \geq 0}, \mathfrak{L}), \tilde{\varrho}_R, \tilde{\vartheta}_R, \tilde{\mathbf{u}}_R, \tilde{W}_R \right]$ is a weak martingale solution of problem (3.5), (3.6) in the sense of Definition 3.1, with the initial law Λ_R specified in Section 4.2. In addition, the triple $[\tilde{\varrho}_R, \tilde{\vartheta}_R, \tilde{\mathbf{u}}_R]$ satisfies the total energy balance (3.41) and the entropy equation (3.42);
- in accordance with (4.12), the initial data satisfy

$$\begin{aligned} \tilde{\varrho}_{0,R} &\rightarrow \tilde{\varrho}_0 \text{ in } C^{2+\nu}(\bar{Q}), \quad \tilde{\vartheta}_{0,R} \rightarrow \tilde{\vartheta}_0 \text{ in } W^{1,2} \cap C(\bar{Q}), \quad \varrho_0, \vartheta_0 > 0, \\ \tilde{\mathbf{u}}_{0,R} &\rightarrow \tilde{\mathbf{u}}_0 \text{ in } H_m \quad \mathfrak{L} - \text{a.s.}; \end{aligned}$$

- the functionals bounded in expectations in (4.8), (4.9), (4.11) and (4.17) are bounded for $(\tilde{\varrho}_R, \tilde{\vartheta}_R, \tilde{\mathbf{u}}_R)_{R>0}$ uniformly for $R \rightarrow \infty$ \mathfrak{L} -a.s.;
- we have

$$\tilde{\varrho}_R \rightarrow \tilde{\varrho} \text{ weakly in } W^{1,p}(0, T; L^p(Q)) \cap L^p(0, T; W^{2,p}(Q)) \text{ for some } p > 1 \quad \mathfrak{L} - \text{a.s.},$$

$$\tilde{\varrho}_R \rightarrow \tilde{\varrho} \text{ in } C([0, T]; W^{-k,2}(Q)) \quad \mathfrak{L} - \text{a.s.}, \quad k > \frac{3}{2};$$

$$\boxed{\text{wWS229}} \quad (4.19) \quad \tilde{\vartheta}_R \rightarrow \tilde{\vartheta} \text{ weakly in } L^2(0, T; W^{1,2}(Q)) \text{ and weakly-} (*) \text{ in } L^\infty(0, T; L^4(Q)) \quad \mathfrak{L} - \text{a.s.},$$

$$\tilde{\mathbf{u}}_R \rightarrow \tilde{\mathbf{u}} \text{ weakly-} (*) \text{ in } L^2(0, T; W^{2,\infty}(Q; R^3)) \quad \mathfrak{L} - \text{a.s.},$$

$$\tilde{\varrho}_R \tilde{\mathbf{u}}_R \rightarrow \tilde{\varrho} \tilde{\mathbf{u}} \text{ in } C([0, T]; W^{-k,2}(Q; R^3)) \quad \mathfrak{L} - \text{a.s.}, \quad k > \frac{3}{2};$$

- the sequence $\left(\tilde{\varrho}_R, \tilde{\vartheta}_R, \tilde{\mathbf{u}}_R, \nabla \tilde{\mathbf{u}}_r, \nabla \tilde{\varrho}_R, \nabla \tilde{\vartheta}_R \right)_{R>0}$ generates a Young measure;
- we have

$$\tilde{W}_R \rightarrow \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \quad \mathfrak{L}\text{-a.s.}$$

Now, we observe that (4.19) yields also *strong* (in L^1) convergence of $(\tilde{\varrho}_R)_{R>0}$. As for the velocity, we have,

$$\text{div } \tilde{\mathbf{u}}_R \text{ bounded in } L^1(0, T; L^\infty(\mathbb{T}^3)) \text{ uniformly in } R \quad \mathfrak{L} - \text{a.s.}$$

Consequently, evoking the well known estimate for the parabolic equation (3.5a)

$$\tilde{\varrho}_R(\tau, x) \geq \min_{\mathbb{T}^3} \tilde{\varrho}_0 \exp \left(- \int_0^\tau \|\text{div } \tilde{\mathbf{u}}_R\|_{L_x^\infty} dt \right)$$

we deduce that $\tilde{\varrho}_R$ is bounded below away from zero in terms of the initial data. Consequently, relation (4.19) implies

$$\tilde{\mathbf{u}}_R \rightarrow \mathbf{u} \text{ in } C([0, T]; W^{-k,2}(Q; R^3));$$

whence, in view of equivalence of norm on H_m ,

$$\boxed{\text{NWS229}} \quad (4.20) \quad \tilde{\mathbf{u}}_R \rightarrow \tilde{\mathbf{u}} \text{ in } C([0, T]; W^{2,\infty}(Q; R^3)) \quad \mathfrak{L} - \text{a.s.}$$

The strong convergence of the temperature,

$$\boxed{\text{NWS229bis}} \quad (4.21) \quad \tilde{\vartheta}_R \rightarrow \vartheta \text{ in, say, } L^2((0, T) \times Q) \text{ } \mathfrak{L} - \text{ a.s.,}$$

can be deduced from the equation (3.5b), exactly as in the deterministic case, see [21, Chapter 3, Section 3.5.3]. Here the proof is particularly simple and may be carried over by means of a variant of Lions–Aubin lemma. We will give a detailed proof under more delicate circumstances in Section 5 below.

Finally, to perform the limit in the stochastic integral, we use the following result proved in [7, Chapter 2, Lemma 2.6.6].

$\boxed{\text{RDL4}}$ **Lemma 4.1.** *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, and $\ell \geq 0$. For $n \in \mathbb{N}$, let W_n be an (\mathfrak{F}_t^n) -cylindrical Wiener process and let \mathbf{G}_n be an (\mathfrak{F}_t^n) -progressively measurable stochastic process such that $\mathbf{G}_n \in L^2(0, T; L^2(\mathfrak{U}; W^{-\ell, 2}(Q)))$ a.s. Suppose that*

$$W_n \rightarrow W \text{ in } C([0, T]; \mathfrak{U}_0) \text{ in probability,}$$

$$\mathbf{G}_n \rightarrow \mathbf{G} \text{ in } L^2(0, T; L^2(\mathfrak{U}; W^{-\ell, 2}(Q))) \text{ in probability,}$$

where $W = \sum_{k=1}^{\infty} e_k W_k$. Let $(\mathfrak{F}_t)_{t \geq 0}$ be the filtration given as

$$\mathfrak{F}_t = \sigma\left(\bigcup_{k=1}^{\infty} \sigma_t[\mathbf{G}e_k] \cup \sigma_t[W_k]\right).$$

Then, after a possible change on a set of zero measure in $\Omega \times (0, T)$, \mathbf{G} is (\mathfrak{F}_t) -progressively measurable, and

$$\int_0^\cdot \mathbf{G}_n dW_n \rightarrow \int_0^\cdot \mathbf{G} dW \text{ in } L^2(0, T; W^{-\ell, 2}(\mathbb{T}^3)) \text{ in probability.}$$

With Lemma 4.1 and the compactness stated in (4.19), (4.20), and (4.21) at hand, it is not difficult to pass to the limit in the equations (3.5a–3.5c) to obtain the following system:

$\boxed{\text{NwW}}$

$$\boxed{\text{NwW3}} \quad (4.22a) \quad d\rho + \operatorname{div}(\rho \mathbf{u}) \, dt = \varepsilon \Delta \rho \, dt, \quad \nabla \rho \cdot \mathbf{n}|_{\partial Q} = 0,$$

$$\begin{aligned} & d\Pi_m[\rho \mathbf{u}] + \Pi_m[\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u})] \, dt + \Pi_m \left[\nabla (p(\rho, \vartheta) + \delta(\rho^2 + \vartheta^\beta)) \right] \, dt \\ & = \Pi_m \left[\varepsilon \Delta(\rho \mathbf{u}) + \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \frac{1}{m} \mathbf{u} \right] \, dt \end{aligned}$$

$\boxed{\text{NwW4}}$

$$(4.22b) \quad + \Pi_m \left[\rho \Pi_m \left[\mathbf{F}_{\varepsilon, \xi}(\rho, \vartheta, \mathbf{u}) \right] \right] \, dW$$

$$\begin{aligned} & d(\rho e_\delta(\rho, \vartheta)) + \left[\operatorname{div}(\rho e_\delta(\rho, \vartheta) \mathbf{u}) - \operatorname{div}(\kappa_\delta(\vartheta) \nabla \vartheta) \right] \, dt \\ & = \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} - p(\rho, \vartheta) \operatorname{div} \mathbf{u} \right] \, dt \end{aligned}$$

$\boxed{\text{Nm119}}$

$$(4.22c) \quad + \left[\varepsilon \delta(\beta \rho^{\beta-2} + 2) |\nabla \rho|^2 + \delta \frac{1}{\vartheta^2} - \varepsilon \vartheta^5 \right] \, dt + \varepsilon \rho |\nabla \mathbf{u}|^2 \, dt, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0.$$

Let us summarize the results obtained so far.

$\boxed{\text{wWP1bis}}$

Theorem 4.3. *Let $\beta > 6$. Let Λ be a Borel probability measure on $C^{2+\nu}(\overline{Q}) \times W^{1,2} \cap C(\overline{Q}) \times H_m$ such that*

$$\Lambda \left\{ 0 < \underline{\rho} \leq \int_Q \rho_0 \, dx \leq \overline{\rho}, \quad \nabla \rho_0 \cdot \mathbf{n}|_{\partial Q} = 0, \quad \rho_0, \vartheta_0 > 0 \right\} = 1,$$

$$\int_{C^{2+\nu} \times W^{1,2} \times H_m} \left[\|\rho_0\|_{C_x^{2+\nu}} + \|\vartheta_0\|_{W_x^{1,2} \cap C_x} + \|\mathbf{u}_0\|_{H_m} \right]^r \, d\Lambda \leq c(r) \text{ for any } r \geq 1.$$

Then the approximate problem (4.22a)–(4.22c) admits a martingale solution in the sense of

Definition 3.1 (with the obvious modifications for $\xi, \varepsilon, \delta > 0$). In addition, the solution satisfies

$$\begin{aligned}
& - \int_0^T \partial_t \psi \left(\int_Q \mathcal{E}_\delta(\varrho, \vartheta, \mathbf{u}) \, dx \right) dt + \int_0^T \psi \int_Q \left(\varepsilon \vartheta^5 + \frac{1}{m} |\mathbf{u}|^2 \right) dt \\
& = \psi(0) \int_Q \mathcal{E}_\delta(\varrho_0, \vartheta_0, \mathbf{u}_0) \, dx + \int_0^T \int_Q \frac{\delta}{\vartheta^2} \psi \, dx \, dt \\
& + \frac{1}{2} \int_0^T \psi \left(\int_Q \sum_{k \geq 1} \varrho |\Pi_m[\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})]|^2 \, dx \right) dt \\
& + \int_0^T \psi \int_Q \varrho \Pi_m[\mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dW \, dx,
\end{aligned} \tag{4.23}$$

with

$$\mathcal{E}_\delta(\varrho, \vartheta, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta - 1} + \varrho^2 \right)$$

for any deterministic smooth test function $\psi \in C_c^\infty[0, T]$.

4.3. Limit $m \rightarrow \infty$. Keeping $\varepsilon > 0$, $\delta > 0$ and $\xi > 0$ fixed, our next goal is to let $m \rightarrow \infty$ in the approximate system (4.22). With the same initial law Λ , this can be done in a similar way as in the preceding step modulo certain modifications due to the lost of regularity of the velocity in the asymptotic limit. In particular, the bound (4.9) and related estimates on the density are no longer valid for $m \rightarrow \infty$.

At this stage, it is also convenient to replace the internal energy equation (4.22c) by the entropy balance

$$\begin{aligned}
& d(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta) \mathbf{u}) \, dt - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] dt \\
& = \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] dt \\
& + \left[\frac{\varepsilon \delta}{\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{\Delta_x \varrho}{\vartheta} \left(\vartheta s_\delta(\varrho, \vartheta) - e_\delta(\varrho, \vartheta) - \frac{p(\varrho, \vartheta)}{\varrho} \right) \right] dt \\
& + \left[-\varepsilon \vartheta^4 + \varepsilon \frac{\varrho}{\vartheta} |\nabla \mathbf{u}|^2 \right] dt
\end{aligned} \tag{4.24}$$

Note carefully that this is possible as $\vartheta > 0$ and equation (4.22c) is satisfied a.a. Thus (4.24) follows by dividing (4.22c) by ϑ .

Now, exactly as in [21, Chapter 3, Section 3.5], we deduce from (4.24) the inequality

$$\begin{aligned}
& d(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta) \mathbf{u}) \, dt - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] dt \\
& - \varepsilon \operatorname{div} \left[\left(\vartheta s_{M, \delta}(\varrho, \vartheta) - e_{M, \delta}(\varrho, \vartheta) - \frac{p_M(\varrho, \vartheta)}{\varrho} \right) \frac{\nabla \varrho}{\vartheta} \right] dt \\
& \geq \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2} (\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] dt \\
& + \left[\frac{\varepsilon \delta}{2\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{1}{\varrho \vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) |\nabla \varrho|^2 - \varepsilon \vartheta^4 \right] dt.
\end{aligned} \tag{4.25}$$

Consequently, adapting the dissipation inequality (4.2) to the present setting, we deduce that (4.9) must be replaced by

$$\mathbb{E} \left[\int_0^T \frac{1}{\vartheta} \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} \, dt \right]^r \leq c(r),$$

which, combined with inequality (4.1), gives rise to

$$\mathbb{E} \left[\int_0^T \|\mathbf{u}\|_{W^{1,2}(Q; \mathbb{R}^3)}^2 \, dt \right]^r \leq c(r). \tag{4.26}$$

At this stage, we choose the initial velocity $\mathbf{u}_0 \in L^2(Q; \mathbb{R}^3)$ and adjust the initial law Λ to $\mathbf{u}_0 = \mathbf{u}_{0,m} = \Pi_m \mathbf{u}_0$. Accordingly, Theorem 4.3 yields a family $(\varrho_m, \vartheta_m, \mathbf{u}_m)$ of approximate solutions. Now, we may repeat step by step the arguments of the preceding part to obtain a martingale solution of the following approximate problem:

$$\boxed{\text{Napde1}} \quad (4.27) \quad d\varrho + \operatorname{div}(\varrho \mathbf{u}) \, dt = \varepsilon \Delta \varrho \, dt, \quad \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0,$$

$$\boxed{\text{Napde2}} \quad (4.28) \quad \begin{aligned} & d(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \, dt + \nabla (p(\varrho, \vartheta) + \delta(\varrho^\beta + \varrho^2)) \, dt \\ & = \left[\varepsilon \Delta(\varrho \mathbf{u}) + \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) \right] dt + \varrho \mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u}) \, dW, \end{aligned}$$

$$\boxed{\text{Napde3}} \quad (4.29) \quad \begin{aligned} & d(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta) \mathbf{u}) \, dt - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] dt \\ & - \varepsilon \operatorname{div} \left[\left(\vartheta s_{M, \delta}(\varrho, \vartheta) - e_{M, \delta}(\varrho, \vartheta) - \frac{p_M(\varrho, \vartheta)}{\varrho} \right) \frac{\nabla \varrho}{\vartheta} \right] dt \\ & \geq \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] dt \\ & + \left[\frac{\varepsilon \delta}{2\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{1}{\varrho \vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) |\nabla \varrho|^2 - \varepsilon \vartheta^4 \right] dt, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0, \end{aligned}$$

$$\boxed{\text{Napde4}} \quad (4.30) \quad \begin{aligned} & d \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta-1} + \varrho^2 \right) \right] dx + \left(\int_Q \left(\varepsilon \vartheta^5 - \frac{\delta}{\vartheta^2} \right) dx \right) dt \\ & = \frac{1}{2} \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})|^2 dx \right) dt + \left(\int_Q \varrho \mathbf{F}_{\varepsilon, \xi}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} dx \right) dW. \end{aligned}$$

Note that the entropy inequality (4.29) as well as the total energy balance (4.30) must be already interpreted in the weak sense as in Definition 2.1. On the other hand, we still recover the strong (a.a. pointwise) convergence of the arguments in the diffusion coefficients $\mathbf{F}_{k, \varepsilon, \xi}$ to perform the limit in the stochastic integral.

4.4. **Limit** $\xi \rightarrow 0$. Our final goal is to perform the limit $\xi \rightarrow 0$. To this end, we choose the cut-off functions h_ξ in (3.4) to approach 1 and the regularizing kernels ω_ξ to approach the Dirac mass. As the stochastic terms in (4.28) and (4.30) do not contain the projection Π_m , we no longer need (3.2). Instead, we simply use

$$\|\mathbf{F}_{k, \varepsilon, \xi}(\varrho, \vartheta, \mathbf{u})\|_{L_x^\infty} \lesssim \|\mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u})\|_{L_x^\infty} \leq c(\varepsilon) f_{k, \varepsilon}$$

in (4.3), (4.5) and all other terms involving the stochastic integral. Summarizing, we record the following result.

$\boxed{\text{wWP1a}}$ **Theorem 4.4.** *Let $\beta > 6$. Let Λ be a Borel probability measure on $C^{2+\nu}(\overline{Q}) \times W^{1,2} \cap C(\overline{Q}) \times L^2(Q; \mathbb{R}^3)$ such that*

$$\begin{aligned} & \Lambda \left\{ 0 < \underline{\varrho} \leq \int_Q \varrho_0 \, dx \leq \overline{\varrho}, \quad \nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0, \quad \varrho_0, \vartheta_0 > 0 \right\} = 1, \\ & \int_{C^{2+\nu} \times W^{1,2} \times L^2} \left[\|\varrho_0\|_{C_x^{2+\nu}} + \|\vartheta_0\|_{W_x^{1,2} \cap C_x} + \|\mathbf{u}_0\|_{L_x^2} \right]^r d\Lambda \leq c(r) \text{ for any } r \geq 1. \end{aligned}$$

Then the approximate problem

$$\boxed{\text{Npde1}} \quad (4.31) \quad d\varrho + \operatorname{div}(\varrho \mathbf{u}) \, dt = \varepsilon \Delta \varrho \, dt, \quad \nabla \varrho \cdot \mathbf{n}|_{\partial Q} = 0,$$

$$\boxed{\text{Npde2}} \quad (4.32) \quad \begin{aligned} & d(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \, dt + \nabla (p(\varrho, \vartheta) + \delta(\varrho^\beta + \varrho^2)) \, dt \\ & = \left[\varepsilon \Delta(\varrho \mathbf{u}) + \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) \right] dt + \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \, dW, \quad \mathbf{u}|_{\partial Q} = 0, \end{aligned}$$

$$\begin{aligned}
& d(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta) \mathbf{u}) \, dt - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] \, dt \\
& - \varepsilon \operatorname{div} \left[\left(\vartheta s_{M,\delta}(\varrho, \vartheta) - e_{M,\delta}(\varrho, \vartheta) - \frac{p_M(\varrho, \vartheta)}{\varrho} \right) \frac{\nabla \varrho}{\vartheta} \right] \, dt \\
& \geq \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] \, dt \\
& + \left[\frac{\varepsilon \delta}{2\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{1}{\varrho \vartheta} \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) |\nabla \varrho|^2 - \varepsilon \vartheta^4 \right] \, dt, \quad \nabla \vartheta \cdot \mathbf{n}|_{\partial Q} = 0,
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
& d \int_{\mathbb{T}^3} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta-1} + \varrho^2 \right) \right] \, dx + \left(\int_{\mathbb{T}^3} \left(\varepsilon \vartheta^5 - \frac{\delta}{\vartheta^2} \right) \, dx \right) \, dt \\
& = \frac{1}{2} \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{F}_{k,\varepsilon}(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) \, dt + \left(\int_{\mathbb{T}^3} \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \right) \, dW.
\end{aligned} \tag{4.34}$$

admits a martingale solution in the sense of Definition 2.1 (with the obvious modification for $\varepsilon, \delta > 0$).

5. THE VANISHING VISCOSITY LIMIT

Our ultimate goal is to perform successively the asymptotic limits $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ in the approximate system (4.31–4.34). It is worth noting that the relations (4.27) and (4.29) are *deterministic* and the same as in [21, Sections 3.6 - 3.7], where a similar limit in the absence of stochastic forcing is performed. Thus, at least in (4.27), (4.29), the limit process is exactly the same as in [21, Chapter 3] as long as suitable uniform bounds are established. Accordingly, we adopt the following strategy:

- Making use of an appropriate form of the total dissipation balance (4.2) we derive the energy bounds.
- We derive all other estimates, in particular for the pressure and the velocity, that require stochastic averaging.
- Changing the probability space we recover (weak) compactness pointwise with respect to the random parameter. Accordingly, we perform the limit passage, in which the equation of continuity and the entropy balance are handled in the same way as in [21, Chapter 3].
- We pass to the limit in the stochastic integral using Lemma 4.1.

We aim to perform the limit $\varepsilon \rightarrow 0$ extending the validity of Theorem 4.4 to the following system.

$$d\varrho + \operatorname{div}(\varrho \mathbf{u}) \, dt = 0, \tag{5.1}$$

$$d(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) \, dt + \nabla (p(\varrho, \vartheta) + \delta(\varrho^2 + \varrho^\beta)) \, dt = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) \, dt + \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \, dW, \tag{5.2}$$

$$\begin{aligned}
& d(\varrho s_\delta(\varrho, \vartheta)) + \operatorname{div}(\varrho s_\delta(\varrho, \vartheta) \mathbf{u}) \, dt - \operatorname{div} \left[\left(\frac{\kappa(\vartheta)}{\vartheta} + \delta(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) \nabla \vartheta \right] \, dt \\
& \geq \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \left(\frac{\kappa(\vartheta)}{\vartheta} + \frac{\delta}{2}(\vartheta^{\beta-1} + \frac{1}{\vartheta^2}) \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] \, dt,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
& d \int_{\mathbb{T}^3} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_\delta(\varrho, \vartheta) + \delta \left(\frac{\varrho^\beta}{\beta-1} + \varrho^2 \right) \right] \, dx - \left(\int_{\mathbb{T}^3} \frac{\delta}{\vartheta^2} \, dx \right) \, dt \\
& = \frac{1}{2} \left(\int_{\mathbb{T}^3} \sum_{k \geq 1} \varrho |\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) \, dt + \left(\int_{\mathbb{T}^3} \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \right) \, dW
\end{aligned} \tag{5.4}$$

with respect to conservative boundary conditions (1.5). The main result of this section is the following.

wWT3

Theorem 5.1. *Let $\beta > 6$. Let Λ be a Borel probability measure on $C^{2+\nu}(\bar{Q}) \times W^{1,2} \cap C(\bar{Q}) \times L^2(Q; \mathbb{R}^3)$ such that*

$$\Lambda \left\{ 0 < \underline{\varrho} \leq \int_{\mathbb{T}^3} \varrho_0 \, dx \leq \bar{\varrho}, \nabla \varrho_0 \cdot \mathbf{n}|_{\partial Q} = 0, \varrho_0, \vartheta_0 > 0 \right\} = 1,$$

$$\int_{C^{2+\nu} \times W^{1,2} \times L^2} \left[\|\varrho_0\|_{C_x^{2+\nu}} + \|\vartheta_0\|_{W_x^{1,2} \cap C_x} + \|\mathbf{u}_0\|_{L_x^2} \right]^r \, d\Lambda \leq c(r) \text{ for all } r \geq 1.$$

Then the approximate problem (5.1)–(5.4) admits a martingale solution in the sense of Definition 2.1.

The proof of Theorem 5.1 requires the full strength of the method developed in the context of the deterministic Navier–Stokes system. Possible oscillations of the density are ruled out thanks to the weak compactness of a quantity called *effective viscous flux*,

$$\left(\frac{4}{3} \mu(\vartheta) + \eta(\vartheta) \right) \operatorname{div} \mathbf{u} - p(\varrho, \vartheta),$$

where μ, η are the viscosity coefficients.

wWS3S1

5.1. Uniform energy bounds. Using (4.33), (4.34), or rather their variational formulation via the appropriate versions of (2.17) and (2.18) containing the artificial pressure, we deduce the total dissipation balance

NtDB

$$(5.5) \quad \begin{aligned} & \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \frac{\delta}{\beta-1} \varrho^\beta + \delta \varrho^2 \right] \, dx + \Theta \int_0^\tau \int_Q \sigma_{\varepsilon, \delta} \, dx + \int_0^\tau \int_Q \varepsilon \vartheta^5 \, dt \\ & \leq \int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\delta, \Theta}(\varrho_0, \vartheta_0) + \frac{\delta}{\beta-1} \varrho_0^\beta + \delta \varrho_0^2 \right] \, dx \\ & + \int_0^\tau \int_Q \left(\frac{\delta}{\vartheta^2} + \varepsilon \Theta \vartheta^4 \right) \, dx \, dt + \int_0^\tau \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW \\ & + \frac{1}{2} \int_0^\tau \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) dt \end{aligned}$$

for any $0 \leq \tau \leq T$ \mathbb{P} -a.s., where

$$\begin{aligned} \sigma_{\varepsilon, \delta} &= \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 + \frac{\delta}{2} \left(\varrho^{\beta-1} + \frac{1}{\vartheta^2} \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right] \\ &+ \frac{\varepsilon \delta}{2\vartheta} (\beta \varrho^{\beta-2} + 2) |\nabla \varrho|^2 + \varepsilon \frac{\partial p_M}{\partial \varrho}(\varrho, \vartheta) \frac{|\nabla \varrho|^2}{\varrho \vartheta}. \end{aligned}$$

Thanks to hypothesis (2.2) we get

$$\int_Q \varrho |\mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u})| |\mathbf{u}| \, dx \lesssim f_k \int_{\mathbb{T}^3} \varrho (1 + |\mathbf{u}|^2) \, dx, \quad \int_Q \varrho |\mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u})|^2 \, dx \lesssim f_k^2 \int_{\mathbb{T}^3} \varrho (1 + |\mathbf{u}|^2) \, dx.$$

Thus we may pass to expectations in (5.5), use Burkholder–Davis–Gundy inequality (note that the process $\int_0^\cdot \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW$ is a real-valued continuous martingale) and apply a Gronwall-type argument to deduce the following bounds, cf. [21, Chapter 3, Section 3.6.1] and recall that we assume $\beta > 6$:

Nbv1

$$(5.6) \quad \mathbb{E} \left[\operatorname{ess\,sup}_{t \in (0, T)} \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \frac{\delta}{\beta-1} \varrho^\beta + \delta \varrho^2 \right] \, dx \right]^r \leq c(r, \Lambda),$$

in particular

$$\begin{aligned}
& \mathbb{E} \left[\left\| \sup_{t \in (0, T)} \|\varrho\|_{L^\beta(Q)}^\beta \right\|^r \right] \leq c(r, \Lambda), \\
\text{Nbv2} \quad (5.7) \quad & \mathbb{E} \left[\left\| \sup_{t \in (0, T)} \|\varrho \mathbf{u}\|_{L^{\frac{2\beta}{\beta+1}}(Q; \mathbb{R}^3)} \right\|^r \right] \leq c(r, \Lambda), \\
& \mathbb{E} \left[\left\| \operatorname{ess\,sup}_{t \in (0, T)} \|\vartheta\|_{L^4(Q)}^4 \right\|^r \right] \leq c(r, \Lambda).
\end{aligned}$$

Moreover, boundedness of the entropy production rate

$$\text{Nbv3} \quad (5.8) \quad \mathbb{E} \left[\|\sigma_{\varepsilon, \delta}\|_{L^1((0, T) \times Q)}^r \right] \leq c(r, \Lambda)$$

gives rise to

$$\text{Nbv4} \quad (5.9) \quad \mathbb{E} \left[\|\nabla \mathbf{u}\|_{L^2(0, T) \times Q; \mathbb{R}^3 \times 3}^r \right] + \mathbb{E} \left[\|\nabla \vartheta\|_{L^2(0, T) \times Q; \mathbb{R}^3}^r \right] \leq c(r, \Lambda);$$

whence, by Poincaré's inequality and (5.7),

$$\text{Nbv5} \quad (5.10) \quad \mathbb{E} \left[\|\mathbf{u}\|_{L^2(0, T; W^{1,2}(Q; \mathbb{R}^3))}^r \right] + \mathbb{E} \left[\|\vartheta\|_{L^2(0, T; W^{1,2}(Q))}^r \right] \leq c(r, \Lambda).$$

Note that we keep the initial law Λ the same as in the previous section.

Finally, we deduce from the equation of continuity (4.22a) that

$$\text{Nbv6} \quad (5.11) \quad \int_{\mathbb{T}^3} \varrho(t, \cdot) \, dx = \int_{\mathbb{T}^3} \varrho_0 \, dx, \quad \mathbb{E} \left[\|\sqrt{\varepsilon} \nabla \varrho\|_{L^2((0, T) \times \mathbb{T}^3)}^r \right] \leq c(r, \Lambda).$$

Note that all estimates are independent of ε .

The above bounds are not strong enough to control the pressure term proportional to ϱ^β that is for the current stage bounded only in the non-reflexive space L_x^1 . The adequate estimates will be derived in the next section.

wWS3S2

5.2. Pressure estimates. Following [21, Chapter 2, Section 2.2.5] we introduce the Bogovskii operator \mathcal{B} enjoying the following properties:

$$\begin{aligned}
\mathcal{B} : L_0^q &\equiv \left\{ f \in L^q(\mathbb{R}^3) \mid (f)_Q = 0 \right\} \rightarrow W_0^{1,q}(Q; \mathbb{R}^3), \quad \operatorname{div} \mathcal{B}[f] = f \text{ in } q, \quad 1 < q < \infty, \\
\|\mathcal{B}[\operatorname{div} \mathbf{g}]\|_{L^r(Q)} &\lesssim \|\mathbf{g}\|_{L^r(Q)} \text{ whenever } \operatorname{div} \mathbf{g} \in L_0^q, \quad 1 < q, r < \infty.
\end{aligned}$$

The idea, borrowed again from [21, Chapter 2, Section 2.2.5], is to use the quantity

$$\mathcal{B}[\varrho - (\varrho)_Q]$$

as a test function in the variational formulation of the momentum balance (4.28) (cf. formula (2.16)). Note that this is not straightforward as the legal test functions allowed have the form $\psi(t)\varphi(x)$, where both ψ and φ are smooth and deterministic. Nevertheless, such a procedure can be rigorously justified by the application of a suitable version of the generalized Itô formula to the functional

$$(\rho, \mathbf{q}) \mapsto \int_Q \mathbf{q} \cdot \mathcal{B}(\varrho - (\varrho)_Q) \, dx$$

(see [9, Sec. 5]). We rewrite (4.28) in the differential form

$$\begin{aligned}
& d \int_Q \varrho \mathbf{u} \cdot \varphi \, dx - \int_Q \left[\varrho \mathbf{u} \otimes \mathbf{u} : \nabla \varphi + p_\delta(\varrho, \vartheta) \operatorname{div} \varphi \right] \, dx \, dt \\
\text{kolmo} \quad (5.12) \quad & = - \int_Q \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \varphi + \varepsilon \nabla(\varrho \mathbf{u}) : \nabla \varphi \right] \, dx \, dt + \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \varphi \, dx \, dW.
\end{aligned}$$

Seeing that

$$d(\mathcal{B}[\varrho - (\varrho)_Q]) = -\mathcal{B} \operatorname{div}(\varrho \mathbf{u}) \, dt + \varepsilon \mathcal{B}[\Delta \varrho] \, dt,$$

we obtain

$$\begin{aligned}
& \int_0^T \int_Q p_\delta(\varrho, \vartheta) [\varrho - (\varrho)_Q] \, dx \, dt \\
&= \left[\int_Q \varrho \mathbf{u} \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \right]_{t=0}^{t=T} - \int_0^T \int_Q \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathcal{B}[\varrho - (\varrho)_Q] \, dx \, dt \\
&\quad + \int_0^T \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathcal{B}[\varrho - (\varrho)_Q] \, dx \, dt + \int_0^T \int_Q \varrho \mathbf{u} \cdot \mathcal{B}[\operatorname{div}(\varrho \mathbf{u})] \, dx \, dt \\
&\quad + \varepsilon \int_Q [\nabla(\varrho \mathbf{u}) \cdot \nabla \mathcal{B}[\varrho - (\varrho)_Q] - \varrho \mathbf{u} \cdot \mathcal{B}(\Delta \varrho)] \, dx \, dt \\
\boxed{\text{wWS310}} \quad (5.13) \quad & - \int_0^T \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \, dW.
\end{aligned}$$

As shown in [21, Chapter 3, Section 3.6.3], all deterministic integrals on the right-hand side of (5.13) are controlled by the energy bounds. This means, in the present setting, their expected values are controlled by the bounds (5.7–5.11).

As for the stochastic integral, we apply the Burkholder–Davis–Gundy inequality:

$$\begin{aligned}
& \mathbb{E} \left[\left| \int_0^\tau \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \, dW \right|^r \right] \\
& \leq \mathbb{E} \left[\sup_{t \in [0, \tau]} \left| \int_0^t \int_Q \varrho \mathbf{F}_\varepsilon(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \, dW \right|^r \right] \\
& \lesssim \mathbb{E} \left[\int_0^\tau \sum_{k=1}^\infty \left| \int_Q \varrho \mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \right|^2 \, dt \right]^{r/2},
\end{aligned}$$

where, due to (2.2) and the properties of \mathcal{B} and $\beta > 3$,

$$\begin{aligned}
\left| \int_Q \varrho \mathbf{F}_{k, \varepsilon}(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho - (\varrho)_Q] \, dx \right| & \lesssim f_k \|\mathcal{B}[\varrho - (\varrho)_Q]\|_{L_x^\infty} \int_Q \varrho(1 + |\mathbf{u}|) \, dx \\
& \lesssim f_k \|\varrho\|_{L_x^\beta} \int_Q \varrho(1 + |\mathbf{u}|) \, dx.
\end{aligned}$$

Hence we conclude that

$$\boxed{\text{wWS316}} \quad (5.14) \quad \mathbb{E} \left[\left| \int_0^T \int_Q p_\delta(\varrho, \vartheta) \varrho \, dx \, dt \right|^r \right] \leq c(r, \delta, \Lambda).$$

$\boxed{\text{CMIT}}$

5.3. Compactness of the momentum in time. Using the same arguments as in Section 4, specifically, the Kolmogorov continuity criterion, we may deduce from the momentum equation (5.12) the estimate

$$\boxed{\text{Nbv7}} \quad (5.15) \quad \mathbb{E} \left[\|\varrho \mathbf{u}\|_{C^s([0, T]; W^{-k, 2}(Q; \mathbb{R}^3))}^r \right] \lesssim c(r) \text{ for a certain } 0 < s(r) < \frac{1}{2}, \, k > \frac{3}{2}.$$

$\boxed{\text{subsub:stochep}}$

5.4. Stochastic compactness method. The uniform bounds derived in the previous section are optimal in view of the energy method. We are ready to perform the limit $\varepsilon \rightarrow 0$. We proceed in two steps. Similarly to Section 4, we make use of the general results stated in Theorems 4.1 and 4.2 to pass to the standard probability space $([0, 1], \mathfrak{B}[0, 1], \mathfrak{L})$. Then we adapt the method known for the deterministic case to show compactness of the temperatures and the densities which is the main issue here.

Applying Theorem 4.4 we get a family of martingale solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon > 0}$ of problem (4.31–4.34). Invoking the compactness method used in Section 4, in particular Theorem 4.1, we get a new family of random variables $(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon)_{\varepsilon > 0}$, together with the processes \tilde{W}_ε and with the associated right-continuous complete filtration $(\tilde{\mathfrak{F}}_t^\varepsilon)_{t \geq 0}$ such that:

- $\left[\left([0, 1], \overline{\mathfrak{B}[0, 1]}, (\tilde{\mathfrak{F}}_t^\varepsilon)_{t \geq 0}, \mathfrak{L} \right), \tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{W}_\varepsilon \right]$ is a weak martingale solution of problem (4.31–4.34), with the initial law Λ ;
- the initial data $(\tilde{\varrho}_{0,\varepsilon}, \tilde{\vartheta}_{0,\varepsilon}, \tilde{\mathbf{u}}_{0,\varepsilon})$ satisfy

$$\begin{aligned} \tilde{\varrho}_{0,\varepsilon} &\rightarrow \tilde{\varrho}_0 \text{ in } C^{2+\nu}(\overline{Q}), \tilde{\vartheta}_{0,\varepsilon} \rightarrow \tilde{\vartheta}_0 \text{ in } W^{1,2} \cap C(\overline{Q}), \varrho_0, \vartheta_0 > 0, \\ \tilde{\mathbf{u}}_{0,\varepsilon} &\rightarrow \tilde{\mathbf{u}}_0 \text{ in } L^2(Q, R^3), \end{aligned}$$

\mathfrak{L} -a.s.;

- the functions $(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon)$ satisfy the bounds (5.6–5.10), (5.14), and (5.15) \mathfrak{L} -a.s. uniformly for $\varepsilon \rightarrow 0$;
- we have

Ncv1

(5.16)

$$\begin{aligned} \tilde{\varrho}_\varepsilon &\rightarrow \tilde{\varrho} \text{ weakly in } L^{\beta+1}((0, T) \times Q) \text{ and weakly-}^* \text{ in } L^\infty(0, T; L^\beta(Q)) \text{ } \mathfrak{L} \text{ - a.s.}, \\ \tilde{\varrho}_\varepsilon &\rightarrow \tilde{\varrho} \text{ in } C([0, T]; W^{-k,2}(Q)) \text{ } k > \frac{3}{2}, \text{ } \mathfrak{L} \text{ - a.s.}; \\ \tilde{\vartheta}_\varepsilon &\rightarrow \tilde{\vartheta} \text{ weakly in } L^2(0, T; W^{1,2}(Q)) \text{ and weakly-}^* \text{ in } L^\infty(0, T; L^4(Q)) \text{ } \mathfrak{L} \text{ - a.s.}, \\ \tilde{\mathbf{u}}_\varepsilon &\rightarrow \tilde{\mathbf{u}} \text{ weakly in } L^2(0, T; W_0^{1,2}(Q; R^3)) \text{ } \mathfrak{L} \text{ - a.s.}, \\ \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon &\rightarrow \tilde{\varrho} \tilde{\mathbf{u}} \text{ and weakly-}^* \text{ in } L^\infty(0, T; L^{\frac{2\beta}{\beta+1}}(Q; R^3)) \text{ } \mathfrak{L} \text{ - a.s.}, \\ \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon &\rightarrow \tilde{\varrho} \tilde{\mathbf{u}} \text{ in } C([0, T]; W^{-k,2}(Q; R^3)) \text{ } k > \frac{3}{2}, \text{ } \mathfrak{L} \text{ - a.s.}; \end{aligned}$$

- the family $(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon \tilde{\varrho}_\varepsilon, \nabla \tilde{\vartheta}_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon)_{\varepsilon > 0}$ generates a Young measure;
- we have

$$\tilde{W}_\varepsilon \rightarrow \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \text{ } \mathfrak{L}\text{-a.s.}$$

Note that a.s. boundedness in all the afore mentioned function spaces, in particular the weak- $(^*)$ convergence in the dual space $L^\infty(0, T; L^q)$, is guaranteed by Theorem 4.1, where the functional $[[\cdot]]$ includes all bounds obtained in (5.7), cf. Remark 4.2. Consequently, the norms

$$\|\tilde{\varrho}_\varepsilon\|_{L^\infty(0, T; L^\beta(Q))}, \|\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0, T; L^{2\beta/\beta+1}(Q))} \text{ and } \|\tilde{\vartheta}_\varepsilon\|_{L^\infty(0, T; L^4(Q))}$$

remain bounded uniformly for $\varepsilon \rightarrow 0$ and the desired result follows.

5.5. Strong convergence of the temperature. We establish the strong convergence of the temperature fields $(\tilde{\vartheta}_\varepsilon)_{\varepsilon > 0}$ exploiting pathwise the piece of information provided by the deterministic entropy inequality (4.29). Indeed repeating step by step the arguments in [21, Chapter 3, Section 3.6.2] for an fixed $\omega \in (0, 1)$ we can show that

Ncv2

$$(5.17) \quad \tilde{\vartheta}_\varepsilon \rightarrow \tilde{\vartheta} \text{ in } L^4((0, T) \times \mathbb{T}^3) \text{ } \mathfrak{L}\text{-a.s.}$$

In a similar way, we recover the weak formulation of the *renormalized equation of continuity*:

Npde1c

$$(5.18) \quad \begin{aligned} &\int_0^T \int_Q [b(\tilde{\varrho}_\varepsilon) \partial_t \psi + b(\tilde{\varrho}_\varepsilon) \tilde{\mathbf{u}}_\varepsilon \cdot \nabla \psi + (b(\tilde{\varrho}_\varepsilon) - \tilde{\varrho}_\varepsilon b'(\tilde{\varrho}_\varepsilon)) \operatorname{div} \tilde{\mathbf{u}}_\varepsilon \psi] \, dx \, dt \\ &= \varepsilon \int_0^T \int_Q b''(\tilde{\varrho}_\varepsilon) |\nabla \tilde{\varrho}_\varepsilon|^2 \, dx \, dt + \varepsilon \int_0^T \int_Q b'(\tilde{\varrho}_\varepsilon) \nabla \tilde{\varrho}_\varepsilon \nabla \psi \, dx \, dt - \int_Q b(\tilde{\varrho}_{0,\varepsilon}) \psi(0) \, dx \end{aligned}$$

for any $\psi \in C_c^\infty([0, T] \times R^3)$ and any sufficiently smooth b .

sec:limstochint

5.6. The limit in the stochastic integral. To perform the limit in the momentum equation (4.32) we have to handle the stochastic integral. In view of Lemma 4.1, we need to show the strong convergence

wWS329a

$$(5.19) \quad \tilde{\varrho}_\varepsilon \mathbf{F}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon) \rightarrow \overline{\tilde{\varrho} \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \text{ in } L^2(0, T; L^2(\mathfrak{U}_0; W^{-k,2}(Q))), \text{ } k > \frac{3}{2}.$$

First, we claim that it is enough to show that

$$\boxed{\text{wWS329b}} \quad (5.20) \quad \tilde{\varrho}_\varepsilon \mathbf{F}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}, \tilde{\mathbf{u}}) \rightarrow \overline{\varrho \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \text{ in } L^2(0, T; L^2(\mathcal{U}_0; W^{-k,2}(Q))), \quad k > \frac{3}{2}.$$

Indeed as the functions $\mathbf{F}_{k,\varepsilon}$ are globally Lipschitz (uniformly for $\varepsilon \rightarrow 0$, recall (2.2)), we have

$$\left| \tilde{\varrho}_\varepsilon \mathbf{F}_{k,\varepsilon}(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon) - \varrho_\varepsilon \mathbf{F}_{k,\varepsilon}(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}, \tilde{\mathbf{u}}) \right| \lesssim f_k \varrho_\varepsilon \left(|\tilde{\vartheta}_\varepsilon - \tilde{\vartheta}| + |\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}}| \right),$$

where, by virtue of (5.16), (5.17),

$$\left\| \tilde{\varrho}_\varepsilon(\tilde{\vartheta}_\varepsilon - \tilde{\vartheta}) \right\|_{L^1(Q)} \leq \|\tilde{\varrho}_\varepsilon\|_{L^{4/3}(Q)} \|\tilde{\vartheta}_\varepsilon - \tilde{\vartheta}\|_{L^4(Q)} \rightarrow 0 \text{ in } L^4(0, T) \quad \mathfrak{L}\text{-a.s.}$$

Similarly,

$$\boxed{\text{Ncv4}} \quad (5.21) \quad \|\tilde{\varrho}_\varepsilon(\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}})\|_{L^1(Q)} \leq \|\sqrt{\tilde{\varrho}_\varepsilon}\|_{L^2(\mathbb{T}^3)} \|\sqrt{\tilde{\varrho}_\varepsilon}(\tilde{\mathbf{u}}_\varepsilon - \tilde{\mathbf{u}})\|_{L^2(Q; \mathbb{R}^3)}.$$

However, in view of (5.16) again, we have

$$\boxed{\text{Ncv5}} \quad (5.22) \quad \int_0^T \int_Q \tilde{\varrho}_\varepsilon |\tilde{\mathbf{u}}_\varepsilon|^2 \, dx \, dt \rightarrow \int_0^T \int_Q \tilde{\varrho} |\tilde{\mathbf{u}}|^2 \, dx \, dt;$$

whence the right-hand side of (5.21) tends to zero in $L^2(0, T)$ \mathfrak{L} -a.s.

Now, convergence in (5.20) follows the fact that $\tilde{\varrho}_\varepsilon$ satisfy the renormalized equation (5.18), specifically,

$$b(\tilde{\varrho}_\varepsilon) \rightarrow \overline{b(\tilde{\varrho})} \text{ in } C_w([0, T]; L^p(Q)) \quad \mathfrak{L} - a.s. \text{ for any bounded continuous } b,$$

see [7, Chapter 4].

In view of Lemma 4.1, we may pass to the limit in the approximate momentum equation (4.32) obtaining

$$\begin{aligned} & \int_0^T \partial_t \psi \int_Q \tilde{\varrho} \tilde{\mathbf{u}} \cdot \boldsymbol{\varphi} \, dx \, dt \\ & + \int_0^T \psi \int_Q \tilde{\varrho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla \boldsymbol{\varphi} \, dx \, dt - \int_0^T \psi \int_Q \mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}}) : \nabla \boldsymbol{\varphi} \, dx \, dt \\ & + \int_0^T \psi \int_Q \overline{p_\delta(\tilde{\varrho}, \tilde{\vartheta})} \operatorname{div} \boldsymbol{\varphi} \, dx \, dt + \int_0^t \psi \int_Q \overline{\tilde{\varrho} \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \cdot \boldsymbol{\varphi} \, dx \, d\tilde{W} \\ & = - \int_Q \tilde{\varrho}_0 \tilde{\mathbf{u}}_0 \cdot \boldsymbol{\varphi} \, dx; \end{aligned} \quad \boxed{\text{LMeq}} \quad (5.23)$$

for all $\psi \in C_c^\infty[0, T]$, $\boldsymbol{\varphi} \in C_c^\infty(Q; \mathbb{R}^3)$ \mathfrak{L} -a.s.

strongconvdensity

5.7. Strong convergence of the density. In the first step, we proceed as in the proof of (5.14) and test the momentum equation (4.32) by $\psi(t)\zeta(x)\nabla\Delta^{-1}[1_Q\tilde{\varrho}_\varepsilon]$, where $\psi \in C_c^\infty(0, T)$, $\zeta \in C_c^\infty(Q)$ and Δ^{-1} is the inverse of the Laplacean on the whole space \mathbb{R}^3 , cf. [21, Chapter 3, Section 3.6.5]. In the stochastic terms, we apply Itô's formula to the function $f(\rho, \mathbf{q}) = \int_Q \mathbf{q} \cdot \zeta \nabla \Delta^{-1}[1_Q \rho] \, dx$. Similarly to (5.13) we obtain the integral identity:

$$\boxed{\text{newEVF}} \quad (5.24) \quad \int_0^T \int_Q \psi \zeta \left(p_\delta(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon) \tilde{\varrho}_\varepsilon - \mathbb{S}(\tilde{\vartheta}_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon) : \mathcal{R}[1_Q \tilde{\varrho}_\varepsilon] \right) \, dx \, dt = \sum_{j=1}^8 I_{j,\varepsilon} \quad \mathfrak{L} - a.s.$$

where

$$\begin{aligned}
I_{1,\varepsilon} &= -\varepsilon \int_0^T \int_Q \psi \zeta \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \nabla \Delta^{-1} [\operatorname{div}(1_Q \nabla \tilde{\varrho}_\varepsilon)] \, dx \, dt \\
I_{2,\varepsilon} &= \varepsilon \int_0^T \int_Q \varepsilon \psi \nabla(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) \cdot \nabla (\zeta \nabla \Delta^{-1} [1_Q \tilde{\varrho}_\varepsilon]) \, dx \, dt \\
I_{3,\varepsilon} &= \int_0^T \int_Q \psi \zeta \left(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \mathcal{R}[1_Q \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon] - (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \mathcal{R}[1_Q \tilde{\varrho}_\varepsilon] \right) \, dx \, dt, \\
I_{4,\varepsilon} &= - \int_0^T \int_Q \psi p_\delta(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon) \nabla \zeta \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}_\varepsilon] \, dx \, dt, \\
I_{5,\varepsilon} &= \int_0^T \int_Q \psi \mathbb{S}(\tilde{\vartheta}_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla \zeta \otimes \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}_\varepsilon] \, dx \, dt, \\
I_{6,\varepsilon} &= - \int_0^T \int_Q \psi (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \nabla \zeta \otimes \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}_\varepsilon] \, dx \, dt, \\
I_{7,\varepsilon} &= - \int_0^T \int_Q \partial_t \psi \zeta \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}_\varepsilon] \, dx \, dt,
\end{aligned}$$

and

$$I_{8,\varepsilon} = - \int_0^T \psi \int_Q \zeta \varrho \mathbf{F}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon) \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}_\varepsilon] \, dx \, d\tilde{W}_\varepsilon$$

Here, the symbol \mathcal{R} stands for the *double Riesz transform*, defined componentwise as $\mathcal{R}_{i,j} = \partial_{x_i} \Delta^{-1} \partial_{x_j}$, cf. [21, Chapter 3, Section 3.6.5].

Applying the same treatment with $\psi(t)\zeta(x)\nabla\Delta^{-1}[1_Q\varrho]$ as the multiplier to the limit equation (5.23), we obtain

$$\boxed{\text{newEVFL}} \quad (5.25) \quad \int_0^T \int_Q \psi \zeta \left(\overline{p_\delta(\tilde{\varrho}, \tilde{\vartheta})} \varrho - \mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}}) : \mathcal{R}[1_Q \tilde{\varrho}] \right) \, dx \, dt = \sum_{j=3}^8 I_j \quad \mathfrak{L}\text{-a.s.}$$

where

$$\begin{aligned}
I_3 &= \int_0^T \int_Q \psi \zeta \left(\tilde{\varrho} \tilde{\mathbf{u}} \cdot \mathcal{R}[1_Q \varrho \mathbf{u}] - (\tilde{\varrho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) : \mathcal{R}[1_Q \varrho] \right) \, dx \, dt, \\
I_4 &= - \int_0^T \int_Q \psi \overline{p_\delta(\tilde{\varrho}, \tilde{\vartheta})} \nabla \zeta \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}] \, dx \, dt, \\
I_5 &= \int_0^T \int_Q \psi \mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}}) : \nabla \zeta \otimes \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}] \, dx \, dt, \\
I_6 &= - \int_0^T \int_Q \psi (\tilde{\varrho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) : \nabla \zeta \otimes \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}] \, dx \, dt, \\
I_7 &= - \int_0^T \int_Q \partial_t \psi \zeta \tilde{\varrho} \tilde{\mathbf{u}} \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}] \, dx \, dt,
\end{aligned}$$

and

$$I_8 = - \int_0^T \psi \int_Q \zeta \overline{\varrho \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \cdot \nabla \Delta_x^{-1} [1_Q \tilde{\varrho}] \, dx \, d\tilde{W}.$$

Now, using the deterministic arguments, exactly as in [21, Chapter 3, Section 3.6.5], we may use the uniform bounds established in Section 5.1, notably (5.11) to show that

$$I_{1,\varepsilon}, I_{2,\varepsilon} \rightarrow 0, \quad I_{j,\varepsilon} \rightarrow I_j \quad \text{for } j = 3, \dots, 7 \quad \mathfrak{L} - \text{a.s.}$$

Finally, we claim convergence of the stochastic integrals $I_{8,\varepsilon} \rightarrow I_8$. To this end, we use Lemma

4.1. To begin, we observe that (5.19), together with the available uniform bounds, implies

$$\boxed{\text{wWS329}'} \quad (5.26) \quad \tilde{\varrho}_\varepsilon \mathbf{F}_\varepsilon(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon) \rightarrow \overline{\tilde{\varrho} \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \quad \text{in } L^2(0, T; L^2(\mathfrak{U}_0; W^{-1,2}(Q)))$$

\mathfrak{L} -a.s. Moreover, we have

$$\boxed{\text{eq:comrho}} \quad (5.27) \quad \zeta \Delta^{-1} \nabla \tilde{\varrho}_\varepsilon \rightarrow \zeta \Delta^{-1} \nabla \tilde{\varrho} \quad \text{in } C([0, T]; W_0^{1,2}(Q; \mathbb{R}^3))$$

by the compactness of the operator $\zeta \Delta^{-1} \nabla : L^p(\mathbb{R}^3) \rightarrow W_0^{1,2}(Q)$ for $p > \frac{6}{5}$. Combining (5.26) and (5.27) with Lemma 4.1 we conclude $I_{8,\varepsilon} \rightarrow I_8$.

Thus we have shown that \mathfrak{L} -a.s

$$\boxed{\text{eq:flux0}} \quad (5.28) \quad \int_0^T \int_Q \psi \zeta \left(p_\delta(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon) \tilde{\varrho}_\varepsilon - \mathbb{S}(\tilde{\vartheta}_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla \Delta^{-1} \nabla 1_Q \tilde{\varrho}_\varepsilon \right) dx dt \\ \rightarrow \int_0^T \int_Q \psi \zeta \left(\overline{p_\delta(\tilde{\varrho}, \tilde{\vartheta})} \tilde{\varrho} - \mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}}) : \nabla \Delta^{-1} \nabla 1_Q \tilde{\varrho} \right) dx dt.$$

Relation (5.28) gives rise to the effective viscous flux identity discovered by Lions [35] and, after a tedious and rather nonstandard manipulation, gives rise to the strong (pointwise) convergence of the density. The arguments are purely deterministic and use only the renormalized equation of continuity (5.18) and compactness (\mathbb{P} -a.s.) of the initial density distribution $\tilde{\varrho}_{\varepsilon,0}$. A detailed proof is given in [21, Chapter 3, Section 3.6.5]; see [7, Chap. 4.4] for the barotropic stochastic case. For the convenience of the reader we recall the main steps in the proof. By decomposing the diffusive part, using commutator estimates and applying Div-Curl lemma (in space-time) we infer from (5.28) that

$$\boxed{\text{eq:fluxa}} \quad (5.29) \quad \int_0^T \int_Q \psi \zeta \left(p_\delta(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon) - \left(\frac{4}{3} \mu(\tilde{\vartheta}_\varepsilon) + \nu(\tilde{\vartheta}_\varepsilon) \right) \operatorname{div} \tilde{\mathbf{u}}_\varepsilon \right) \tilde{\varrho}_\varepsilon dx dt \\ \longrightarrow \int_0^T \int_Q \psi \zeta \left(\overline{p_\delta(\tilde{\vartheta}, \tilde{\varrho})} - \left(\frac{4}{3} \mu_\delta(\tilde{\vartheta}) + \nu(\tilde{\vartheta}) \right) \operatorname{div} \tilde{\mathbf{u}} \right) \tilde{\varrho} dx dt$$

\mathfrak{L} -a.s. As the next step, we intend to prove the following

$$\boxed{\text{eq:fluxfinal}} \quad (5.30) \quad \limsup_{\varepsilon \rightarrow \infty} \int_Q p_\delta(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon) \tilde{\varrho}_\varepsilon dx dt \leq \int_Q \overline{p_\delta(\tilde{\varrho}, \tilde{\vartheta})} \tilde{\varrho} dx dt.$$

\mathfrak{L} -a.s. Comparing the renormalized continuity equation ((5.18) with the choice $b(z) = z \ln z$) for $(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon)$ with the continuity equation for $(\tilde{\varrho}, \tilde{\mathbf{u}})$ (to which DiPerna–Lions theory of renormalized solutions [17] applies) shows

$$\boxed{\text{eq:renormlimit2}} \quad (5.31) \quad \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^3} \tilde{\varrho}_\varepsilon \operatorname{div} \tilde{\mathbf{u}}_\varepsilon dx dt \leq \int_0^T \int_{\mathbb{T}^3} \tilde{\varrho} \operatorname{div} \tilde{\mathbf{u}} dx dt.$$

Using this in shows (5.29) shows (5.30). The rest of the proof uses monotonicity of the mapping $t \mapsto p(t, \vartheta)$ for fixed ϑ and strong convergence of $\tilde{\vartheta}_\varepsilon$ (recall (5.17)). We finally deduce (up to a subsequence) strong (a.a. pointwise) convergence of $(\tilde{\varrho}_\varepsilon)_{\varepsilon > 0}$.

5.8. Conclusion. As we have established strong convergence of the families $(\tilde{\varrho}_\varepsilon)_{\varepsilon > 0}$, $(\tilde{\vartheta}_\varepsilon)_{\varepsilon > 0}$ it is a routine matter to pass to the limit in the weak formulation of (4.31–4.34) to obtain the limit system (5.1–5.4). We have shown Theorem 5.1.

6. THE LIMIT IN THE ARTIFICIAL PRESSURE

$\boxed{\text{vanishingpressure}}$

In this final section we let $\delta \rightarrow 0$ in the approximate system (5.1–5.4) and complete the proof of the existence of solutions stated in Theorem 2.1. As in the preceding sections, the proof consists in (i) showing uniform bounds independent of δ , (ii) applying the stochastic compactness method based on Skorokhod representation theorem, (iii) showing compactness of the temperature and the density by means of deterministic arguments.

6.1. Initial data. The initial data considered in Theorem 5.1 are quite regular. In order to achieve the generality of the initial law in Theorem 2.1, we consider a family of Borel probability measure $(\Lambda_\delta)_{\delta>0}$ defined on $L^1(Q) \times L^1(Q) \times L^1(Q; \mathbb{R}^3)$ such that

$$\Lambda_\delta = \left\{ \varrho_0 > 0, 0 < \underline{\varrho} \leq \int_{\mathbb{T}^3} \varrho_0 \, dx \leq \bar{\varrho}, \nabla \varrho_0 \cdot \mathbf{n} = 0, \varrho_0, \vartheta_0 > 0 \right\} = 1,$$

$$\int_{L^1 \times L^1 \times L^1} \left[\|\varrho_0\|_{C_x^{2+\nu}}^r + \|\vartheta_0\|_{W_x^{1,2} \cap C_x}^r + \|\mathbf{u}_0\|_{L_x^2}^r \right] d\Lambda_\delta \leq c(r, \delta) \text{ for all } r \geq 1.$$

Let Λ be the law specified in (2.1). We consider a sequence $(\Lambda_\delta)_{\delta>0}$ such that

$$\Lambda_\delta \rightarrow \Lambda \text{ weakly-}^*(*) \text{ in } \mathcal{M}^+(L^1(Q) \times L^1(Q) \times L^1(Q; \mathbb{R}^3)),$$

$$\int_{L_x^1 \times L_x^1 \times L_x^1} \left\| \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e_\delta(\varrho_0, \vartheta_0) + \varrho_0 s_\delta(\varrho_0, \vartheta_0) + \delta \varrho_0^\beta \right\|_{L_x^1}^r d\Lambda_\delta \leq c(r) \text{ for all } r \geq 1$$

uniformly for $\delta \rightarrow 0$.

wWS4S1

6.2. Uniform energy bounds. We start with the energy estimates that basically mimic those obtained in Section 5.1. Let $(\varrho, \vartheta, \mathbf{u})$ be a weak martingale solution to (5.1-5.4) constructed by means of Theorem 5.1. We intend to derive estimates which hold true uniformly in δ . Although the arguments are exactly the same as in Section 5.1, we collect them here for reader's convenience.

We have the δ -dependent variant of the total dissipation balance (5.5), specifically,

$$\begin{aligned} & \int_Q \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H_{\delta, \Theta}(\varrho, \vartheta) + \frac{\delta}{\beta-1} \varrho^\beta + \delta \varrho^2 \right] dx + \Theta \int_0^\tau \int_Q \sigma_\delta \, dx \\ & \leq \int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + H_{\delta, \Theta}(\varrho_0, \vartheta_0) + \frac{\delta}{\beta-1} \varrho_0^\beta + \delta \varrho_0^2 \right] dx \\ & + \int_0^\tau \int_Q \frac{\delta}{\vartheta^2} \, dx \, dt + \int_0^\tau \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW \\ & + \frac{1}{2} \int_0^\tau \left(\int_Q \sum_{k \geq 1} \varrho |\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})|^2 \, dx \right) dt \end{aligned} \quad (6.1)$$

NTDBbis

for any $0 \leq \tau \leq T$ \mathbb{P} -a.s., where

$$H_{\delta, \Theta} = \varrho e_\delta(\varrho, \vartheta) - \Theta \varrho s_\delta(\varrho, \vartheta),$$

and

$$\sigma_\delta = \frac{1}{\vartheta} \left[\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta} |\nabla \vartheta|^2 + \frac{\delta}{2} \left(\varrho^{\beta-1} + \frac{1}{\vartheta^2} \right) |\nabla \vartheta|^2 + \delta \frac{1}{\vartheta^2} \right].$$

Note that (6.1) follows directly from the energy balance (5.4) and the entropy inequality (5.3) whereas validity of both is guaranteed by Theorem 5.1.

The stochastic integral in (6.1) can be controlled, exactly as in (4.5), by means of the Burkholder–Davis–Gundy inequality,

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq \tau} \left| \int_0^t \int_Q \varrho [\mathbf{F}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \, dW \right|^r \right] \\ & \lesssim \mathbb{E} \left[\int_0^\tau \sum_{k \geq 0} \left| \int_Q \varrho [\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \right|^2 \right]^{r/2}, \quad r \geq 1, \end{aligned} \quad (6.2)$$

wWS26bis

where, in view of hypotheses (2.2),

$$\left| \int_Q \varrho [\mathbf{F}(\varrho, \vartheta, \mathbf{u})] \cdot \mathbf{u} \, dx \right|^2 \leq c(\bar{\varrho}) \|\sqrt{\varrho} \mathbf{u}\|_{L_x^2}^2.$$

The correction term can be bounded similarly. Consequently, in accordance with the coercivity properties (2.19) of the function $H_{\delta, \Theta}$, along with the observation made in Remark 2.3, we may

pass to expectation in (6.1), and, applying Gronwall's lemma, to obtain the following uniform bounds that may be seen as stochastic counterparts of the estimates in [21, Chapter 3, Section 3.7.1]:

$$\boxed{\text{wWS47}} \quad (6.3) \quad \mathbb{E} \left[\left| \sup_{t \in [0, T]} \|\varrho\|_{L_x^{5/3}}^{5/3} \right|^r \right] + \mathbb{E} \left[\left| \sup_{t \in [0, T]} \delta \|\varrho\|_{L_x^\beta}^\beta \right|^r \right] \lesssim c_1(r, \Lambda);$$

$$\boxed{\text{wWS48}} \quad (6.4) \quad \mathbb{E} \left[\left| \text{ess sup}_{t \in [0, T]} \|\varrho \mathbf{u}\|^2 \right|_{L_x^1} \right]^r + \left| \sup_{t \in [0, T]} \|\varrho \mathbf{u}\|_{L_x^{5/4}}^{5/4} \right|^r \lesssim c_1(r, \Lambda);$$

$$\boxed{\text{wWS49}} \quad (6.5) \quad \mathbb{E} \left[\|\mathbf{u}\|_{L_t^2 W_x^{1,2}}^{2r} \right] \lesssim c_2(r, \Lambda),$$

$$\boxed{\text{wWS217aSS}} \quad (6.6) \quad \mathbb{E} \left[\left| \text{ess sup}_{t \in [0, T]} \|\vartheta\|_{L_x^4}^4 \right|^r \right] + \mathbb{E} \left[\|\nabla \vartheta\|_{L^2((0, T) \times Q)}^r \right] \lesssim c(r, \Lambda);$$

$$\boxed{\text{wWS217SS}} \quad (6.7) \quad \mathbb{E} \left[\left\| \frac{\kappa_\delta(\vartheta)}{\vartheta} \nabla \vartheta \right\|_{L^2((0, T) \times Q)}^r \right] \lesssim c(r, \Lambda).$$

Finally, we report the conservation of mass principle

$$\boxed{\text{wWS411}} \quad (6.8) \quad \|\varrho(\tau, \cdot)\|_{L_x^1} = \int_Q \varrho(\tau, \cdot) \, dx = \int_Q \varrho_0 \, dx \leq \bar{\varrho} \text{ for all } \tau \in [0, T].$$

$\boxed{\text{wWS4S1S}}$

6.3. Pressure estimates. In order to derive refined estimates of the pressure, we apply a method similar to Section 5.2. We consider

$$\mathcal{B}[b(\varrho) - (b(\varrho))_Q]$$

as test function in the momentum equation (5.2). Here b is a smooth function with moderate growth specified below. Next, using the regularization method of DiPerna and Lions [17] we can show that ϱ, \mathbf{u} satisfy the renormalized continuity equation (cf. [21, Chapter 2.2.5]) holds

$$\boxed{\text{Npd1c}} \quad (6.9) \quad \begin{aligned} & \int_0^T \int_Q [b(\varrho) \partial_t \psi + b(\varrho) \mathbf{u} \cdot \nabla \psi + (b(\varrho) - \varrho b'(\varrho)) \operatorname{div} \mathbf{u} \psi] \, dx \, dt \\ & = - \int_Q b(\varrho_0) \psi(0) \, dx \end{aligned}$$

for any test function $\psi \in C_c^\infty([0, T] \times R^3)$. In other words, the renormalized equation (6.9) provided \mathbf{u} has been extended to be zero outside Q . Consequently, we get

$$\boxed{\text{wWS412}} \quad (6.10) \quad \begin{aligned} d\mathcal{B}[b(\varrho) - (b(\varrho))_Q] & = -\mathcal{B}[\operatorname{div}(b(\varrho)\mathbf{u})] \, dt \\ & + \mathcal{B} \left[(b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u} - ((b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u})_Q \right] \, dt. \end{aligned}$$

Exactly as in Section 5.2 we deduce

$$\boxed{\text{wWS413}} \quad (6.11) \quad \begin{aligned} & \int_0^\tau \int_Q p_\delta(\varrho, \vartheta) [b(\varrho) - (b(\varrho))_Q] \, dx \, dt \\ & = \left[\int_Q \varrho \mathbf{u} \cdot \mathcal{B}[b(\varrho) - (b(\varrho))_Q] \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_Q \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \mathcal{B}[b(\varrho) - (b(\varrho))_Q] \, dx \, dt \\ & + \int_0^\tau \int_Q \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathcal{B}[b(\varrho) - (b(\varrho))_Q] \, dx \, dt - \int_0^\tau \int_Q \varrho \mathbf{u} \cdot \mathcal{B}[\operatorname{div}(b(\varrho)\mathbf{u})] \, dx \, dt \\ & + \int_0^\tau \int_Q \varrho \mathbf{u} \cdot \Delta^{-1} \mathcal{B} \left[(b'(\varrho)\varrho - b(\varrho)) \operatorname{div} \mathbf{u} - ((b'(\varrho)\varrho - b(\varrho)) \operatorname{div} \mathbf{u})_Q \right] \, dx \, dt \\ & - \int_0^\tau \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[b(\varrho) - (b(\varrho))_Q] \, dx \, dW \equiv \sum_{i=1}^6 I_i. \end{aligned}$$

Now, consider $b(\varrho) = \varrho^\alpha$ where $0 < \alpha < \frac{1}{3}$.

The estimates for the “deterministic” integrals can be obtained exactly as in [7, Sec. 4.5.2] using the uniform bounds from the previous section. We only give the details for the stochastic integral. As a consequence of (6.8), standard L^q -estimates for the inverse Laplacian, and the embedding relation $W^{1,q}(Q) \hookrightarrow C(\overline{Q})$ for $q > 3$,

$$\boxed{\text{wWS414}} \quad (6.12) \quad \sup_{t \in [0, T]} \|\mathcal{B}[b(\varrho) - (b(\varrho))_Q]\|_{L^\infty} \lesssim 1 \text{ } \mathbb{P}\text{-a.s.}$$

where the norm is controlled by a deterministic constant proportional to $\bar{\varrho}$. Next, we have by the Burgholder–Davis–Gundy inequality

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^\tau \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[b(\varrho) - (b(\varrho))_Q] \, dx \, dW \right|^r \right] \\ & \leq \mathbb{E} \left[\sup_{t \in [0, \tau]} \left| \int_0^t \int_Q \varrho \mathbf{F}(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho^\alpha - (\varrho^\alpha)_Q] \, dx \, dW \right|^r \right] \\ & \lesssim \mathbb{E} \left[\int_0^\tau \sum_{k=1}^\infty \left| \int_Q \varrho \mathbf{F}_k(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho^\alpha - (\varrho^\alpha)_Q] \, dx \right|^2 dt \right]^{r/2}, \end{aligned}$$

where, due to (2.2) and (6.8), (6.12),

$$\begin{aligned} \left| \int_Q \varrho \mathbf{F}_k(\varrho, \vartheta, \mathbf{u}) \cdot \mathcal{B}[\varrho^\alpha - (\varrho^\alpha)_Q] \, dx \right| & \lesssim f_k \|\mathcal{B}[\varrho^\alpha - (\varrho^\alpha)_Q]\|_{L^\infty} \int_Q (\varrho + \varrho|\mathbf{u}|) \, dx \\ & \lesssim c(\bar{\varrho}) f_k \int_Q (\varrho + \varrho|\mathbf{u}|) \, dx. \end{aligned}$$

Consequently, we may infer, similarly to Section 5.2, that

$$\boxed{\text{wWS422}} \quad (6.13) \quad \mathbb{E} \left[\left| \int_0^T \int_Q p_\delta(\varrho, \vartheta) \varrho^\alpha \, dx \, dt \right|^r \right] \lesssim c(\bar{\varrho}, r)$$

for a certain $\alpha > 0$.

$\boxed{\text{wWS4S2}}$

6.4. Stochastic compactness method. Exactly as in Section 5.3, we show the estimate

$$\boxed{\text{Nbv7bis}} \quad (6.14) \quad \mathbb{E} \left[\|\varrho \mathbf{u}\|_{C^s([0, T]; W^{-k, 2}(\mathbb{T}^3))}^r \right] \lesssim c(r) \text{ for a certain } 0 < s(r) < \frac{1}{2}, \, k > \frac{3}{2}.$$

Now, following the arguments of Section 5.4 based on Theorems 4.1 and 4.2 we pass to the Skorokhod representation on the standard probability space $([0, 1], \mathfrak{B}[0, 1], \mathfrak{L})$ obtaining a family of random variables $(\tilde{\varrho}_\delta, \tilde{\vartheta}_\delta, \tilde{\mathbf{u}}_\delta)_{\delta > 0}$, together with the processes \tilde{W}_δ and with the associated right-continuous complete filtration $(\tilde{\mathfrak{F}}_t^\delta)_{t \geq 0}$ such that:

- $\left(\left([0, 1], \overline{\mathfrak{B}[0, 1]}, (\tilde{\mathfrak{F}}_t^\delta)_{t \geq 0}, \mathfrak{L} \right), \tilde{\varrho}_\delta, \tilde{\vartheta}_\delta, \tilde{\mathbf{u}}_\delta, \tilde{W}_\delta \right)$ is a weak martingale solution of problem (5.1–5.4), with the initial law Λ_δ ;
- the initial data satisfy $(\tilde{\varrho}_{0, \delta}, \tilde{\vartheta}_{0, \delta}, \tilde{\mathbf{u}}_{0, \delta})$ satisfy

$$\begin{aligned} \tilde{\varrho}_{0, \delta} & \rightarrow \tilde{\varrho}_0 \text{ in } L^1(Q), \, \varrho_0 \geq 0 \\ \tilde{\vartheta}_{0, \delta} & \rightarrow \tilde{\vartheta}_0 \text{ in } L^1(Q), \, \vartheta_0 > 0, \\ \tilde{\mathbf{u}}_{0, \delta} & \rightarrow \tilde{\mathbf{u}}_0 \text{ in } L^1(Q, R^3); \end{aligned}$$

\mathfrak{L} -a.s.;

- the functions $(\tilde{\varrho}_\delta, \tilde{\vartheta}_\delta, \tilde{\mathbf{u}}_\delta)$ satisfy the bounds (6.3–6.8), (6.13), and (6.14) \mathfrak{L} \mathbb{P} -a.s. uniformly for $\delta \rightarrow \infty$;

- we have

$$\tilde{\varrho}_\delta \rightarrow \tilde{\varrho} \text{ weakly in } L^q((0, T) \times Q) \text{ for some } q > \frac{5}{3}, \text{ weakly-}^*(*) \text{ in } L^\infty(0, T; L^{5/3}(Q)),$$

$$\tilde{\varrho}_\delta \rightarrow \tilde{\varrho} \text{ in } C([0, T]; W^{-k, 2}(Q)) \text{ } k > \frac{3}{2},$$

Ncv1bis

$$(6.15) \quad \tilde{\vartheta}_\delta \rightarrow \tilde{\vartheta} \text{ weakly in } L^2(0, T; W^{1, 2}(Q)), \text{ weakly-}^*(*) \text{ in } L^\infty(0, T; L^4(Q)),$$

$$\tilde{\mathbf{u}}_\delta \rightarrow \tilde{\mathbf{u}} \text{ weakly in } L^2(0, T; W_0^{1, 2}(Q; R^3)),$$

$$\tilde{\varrho}_\delta \tilde{\mathbf{u}}_\delta \rightarrow \tilde{\varrho} \tilde{\mathbf{u}} \text{ weakly-}^*(*) \text{ in } L^\infty(0, T; L^{5/4}(Q; R^3)),$$

$$\tilde{\varrho}_\delta \tilde{\mathbf{u}}_\delta \rightarrow \tilde{\varrho} \tilde{\mathbf{u}} \text{ in } C([0, T]; W^{-k, 2}(Q; R^3)) \text{ } k > \frac{3}{2},$$

\mathfrak{L} -a.s.;

- the sequence $(\tilde{\varrho}_\delta, \tilde{\vartheta}_\delta, \tilde{\mathbf{u}}_\delta, \nabla \tilde{\vartheta}_\delta, \nabla \tilde{\mathbf{u}}_\delta)_{\delta > 0}$ generates a Young measure;
- we have

$$\tilde{W}_\delta \rightarrow \tilde{W} \text{ in } C([0, T]; \mathfrak{U}_0) \text{ a.s.}$$

We remark that weak continuity in time of $\tilde{\varrho}$ and $\tilde{\varrho} \tilde{\mathbf{u}}$ follows from (6.15) and the embedding

$$L^\infty(0, T; L^p(Q)) \cap C([0, T]; W^{\ell, 2}(Q)) \hookrightarrow C_w([0, T]; L^p(Q))$$

which holds for all $p > 1$ and all $\ell \in R$.

Also note that a.s. boundedness in all the afore mentioned function spaces, in particular the weak- $(*)$ convergence in the dual space $L^\infty(0, T; L^q)$, is guaranteed by Theorem 4.1, where the functional $[[\cdot]]$ includes all bounds obtained in (6.3), (6.4) and (6.6), cf. Remark 4.2. Consequently, the norms

$$\|\tilde{\varrho}_\delta\|_{L^\infty(0, T; L^{5/3}(Q))}, \|\tilde{\varrho}_\delta \tilde{\mathbf{u}}_\delta\|_{L^\infty(0, T; L^{5/4}(Q))} \text{ and } \|\tilde{\vartheta}_\delta\|_{L^\infty(0, T; L^4(Q))}$$

remain bounded uniformly for $\delta \rightarrow 0$ and the desired result follows.

The limit in the convective terms follows again immediately. Now, following step by step the arguments used in Section 5.6 we can show

wWS433

$$(6.16) \quad \tilde{\varrho}_\delta \mathbf{F}(\tilde{\varrho}_\delta, \tilde{\vartheta}_\delta, \tilde{\mathbf{u}}_\delta) \rightarrow \overline{\tilde{\varrho} \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \text{ in } L^2(0, T; L^2(\mathfrak{U}_0, W^{-k, 2}(Q))), \text{ } k > \frac{3}{2},$$

\mathfrak{L} -a.s. Having established compactness of the stochastic integrals, we may perform the limit for $\varepsilon \rightarrow 0$ in continuity and momentum equation obtaining

wWS438

$$(6.17) \quad \int_0^T \int_Q [\tilde{\varrho} \partial_t \varphi + \tilde{\varrho} \tilde{\mathbf{u}} \cdot \nabla \varphi] \, dx \, dt = - \int_Q \tilde{\varrho}_0 \psi(0, \cdot) \, dx$$

for any $\psi \in C_c^\infty([0, T] \times R^3)$ \mathfrak{L} -a.s.;

wWS439

$$(6.18) \quad \begin{aligned} & \int_0^T \partial_t \psi \int_Q \tilde{\varrho} \tilde{\mathbf{u}} \cdot \varphi \, dx \, dt + \int_Q \tilde{\varrho}_0 \tilde{\mathbf{u}}_0 \cdot \varphi \, dx \\ & + \int_0^T \psi \int_Q [\tilde{\varrho} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla \varphi + \overline{p(\tilde{\varrho}, \tilde{\vartheta})} \operatorname{div} \varphi] \, dx \, dt \\ & = \int_0^T \psi \int_Q \overline{\mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}})} : \nabla \varphi \, dx \, dt - \int_0^T \psi \int_Q \overline{\tilde{\varrho} \mathbf{F}(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})} \cdot \varphi \, dx \, d\tilde{W} \end{aligned}$$

for any $\varphi \in C_c^\infty(Q; R^N)$ and $\psi \in C_c^\infty[0, T]$, $\psi(0) = 1$, \mathfrak{L} -a.s.

6.5. Compactness of the temperature and the density. Following the strategy from Section 5 we can use the deterministic arguments specified in [21, Chapter 3, Section 3.7.3] to show that

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$$(6.19) \quad \tilde{\vartheta}_\delta \rightarrow \tilde{\vartheta} \text{ in } L^4((0, T) \times Q) \text{ } \mathfrak{L} - \text{a.s.}$$

Consequently we have $\overline{\mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}})} = \mathbb{S}(\tilde{\vartheta}, \nabla \tilde{\mathbf{u}})$ in (6.18). Moreover, we can pass to the limit in the entropy balance.

Next, we easily adapt the arguments of Section 5.7 to the multipliers of the form

$$\xi(t)\zeta(x)\Delta^{-1}\nabla(1_Q T_k(\tilde{\varrho}_\delta)), \text{ and } \xi(t)\zeta(x)\Delta^{-1}\nabla(1_Q \overline{T_k(\varrho)}),$$

where T_k is a family of cut-off functions,

$$T_k(r) = kT\left(\frac{r}{k}\right),$$

$$T \in C^\infty[0, \infty), \quad T(r) = \begin{cases} r & \text{for } 0 \leq r \leq 1, \\ T''(r) \leq 0 & \text{for } r \in (1, 3), \\ 2 & \text{for } r \geq 3. \end{cases}$$

Using the same arguments as in Section 5.7, in particular when handling the limit in the stochastic integral, we deduce the effective viscous flux identity in the form:

$$\begin{aligned} \overline{p_M(\varrho, \vartheta) T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right) \overline{T_k(\varrho) \operatorname{div} \mathbf{u}} \\ \text{evfid} \quad (6.20) \quad = \overline{p_M(\varrho, \vartheta) T_k(\varrho)} - \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right) \overline{T_k(\varrho) \operatorname{div} \mathbf{u}} \quad \mathfrak{L} - \text{a.s.} \end{aligned}$$

see also [21, Chapter 3, Section 3.7.4] and [7, Chap. 4.5].

Relation (6.20) in fact yields strong (a.a. pointwise) convergence of the density. Here, the argument is more involved than in Section 5.7, however, still purely deterministic. The reader may find a detailed proof in [21, Chapter 3.7.5].

Having established the strong convergence of the approximate densities, we easily perform the limit $\delta \rightarrow 0$ in the system (5.1–5.4) to recover the original problem. We have proved Theorem 2.1.

7. CONCLUSION, POSSIBLE EXTENSIONS, ABSENCE OF STATIONARY SOLUTIONS

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For the sake of simplicity, we have assumed that the viscosity coefficients behave like linear functions of the temperature, see (2.11). In particular, this gives rise to the velocity field living in the “standard” energy space $L^2(0, T; W_0^{1,2}(Q; \mathbb{R}^3))$. A more elaborate treatment in the spirit of [21, Chapter 3] would yield the existence result in the physically more relevant range of viscosities, namely,

$$0 < \underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \bar{\mu}(1 + \vartheta^\alpha), \quad \frac{2}{5} < \alpha \leq 1.$$

Further generalizations of the constitutive assumptions as well as the underlying spatial domain and boundary conditions in the spirit of [21, Chapter 3] are possible.

Finally, it is worth-noting that, unlike the simplified barotropic problem (see [8]), the full Navier–Stokes–Fourier system does not admit, in general, *stationary solutions* (that is, solutions which law is independent of time). This is a simple consequence of the total energy balance (1.9). Indeed passing to expectations and assuming the total energy to be stationary, we get

$$\mathbb{E} \left[\int_{\tau_1}^{\tau_2} \int_Q \frac{1}{2} \varrho \sum_{k \geq 1} |\mathbf{F}_k(\varrho, \vartheta, \mathbf{u})|^2 \, dx \, dt + \int_{\tau_1}^{\tau_2} \int_Q \varrho H \, dx \, dt \right] = 0$$

for a.a. $0 \leq \tau_1 < \tau_2$ \mathbb{P} -a.s.. Thus if $H \geq 0$, the driving force must vanish identically a.s. along the paths of any stationary solution. This is in strong contrast to the barotropic case, where the existence of (non-trivial) stationary martingale solutions has been recently shown in [8].

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