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Stochastic exponential integrators for a finite element discretisation of SPDEs with additive noise

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Abstract

We consider the numerical approximation of the general second order semilinear parabolic stochastic partial differential equations (SPDEs) driven by additive space-time noise. Our goal is to build two numerical algorithms with strong convergence rates higher than that of the standard semi-implicit scheme. In contrast to the standard time stepping methods which use basic increments of the noise, we introduce two schemes based on the exponential integrators, designed for finite element, finite volume or finite difference space discretisations. We prove the convergence in the root mean square L^2 norm for a general advection diffusion reaction equation and a family of new Lipschitz nonlinearities. We observe from both the analysis and numerics that the proposed schemes have better convergence properties than the current standard semi-implicit scheme.

Keywords: Parabolic stochastic partial differential equations, Finite element method, Exponential integrators, Higher order approximation, Strong numerical approximation, Additive noise, Transport in porous media.

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1. Introduction

Stochastic Partial Differential Equations (SPDEs) model numerous phenomena in engineering and biological sciences (eg. [4, 31, 6]). As analytical solutions are not available, the study of numerical solutions of SPDEs is therefore an active research area and there is an extensive literature on numerical methods for SPDEs (see [14, 13, 15] and references therein).

In this work, our goal is to build two numerical algoritms with high strong convergence rates [1] of the following SPDEs in $\Omega \subset \mathbb{R}^d$, $d = \{1, 2, 3\}$

$$dX(t,x) = (\nabla \cdot (\mathbf{D}\nabla X(t,x)) - \mathbf{q}(x) \cdot \nabla X(t,x) + f(x, X(t,x), \nabla X(t,x)))\, dt + dW(t,x), \tag{1}$$

$x \in \Omega, t \in [0,T]$ where $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz continuous function, W is a Q-Wiener process and $\mathbf{q} \in (L^\infty(\Omega))^d$. The initial data $X(0) = X_0$ is given. In the abstract setting, the linear operator considered is given by

$$A = \nabla \cdot \mathbf{D}\nabla(.) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right), \tag{2}$$

where $\mathbf{D} = (a_{i,j})_{1\leq i,j\leq d}$, is symmetric and satisfies the following ellipticity condition

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\xi_j \geq c_1|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Omega}, \quad c_1 > 0, \tag{3}$$

and the nonlinear function is defined by $F(u)(x) = f(x, u(x), \nabla u(x)) - \mathbf{q}(x) \cdot \nabla u(x)$. This is in contrast of the work in [26, 37] where the linear operator is non-self-adjoint as the advection term [2] is also included in the operator A. In our abstract setting, (1) is equivalent to

$$dX = (AX + F(X))dt + dW, \tag{4}$$

in the Hilbert space $H = L^2(\Omega)$. Under the ellipticity condition (3), it is well known that the linear operator A is self adjoint, positive definite and is the generator of an analytic semigroup $S(t) := e^{tA}, t \geq 0$ with eigenfunctions e_i and eigenvalues λ_i, $i \in \mathbb{N}^d$. The Q-Wiener process W is white in time and defined on a filtered probability space $(\mathbb{D}, \mathcal{F}, \mathbb{P}, \{F_t\}_{t\geq 0})$. The noise can be represented as

$$W(x,t) = \sum_{i\in\mathbb{N}^d} \sqrt{q_i} e_i(x)\beta_i(t), \tag{5}$$

[1] Indeed for stochastic diffusion ($\mathbf{q} = 0$ in (1)), high orders schemes have been obtained in [14, 13, 15]. Our goal is to update such schemes for $\mathbf{q} \neq 0$.

[2] The term with $\mathbf{q}$ in (1).

where $q_i \geq 0,\ i \in \mathbb{N}^d$ are the eigenvalues of the covariance operator Q and β_i are independent and identically distributed standard Brownian motions. Here, we assume that the linear operator A and Q have the same eigenfunctions[3]. The time stepping methods in this paper are based on the mild solution of (4). Precise assumptions on F, Q, X_0 and A will be given in the next section to ensure the existence of the unique mild solution X of (4) in the form

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + O(t), \qquad t \in (0,T] \tag{6}$$

where O is the stochastic process given by the stochastic convolution

$$O(t) = \int_0^t S(t-s)dW(s). \tag{7}$$

We build our numerical algorithms on recent works by Jentzen and co-workers [14, 13, 15, 16] that use Taylor expansion and linear functionals of the noise for a spectral Galerkin discretisation of (4). We now briefly describe these schemes. Let P_N, $N \in \mathbb{N}$ be the spectral projection defined for $u \in L^2(\Omega)$ by

$$P_N u = \sum_{i \in \mathcal{I}_N} (e_i, u)e_i, \qquad \mathcal{I}_N = \{1, 2, ..., N\}^d, \tag{8}$$

where $(\cdot,\cdot)$ is the standard inner product on H. Assume that F is independent of ∇X. The spectral Galerkin discretisation of (4) yields the following semi-discrete form

$$dX^N = (A_N X^N + F_N(X^N))dt + dW^N, \tag{9}$$

with $A_N = P_N A,\ F_N = P_N F$ and $W^N = P_N W$. Note that (9) is a diagonal system to be solved in each Fourier mode. Jentzen and co-workers [15, 16] examine the following two high order time stepping schemes which overcome the order barrier (see [15]) of numerical schemes approximating (4)

$$X^N_{m+1} = e^{\Delta t A_N} X^N_m + \Delta t \varphi_1(\Delta t A_N) F_N(X^N_m) + P_N O_m \tag{10}$$

and

$$Y^N_{m+1} = \varphi_0(\Delta t A_N)\left(Y^N_m + \Delta t F_N(Y^N_m)\right) + P_N O_m, \tag{11}$$

where the standard $\varphi-$functions are defined by

$$\varphi_0(\Delta t A_N) = e^{\Delta t A_N}, \qquad \varphi_1(\Delta t A_N) = (\Delta t\, A_N)^{-1}\left(e^{\Delta t A_N} - I\right) = \frac{1}{\Delta t}\int_0^{\Delta t} e^{(\Delta t - s)A_N} ds.$$

The process

$$O_m = \int_{t_m}^{t_{m+1}} e^{(t_{m+1}-s)A} dW, \tag{12}$$

[3]See [25] for a case where the eigenfunctions are different

has the exact variance in each Fourier mode as an Ornstein–Uhlenbeck process. More precisely, by assuming that the linear operator A and the covariance operator Q have the same eigenbasis, applying the Itô isometry in each mode yields

$$(e_i, O_m) = \left(\frac{q_i}{2\lambda_i}\left(1 - e^{-2\lambda_i \Delta t}\right)\right)^{1/2} R_{i,m}, \tag{13}$$

$i \in \mathcal{I}_N = \{1, 2, 3, ..., N\}^d$, $m = 0, 1, 2..., M-1$ and $R_{i,m}$ are independent, standard normally distributed random variables with means 0 and variance 1. In (13), the noise is said to be computed using its linear functionals. Note that the equality (13) is understood in the sense of probability law. The optimal strong orders for scheme (10) have been obtained in [38] under more relaxed assumptions on the nonlinear function F. Although schemes (10)-(11) are high orders in time, they are limited in real practical applications. Our aim is a first step to address this issue. For complex domains, advection problems or problems with mixed boundary conditions, the spectral Galerkin approach is not feasible and preference is usually given to finite element (mostly its mixed form), finite difference or finite volume methods even though the diagonalization of the linear operator is destroyed. We analyse here a finite element discretisation, examine its implementation and in addition illustrate a finite volume implementation. Our main motivation is flow and transport in heterogeneous porous media. More precisely our new schemes solve [4]the equation

$$dX = \left(D\varDelta X - \nabla \cdot (\mathbf{q}X) + f(X)\right) dt + dW, \tag{14}$$

without requiring information on the eigenvalues and eigenfunctions of the corresponding linear operator $D\varDelta$ with homogeneous mixed boundary conditions, which can be expensive to compute. Note that the Dirichlet boundary condition is applied on Γ and the homogeneous Neumann boundary on $\partial\Omega \setminus \Gamma$. Indeed the operator $D\varDelta$ with mixed Neumann-Dirichlet boundary conditions is decomposed as a sum of two operators, one linear unbounded operator in H with homogeneous Neumann boundary conditions and an operator related to the trace operator. More precisely, using the trace operator (see [18]) in Green's theorem yields the following decomposition

$$dX = (AX + F_1(X) + \mathbb{T}(X))dt + dW, \tag{15}$$

where for $v \in H^1(\Omega)$

$$(Au, v) = -\int_\Omega D\nabla u \nabla v \, dx,$$

and

$$(\mathbb{T}u, v) = \int_\Gamma \frac{\partial u}{\partial \nu} \gamma_0 v \, d\sigma, \qquad \gamma_0 v = v \mid_{\partial\Omega}, \; v \in H^1(\Omega).$$

In this abstract setting (4), the linear operator is $A = D\varDelta$ with homogeneous Neumann boundary conditions, the nonlinear term is then $F = F_1 + \mathbb{T}$. If the noise W and the operator

[4]Numerically and in some cases both numerically and rigorously

A have the same eigenfunctions, our schemes can then be used for (15). The velocity $\mathbf{q}$ in (14) is obtained from the following steady state mass convervation equation and Darcy's law

$$\nabla \cdot \mathbf{q} = q_{in}, \qquad \mathbf{q} = -\frac{\mathbf{k}}{\mu}\nabla p, \tag{16}$$

where $\mathbf{k}$ is the heterogeneous permeability tensor, p is the pressure, μ is the dynamic viscosity of the fluid [2] and q_{in} the fluid injection rate. In (14), f is the reaction function which can be the Langmuir adsorption function, and $D > 0$ is the diffusion coefficient. Typically, the deterministic case (14)-(16) are solved using either a finite element (mostly its mixed form) or finite volume discretisation in space due to the heterogeneous nature of the permeability as a spectral Galerkin approach is not feasible in such applications.

In this paper, we introduce and analyse the convergence of two new schemes by combining the finite element discretisation with the exponential time stepping and linear functionals of the noise. We prove convergence in the root mean square L^2 norm for the general advection diffusion reaction equation and a new family of Lipschitz nonlinear functions (see Assumption 2.1). Our approach, based on the projection of the noise onto a standard finite element grid, allows practitioners to simply adapt existing codes to examine the effects of stochastic forcing. In [25], the use of linear functionals of the noise is extended to finite–element discretisations with a semi-implicit Euler–Maruyama method. In contrast to [25], we consider here two exponential based methods for time-stepping as in [26, 37, 23, 24, 15, 16, 17] where the discrete semi-group is no longer approximated by a rational function. Our new schemes in this work solve more general second order semilinear parabolic stochastic partial differential equations with additive noise (1), which is part of [26], but in general, the eigenfunctions of the self adjoint operator (or a related operator [5]) should be known in contrast to the schemes in [26]. The reward is that the new schemes are more accurate than schemes in [26] as the strong orders of convergence in time have double. The new schemes are also more accurate than the scheme in [26], this accuracy comes from the fact that we need to compute the exponential of a non–diagonal matrix, which is a notoriously hard problem in numerical analysis [28]. However, new developments for both Léja points and Krylov subspace techniques [12, 30, 36, 3, 1] have led to efficient methods to compute the matrix exponential functions.

The paper is organized as follows. In Section 2, some properties of the mild solution and assumptions on SPDE (4) are provided. In Section 3, we present the two numerical schemes based on the exponential integrators and linear functionals of the noise. We also present and comment on our convergence results. In Section 4, we present some simulations, and also show that equipped with the well known eigenvalues and eigenfunctions of the operator Δ with Neumann or Dirichlet boundary conditions, we can apply the new schemes with mixed boundary conditions for the operator $A = D\Delta$ as indicated in (14)-(16). The proofs

[5] In (14), the linear operator is $D\Delta$ with mixed Neumann-Dirichlet boundary conditions. It is related to the operator Δ with homogeneous Neumann boundary conditions where the eigenfunctions are well known.

of our convergence theorems (SETD1 and SETD0 schemes) are presented in Section 5. We conclude by summarizing our findings in Section 6.

2. Assumptions and properties of the mild solution

We start by presenting briefly the notation for the main function spaces and norms that we will use in this paper. We denote by $\|\cdot\|$ the norm associated to the inner product $(\cdot,\cdot)$ of a separable Hilbert space H. For a Banach space $\mathcal{V}$, we denote by $\|.\|_{\mathcal{V}}$ the norm of $\mathcal{V}$, $L(\mathcal{V})$ the set of bounded linear mapping from $\mathcal{V}$ to $\mathcal{V}$ and by $L_2(\mathbb{D},\mathcal{V})$ the Hilbert space of all equivalence classes of square integrable $\mathcal{V}-$valued random variables. Note that $\mathbb{D}$ is the sample space.

Throughout the paper, we assume that Ω is bounded and has a smooth boundary or is a convex polygon of $\mathbb{R}^d$, $d=\{1,2,3\}$. Although in our practical implementation we will restrict to the operator $A = D\,\Delta,\quad D>0$ in a rectangular domain Ω [6], our analysis will focus on the general second order semi–linear parabolic stochastic partial differential equation given in (1)

Let $\mathbb{H}\subset V\subset H=L^2(\Omega)$ be a space that depends on the choice of the boundary conditions. For Dirichlet boundary conditions, we set

$$V=\mathbb{H}=H_0^1(\Omega)=\{v\in H^1(\Omega): v=0 \ \text{ on } \ \partial\Omega\}.$$

For Robin boundary conditions (Neumann conditions being a particular case), we set $V=H^1(\Omega)$ and

$$\mathbb{H}=\left\{v\in H^2(\Omega):\partial v/\partial\nu_A+\sigma v=0\quad\text{on}\quad\partial\Omega\right\},\qquad \sigma\in\mathbb{R}.$$

Note that $\partial v/\partial\nu_A$ is the normal derivative of v and ν_A is the exterior pointing normal $\mathbf{n}=(n_i)$ to the boundary of Ω given by

$$\partial v/\partial\nu_A=\sum_{i,j=1}^{d} n_i(x)a_{i,j}(x)\frac{\partial v}{\partial x_j}. \tag{17}$$

The corresponding bilinear form of $-A$ is given by

$$a(u,v)=\int_\Omega\left(\sum_{i,j=1}^{d}a_{i,j}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\right)dx\qquad u,v\in V \tag{18}$$

for Dirichlet and Neumann boundary conditions, and by

$$a(u,v)=\int_\Omega\left(\sum_{i,j=1}^{d}a_{i,j}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\right)dx+\int_{\partial\Omega}\sigma u\,v\,dx,\qquad u,v\in V, \tag{19}$$

[6]Since the eigenfunctions are well known for Dirichlet and Neumann boundary conditions.

for Robin boundary conditions. For $r \in \{1,2\}$, with the space $\mathbb{H}$ in hand, we can characterize the domain of the operator $(-A)^{r/2}$, denoted by $\mathcal{D}((-A)^{r/2})$ and have the following norm equivalence results [9, 7], which will be used in our convergence proofs

$$\begin{aligned}
\|v\|_{H^r(\Omega)} &\equiv \|(-A)^{r/2}v\| =: \|v\|_r \qquad \forall v \in \mathcal{D}((-A)^{r/2}),\\
\mathcal{D}((-A)^{r/2}) &= \mathbb{H} \cap H^r(\Omega) \qquad\qquad \text{(Dirichlet boundary conditions)},\\
\mathcal{D}(-A) &= \mathbb{H},\ \ \mathcal{D}((-A)^{1/2}) = H^1(\Omega) \qquad \text{(Robin boundary conditions)}.
\end{aligned}$$

In the Banach space $\mathcal{D}((-A)^{\alpha/2})$, $\alpha \in \mathbb{R}$, we use the notation $\|.\|_\alpha := \|(-A)^{\alpha/2}.\|$.

Under condition (3), it is well known (see [9]) that the linear operator A generates an analytic semigroup $S(t) \equiv e^{tA}$.

As we can observe in (13), our schemes use in their implementation the eigenvalues of the linear operator A. The following example shows that the linear operator $D\Delta$ can be of interest in realistic applications.

Example 2.1. *When dealing with heat transfer in geothermal subsurface, for a low enthalpy reservoir, where the rock and fluid heat capacities are almost constant, we can set*

$$\mathbf{D} = \frac{\lambda_{\mathbf{g}}\mathbf{I}}{(\rho c)_{\mathbf{g}}}, \qquad \mathbf{q}(x) = \frac{(\rho c)_{\mathbf{f}}}{(\rho c)_{\mathbf{g}}}\left(-\frac{\mathbf{k}}{\mu}\nabla p\right). \tag{20}$$

Equation (1) *models the heat transfer with deterministic known sink/source f and random sink/source dW. Note that the subscripts* $\mathbf{f}$ *and* $\mathbf{g}$ *denote fluid and bulk properties, respectively, ρ ($Kg \cdot m^{-3}$) is the density, c ($J \cdot kg^{-1} \cdot K^{-1}$) is the specific heat capacity and λ ($W \cdot m^{-1} \cdot K^{-1}$) is the thermal conductivity. Note also that the unknown X is the stochastic temperature distribution. The range of documented hydraulic conductivity $\mathbf{K} = \rho g \mathbf{k}/\mu$ [7] values of clastic sedimentary rocks is typically between $10^{-3} m \cdot s^{-1}$ and $10^{-12} m \cdot s^{-1}$.* $\mathbf{K}$ *is therefore an extremely multiscale parameter compared to the associated thermal conductivities, which are normally between $0.5\, W \cdot m^{-1} \cdot K^{-1}$ and $4.5\, W \cdot m^{-1} \cdot K^{-1}$ (see [29]). Since the thermal conductivity does not vary so much, in some low enthalpy reservoirs it is sometimes assumed to be constant, while the permeabilities remain multiscale. In such cases, the diffusion part of* (1) *is just $D\Delta$ with $D = \lambda_{\mathbf{g}}/(\rho c)_{\mathbf{g}}$.*

We recall some basic properties of the semigroup $S(t)$ generated by the linear operator A.

Proposition 2.1. ***[Smoothing properties of the semigroup ([11])]***
Let $\alpha > 0$, $\beta \geq 0$ and $0 \leq \gamma \leq 1$, then there exists $C > 0$ such that

$$\begin{aligned}
\|(-A)^{\beta}S(t)\|_{L(H)} &\leq Ct^{-\beta} \quad \text{for} \quad t > 0\\
\|(-A)^{-\gamma}(I - S(t))\|_{L(H)} &\leq Ct^{\gamma} \quad \text{for} \quad t \geq 0.
\end{aligned}$$

[7]Note that g is the gravity.

In addition,

$$\begin{aligned}(-A)^{\beta}S(t) &= S(t)(-A)^{\beta} \quad on \quad \mathcal{D}((-A)^{\beta})\\ If \;\; \beta &\geq \gamma \quad then \quad \mathcal{D}((-A)^{\beta}) \subset \mathcal{D}((-A)^{\gamma}),\\ \|D_t^l S(t)v\|_{\beta} &\leq Ct^{-l-(\beta-\alpha)/2}\,\|v\|_{\alpha}, \;\; t>0, \;\; v\in\mathcal{D}((-A)^{\alpha/2}) \;\; l=0,1, \; \beta\geq\alpha\end{aligned}$$

where $D_t^l := \dfrac{d^l}{dt^l}$.

We investigate our convergence proofs with the following new assumptions .

Assumption 2.1. ***[Nonlinearity]*** *We assume that there exists a positive constant $L > 0$ such that F satisfies one of the following.*

(a) The nonlinear function F satisfies the following globally Lipschitz condition

$$\|F(Z)-F(Y)\|_{-1} \leq L\|Z-Y\| \qquad \forall Z, Y \in L^2(\Omega).$$

(b) F is Lipschitz, twice continuously differentiable and satisfies

$$\begin{aligned}\|F(Z)-F(Y)\|_{-1} &\leq L\|Z-Y\|,\\ \|F'(Z)(X)\|_{-1} &\leq L\|X\|,\\ \|(-A)^{-\eta/2}F''(Z)(X,Y)\| &\leq L\|X\|\|Y\| \quad for\ some\ \eta\in[1,2), \qquad \forall Z, Y \in L^2(\Omega).\end{aligned}$$

Remark 2.1. *In the abstract setting* (4)*, if the nonlinear function F is expressed as $F(u)(x) = f(x,u(x)) - \boldsymbol{q}(x)\cdot\nabla u(x)$ where $f:\Omega\times\mathbb{R}\rightarrow\mathbb{R}$ is twice continuously differentiable function with the bounded partial derivatives and $\boldsymbol{q}\in L^{\infty}(\Omega)$, given by* (16)*, then Assumption 2.1 is satisfied as we have*

$$\|F(Z)-F(Y)\| \leq L\|Z-Y\|_{H^1(\Omega)} \qquad \forall Z, Y \in L^2(\Omega). \tag{21}$$

Note that for

$$F(u)(x) = f(x,u(x)) - \boldsymbol{q}(x)\cdot\nabla u(x) =: G(u)(x) - \boldsymbol{q}(x)\cdot\nabla u(x), \tag{22}$$

if the nonlinear function G satisfying for X, Y, $Z \in H$

$$\|G(X)\| \leq L(1+\|X\|), \tag{23}$$
$$\|G'(Z)(X)\| \leq L\|X\| \tag{24}$$
$$\|(-A)^{-\eta/2}G''(Z)(X,Y)\| \leq L\|X\|\|Y\| \quad for\ some\ \eta\in[1,2), \tag{25}$$

then F satisfies Assumption 2.1(b). Details on functions G satisfying (23)-(25) *can be found in [42, Example 3.2].*

We now turn our attention to the noise. We introduce the spaces and notation we need for the Q-Wiener process W. An operator $l \in L(H)$ is Hilbert-Schmidt if

$$\|l\|_{HS}^2 := \sum_{i\in\mathbb{N}^d} \|le_i\|^2 < \infty,$$

where (e_i) is an orthonormal basis in H. The sum in $\|.\|_{HS}^2$ is independent of the choice of the orthonormal basis in H. We denote by $L_2^0 := HS(Q^{1/2}(H), H)$, the space of Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H with the corresponding norm $\|.\|_{L_2^0}$ defined by

$$\|l\|_{L_2^0} := \|lQ^{1/2}\|_{HS} = \left(\sum_{i\in\mathbb{N}^d} \|lQ^{1/2}e_i\|^2\right)^{1/2}, \qquad l \in L_2^0.$$

Let $\varphi : [0,T] \times \mathbb{D} \to L_2^0$ be a L_2^0-valued predictable stochastic process with

$$\int_0^t \mathbf{E}\|\varphi Q^{1/2}\|_{HS}^2 ds < \infty.$$

Then Itô's isometry (see e.g. [5, Step 2 in Section 2.3.2]) gives

$$\mathbf{E}\|\int_0^t \varphi dW\|^2 = \int_0^t \mathbf{E}\|\varphi\|_{L_2^0}^2 ds = \int_0^t \mathbf{E}\|\varphi Q^{1/2}\|_{HS}^2 ds, \qquad t \in [0,T].$$

For the noise, we use the following assumption

Assumption 2.2. *We assume that the covariance operator Q satisfies*

$$\|(-A)^{(r-1)/2}Q^{1/2}\|_{HS} < \infty, \quad \textit{for some } 1 \leq r \leq 2. \tag{26}$$

As a consequence

$$O(t) \in L_2(\mathbb{D}, \mathcal{D}((-A)^{r/2})), \;\; 0 \leq t \leq T, \;\; \textit{for some } 1 \leq r \leq 2.$$

Remark 2.2. *By using [38, Lemma 2.3], we can easily check that if* (26) *is satisfied, we therefore have*

$$\mathbf{E}\|O(t)\|_r^2 = \int_0^t \|(-A)^{r/2}S(t-s)\|_{L_2^0}^2 ds \leq C\|(-A)^{\frac{r-1}{2}}Q^{\frac{1}{2}}\|_{HS} < \infty. \tag{27}$$

Finally we make the following assumption for the initial data.

Assumption 2.3. ***[Initial data** X_0**]***
Let r be the noise's parameter given in (26)*, we assume that the initial data satisfies* $\mathbf{E}\|(-A)^{r/2}X_0\|^2 < \infty, \;\; 1 \leq r \leq 2.$

Theorem 2.1. ***[Existence and uniqueness]***
Let Assumption 2.1, Assumption 2.2 and Assumption 2.3 be fulfilled. Then there exists a unique mild solution $X : [0,T] \times \mathbb{D} \to \mathcal{D}((-A)^{r/2})$ of (4) in the form (6) such that:

$$\sup_{0 \leq t \leq T} \mathbf{E}[\|(-A)^{r/2} X(t)\|^2] < \infty, \qquad 1 \leq r < 2.$$

The parameter r being defined in (26).

Proof. The proof can be found in [21, Theorem 2.27]. Note that this proof uses the following condition

$$\|F(Z)\|_{-2+r} \leq C(1 + \|Z\|_{r-1}), \quad Z \in \mathcal{D}((-A)^{r-1}), \qquad 1 \leq r < 2. \tag{28}$$

which is obviously satisfied. Indeed from Assumption 2.1 (a) and (21), the condition (28) is satisfied for $r = 1$ and $r = 2$ respectively. By interpolation, the condition (28) is therefore satisfied. ■

3. Numerical schemes and main results

3.1. Numerical schemes

We consider the discretisation of the spatial domain by a finite element triangulation. Let $\mathcal{T}_h$ be a set of disjoint intervals of Ω (for $d = 1$), a triangulation of Ω (for $d = 2$) or a set of tetrahedra (for $d = 3$) satisfying the standard regularity assumptions (see [9]). Let $V_h \subset V$ denotes the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_h$. To discretise in space, we use two projections. The first projection operator P_N (8) projects onto a finite dimensional spectral set. The second projection operator P_h is the $L^2(\Omega)$ projection onto the finite element space V_h defined for $u \in L^2(\Omega)$ by

$$(P_h u, \chi) = (u, \chi) \qquad \forall\, \chi \in V_h. \tag{29}$$

Then $A_h : V_h \to V_h$ is the discrete analogue of A defined by

$$(A_h \varphi, \chi) = -a(\varphi, \chi) \qquad \varphi, \chi \in V_h, \tag{30}$$

where $a(\ ,\ )$ is the corresponding bilinear form associated to the operator A. We denote by S_h the semigroup generated by the operator A_h.

The semi–discrete version of the problem (4) is to find the process $X_h(t) = X_h(.,t) \in V_h$ such that for $t \in [0,T]$,

$$dX_h = (A_h X_h + P_h F(X_h))dt + P_h P_N dW, \qquad X_h(0) = P_h X_0. \tag{31}$$

The mild solution of (31) is given by

$$X_h(t) = S_h(t) X_h(0) + \int_0^t S_h(t-s) F(X_h(s)) ds + \int_0^t S_h(t-s) P_h P_N dW.$$

Given the mild solution at time t_m, we construct the corresponding solution at t_{m+1} by

$$\begin{aligned} X^h(t_{m+1}) &= S_h(\Delta t)X^h(t_m) + \int_0^{\Delta t} S_h(\Delta t - s)P_h F(X^h(s+t_m))ds \\ &\quad + \int_{t_m}^{t_{m+1}} S_h(t_{m+1}-s)P_h P_N dW(s). \end{aligned} \tag{32}$$

Let $O_{h,N}^m$ and $O_m^{h,N}$ be two V_h-valued stochastic convolutions defined by

$$O_{h,N}^m = \int_{t_m}^{t_{m+1}} S_h(t_{m+1}-s)P_h P_N dW \tag{33}$$

$$O_m^{h,N} = P_h P_N\, O_m, \quad where \quad O_m = \int_{t_m}^{t_{m+1}} S(t_{m+1}-s)dW. \tag{34}$$

To build our schemes, we use the following approximation for the noise $O_{h,N}^m \approx O_m^{h,N}$. For our first numerical scheme SETD1, we use the following approximations

$$F(X^h(t_m+s)) \approx F(X^h(t_m)) \qquad s \in [0,\ \Delta t].$$

Then, we approximate X_m^h of $X(m\Delta t)$ by

$$X_{m+1}^h = e^{\Delta t A_h}X_m^h + \Delta t \varphi_1(\Delta t A_h)P_h F(X_m^h) + O_m^{h,N}. \tag{35}$$

For efficiency, to avoid computing two matrix exponential functions, we rewrite (35) as

$$X_{m+1}^h = X_m^h + \Delta t \varphi_1(\Delta t A_h)\left(A_h X_m^h + P_h F(X_m^h)\right) + O_m^{h,N}.$$

We call this scheme (SETD1). Our second numerical method called SETD0 is similar to the one in [23, 24, 17]. It is based on approximating the deterministic integral in (32) at the left–hand endpoint of each partition. We can therefore define the approximation Y_m^h of $X(m\Delta t)$ by

$$Y_{m+1}^h = \varphi_0(\Delta t A_h)\left(Y_m^h + \Delta t P_h F(Y_m^h)\right) + O_m^{h,N}. \tag{36}$$

Note that the standard semi-implicit Euler-Maruyama scheme applied to the semi-discrete problem (31) yields

$$\begin{aligned} Z_{m+1}^h &= (\mathrm{I} - \Delta t\ A_h)^{-1}\left(Z_m^h + \Delta t\, P_h F(Z_m^h) + P_h \Delta W_m^N\right) \\ \Delta W_m^N &= \sqrt{\Delta t}\sum_{i\in\mathcal{I}_N}\sqrt{q_i}R_{i,m}e_i, \quad \mathcal{I}_N = \{1,2,...,N\}^d, \end{aligned} \tag{37}$$

where $R_{i,m}$ are independent, standard normally distributed random variables with mean 0 and variance 1. In [25], it has been proved that this standard scheme is less accurate than the modified implicit scheme developed in [25]. We will therefore compare our new schemes with the modified implicit scheme developed in [32, 25]. This modified implicit scheme is given by

$$K_{m+1}^h = (\mathrm{I} - \Delta t\, A_h)^{-1} \left(K_m^h + \Delta t\, P_h F(K_m^h) - P_h P_N O(t_m)\right) + P_h P_N O(t_{m+1}). \qquad (38)$$

We use the Monte Carlo method to approximate the discrete root mean square L^2 norm of the error on a regular mesh with size h at the final time $T = M\Delta t$

$$\begin{aligned} \left(\mathbf{E}\|X(T) - \xi_M^h\|^2\right)^{1/2} &= \left(\mathbf{E}\|X(.,T) - \xi_M^h(.)\|^2\right)^{1/2} \\ &\approx \left(\frac{h^d}{K}\sum_{\ell=1}^{K}\sum_{i=1}^{N_h}\left(X(a_i,T) - \xi_M^h(a_i)\right)^2\right)^{1/2}, \end{aligned} \qquad (39)$$

where ξ_M^h is either X_M^h, Y_M^h, or K_M^h (the numerical solutions from the final step respectively in (35), (36), (37) or (38) for each sample ℓ), K is the number of sample solutions and $X(T)$ is the 'exact' solution for the sample ℓ that we will specify in Section 4.

3.2. Main results

Throughout the paper we let N be the number of terms of truncated noise, $\mathcal{I}_N = \{1, 2, ..., N\}^d$ and take $t_m = m\Delta t \in (0, T]$, where $T = M\Delta t$ for $m, M \in \mathbb{N}$. We take C to be a constant that may depend on T and other parameters but not on Δt, N or h. The convergence results of SETD1 and SETD0 are given by the following theorem. In particular this theorem covers the case of the advection-diffusion-reaction SPDEs arising in our examples from porous media flow.

Theorem 3.1. *Suppose that Assumption 2.1, Assumption 2.2 and Assumption 2.3 are satisfied. Let X be the mild solution of equation (4) represented by equation (6)and ζ_m^h be the numerical approximations through scheme* (35) *or* (36) *($\zeta_m^h = X_m^h$ for scheme SETD1 and $\zeta_m^h = Y_m^h$ for scheme SETD0). Let r_0 be defined as $r_0 = r$ if $1 \leq r < 2$ and $r_0 = 2 - \epsilon$, ϵ small enough if $r = 2$. If Assumption 2.1(a) is satisfied, then*

$$(\mathbf{E}\|X(t_m) - \zeta_m^h\|^2)^{1/2} \leq C\left(h^r + \Delta t^\beta + \left(\inf_{j \in \mathbb{N}^d \setminus \mathcal{I}_N} \lambda_j\right)^{-r/2}\right),$$

where $\beta = \min(1/2, r/2)$ and r is defined in Assumption 2.2 via (26)*.*

If Assumption 2.1(b) is satisfied, then

$$\begin{aligned} (\mathbf{E}\|X(t_m) - X_m^h\|^2)^{1/2} &\leq C\left(h^{r_0} + \Delta t^{r/2} + \left(\inf_{j \in \mathbb{N}^d \setminus \mathcal{I}_N} \lambda_j\right)^{-r/2}\right), \\ (\mathbf{E}\|X(t_m) - Y_m^h\|^2)^{1/2} &\leq C\left(h^{r_0} + \Delta t^{r/2} + \Delta t|\ln(\Delta t)| + \left(\inf_{j \in \mathbb{N}^d \setminus \mathcal{I}_N} \lambda_j\right)^{-r/2}\right). \end{aligned}$$

However if Assumption 2.1(b) is satisfied with

$$\begin{aligned} \|(-A)^{-\frac{\delta}{2}} F'(Z)(X)\| &\leq L(1+\|Z\|_{\min(r,1)})\|X\|_{-\min(r,1)}, \\ &\qquad X \in H, Z \in \mathcal{D}((-A)^{\frac{\min(r,1)}{2}}), \delta \in [1,2), \end{aligned} \tag{40}$$

then

$$(\mathbf{E}\|X(t_m)-X_m^h\|^2)^{1/2} \leq C\left(h^{r_0}+\Delta t+\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-r/2}\right), \tag{41}$$

$$(\mathbf{E}\|X(t_m)-Y_m^h\|^2)^{1/2} \leq C\left(h^{r_0}+\Delta t+\Delta t|\ln(\Delta t)|+\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-r/2}\right) \tag{42}$$

We remark that if we denote by N_h the number of vertices in the finite element mesh then it is well known (see for example [10]) [8] that if $N \geq N_h$ then

$$\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-r/2} \leq Ch^r. \tag{43}$$

As a consequence the estimates in Theorem 3.1 can be expressed as functions of h and Δt only, and the error from the finite element approximation is dominated. If $N \leq N_h$ then the error from the projection P_N of the noise onto a finite number of modes is dominated.

Remark 3.1. *From Theorem 3.1, we can observe that our new schemes are more accurate than the schemes in [26] as the orders of strong convergence have double when F satisfies Assumption 2.1 (b). We can also observe that the SETD1 scheme is more accurate than SETD0 scheme as the error estimate in SETD0 depends on an infinitesimal factor ϵ. This accuracy can also be observed in Figure 1.*

4. Numerical simulations and applications

4.1. Implementation

The key step in our stochastic exponential schemes is the computation of the action of matrix exponential functions on a vector. This will be done using either the real fast Léja points (with a tol = 10^{-6}) or Krylov subspace techniques with tol = 10^{-6} and 10 for the dimension of the subspace. More details can be found in [3, 1, 12, 30, 36, 34, 35]. In our graphs, we use the following notations

- 'SETD0 $r = a$' and 'SETD1 $r = a$', $a \in \{1, 2\}$, the errors graphs for our new schemes SETD0 and SETD1 where r the noise's parameter (see (48)).

[8] In one dimension see [39]

- 'Modified Implicit $r = a$' , $a \in \{1, 2\}$ is used to denote the errors graphs of the modified scheme (38). Once again r is a parameter used in the noise (see (48)).

The noise is projected onto a finite number of modes by P_N and we take $|\mathcal{I}_N| = N_h = \dim(V_h)$, then $N \geq N_h^{1/d}$ as suggested in [39, 20, 19] to avoid order reduction. As noted in the introduction, to compute $O_m^{h,N} = P_h P_N O_m$, the process $P_N O_m$ is projected onto the finite element space by P_h. If the noise is not smooth, then $P_h P_N O_m$ is evaluated following the work in [39, Section 5] for $P_h W$. Indeed, by setting $P_h P_N O_m = \sum_{i=1}^{N_h} \alpha_i^{1/2} \varphi_i$, as (e_i, O_m) is known from (13), the coefficients α_i are found by solving the linear system

$$\sum_{i=1}^{N_h} (e_i, O_m))^2 (e_i, \varphi_j)^2 = \sum_{i=1}^{N_h} \alpha_i (\varphi_i, \varphi_j)^2, \qquad j = 1, 2, ..., N_h, \tag{44}$$

where $(\varphi_i)_{1 \leq i \leq N_h}$ is the nodal basis with $\varphi_i(a_j) = \delta_{i,j}$. For problems without exact solutions, "the exact solution" or "reference solution" is the numerical solution with smaller time step δt. The numerical solution with the time step $\Delta t = R\delta t = t_{m+1} - t_n$, $R \in \mathbb{N}$ uses the following decomposition of the convolution operator O_m.

$$O_m = O(t_m) = \int_{t_m}^{t_{m+1}} e^{(t_{m+1}-s)A} dW = \sum_{j=1}^{R} \int_{\tau_j}^{\tau_{j+1}} e^{(t_{m+1}-s)A} dW, \tag{45}$$

where (τ_j) is such that $\tau_1 = t_m$, $\tau_{R+1} = t_{m+1}$ and $\delta t = \tau_{j+1} - \tau_j$.

So, using the Itô's isometry yields

$$\begin{aligned} (e_i, O_m) &= \sum_{j=1}^{R} \frac{q_i}{2\lambda_i} [e^{-2\lambda_i(\tau_{j+1} - t_{m+1})} - e^{-2\lambda_i(\tau_j - t_{m+1})}] R_{i,m}^j && (46) \\ &= \sum_{j=1}^{R} \frac{q_i}{2\lambda_i} [e^{-2\lambda_i(j-1)\delta t} - e^{-2\lambda_i(j\delta t)}] R_{i,m}^j, && (47) \end{aligned}$$

where $R_{i,m}^j$ are independent, standard normally distributed random variables with mean 0 and variance 1.

The covariance operator Q used for the noise has the same eigenfunctions as Δ with homogeneous Neumann boundary conditions in the domain $\Omega = [0, 1] \times [0, 1]$. The eigenfunctions $\{e_i^{(1)} e_j^{(2)}\}_{i,j \geq 0}$ of the operator Δ with homogeneous Neumann boundary conditions are given by

$$e_0^{(l)}(x) = 1 \qquad e_i^{(l)}(x) = \sqrt{2} \cos(\lambda_i^{(l)} x), \qquad \lambda_0^{(l)} = 0, \qquad \lambda_i^{(l)} = i\pi$$

where $l \in \{1, 2\}$, $x \in \Omega$ and $i \in \mathbb{N}^d$ with the corresponding eigenvalues $\{\lambda_{i,j}\}_{i,j \geq 0}$ given by $\lambda_{i,j} = (\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2$.

4.2. Stochastic advection diffusion reaction equations in heterogeneous porous media

As a more challenging example, we consider the stochastic advection diffusion reaction SPDE (14) in the domain $\Omega = [0,1] \times [0,1]$ with two types of boundary conditions:

- (a) Mixed Neumann-Dirichlet boundary condition. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x,y) : \; x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere.
- (b) Homogeneous Neumann boundary conditions in the entire boundary.

The first goal is to prove that our theoretical results are in agreement with our numerical results. Our second goal is to show that with the well known eigenvalues and eigenfunctions of the operator Δ with Neumann (or Dirichlet) boundary conditions, we can apply our new schemes to mixed boundary conditions for the operator $D\Delta$ without explicitly having eigenvalues and eigenfunctions[9]. In the decomposition (5), we have used

$$q_{i,j} = \left(i^2 + j^2\right)^{-(r+\delta)}, \qquad r > 0 \text{ and } \delta > 0 \quad \text{small enough.} \tag{48}$$

We obviously have

$$\sum_{(i,j)\in\mathbb{N}^2} \lambda_{i,j}^{r-1} q_{i,j} < \pi^2 \sum_{(i,j)\in\mathbb{N}^2} \left(i^2 + j^2\right)^{-(1+\delta)} < \infty \qquad 0 \leq r \leq 2,$$

thus Assumption 2.2 is satisfied. Note that r is the noise's parameter which influences the order of convergence. Using in (14) the trace operator $\gamma_1 \equiv \dfrac{\partial}{\partial \nu}$ (see [18]) and Green's theorem yields

$$dX = (AX + F_1(X) + \mathbb{T}(X))dt + dW, \tag{49}$$

where

$$\begin{aligned} (Au, v) &= -\int_\Omega D\nabla u \nabla v \, dx, \;\; (\mathbb{T}u, v) = \int_\Gamma \gamma_1 u \gamma_0 v \, d\sigma, \;\; \gamma_0 v = v \mid_{\partial\Omega}, \;\; v \in H^1(\Omega), \\ u &\in \{x \in H^2(\Omega) : \frac{\partial x}{\partial \nu} = 0 \;\text{ in }\; \Gamma_1\}, \quad \Gamma_1 = \partial\Omega \setminus \Gamma. \end{aligned} \tag{50}$$

In the abstract setting of (4), we take the linear operator to be $A = D\Delta$ using only homogeneous Neumann boundary. The explicit expression of $\mathbb{T}$ is unknown, however it may be approximated numerically, (see for example[8, 33, 18] for finite volumes).

- For boundary condition (a), the nonlinear term is now $F = F_1 + \mathbb{T}$ where

$$F_1(u) = -\nabla \cdot (\mathbf{q}u) - \frac{u}{(|u| + 1)}, \quad u \in \mathbb{R}^+. \tag{51}$$

[9] Rather we require the eigenfunctions of a related operator.

Indeed here, the operator $D\Delta$ with mixed Neumann-Dirichlet boundary conditions has been decomposed as a sum of two operators, one linear unbounded operator with homogeneous Neumann boundary conditions A and $\mathbb{T}$. Note that Assumption 2.1 (a) is not satisfied since the domain of the operateur $\mathbb{T}$ is $H^2(\Omega)$.

- For bounday condition (b), the nonlinear term is $F = F_1$ as $\mathbb{T} = 0$ for homogeneous Neumann condition. Assumption 2.1 (a) is clearly satisfied as soon as $q_i \in L^\infty(\Omega)$, $\mathbf{q} = (q_i)$.

We use a heterogeneous medium with three parallel high permeability streaks, 100 times higher compared to the other part of the medium. This could represent, for example, a highly idealized fracture pattern. We obtain the Darcy velocity field $\mathbf{q}$ by solving (16) with Dirichlet boundary conditions $\Gamma_D^1 = \{0, 1\} \times [0, 1]$ and Neumann boundary $\Gamma_N^1 = (0, 1) \times \{0, 1\}$ such that

$$p = \begin{cases} 1 & \text{in} \quad \{0\} \times [0,1] \\ 0 & \text{in} \quad \{L_1\} \times [0,1] \end{cases} \qquad -k\,\nabla p(\mathbf{x}, t) \cdot \mathbf{n} = 0 \quad \text{in} \quad \Gamma_N^1, \quad q_{in} = 0.$$

To deal with high Péclet flows we discretise in space using the finite volume method, viewed as a finite element method (see [8, 33, 32]). We can write the semi-discrete finite volume method as

$$dX^h = (A_h X^h + P_h F_1(X^h) + P_h \mathbb{T}(X^h)) + P_h P_N dW, \tag{52}$$

where here A_h is the space discretisation of A and $P_h\mathbb{T}(X^h)$ comes from the approximation of diffusion flux on the Dirichlet boundary condition side (see [8, 32]). Remember that for homogeneous Neumann condition, $\mathbb{T} = 0$. Thus, we can form the noise as in Section 4.1 with the eigenvalues function of Δ with full Neumann boundary conditions and (48).

In all our simulations in this section, the number of realizations used is 50 and $\Delta x = \Delta y = 1/150$. For boundary condition (a), the diffusion coefficient used is $D = 0.1$ in Figure 1(c), while for the boundary condition (b), the diffusion coefficient is $D = 10^{-2}$ in Figure 1(b) and $D = 1$ in Figure 1(a). The "reference solution" or 'exact solution" in each graph is the numerical solution with the smaller time step $\delta t = 1/15360$. Note that the numerical solution with time step $\Delta t = R\delta t$, $R \in \mathbb{N}$ is linked with the reference solution by (46).

From [42, Example 3.2], we can observe that Assumption 2.1(b) is satisfied for boundary condition (b) in Figure 1 (a). For noise parameters, we used $\delta = 0.0001$ [10], $r = 1$ and $r = 2$ in our convergence graphs. According to Theorem 3.1, the orders of convergence expected should be 0.5 for $r = 1$ and 1 for $r = 2$. In Figure 1 (a), we have observed for orders of convergence 0.55 with SETD1, 0.56 with SETD0 and modified scheme for $r = 1$, and 0.95 with SETD1, 0.97 with SETD0, 1.05 modified scheme for $r = 2$, which are close to the expected orders.

[10] This parameter should be small to provide the true order of convergence numerically

In Figure 1 (b) where the boundary condition (b) is used with $D = 10^{-2}$, we have observed high orders of convergence in both $r = 1$ and $r = 2$. More precisely, we have observed 1.08 with SETD1, SETD0 and modified scheme for $r = 1$ and $r = 2$. It seems as the extra condition (40) tends to be satisfied. Indeed, as the convective part of the nonlinear function F is linear, it is equal to its Frechet derivative. For $\delta_1 \in [1, 2)$, if the operator $(-A)^{-\frac{\delta_1}{2}}$ and the convective part of F commute, we can prove that the extra condition (40) is satisfied.

In Figure 1 (c) where the boundary condition (a) is used, Assumption 2.1 (a) and Assumption 2.1(b) are not satisfied as the domain of $\mathbb{T}$ is $H^2(\Omega)$. We have also observed high orders of convergence both for $r = 1$ and $r = 2$. More precisely, we have observed roughly 1.02 for $r =$ and 1.1 for $r = 2$. This result also suggests that our convergence results can also be extended to larger family of nonlinear functions F.

To sum up in Figure 1, for boundary condition (a) or boundary condition (b), we can observe that the schemes SETD1 or SETD0 are more accurate or have similar accuracy that the modified implicit scheme developed in [25]. This modified implicit scheme has been proved in [25] to be very accurate than the standard semi-implicit Euler-Maruyama scheme given in (37).

5. Proofs of the main results

5.1. Two preparatory results

We introduce the Riesz representation operator $R_h : V \to V_h$ defined by

$$(-AR_h v, \chi) = (-Av, \chi) = a(v, \chi), \qquad v \in V, \ \forall \chi \in V_h. \tag{53}$$

Under the regularity assumptions on the triangulation and in view of the $V-$ellipticity (3), it is well known (see [9]) that the following error bounds holds for $v \in V \cap H^r(\Omega)$,

$$\|R_h v - v\| + h\|R_h v - v\|_{H^1(\Omega)} \leq Ch^r \|v\|_{H^r(\Omega)}, \ 1 \leq r \leq 2. \tag{54}$$

It follows that

$$\|P_h v - v\| \leq Ch^r \|v\|_{H^r(\Omega)} \quad \forall v \in V \cap H^r(\Omega), \quad 1 \leq r \leq 2. \tag{55}$$

Since

$$\|P_h v - v\| \leq C\|v\|, \ v \in H,$$

we therefore have by interpolation theory

$$\|P_h v - v\| \leq Ch^r \|v\|_{H^r(\Omega)} \quad \forall v \in V \cap H^r(\Omega), \quad 0 \leq r \leq 2. \tag{56}$$

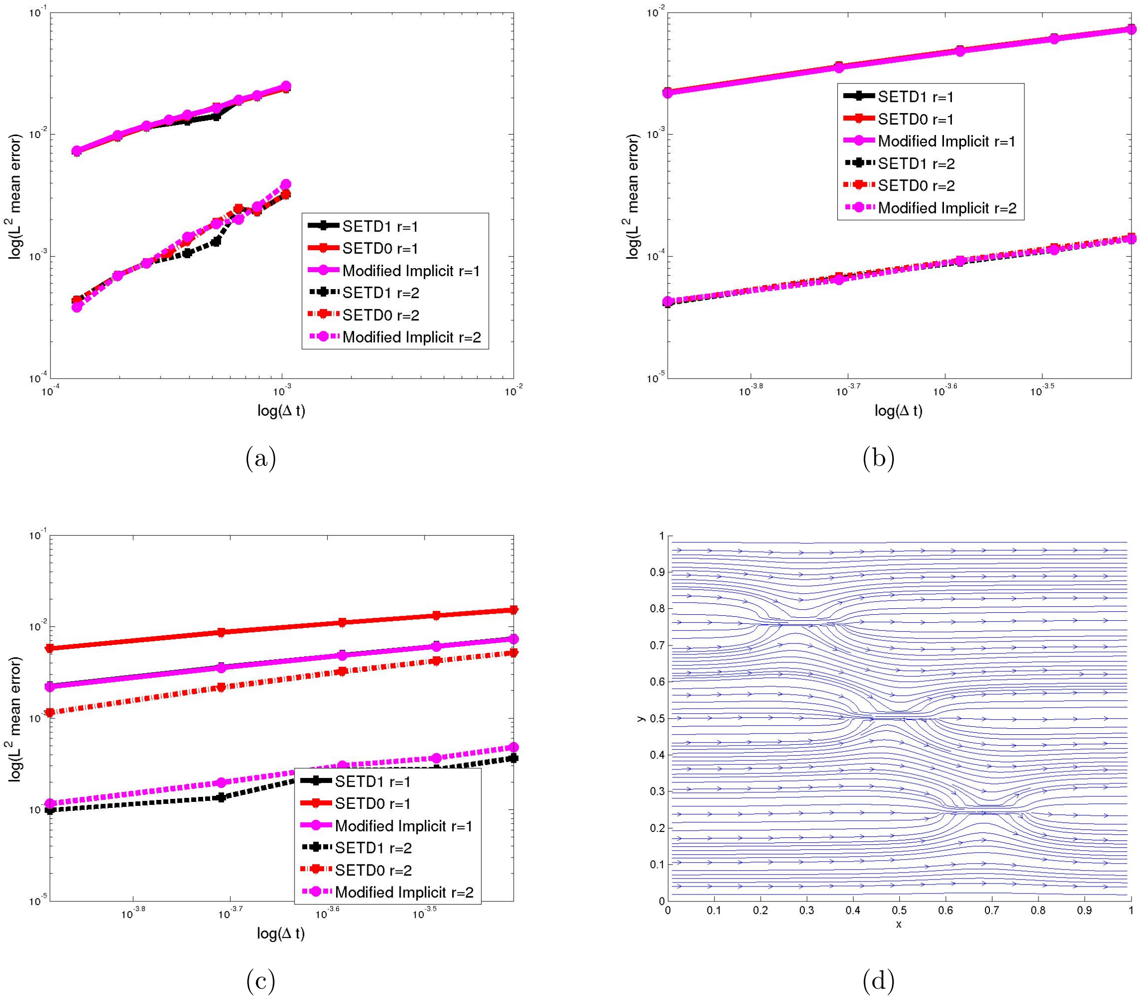

Figure 1: Convergence in the root mean square L^2 norm at $T = 1$ as a function of Δt with 50 realizations and $\Delta x = \Delta y = 1/150$, $X_0 = 0$. Graphs in (a) ($D = 1$) and (b) ($D = 0.01$) are for boundary condition (b)(homogeneous Neumann boundary condition), while graphs in (c) ($D = 0.1$) are for boundary condition (a) (mixed boundary conditions). The noise is white in time and the stochastic process $O(t) \in H^r(\Omega)$ respect to space variable with $r = 1$, $r = 2$ and $\delta = 0.0001$ in relation (48). The streamline of the velocity field is in (d). The reference solution or true solution for each realization is the numerical solution with smaller time step 1/15360. Note that the numerical solution with time step $\Delta t = R\delta t$, $R \in \mathbb{N}$ is linked with the reference solution by (46).

This inequality plays a key role in our convergence proofs. Let us consider the following deterministic linear problem: find $u \in V$ such that such that

$$\frac{du}{dt} = Au \qquad \text{given} \quad u(0) = v \qquad\qquad t \in (0, T]. \tag{57}$$

The corresponding semi-discretisation in space is to find $u_h \in V_h$ such that $\dfrac{du_h}{dt} = A_h u_h$ where $u_h^0 = P_h v$. From the continuous and semi-discrete problems, we define the operator

$$T_h(t) := u(t) - u_h(t) = S(t) - S_h(t)P_h = e^{tA} - e^{tA_h}P_h. \tag{58}$$

The following lemma is key in our convergence proofs.

Lemma 5.1. *The following estimates hold on the semi-discrete approximation of* (57)

$$\begin{aligned} \|u(t) - u_h(t)\| &= \|T_h(t)v\| \leq Ch^r t^{-(r-\beta)/2}\|v\|_\beta, \quad v \in \mathcal{D}((-A)^{\beta/2}), && (59) \\ \|\int_0^t T_h(s)vds\| &\leq Ch^{2-\rho}\|v\|_{-\rho} \quad v \in \mathcal{D}((-A)^{-\frac{\rho}{2}}), && (60) \end{aligned}$$

for $1 \leq r \leq 2$ *linked to* (54) *and* $0 \leq \beta \leq r$.

Proof. The proof of the estimate (59) can be found in [32, 26, 37], while the proof of the estimate (59) can be found in [41, Lemma 4.2 (i)]. ■

To prove our convergence results, we will also need the following lemma.

Lemma 5.2. *Let X be the mild solution of (4) given in* (6)*, such that* (26) *of Assumption 2.2 is satisfied for $r \in [1,2)$. Let $t_1, t_2 \in [0,T]$, $t_1 < t_2$, assume that $X_0 \in L_2(\mathbb{D}, \mathcal{D}((-A)^{r/2}))$ If X is a $H^1(\Omega)$-valued process and F satisfies the following linear growth condition*

$$\|F(X)\| \leq C\left(1 + \|X\|_{H^1(\Omega)}\right), \tag{61}$$

then

$$\mathbf{E}\|X(t_2) - X(t_1)\|^2 \leq C(t_2 - t_1)^r\left(\mathbf{E}\|X_0\|_r^2 + \sup_{0\leq s\leq T}\mathbf{E}\|X(s)\|^2_{H^1(\Omega)} + 1\right).$$

Proof. See a similar proof in [38, (2.13) of Theorem 2.4]. This proof can easily be updated for part (ii) as we can bound $\|F(X(s))\|$ by (61). ■

5.2. Proof of Theorem 3.1 for scheme SETD1

The proof follows the same basic steps as in [40], however here the discrete semigroup is an exponential. Set

$$\begin{aligned} X(t_m) &= S(t_m)X_0 + \sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m - s)F(X(s))ds + O(t_m) \\ &= \overline{X}(t_m) + O(t_m). \end{aligned}$$

Recall that by construction

$$\begin{aligned} X_m^h &= e^{\Delta t A_h}X_{m-1}^h + \int_0^{\Delta t} e^{(\Delta t - s)A_h}P_hF(X_{m-1}^h)ds + P_hP_N\int_{t_{m-1}}^{t_m} e^{(t_m - s)A}dW(s) \\ &= S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\left(\int_{t_k}^{t_{k+1}} S_h(t_m - s)P_hF(X_k^h)ds\right) + P_hP_NO(t_m) \\ &= Z_m^h + P_hP_NO(t_m), \end{aligned}$$

where

$$\begin{aligned} Z_m^h &= S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\left(\int_{t_k}^{t_{k+1}} S_h(t_m-s)P_hF(X_k^h)ds\right) \\ &= S_h(t_m)P_hX_0 + \sum_{k=0}^{m-1}\left(\int_{t_k}^{t_{k+1}} S_h(t_m-s)P_hF(Z_k^h+P_hP_NO(t_k))ds\right). \end{aligned}$$

We are now estimating $\left(\mathbf{E}\|X(t_m)-X_m^h\|^2\right)^{1/2}$. We obviously have

$$\begin{aligned} X(t_m)-X_m^h &= \overline{X}(t_m)+O(t_m)-X_m^h \\ &= \overline{X}(t_m)+O(t_m)-\left(Z_m^h+P_hP_NO(t_m)\right) \\ &= \left(\overline{X}(t_m)-Z_m^h\right)+\left(P_N(O(t_m))-P_hP_N(O(t_m))\right) \qquad (62)\\ &\quad +\left(O(t_m)-P_N(O(t_m))\right) \\ &= I+II+III. \qquad (63) \end{aligned}$$

Then

$$\left(\mathbf{E}\|X(t_m)-X_m^h\|^2\right)^{1/2} \leq \left(\mathbf{E}\|I\|^2\right)^{1/2}+\left(\mathbf{E}\|II\|^2\right)^{1/2}+\left(\mathbf{E}\|III\|^2\right)^{1/2}.$$

Since the first term requires the most work, we first estimate the other two.

Let us estimate $(\mathbf{E}\|II\|^2)^{1/2}$. Using the property (56) of the projection P_h, the fact that the semigroup and the spectral projection are bounded operators, yields

$$\mathbf{E}\|II\|^2 \leq Ch^{2r}\mathbf{E}\|O(t_m)\|_{H^r(\Omega)}, \quad 1\leq r\leq 2.$$

Using Remark 2.2 and the equivalence $\|.\|_{H^r(\Omega)} \equiv \|(-A)^{r/2}.\|$ in $\mathcal{D}((-A)^{r/2})$ yields

$$\begin{aligned} \mathbf{E}\|II\|^2 &\leq Ch^{2r}\int_0^{t_m}\|(-A)^{r/2}S(t_m-s)Q^{1/2}\|_{HS}^2ds \\ &\leq Ch^{2r}\|(-A)^{(r-1)/2}Q^{1/2}\|_{HS}^2. \end{aligned}$$

For the third term, we have

$$\mathbf{E}\|III\|^2=\mathbf{E}\|(\mathrm{I}-P_N)O(t_m)\|^2=\mathbf{E}\|(\mathrm{I}-P_N)(-A)^{-r/2}(-A)^{r/2}O(t_m)\|^2,$$

and so

$$\mathbf{E}\|III\|^2 \leq \|(\mathrm{I}-P_N)(-A)^{-r/2}\|_{L(L^2(\Omega))}^2\mathbf{E}\|(-A)^{r/2}O(t_m)\|^2 \leq C\left(\inf_{j\in\mathbb{N}^d\setminus\mathcal{I}_N}\lambda_j\right)^{-r}.$$

We now turn our attention to the first term $\mathbf{E}\|I\|^2$. Using the definition of T_h from (58), the first term I can be expanded as

$$
\begin{aligned}
I &= T_h X_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S(t_m - s)F(X(s)) - S_h(t_m - s)P_h F(Z_k^h + P_h P_N O(t_k))ds \\
&= T_h X_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_h(t_m - s)P_h(F(X(t_k)) - F(Z_k^h + P_h P_N O(t_k))))ds \\
&\quad + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S(t_m - s)(F(X(s)) - F(X(t_k)))ds \\
&\quad + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (S(t_m - s) - S_h(t_m - s)P_h)F(X(t_k))ds \\
&= I_1 + I_2 + I_3 + I_4. \qquad (64)
\end{aligned}
$$

Then

$$
\left(\mathbf{E}\|I\|^2\right)^{1/2} \leq \left(\mathbf{E}\|I_1\|^2\right)^{1/2} + \left(\mathbf{E}\|I_2\|^2\right)^{1/2} + \left(\mathbf{E}\|I_3\|^2\right)^{1/2} + \left(\mathbf{E}\|I_4\|^2\right)^{1/2}.
$$

For I_1, from (59) of Lemma 5.1 if $X_0 \in L_2(\mathbb{D}, \mathcal{D}((-A)^{r/2})),\ 1 \leq r \leq 2$, we have

$$
(\mathbf{E}\|I_1\|^2)^{1/2} \leq Ch^r \left(\mathbf{E}\|X_0\|_r^2\right)^{1/2}.
$$

If F satisfies Assumption 2.1 (a), then using the Lipschitz condition, the triangle inequality, the fact that P_h is an bounded operator and S_h satisfies the smoothing property analogous to $S(t)$ independently of h [22], i.e.

$$
\|S_h(t)v\|^2 \leq Ct^{-1/2}\|v\|_{-1} \qquad v \in V_h \quad t > 0,
$$

we have

$$
\begin{aligned}
&(\mathbf{E}\|I_2\|^2)^{1/2} \\
&\leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left(\mathbf{E}\|S_h(t_m - s)P_h\left(F(X(t_k)) - F(Z_k^h + P_h P_N O(t_k))\right)\|^2\right)^{1/2} ds \\
&\leq C\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-1/2} \left(\mathbf{E}\|F(X(t_k)) - F(Z_k^h + P_h P_N O(t_k))\|_{-1}^2\right)^{1/2} ds \\
&\leq C\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-1/2} \left(\mathbf{E}\|X(t_k) - X_k^h\|^2\right)^{1/2} ds.
\end{aligned}
$$

As the estimation of I_3 requires more work, let us first estimate I_4. From Lemma 5.2, more precisely (59) with $\beta = 0$, for $1 \leq r < 2$, $X(t)$ is a $H^1(\Omega)$- valued process, we have

$$
\begin{aligned}
(\mathbf{E}\|I_4\|^2)^{1/2} &\leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left(\mathbf{E}\|T_h(t_m - s)F(X(t_k))\|^2\right)^{1/2} ds \\
&\leq Ch^r \left(\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-r/2} ds\right) \left(\sup_{0 \leq s \leq T} \mathbf{E}\|F(X(s))\|^2\right)^{1/2} \\
&\leq Ch^r \left(1 + \left(\sup_{0 \leq s \leq T} \mathbf{E}\|X(s)\|^2_{H^1(\Omega)}\right)^{1/2}\right) \\
&\leq Ch^r.
\end{aligned}
$$

For $r = 2$, we have

$$
(\mathbf{E}\|I_4\|^2)^{1/2} \leq Ch^{2-\epsilon},
$$

where $\epsilon > 0$ small enough.

With only the Lipschitz condition in Assumption 2.1(a) and Lemma 5.2, the estimation of I_3 is given by

$$
\begin{aligned}
(\mathbf{E}\|I_3\|^2)^{1/2} &\leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (\mathbf{E}\|S(t_m - s)\left(F(X(s)) - F(X(t_k))\|^2\right)^{1/2}) ds \\
&\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-1/2} \left(\mathbf{E}\|F(X(s)) - F(X(t_k))\|_{-1}\right)^{1/2} ds \\
&\leq C \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-1/2} (\mathbf{E}\|X(s) - X(t_k)\|^2)^{1/2} ds.
\end{aligned}
$$

Since

$$
\sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (t_m - s)^{-1/2} ds \leq 2\sqrt{T},
$$

then if $X_0 \in L_2(\mathbb{D}, \mathcal{D}((-A)^{r/2}))$, as $X(t)$ is a $H^1(\Omega)$- valued process

$$
(\mathbf{E}\|I_3\|^2)^{1/2} \leq C\Delta t^{\frac{r}{2}} \left(\mathbf{E}\|X_0\|^2_r + \sup_{0 \leq s \leq T} \mathbf{E}\|X(s)\|^2_{H^1(\Omega)} + 1\right)^{1/2}.
$$

To obtain a higher rate, Assumption 2.1(b) is needed. If Assumption 2.1(b) is satisfied, we follow closely the proof in [40, Theorem 4.1 (I_{11})], but with our new assumption.

$$
\begin{aligned}
(\mathbf{E}\|I_3\|^2)^{1/2} &\leq \left(\mathbf{E}\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)F'(X(t_k))(S(t_m-t_k)-I)X(t_k))ds\|^2\right)^{1/2}\\
&+ \left(\mathbf{E}\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)F'(X(t_k))\int_{t_k}^{s} S(s-\sigma)F(X(\sigma))d\sigma ds\|^2\right)^{1/2}\\
&+ \left(\mathbf{E}\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)F'(X(t_k))\int_{t_k}^{s} S(s-\sigma)dW(\sigma)ds\|^2\right)^{1/2}\\
&+ \left(\mathbf{E}\|\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} S(t_m-s)R(X(t_k),X(s))ds\|^2\right)^{1/2}\\
&=: I_3^{(1)}+I_3^{(2)}+I_3^{(3)}+I_3^{(4)},
\end{aligned}
$$

where

$$
R(X(t_k),X(s)) := \int_0^1 F''(X(t_k)+\lambda(X(s)-X(t_k)))(X(s)-X(t_k),X(s)-X(t_k))(1-\lambda)d\lambda.
$$

The estimation of $I_3^{(4)}$ is the same as the one in [40, Proof of Theorem 4.1, $I_{11}^{(4)}$]. For the estimation of $I_4^{(1)}$, using the fact that Assumption 2.1(b) is satisfied, Proposition 2.1 and the regularity of the solution, we have

$$
\begin{aligned}
I_3^{(1)} &\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}} t_m-s)^{-1/2}\|S(t_m-t_k)-I)(-A)^{-\frac{r}{2}}\|_{L(H)}\left(\mathbf{E}\|(-A)^{\frac{r}{2}}X(t_k))\|^2\right)^{1/2}ds\\
&\leq C\Delta t^{r/2}\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}(t_m-s)^{-1/2}ds.\\
&\leq C\Delta t^{r/2}.
\end{aligned}
$$

Again, using Assumption 2.1(b), Proposition 2.1, the regularity of the solution, the linear growth (61) yields

$$
\begin{aligned}
I_3^{(2)} &\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}(t_m-s)^{-1/2}\left(\mathbf{E}\|F(X(\sigma))\|^2\right)^{1/2}d\sigma ds\\
&\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}(t_m-s)^{-1/2}\left(1+\sup_{0\leq\sigma\leq T}\mathbf{E}\|(X(\sigma))\|_1^2\right)^{1/2}d\sigma ds\\
&\leq C\sum_{k=0}^{m-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}(t_m-s)^{-1/2}d\sigma ds\\
&\leq C\Delta t.
\end{aligned}
$$

The estimation of $I_3^{(3)}$ follows the one in [40, Proof of Theorem 4.1, $I_{11}^{(3)}$] but with Assumption 2.1(b). Indeed using Burkholder-Davis-Gundy-type inequality [40, Lemma 4.2] gives

$$I_3^{(3)} = \left(\mathbf{E}\|\sum_{k=0}^{m-1} Z_k\|^2\right)^{1/2} \leq C\left(\sum_{k=0}^{m-1}\mathbf{E}\|Z_k\|^2\right)^{1/2}, \tag{65}$$

where

$$Z_k = \int_{t_k}^{t_{k+1}} S(t_m - s)F'(X(t_k))\int_{t_k}^{s} S(s-\sigma)dW(\sigma).$$

As in [40], using Assumption 2.1(b) and Assumption 2.2, we have

$$\begin{aligned}
\mathbf{E}\|Z_k\|^2 &\leq \Delta t\int_{t_k}^{t_{k+1}}\mathbf{E}\|\int_{t_k}^{s} S(t_m-s)F'(X(t_k))S(s-\sigma))dW(\sigma)\|^2 ds \\
&\leq \Delta t\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}\mathbf{E}\|S(t_m-s)F'(X(t_k))S(s-\sigma))\|^2_{L_2^0} d\sigma ds \\
&\leq C\Delta t\int_{t_k}^{t_{k+1}}(t_m-s)^{-1/2}(s-t_k)^{\min(1,r)}ds \\
&\leq C\Delta t^{\min(2,r+1)}\int_{t_k}^{t_{k+1}}(t_m-s)^{-1/2}ds.
\end{aligned} \tag{66}$$

Using (66) in (65) yields

$$I_3^{(3)} \leq C\Delta t^{\min(1,(r+1)/2)}. \tag{67}$$

Then we find

$$I_3 \leq C\Delta t^{r/2}. \tag{68}$$

As we can observe, to improve the estimation of I_3, we need to improve the estimation of $I_3^{(1)}$. If Assumption 2.1(b) and (40) are satisfied, the estimation of $I_3^{(1)}$ is done as in [38, Proof of Theorem 3.1, I_{31}] and we have

$$I_{31} \leq \Delta t^{\min(1,r)}. \tag{69}$$

Combining our estimates $(\mathbf{E}\|I\|^2)^{1/2}, (\mathbf{E}\|II\|^2)^{1/2}$ and $(\mathbf{E}\|III\|^2)^{1/2}$ and using the discrete Gronwall lemma concludes the proof.

5.3. Proof of Theorem 3.1 for SETD0 scheme

Recall that

$$\begin{aligned}
Y_m^h &= e^{\Delta t A_h}\left(Y_{m-1}^h + \Delta t P_h F(Y_{m-1}^h)\right) + P_h P_N \int_{t_{m-1}}^{t_m} e^{(t_m-s)A} dW(s) \\
&= S_h(t_m) P_h X_0 + \sum_{k=0}^{m-1} \left(\int_{t_k}^{t_{k+1}} S_h(t_m - t_k) P_h F(Y_k^h) ds \right. \\
&\quad \left. + P_h P_N \int_{t_k}^{t_{k+1}} S(t_m - s) dW(s) \right) \\
&= S_h(t_m) P_h X_0 + \sum_{k=0}^{m-1} \left(\int_{t_k}^{t_{k+1}} S_h(t_m - t_k) P_h F(Y_k^h) ds \right) + P_h P_N O(t_m) \\
&= Z_m^h + P_h P_N O(t_m).
\end{aligned}$$

As in the proof of SETD1 scheme, we obviously have

$$\begin{aligned}
&X(t_m) - Y_m^h \\
&\quad = \overline{X}(t_m) + O(t_m) - Y_m^h \\
&\quad = \overline{X}(t_m) + O(t_m) - \left(Z_m^h + P_h P_N O(t_m)\right) \\
&\quad = \left(\overline{X}(t_m) - Z_m^h\right) + \left(P_N(O(t_m)) - P_h P_N(O(t_m))\right) + \left(O(t_m) - P_N(O(t_m))\right) \\
&\quad = I + II + III.
\end{aligned}$$

The estimations of $(\mathbf{E}\|II\|^2)^{1/2}$ and $(\mathbf{E}\|III\|^2)^{1/2}$ can be found in the analysis of the SETD1 scheme. We also have

$$\begin{aligned}
I &= T_h X_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S(t_m - s) F(X(s)) - S_h(t_m - t_k) P_h F(Z_k^h + P_h P_N O(t_k)) ds \\
&= T_h X_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S_h(t_m - t_k) P_h (F(X(t_k)) - F(Z_k^h + P_h P_N O(t_k)))) ds \\
&\quad + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} S(t_m - s)(F(X(s)) - F(X(t_k))) ds \\
&\quad + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} (S(t_m - s) - S_h(t_m - t_k) P_h) F(X(t_k)) ds \\
&= I_1 + I_2 + I_3 + I_4. \qquad (70)
\end{aligned}$$

The estimation of $(\mathbf{E}\|I\|^2)^{1/2}$ is therefore performed as for the SETD1 scheme, but the estimation of $(\mathbf{E}\|I_4\|^2)^{1/2}$ is closer to [40, Proof of Theorem 4.1, I_{12}]. Due to the nature of the nonlinear function F in Assumption 2.1(b), we should not follow [40] for Theorem 3.1 in the estimation of I_4 as an extra term I_5 arises,

$$I_5 = \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \left(S(t_m - s) - S(t_m - t_k)\right) F(X(t_k)) ds.$$

As in [25] that extra term can be estimated by

$$\left(\mathbf{E}\|I_5\|^2\right)^{1/2} \leq C(\Delta t + \Delta t |\ln(\Delta t)|).$$

6. Conclusion

In this work, we have considered the numerical approximation of general second order semi linear parabolic stochastic partial differential equations (SPDEs) driven by additive space-time noise and have designed two novel schemes for finite element method, finite volume method and finite difference method using linear functionals of the noise and the exponential time stepping methods. We have provided rigorous convergence proofs for a new family of Lipschitz nonlinear functions and obtained high orders of convergence. Numerical simulations to sustain our theoretical results are provided. Those numerical simulations cover realistic flow problems in porous media and also reveal that our theoretical results can be extended to larger family of nonlinear functions. This will be our interest for future work.

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