



Heriot-Watt University  
Research Gateway

## Double Field Theory for the A/B-Models and Topological S-Duality in Generalized Geometry

### Citation for published version:

Kökényesi, Z, Sinkovics, A & Szabo, RJ 2018, 'Double Field Theory for the A/B-Models and Topological S-Duality in Generalized Geometry', *Fortschritte der Physik*. <https://doi.org/10.1002/prop.201800069>

### Digital Object Identifier (DOI):

[10.1002/prop.201800069](https://doi.org/10.1002/prop.201800069)

### Link:

[Link to publication record in Heriot-Watt Research Portal](#)

### Document Version:

Peer reviewed version

### Published In:

Fortschritte der Physik

### Publisher Rights Statement:

This is the peer reviewed version of the following article: Z. Kökényesi, A. Sinkovics, R. J. Szabo, *Fortschr. Phys.* 2018, 1800069, which has been published in final form at <https://doi.org/10.1002/prop.201800069>. This article may be used for non-commercial purposes in accordance with Wiley Terms and Conditions for Use of Self-Archived Versions.

### General rights

Copyright for the publications made accessible via Heriot-Watt Research Portal is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

### Take down policy

Heriot-Watt University has made every reasonable effort to ensure that the content in Heriot-Watt Research Portal complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [open.access@hw.ac.uk](mailto:open.access@hw.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.

# Double field theory for the A/B-models and topological S-duality in generalized geometry

Zoltán Kökenyesi<sup>(a),(b)</sup>, Annamária Sinkovics<sup>(b)</sup> and Richard J. Szabo<sup>(c)</sup>

<sup>(a)</sup> *MTA Lendület Holographic QFT Group  
Wigner Research Center for Physics*

*Konkoly-Thege Miklós út 29-33, 1121 Budapest, Hungary*  
Email: [kokenyesi.zoltan@wigner.mta.hu](mailto:kokenyesi.zoltan@wigner.mta.hu)

<sup>(b)</sup> *Institute of Theoretical Physics  
MTA-ELTE Theoretical Research Group  
Eötvös Loránd University*

*Pázmány s. 1/A, 1117 Budapest, Hungary*  
Email: [sinkovics@general.elte.hu](mailto:sinkovics@general.elte.hu)

<sup>(c)</sup> *Department of Mathematics  
Heriot-Watt University*

*Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, UK  
Maxwell Institute for Mathematical Sciences, Edinburgh, UK  
The Higgs Centre for Theoretical Physics, Edinburgh, UK*  
Email: [R.J.Szabo@hw.ac.uk](mailto:R.J.Szabo@hw.ac.uk)

## Abstract

We study AKSZ-type BV constructions for the topological A- and B-models within a double field theory formulation that incorporates backgrounds with geometric and non-geometric fluxes. We relate them to a Courant sigma-model, on an open membrane, corresponding to a generalized complex structure, which reduces to the A- or B-models on the boundary. We introduce S-duality at the level of the membrane sigma-model based on the generalized complex structure, which exchanges the related AKSZ field theories, and interpret it as topological S-duality of the A- and B-models. Our approach leads to new classes of Courant algebroids associated to (generalized) complex geometry.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>AKSZ constructions and dimensional reductions</b>	<b>4</b>
2.1	AKSZ sigma-models . . . . .	4
2.2	Dimensional reduction by gauge fixing . . . . .	6
2.3	Dimensional reduction by effective actions . . . . .	8
<b>3</b>	<b>String sigma-models</b>	<b>9</b>
3.1	Poisson sigma-model . . . . .	9
3.2	AKSZ formulations of the A-model . . . . .	10
3.3	AKSZ formulations of the B-model . . . . .	12
<b>4</b>	<b>Courant sigma-models</b>	<b>13</b>
4.1	Courant algebroids . . . . .	13
4.2	Standard Courant sigma-model . . . . .	15
4.3	Contravariant Courant sigma-model . . . . .	17
<b>5</b>	<b>DFT membrane sigma-models</b>	<b>19</b>
5.1	DFT algebroids . . . . .	19
5.2	AKSZ construction of DFT membrane sigma-models . . . . .	21
<b>6</b>	<b>Double field theory for the A- and B-models</b>	<b>22</b>
6.1	Doubled Poisson sigma-model . . . . .	22
6.2	Large contravariant Courant sigma-model . . . . .	24
6.3	Courant sigma-model for generalized complex geometry . . . . .	26
6.4	Dimensional reductions to the A- and B-models . . . . .	28
<b>7</b>	<b>Topological S-duality in generalized complex geometry</b>	<b>30</b>
7.1	Duality between Poisson and complex structure Courant algebroids . . . . .	30
7.2	Topological S-duality . . . . .	31
<b>8</b>	<b>Conclusions and outlook</b>	<b>31</b>
<b>A</b>	<b>Differential calculus of graded functionals</b>	<b>32</b>

## 1 Introduction

The topological A- and B-models were introduced originally by Witten [1–3] as twists of two-dimensional  $\mathcal{N} = 2$  supersymmetric sigma-models, and were subsequently used to construct topological string theory [4] by coupling them to two-dimensional topological gravity through integration over the moduli space of the target Calabi-Yau manifold. In this paper we reformulate the AKSZ constructions of the A- and B-models in the framework of double field theory as a single membrane sigma-model, and study its relation to generalized complex geometry and S-duality.

AKSZ formulations provide natural geometric methods for constructing BV quantized sigma-models [5–8], and they produce examples of topological field theories of Schwarz-type in arbitrary dimensionality such as the Poisson sigma-model, Chern-Simons theory and BF-theory; special gauge fixing action functionals also yield examples of topological field theories of Witten-type, such as the A- and B-models. One motivation for studying them comes from flux compactifications of type II string theory, where fluxes appear as twists of the Courant algebroid on the T-duality inspired generalized tangent bundle [9–12]. Courant algebroids are in one-to-one correspondence to three-dimensional AKSZ sigma-models with target QP-manifold of degree 2, called Courant sigma-models [13–19], which geometrize fluxes in the sense that they are uplifts of string sigma-models to one higher dimension that can accommodate the fluxes [7, 8, 20–22] (see [23] for a review). A more natural framework for describing fluxes is double field theory (DFT) [24–26], where the original coordinates conjugate to closed string momentum modes are extended with dual coordinates conjugate to winding modes, and the continuous version of T-duality becomes a manifest symmetry of the theory (see [27–29] for reviews). The algebroid structure of double field theory was studied in [12, 30–36]; in particular, the notion of DFT algebroid was introduced in [35] whose derived bracket is the C-bracket of double field theory, and whose corresponding membrane sigma-model naturally captures the T-duality orbit of geometric and non-geometric flux backgrounds in a single unified description.

The topological A- and B-model string theories in backgrounds with  $H$ -flux are captured by generalized complex geometry [37–39]. They have been extensively studied by introducing AKSZ string and open membrane sigma-models with generalized complex structures, which reduce upon gauge fixing to the A- and B-models [40–44]. The novelty of our approach is that their AKSZ sigma-models are reformulated within double field theory on both the string and membrane levels, which gives a more natural explanation of how their AKSZ constructions are related to generalized complex structures. It also highlights some new aspects, such as how topological S-duality appears on the level of AKSZ sigma-models and can be traced back to generalized complex geometry.

Based on our observation that the Poisson sigma-model on doubled spaces captures both the A- and B-models with different choices of the doubled Poisson structure, we propose an open AKSZ membrane sigma-model, inspired by the approach of [45] to T-duality between geometric and non-geometric fluxes, which gives back the doubled Poisson sigma-model on the boundary in a specific gauge. Then we reduce the fields in a way which can be interpreted as the same reduction performed in [35], where it was called a DFT projection. The resulting AKSZ membrane sigma-model captures a particular class of generalized complex structures given by an initial Poisson and complex structure. It therefore corresponds to a Courant algebroid for the generalized complex structure with the identities of its integrability condition; to the best of our knowledge this is a new example of Courant algebroid. We also show that the AKSZ membrane theory can be reduced through gauge fixing to the A- or B-models on the boundary if the generalized complex structure is given by a purely Poisson or complex structure respectively. Furthermore, we find a realization of topological S-duality, which exchanges the weakly and strongly coupled sectors of the topological A- and B-model string theories [46], on the level of the AKSZ construction. Our result is based on an S-duality which maps Poisson and complex structure Courant algebroids into each other, and

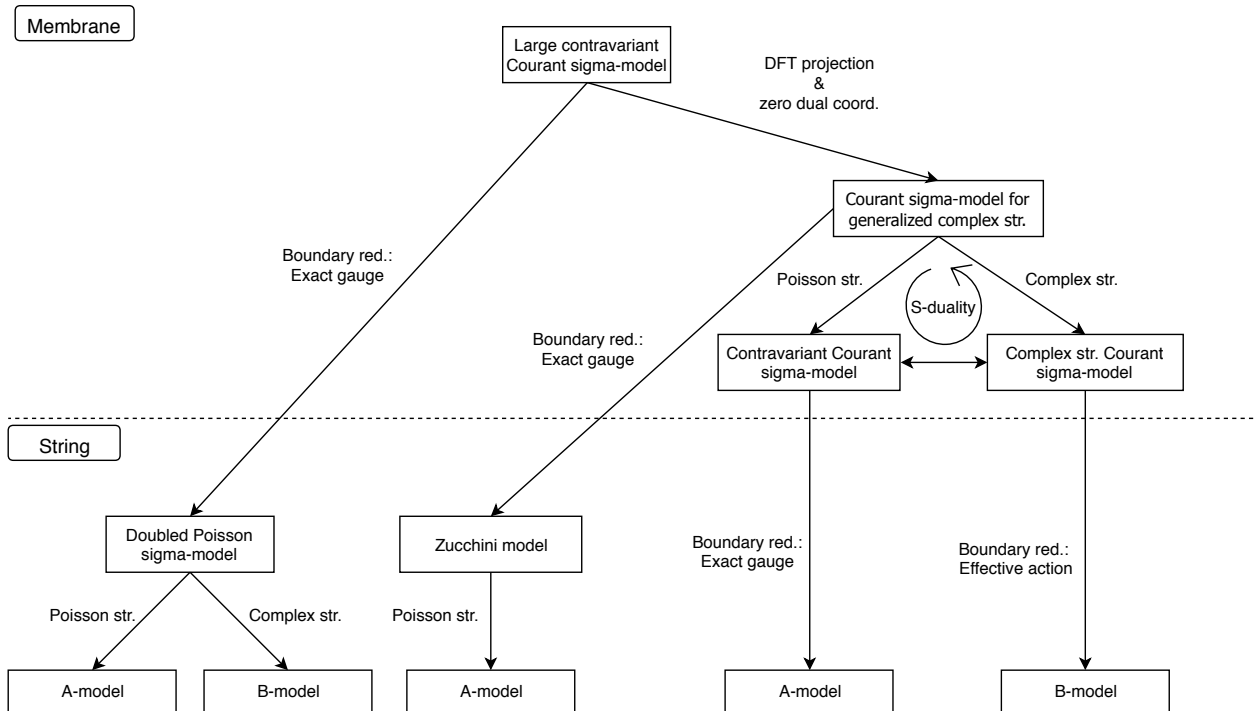


Figure 1: Schematic presentation of the different reductions and connections between the AKSZ string and membrane sigma-models related to the topological A- and B-models.

lies within the Courant algebroid for the generalized complex structure. This duality is promoted to the AKSZ membrane sigma-model and interpreted as S-duality which relates the couplings  $g_A$  and  $g_B$  of the A- and B-models in the usual way:  $g_A = 1/g_B$ .

In Figure 1 we summarize the relations between the different AKSZ string and membrane sigma-models appearing in this paper. In the following we also discuss some additional new results: We introduce a gauge fixing method which is a natural way to perform boundary reductions, and we show that the contravariant Courant sigma-model of [45] reduces to the Poisson sigma-model in this gauge; we also relate its twisting by (geometric)  $R$ -flux to the membrane sigma-model introduced in [21] which describes the nonassociative deformations of (locally non-geometric)  $R$ -flux backgrounds. We further show that the standard Courant sigma-model [8] twisted by  $H$ -flux is related to the B-model in a particular gauge.

This paper is organized as follows. In §2 we give a relatively detailed overview of the AKSZ construction, together with two techniques for dimensional reduction: the first method is new and is specific to boundary reductions, whereas the second method is based on the more general reductions through effective actions discussed in [43, 47]. In §3 we survey some aspects of string sigma-models constructed by the AKSZ method, and in particular the various AKSZ formulations of the A- and B-models. In §4 we review Courant sigma-models together with two relevant examples, the standard and the contravariant Courant sigma-model, and we describe their boundary reductions. In §5 we recall the DFT projection which we use in later sections to derive the Courant sigma-model for generalized complex geometry. In §6 we describe how to reformulate the A- and B-models within double field theory, and apply the DFT projection to show how it reduces to the A- and B-models on the boundary. In §7 we introduce the duality between the Poisson and complex structure Courant algebroids, and connect it to topological S-duality. In §8 we close with some concluding remarks, and directions for further study of our geometrical structures and constructions, while in Appendix A we summarize some key formulas from the differential calculus of graded functionals

relevant for the AKSZ construction which we have not found in the literature.

## 2 AKSZ constructions and dimensional reductions

The AKSZ construction is a BV quantized sigma-model formulation which produces a geometric solution to the classical master equation, called the AKSZ action. In this section we survey some pertinent background about the AKSZ construction and BV quantization; a more complete review can be found in [8]. We then introduce a new boundary reduction method using a BV gauge fixing procedure, and review another known dimensional reduction technique based on effective actions by integration over one coordinate direction which is not specific to boundary reductions.

### 2.1 AKSZ sigma-models

The basic ingredients of AKSZ theory consists of two classes of supermanifolds. The ‘source’  $(\mathcal{W}, Q_{\mathcal{W}}, \mu)$  consists of a differential graded (dg-)manifold, which is a graded manifold  $\mathcal{W}$  equipped with a cohomological vector field  $Q_{\mathcal{W}}$ , i.e.  $Q_{\mathcal{W}}$  is of degree 1 and its Lie derivative  $\mathcal{L}_{Q_{\mathcal{W}}}$  squares to zero, and a measure  $\mu$  which is invariant under  $Q_{\mathcal{W}}$ . In this paper we take  $\mathcal{W} = T[1]\Sigma_d$ , the tangent bundle of a  $d$ -dimensional oriented worldvolume manifold  $\Sigma_d$  with the degree of its fibers shifted by 1, which is isomorphic to the exterior algebra of differential forms  $(\Omega(\Sigma_d), \wedge)$ . We choose the cohomological vector field  $Q_{\mathcal{W}}$  corresponding to the de Rham differential, which in local affine coordinates  $\hat{z}^{\hat{\mu}} = (\sigma^{\mu}, \theta^{\mu}) \in T[1]\Sigma_d$ , with degree 0 coordinates  $\sigma^{\mu}$  on  $\Sigma_d$  and degree 1 fiber coordinates  $\theta^{\mu}$ , has the form  $Q_{\mathcal{W}} = \theta^{\mu} \frac{\partial}{\partial \sigma^{\mu}} =: \mathbf{D}$ , where repeated upper and lower indices are always implicitly understood to be summed over. The measure in local coordinates can be written in the form  $\mu = d^d \hat{z} := d^d \sigma d^d \theta$ .

The ‘target’  $(\mathcal{M}, Q_{\gamma}, \omega)$  is a symplectic dg-manifold, which is a graded manifold  $\mathcal{M}$  with a cohomological vector field  $Q_{\gamma}$ , and a graded symplectic form  $\omega$  for which  $Q_{\gamma}$  is a Hamiltonian vector field:  $\iota_{Q_{\gamma}} \omega = d\gamma$  for some Hamiltonian function  $\gamma$  on  $\mathcal{M}$ , where  $\iota_Q$  denotes contraction of a differential form along the vector field  $Q$ . In order to reproduce the BV formalism, the symplectic structure  $\omega$  is taken to be of degree  $d+1$ , so that the Hamiltonian function  $\gamma$  is of degree  $d$ . A common choice of target for the AKSZ construction is to take  $\mathcal{M}$  to be an N-manifold, which is a graded manifold with no coordinates of negative degree. In this case the triple  $(\mathcal{M}, Q_{\gamma}, \omega)$  is called a QP-manifold of degree  $n = d-1$ , or QP $n$ -manifold for short. They typically arise from  $n$ -graded vector bundles over the degree 0 body  $M = \mathcal{M}_0$  of  $\mathcal{M}$  [8], and in particular functions of degree  $n-1$  can be identified with sections of a vector bundle  $E \rightarrow M$  equipped with the structure of a Leibniz algebroid. In subsequent sections we describe the AKSZ topological field theories associated with the first two non-trivial members in the hierarchy of QP-structures on the target manifold for dimensions  $d = 2, 3$ , in the context of the string and membrane sigma-models of interest in this paper.

The AKSZ space of fields is the mapping space

$$\mathcal{M} = \text{Map}(T[1]\Sigma_d, \mathcal{M}) \quad (2.1)$$

consisting of smooth maps from  $(T[1]\Sigma_d, \mathbf{D}, \mu)$  to  $(\mathcal{M}, Q_{\gamma}, \omega)$ , which we refer to as superfields in the following. We can introduce local coordinates on  $\mathcal{M}$  via the superfields

$$\hat{X}^i(\hat{z}^{\hat{\mu}}) = \phi^*(\hat{X}^i)(\hat{z}^{\hat{\mu}}), \quad (2.2)$$

for local coordinates  $\hat{z}^{\hat{\mu}} \in \mathcal{W}$ ,  $\hat{X}^i \in \mathcal{M}$  and  $\phi \in \mathcal{M}$ . The cohomological vector fields  $Q_{\mathcal{W}} = \mathbf{D}$  and  $Q_{\gamma}$  induce a cohomological vector field  $\mathbf{Q}$  on  $\mathcal{M}$  in the following way. For  $\phi \in \mathcal{M}$  and  $\hat{z} \in \mathcal{W}$ , use local coordinates to define

$$\mathbf{Q}_0 = \int_{T[1]\Sigma_d} d^d \hat{z} \mathbf{D} \hat{X}^i(\hat{z}) \frac{\delta}{\delta \hat{X}^i(\hat{z})} \quad \text{and} \quad \mathbf{Q}_{\gamma} = \int_{T[1]\Sigma_d} d^d \hat{z} Q_{\gamma}^i(\hat{X}(\hat{z})) \frac{\delta}{\delta \hat{X}^i(\hat{z})}, \quad (2.3)$$

where  $Q_\gamma^i(\hat{X}) \partial/\partial \hat{X}^i$  is the local form of the vector field  $Q_\gamma$  on  $\mathcal{M}$ ; relevant definitions and formulas in differential calculus on mapping superspaces are summarized in Appendix A. Then  $\mathcal{M}$  is a dg-manifold with the cohomological vector field

$$Q = Q_0 + Q_\gamma . \quad (2.4)$$

We note that  $Q_0$  has ghost number<sup>1</sup>  $|D| - d = 1 - d$  and  $Q_\gamma$  has ghost number  $|Q_\gamma| - d = 1 - d$  as well, where  $|Q_\gamma|$  denotes the degree of  $Q_\gamma$ . If a vector field on  $\mathcal{M}$  acts as a derivative, its ghost number is shifted by  $d - 1$ , because a vector field based with coordinate  $\hat{X}^i(\hat{z})$  has ghost number  $|\hat{X}^i| - 1$ , but a functional derivative with respect to  $\hat{X}^i(\hat{z})$  has ghost number  $|\hat{X}^i| + d$ .

Given an  $n$ -form  $\alpha \in \Omega^n(\mathcal{M})$ , we can lift it to an  $n$ -form  $\alpha \in \Omega^n(\mathcal{M})$  by transgression to the mapping space as

$$\alpha = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ ev}^*(\alpha) , \quad (2.5)$$

where  $\text{ev} : T[1]\Sigma_d \times \mathcal{M} \rightarrow \mathcal{M}$  is the evaluation map. As we see,  $\alpha$  is an  $n$ -form functional of the fields in  $\mathcal{M}$ , and due to the integration  $\alpha$  has ghost number  $|\alpha| - d$ , where  $|\alpha|$  denotes the total degree of  $\alpha$  (i.e. the form degree coming from the grading of  $\delta$  plus the degree of the graded coordinates). In particular, since transgression is a chain map, from the degree  $d + 1$  symplectic form  $\omega$  on  $\mathcal{M}$  and a Liouville potential  $\vartheta$ , such that  $\omega = d\vartheta$ , we get the symplectic form  $\omega$  of ghost number 1 and Liouville potential  $\vartheta$  on  $\mathcal{M}$ , such that  $\omega = \delta\vartheta$ . Furthermore, the cohomological vector field  $Q$  on  $\mathcal{M}$  is also Hamiltonian with Hamiltonian function  $-\iota_{Q_0}\vartheta + \gamma$  of degree 0:  $\iota_{Q_\gamma}\omega = \delta\gamma$ , where  $\iota_{Q_0}$  and  $\iota_{Q_\gamma}$  have ghost number 0, while  $\gamma$  has ghost number  $|\gamma| - d = 0$ . In other words, the mapping space of superfields  $\mathcal{M}$  is itself a symplectic dg-manifold.

The BV bracket  $(\cdot, \cdot)_{\text{BV}}$  is the graded Poisson bracket of ghost number 1 on  $\mathcal{M}$  defined from  $\omega$ , and it corresponds to the graded Poisson bracket  $\{\cdot, \cdot\}$  of degree  $-d + 1$  on  $\mathcal{M}$  defined from  $\omega$ , since the transgression map  $\int_{T[1]\Sigma_d} d^d \hat{z} \text{ ev}^*$  is a Lie algebra homomorphism from  $(\mathcal{M}, \{\cdot, \cdot\})$  to  $(\mathcal{M}, (\cdot, \cdot)_{\text{BV}})$ . The cohomological vector fields as derivatives can be represented through derived brackets as

$$Q_\gamma = \{\gamma, \cdot\} \quad \text{and} \quad \vec{Q} = (\mathcal{S}, \cdot)_{\text{BV}} , \quad (2.6)$$

where the Hamiltonian  $\mathcal{S}$  on  $\mathcal{M}$  is defined to be the AKSZ action, which is the desired BV action. To explicitly specify it, we choose a Liouville potential  $\vartheta$  on  $\mathcal{M}$ , and consider its zero locus  $\mathcal{L}$  which is a Lagrangian submanifold of  $\mathcal{M}$ . We pick a submanifold  $\mathcal{L}' \subseteq \mathcal{L}$  and restrict the space of fields  $\mathcal{M}$  to the subspace  $\mathcal{M}_{\mathcal{L}'} \subset \mathcal{M}$  consisting of maps that send the boundary  $\partial\mathcal{W} = T[1]\partial\Sigma_d$  into  $\mathcal{L}'$ . This assigns boundary conditions on our fields, and now we can write the degree 0 AKSZ action  $\mathcal{S}$  on  $\mathcal{M}_{\mathcal{L}'}$  in the form

$$\mathcal{S} = \mathcal{S}_0 + \gamma , \quad (2.7)$$

where

$$\mathcal{S}_0 = -\iota_{Q_0}\vartheta \quad (2.8)$$

is the kinetic term and the Hamiltonian function

$$\gamma = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ ev}^*(\gamma) \quad (2.9)$$

is the interaction term. The cocycle conditions  $Q_\gamma^2 = 0$  and  $\vec{Q}^2 = 0$  are equivalent to  $\{\gamma, \gamma\} = 0$  and  $(\gamma, \gamma)_{\text{BV}} = 0$ , hence the AKSZ action is a solution of the classical master equation  $(\mathcal{S}, \mathcal{S})_{\text{BV}} = 0$ . In the BV formalism, the cohomological vector field  $\vec{Q}$  corresponds to the BRST charge.

---

<sup>1</sup>We use the terminology ‘ghost number’ for the degree of a superfield  $\phi$  in  $\mathcal{M}$ .

A canonical transformation is associated to a degree  $d - 1$  function  $\alpha$  on  $\mathcal{M}$ . We use the notation  $\delta_\alpha$  for the corresponding Hamiltonian vector field, and  $e^{\delta_\alpha}$  and  $e^{\delta_\alpha}$  for the respective canonical transformations. The action of the canonical transformation on  $\gamma$  is given by  $e^{\delta_\alpha}\gamma = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(e^{\delta_\alpha}\gamma)$ , which preserves the classical master equation as

$$\{e^{\delta_\alpha}\gamma, e^{\delta_\alpha}\gamma\} = e^{\delta_\alpha}\{\gamma, \gamma\} = 0, \quad (2.10)$$

due to  $\{\gamma, \gamma\} = 0$ . If  $\alpha|_{\mathcal{L}'} = 0$ , then the AKSZ action  $\mathcal{S}_0 + \gamma$  is equivalent to  $\mathcal{S}_0 + e^{\delta_\alpha}\gamma$  up to a canonical transformation. The canonical transformation  $e^{\delta_\alpha}$  is an example of a duality transformation, which in the AKSZ formalism is defined to be a symplectomorphism, i.e. a diffeomorphism between underlying symplectic manifolds which preserves the symplectic structures.

We can then introduce a boundary term in the AKSZ action using the ingredients of a canonical transformation. Let  $\beta$  be a degree  $d-1$  function on  $\mathcal{M}$  as before, and further assume that  $\{\beta, \beta\} = 0$  and  $e^{\delta_\beta}\gamma|_{\mathcal{L}'} = 0$ . Then the AKSZ action  $\mathcal{S}_0 + \gamma$  on  $\mathcal{M}_{\mathcal{L}'}$  is equivalent to the AKSZ action

$$\mathcal{S} = \mathcal{S}_0 + \gamma - \oint_{T[1]\partial\Sigma_d} d^{d-1} \hat{z} \text{ev}^*(\beta) \quad (2.11)$$

on  $\mathcal{M}_{\mathcal{L}'_\beta}$ , which is given by shifting the Liouville potential  $\vartheta$  to  $\vartheta - d\beta$  with  $\mathcal{L}'_\beta = e^{\delta_\beta}\mathcal{L}'$  the new zero locus of the shifted Liouville potential.

## 2.2 Dimensional reduction by gauge fixing

In AKSZ constructions the fields and antifields are not distinguished from the onset. The theory is specified once the antifields are assigned in the entire field content, and different choices yield different field theories. In [48] we also studied the gauge fixings of AKSZ theories from a different perspective.

In the usual BV quantized theories, the fields and antifields are distinguished from the start, with the former including the physical and ghost fields from the BRST picture, while the latter define canonically conjugate variables with respect to the symplectic phase space structure on the space of all fields  $\mathcal{M}$ . Gauge fixing is then equivalent to a choice of a Lagrangian submanifold  $\mathcal{L}$  of  $\mathcal{M}$ . The Batalin-Vilkovisky theorem [49] ensures that the path integral over  $\mathcal{L}$  is independent of the choice of representative for the homology class of the Lagrangian submanifold  $\mathcal{L}$ . The Lagrangian submanifold intersects the gauge orbits transversally, i.e. the action of the BV-BRST charge  $(\mathcal{S}, \cdot)_{\text{BV}}$  vanishes on Lagrangian submanifolds, as the BV bracket acts as zero there. Thus the BV gauge symmetry is completely fixed on Lagrangian submanifolds.

In the following we introduce a particular gauge fixing as a dimensional reduction technique which reduces a given AKSZ theory on the superworldvolume  $\mathcal{W} = T[1]\Sigma_d$  to an AKSZ theory on its boundary  $\partial\mathcal{W} = T[1]\partial\Sigma_d$ . We consider the case when the fields (but not the antifields) occur in even number and can be paired: a superfield  $\phi^a(\hat{z})$  with ghost degree  $|a|$  is paired with another superfield  $\chi_a(\hat{z})$  with ghost degree  $d - 2 - |a|$ , and vice versa. The BV symplectic form is written in the canonical form

$$\omega = \int_{T[1]\Sigma_d} d^d \hat{z} \left( (-1)^{d+|a|} \delta\phi_a^+(\hat{z}) \delta\phi^a(\hat{z}) + (-1)^{|a|} \delta\chi^{a+}(\hat{z}) \delta\chi_a(\hat{z}) \right), \quad (2.12)$$

where we chose a convenient ordering of antifields  $\phi_a^+$ ,  $\chi^{a+}$  and fields  $\phi^a$ ,  $\chi_a$  in this way. The ghost degrees of the antifields  $\phi_a^+$  and  $\chi^{a+}$  are  $d - 1 - |a|$  and  $|a| + 1$  respectively. An arbitrary superfield  $\phi^a$  can be expanded in terms of the degree 1 fiber coordinates  $\theta^\mu$  of  $\mathcal{W} = T[1]\Sigma_d$  in the form

$$\phi^a(\hat{z}) = \phi^{(0)a}(\sigma) + \sum_{p=1}^d \frac{1}{p!} \phi_{\mu_1 \dots \mu_p}^{(p)a}(\sigma) \theta^{\mu_1} \dots \theta^{\mu_p}, \quad (2.13)$$



where  $\phi^{(p)a}$  are the degree  $|a|-p$  coefficients of  $\phi^a$  which can be identified with  $p$ -forms on  $\Sigma_d$ .

We choose a submanifold as gauge fixing on the space of superfields  $\mathcal{M}$ . It is given by the constraints<sup>2</sup>

$$\phi_a^+ = D\chi_a \quad \text{and} \quad \chi^{a+} = (-1)^{|a|(d+1)+1} D\phi^a, \quad (2.14)$$

which reduces the BV symplectic form to

$$\omega_{\text{gf}} = \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} (-1)^{d+|a|+1} \delta\chi_a(\hat{z}) \delta\phi^a(\hat{z}). \quad (2.15)$$

In the following we refer to this gauge as the *exact gauge*. If the worldvolume has no boundaries or the boundary conditions give  $\omega_{\text{gf}} = 0$ , the submanifold is a Lagrangian submanifold as well, and hence it gives a full gauge fixing. Otherwise the submanifold is not Lagrangian, and therefore it only gives a full gauge fixing in the bulk  $\mathcal{W} \setminus \partial\mathcal{W}$  but not on the boundary  $\partial\mathcal{W}$ .

If the Liouville potential is chosen as

$$\vartheta = \int_{T[1]\Sigma_d} d^d\hat{z} \left( (-1)^{d+|a|} \phi_a^+(\hat{z}) \delta\phi^a(\hat{z}) + (-1)^{(d+1)|a|} \chi_a(\hat{z}) \delta\chi^{a+}(\hat{z}) \right), \quad (2.16)$$

then the kinetic part of the AKSZ action is given by

$$\mathcal{S}_0 = -\iota_{\mathbf{Q}_0} \vartheta = \int_{T[1]\Sigma_d} d^d\hat{z} \left( (-1)^{d+|a|+1} \phi_a^+(\hat{z}) D\phi^a(\hat{z}) + (-1)^{(d+1)|a|+1} \chi_a(\hat{z}) D\chi^{a+}(\hat{z}) \right). \quad (2.17)$$

We have not specified any boundary conditions yet. They are needed in order to derive consistent equations of motion. The variation of the action  $\mathcal{S}_0$  gives

$$\begin{aligned} \delta\mathcal{S}_0 = \int_{T[1]\Sigma_d} d^d\hat{z} & \left( (-1)^{d+|a|+1} \delta\phi_a^+(\hat{z}) D\phi^a(\hat{z}) + (-1)^{d+|a|+1} \phi_a^+(\hat{z}) D\delta\phi^a(\hat{z}) \right. \\ & \left. + (-1)^{(d+1)|a|+1} \chi_a(\hat{z}) D\delta\chi^{a+}(\hat{z}) + (-1)^{(d+1)|a|+1} \chi_a(\hat{z}) D\delta\chi^{a+}(\hat{z}) \right). \end{aligned} \quad (2.18)$$

The equations of motion for  $\phi^a$  and  $\chi_a$  are obtained via integration by parts. The boundary terms of the variation

$$\delta\mathcal{S}_0|_{T[1]\partial\Sigma_d} = \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} \left( \phi_a^+(\hat{z}) \delta\phi^a(\hat{z}) - (-1)^{d(|a|+1)} \chi_a(\hat{z}) \delta\chi^{a+}(\hat{z}) \right) \quad (2.19)$$

must vanish on their own. The straightforward boundary conditions  $\phi_a^+|_{T[1]\partial\Sigma_d} = 0$ ,  $\chi^{a+}|_{T[1]\partial\Sigma_d} = 0$  and  $\delta\phi^a|_{T[1]\partial\Sigma_d} = 0$ ,  $\delta\chi_a|_{T[1]\partial\Sigma_d} = 0$  result in a vanishing reduced kinetic action on the boundary, so they are not suitable for us. On the other hand, the boundary variation term  $\delta\mathcal{S}_0|_{T[1]\partial\Sigma_d}$  in the partial exact gauge fixing reduces to

$$\begin{aligned} \delta\mathcal{S}_{0,\text{gf}}|_{T[1]\partial\Sigma_d} &= \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} \left( D\chi_a(\hat{z}) \delta\phi^a(\hat{z}) + (-1)^{d+|a|} \phi^a(\hat{z}) D\delta\chi_a(\hat{z}) \right) \\ &= \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} D(\chi_a(\hat{z}) \delta\phi^a(\hat{z})) \\ &= 0. \end{aligned} \quad (2.20)$$

As we see, the exact gauge is consistent with the necessary boundary conditions, which means the equations of motion are well-defined in this gauge, and the master equation also holds because the interaction term reduces to the boundary as well. This is not true without the exact

---

<sup>2</sup>It is important to note that the fields and antifields in this gauge are assigned in the bulk  $\mathcal{W} \setminus \partial\mathcal{W}$ .

gauge or suitable boundary conditions. Hence the exact gauge fixing appears here as a boundary condition.

The gauge fixed kinetic action

$$\mathcal{S}_{0,\text{gf}} = \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} (-1)^{d+|a|+1} \chi_a(\hat{z}) D\phi^a(\hat{z}) \quad (2.21)$$

can be derived from the Liouville potential

$$\vartheta_{\text{b}} = \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} (-1)^{d+|a|+1} \chi_a(\hat{z}) \delta\phi^a(\hat{z}) , \quad (2.22)$$

with  $\omega_{\text{gf}} = \delta\vartheta_{\text{b}}$ , but with the opposite sign:

$$\mathcal{S}_{0,\text{gf}} = \iota_{\mathbf{Q}_{0,\text{b}}}\vartheta_{\text{b}} , \quad (2.23)$$

where  $\mathbf{Q}_{0,\text{b}}$  is the cohomological vector field on  $T[1]\partial\Sigma_d$ . The interaction term enters into the picture in a simpler way. Let us assume that the Hamiltonian functional  $\gamma = \int_{T[1]\Sigma_d} d^d\hat{z} \text{ev}^*(\gamma)$ , which satisfies the equation  $\iota_{\mathbf{Q}_\gamma}\omega = \delta\gamma$ , reduces to the boundary in the exact gauge as

$$\gamma_{\text{gf}} = - \int_{T[1]\Sigma_d} d^d\hat{z} D \text{ev}^*(\beta) = - \oint_{T[1]\partial\Sigma_d} d^{d-1}\hat{z} \text{ev}^*(\beta) =: -\beta \quad (2.24)$$

for a function  $\beta$  on the target graded manifold  $\mathcal{M}$  with degree  $d-1$ . Then the full action (2.7) reduces in the exact gauge to

$$\mathcal{S}_{\text{gf}} = \iota_{\mathbf{Q}_{0,\text{b}}}\vartheta_{\text{b}} - \beta , \quad (2.25)$$

which satisfies the BV master equation, and thus gives an AKSZ action on the boundary.

### 2.3 Dimensional reduction by effective actions

In this paper we shall also apply another dimensional reduction method, called ‘Losev’s trick’ [50], which is not specific to boundary reductions. We briefly recall the technique following [47], see also [43] where a similar technique is employed.

The symplectic structure  $\omega$  on the target supermanifold  $\mathcal{M}$  induces a natural second order differential operator  $\Delta$ , which in local coordinates is given by

$$\Delta = \frac{1}{2} \omega^{\hat{i}\hat{j}} \frac{\vec{\partial}}{\partial\hat{X}^{\hat{i}}} \frac{\vec{\partial}}{\partial\hat{X}^{\hat{j}}} , \quad (2.26)$$

where  $\omega^{\hat{i}\hat{j}}$  is the inverse of  $\omega_{\hat{i}\hat{j}}$ . This pulls back to give the BV Laplacian  $\Delta$  for the BV bracket  $(\cdot, \cdot)_{\text{BV}}$  on the space of AKSZ fields  $\mathcal{M}$ . The AKSZ action  $\mathcal{S}$  satisfies the BV quantum master equation  $\Delta e^{-\mathcal{S}/\hbar} = 0$  on  $\mathcal{M}$ , which is equivalent to  $\frac{1}{2}(\mathcal{S}, \mathcal{S})_{\text{BV}} = \hbar \Delta\mathcal{S}$ . This ensures independence of the BRST-invariant quantum field theory on the choice of gauge fixing, provided we define the path integral by equipping  $\mathcal{M}$  with a measure  $\mu$  which is compatible with  $\omega$  [49].

Borrowing standard terminology from renormalization of quantum field theory, let us now assume that the space of AKSZ fields can be decomposed into a direct product  $\mathcal{M} = \mathcal{M}_{\text{UV}} \times \mathcal{M}_{\text{IR}}$  of ultraviolet (UV) and infrared (IR) degrees of freedom, with a compatible decomposition of the canonical symplectic form  $\omega = \omega_{\text{UV}} + \omega_{\text{IR}}$ . Then the BV Laplacian also decomposes as  $\Delta = \Delta_{\text{UV}} + \Delta_{\text{IR}}$ . One now ‘integrates out’ the ultraviolet degrees of freedom to get an effective action. The integration requires a gauge fixing on the ultraviolet sector  $\mathcal{M}_{\text{UV}}$  of the space of

superfields, which means a choice of a Lagrangian submanifold  $\mathcal{L} \subset \mathcal{M}_{\text{UV}}$ . Then the effective BV action  $\mathcal{S}_{\text{eff}}$  in the infrared sector is defined as

$$e^{-\mathcal{S}_{\text{eff}}/\hbar} := \int_{\mathcal{L}} \sqrt{\mu_{\mathcal{L}}} e^{-\mathcal{S}/\hbar}, \quad (2.27)$$

where  $\sqrt{\mu_{\mathcal{L}}}$  is the measure on  $\mathcal{L}$  induced by  $\mu$ . Therefore the effective action satisfies the quantum master equation  $\Delta_{\text{IR}} e^{-\mathcal{S}_{\text{eff}}/\hbar} = 0$ . A change of gauge fixing in the ultraviolet sector only changes  $e^{-\mathcal{S}_{\text{eff}}/\hbar}$  by a  $\Delta_{\text{IR}}$ -exact term. Similarly, the value of the partition function is independent of the particular choice of splitting  $\mathcal{M} = \mathcal{M}_{\text{UV}} \times \mathcal{M}_{\text{IR}}$  by the Batalin-Vilkovisky theorem [49]. In the following we use this method to reduce three-dimensional AKSZ sigma-models to AKSZ sigma-models in two dimensions.

### 3 String sigma-models

In this section we describe several relevant examples of two-dimensional AKSZ sigma-models which are related to the topological A- and B-models.

#### 3.1 Poisson sigma-model

In dimension  $d = 2$ , the AKSZ theory with target space a degree 1 QP-manifold describes the topological sigma-model for strings in an NS-NS  $B$ -field background, whose first order formalism is the Poisson sigma-model [51, 52] whereby BV quantization yields the Cattaneo-Felder path integral approach [53]. The AKSZ formulation of the Poisson sigma-model is studied in [6]. We take  $\mathcal{W} = T[1]\Sigma_2$  for an oriented Riemann surface  $\Sigma_2$ , and  $\mathcal{M} = T^*[1]M$  with degree 0 base coordinates  $X^i$  on the target space  $M$  and degree 1 fiber coordinates  $\chi_i$ . The canonical symplectic form on  $\mathcal{M}$  is

$$\omega_2 = d\chi_i \wedge dX^i, \quad (3.1)$$

which leads to the canonical graded Poisson bracket  $\{X^i, \chi_j\} = \delta^i_j$  on the local coordinates of  $\mathcal{M}$ . We choose the Liouville potential to be  $\vartheta_2 = \chi_i dX^i$ . Its zero locus is  $\mathcal{L}_2 = \{\chi_i = 0\}$ . The most general form of a degree 2 Hamiltonian function  $\gamma_\pi$  on  $\mathcal{M}$  is given by a  $(0, 2)$ -tensor  $\pi$  on  $M$  as

$$\gamma_\pi = \frac{1}{2} \pi^{ij}(X) \chi_i \chi_j. \quad (3.2)$$

Compatibility of the corresponding cohomological vector field  $Q_{\gamma_\pi}$  with  $\omega_2$  and the classical master equation  $\{\gamma_\pi, \gamma_\pi\} = 0$  implies that  $\pi$  must be a Poisson bivector on  $M$ , i.e.  $[\pi, \pi]_{\text{S}} = 0$ , where  $[\cdot, \cdot]_{\text{S}}$  denotes the Schouten bracket on multivectors. In other words, a QP1-manifold is the same thing as a Poisson manifold  $(M, \pi)$ , which by construction is also a Lie algebroid on the cotangent bundle  $T^*M$ . The Hamiltonian function determines a derived bracket which defines a Poisson bracket on  $C^\infty(M)$  through

$$\{f, g\}_\pi = \pi(df \wedge dg) = -\{\{f, \gamma\}, g\}. \quad (3.3)$$

The kinetic part of the AKSZ action is inherited from the cohomological vector field  $Q_{\mathcal{W}}$  on  $\mathcal{W} = T[1]\Sigma_2$ , and is given by

$$\mathcal{S}_0^{(2)} = \int_{T[1]\Sigma_2} d^2 \hat{z} \chi_i \mathbf{D} X^i, \quad (3.4)$$

where as before the superworldsheet differential is  $\mathbf{D} = \theta^\mu \frac{\partial}{\partial \sigma^\mu} = Q_{\mathcal{W}}$ . Together these ingredients give the AKSZ action for the Poisson sigma-model as

$$\mathcal{S}_\pi^{(2)} = \int_{T[1]\Sigma_2} d^2 \hat{z} \left( \chi_i \mathbf{D} X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right), \quad (3.5)$$

where  $\mathbf{f} = \phi^*(f) = f(\phi)$  for a function  $f$  on  $\mathcal{M}$  and  $\phi \in \mathcal{M}$ . Integrating over the odd coordinates  $\theta^\mu$  and restricting to the degree 0 fields in (3.5) yields the first order string sigma-model action

$$\int_{\Sigma_2} \left( \chi_i \wedge dX^i + \frac{1}{2} \pi^{ij} \chi_i \wedge \chi_j \right), \quad (3.6)$$

and if  $\pi$  is non-degenerate it is classically equivalent to the topological bosonic string  $B$ -field coupling

$$\int_{\Sigma_2} X^*(B) = \frac{1}{2} \int_{\Sigma_2} B_{ij} dX^i \wedge dX^j, \quad (3.7)$$

where the flat Kalb-Ramond two-form field  $B$  on  $M$  is the inverse of the bivector  $\pi$ .

### 3.2 AKSZ formulations of the A-model

The topological A- and B-models coupled to gravity are the topological A- and B-model string theories, which have been widely studied for more than 20 years. They were also one of the first examples of the AKSZ construction in [5]. In the following we review their relevant AKSZ constructions, which reduce to the A- or B-model in a particular gauge. The reader can find details about their field-antifield choices and gauge fixing in the indicated references, and therefore we only define their AKSZ sigma-models.

We begin with the A-model, whose AKSZ constructions were mostly related to the Poisson sigma-model or the  $B$ -field coupling. Recently a different approach related to the AKSZ sigma-model of topological membranes on  $G_2$ -manifolds appeared in [48].

A1. The original AKSZ construction [5] is formulated in the same way as the Poisson sigma-model in §3.1 but with zero kinetic term. Thus it has the same target QP1-manifold with the same symplectic structure and Hamiltonian as those of the Poisson sigma-model, where the Poisson bivector  $\pi$  is given by the inverse of the Kähler form on the target Calabi-Yau manifold. The AKSZ action thus constructed is

$$\mathcal{S}_{A1}^{(2)} = \frac{1}{2} \int_{T[1]\Sigma_2} d^2\hat{z} \pi^{ij} \chi_i \chi_j. \quad (3.8)$$

A2. A complete Poisson sigma-model formulation for the A-model with kinetic term (3.5) appeared in [42] as

$$\mathcal{S}_{A2}^{(2)} = \mathcal{S}_\pi^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i dX^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right). \quad (3.9)$$

The equation of motion for  $\chi_i$  reduces it to the AKSZ action (3.8) up to a sign, so they are classically equivalent.

A3. The BV quantized topological NS–NS  $B$ -field coupling is not strictly speaking constructed by the AKSZ formalism, but it is nevertheless worth mentioning as a BV action which gives the A-model [40] with the same field definitions as those of the Poisson sigma-model:

$$\mathcal{S}_{A3}^{(2)} = \frac{1}{2} \int_{T[1]\Sigma_2} d^2\hat{z} B_{ij} dX^i dX^j, \quad (3.10)$$

where the two-form  $B$  is the Kähler form, and the flat condition  $dB = 0$  is equivalent to the Poisson condition of its inverse  $\pi$ . It is of course not surprising that the  $B$ -field coupling is classically equivalent to the Poisson sigma-model as well.

A4. In [40, 44] an AKSZ Poisson sigma-model together with the topological  $B$ -field coupling is used as an AKSZ formulation of the A-model with action

$$\mathcal{S}_{A4}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i \mathbf{D}X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j + \frac{1}{4} B_{ij} \mathbf{D}X^i \mathbf{D}X^j \right), \quad (3.11)$$

where  $B_{ij}$  is the inverse of  $\pi^{ij}$ . The last term has no effect in the BV bracket since  $dB = 0$ .

A5. Another simple construction also gives the A-model [48]. Let the target supermanifold be  $\mathcal{M} = T^*[1]T[1]M$  where the local coordinates are given by  $(X^i, \chi_i, \zeta^i, y_i)$  with degrees  $(0, 1, 1, 0)$ . The symplectic form is

$$\omega_{A5} = d\chi_i \wedge dX^i + dy_i \wedge d\zeta^i. \quad (3.12)$$

The Hamiltonian

$$\gamma_{A5} = \chi_i \zeta^i \quad (3.13)$$

gives an AKSZ action with zero kinetic term as

$$\mathcal{S}_{A5}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \chi_i \zeta^i. \quad (3.14)$$

An interesting feature of this AKSZ construction is that it can be reproduced from an AKSZ membrane theory, namely the standard Courant sigma-model which we discuss in §4. The reduction can be performed by taking all fields to be independent of the extra coordinate direction.

A6. A somewhat different construction was proposed in [48], which uses degree  $-1$  target space coordinates. Thus the target space is not strictly speaking a QP-manifold anymore, and this takes us out of the realm of graded geometry into derived geometry, but its AKSZ construction is still applicable wherein using negative degree fields such as ghosts and antifields is natural. The target derived manifold is  $\mathcal{M} = T^*[1]T[-1]T[1]M$  on which the local coordinates of  $T[-1]T[1]M$  are  $(X^i, \zeta^i, b^i, \eta^i)$  with degrees  $(0, 1, 0, -1)$ , and  $X^i$  associated to  $M$ . The cotangent fiber coordinates are  $(\chi_i, y_i, n_i, h_i)$  with degrees  $(1, 0, 1, 2)$ , and the symplectic structure is

$$\omega_{A6} = d\chi_i \wedge dX^i + dy_i \wedge d\zeta^i + dh_i \wedge d\eta^i + dn_i \wedge db^i. \quad (3.15)$$

The Hamiltonian

$$\gamma_{A6} = \chi_i \zeta^i + b^i h_i \quad (3.16)$$

gives the AKSZ action

$$\mathcal{S}_{A6}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} (\chi_i \zeta^i + b^i h_i) \quad (3.17)$$

with zero kinetic term. This construction can be obtained from a similar reduction of the standard Courant sigma-model as in the previous construction. These latter two AKSZ constructions can be obtained by dimensional reductions of AKSZ built topological membrane theories on  $G_2$ -manifolds, which can be reformulated as an AKSZ threbrane theory related to exceptional generalized geometry of M-theory [48].

Zucchini model. The BV sigma-model of [40] is not strictly speaking given by an AKSZ construction, since it involves BV quantized kinetic terms which do not arise from a Louville potential of the BV symplectic form. It has the same field content and BV symplectic form as those of the Poisson sigma-model, and the BV action is given by

$$\mathcal{S}_Z^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i \mathbf{D}X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j + \frac{1}{2} \omega_{ij} \mathbf{D}X^i \mathbf{D}X^j + J^i_j \chi_i \mathbf{D}X^j \right), \quad (3.18)$$

where  $\pi$  is a bivector and  $\omega$  is a two-form, and together with the  $(1,1)$ -tensor  $J$  they satisfy the identities

$$\begin{aligned} J^i_k J^k_j + \pi^{ik} \omega_{kj} + \delta^i_j &= 0, \\ J^i_k \pi^{kj} + J^j_k \pi^{ki} &= 0, \\ \omega_{ik} J^k_j + \omega_{jk} J^k_i &= 0. \end{aligned} \tag{3.19}$$

The master equation imposes further constraints

$$\begin{aligned} \pi^{[i|l} \partial_l \pi^{jk]} &= 0, \\ J^l_i \partial_l \pi^{jk} + 2 \pi^{jl} \partial_{[i} J^k_{|l]} + \pi^{kl} \partial_l J^j_i - J^j_l \partial_i \pi^{lk} &= 0, \\ 2 J^l_{[i} \partial_l J^k_{|j]} - 2 J^k_l \partial_{[i} J^l_{|j]} + 3 \pi^{kl} \partial_{[l} \omega_{ij]} &= 0, \\ J^l_i \partial_{[l} \omega_{jk]} + J^l_j \partial_{[l} \omega_{ki]} + J^l_k \partial_{[l} \omega_{ij]} - \partial_{[i} (\omega_{j|l} J^l_{|k]}) &= 0, \end{aligned} \tag{3.20}$$

where  $\partial_i = \partial/\partial X^i$ , which are the same identities as the integrability condition of a generalized complex structure  $\mathbb{J}$  in the form

$$\mathbb{J}^I_J = \begin{pmatrix} J^i_j & \pi^{ij} \\ \omega_{ij} & -J^j_i \end{pmatrix}, \tag{3.21}$$

where the doubled indices  $I, J$  have been introduced. The Zucchini model reduces to the Poisson sigma-model upon setting  $J = 0$  and  $\omega = 0$ , which is the A-model. If in addition  $\omega$  is non-zero it adds a  $B$ -field coupling, which is just another copy of the A-model.

### 3.3 AKSZ formulations of the B-model

AKSZ constructions for the topological B-model are more diverse and have different superfield contents. We do not enumerate all of them here, nor the original construction from [5], since they are similar to the ones described below.

B1. The base degree 0 manifold of the target QP-manifold, which is a Calabi-Yau threefold  $M$ , is equipped with a complex structure which splits the local coordinate indices to  $i = (a, \bar{a})$ , where  $a = 1, 2, 3$ . The target QP-manifold  $\mathcal{M}$  is defined by its coordinates:  $X^a, X^{\bar{a}}, \tilde{X}_{\bar{a}}$  have degree 0, and  $\chi_a, \chi_{\bar{a}}, \tilde{\chi}^{\bar{a}}$  have degree 1. The symplectic form on  $\mathcal{M}$  is

$$\omega_{B1} = dX^a \wedge d\chi_a + dX^{\bar{a}} \wedge d\chi_{\bar{a}} + d\tilde{X}_{\bar{a}} \wedge d\tilde{\chi}^{\bar{a}}. \tag{3.22}$$

The B-model is constructed in [54] by the AKSZ action

$$\mathcal{S}'_{B1}{}^{(2)} = \int_{T[1]\Sigma_2} d^2 \hat{z} (\chi_a D X^a + \chi_{\bar{a}} D X^{\bar{a}} + \tilde{X}_{\bar{a}} D \tilde{\chi}^{\bar{a}} + \chi_{\bar{a}} \tilde{\chi}^{\bar{a}}). \tag{3.23}$$

We can enlarge its field content with the addition of new coordinates  $\tilde{\chi}^a$  and  $\tilde{X}_a$  whose contribution to the symplectic structure is defined by the term

$$d\tilde{X}_a \wedge d\tilde{\chi}^a, \tag{3.24}$$

and furthermore we also add the term  $\tilde{X}_a D \tilde{\chi}^a + \chi_a \tilde{\chi}^a$  to the AKSZ action (3.23) which can be set to zero with gauge fixing  $\tilde{\chi}^a = 0$ . Introducing the new fields leads to an extended AKSZ action for the B-model given by

$$\mathcal{S}_{B1}{}^{(2)} = \int_{T[1]\Sigma_2} d^2 \hat{z} (\chi_i D X^i + \tilde{X}_i D \tilde{\chi}^i + \chi_i \tilde{\chi}^i). \tag{3.25}$$

B2. The AKSZ construction of the B-model with an explicit complex structure  $J$  was studied in [43], see also [8]. It has the same field content as the first construction of the B-model:  $X^i, \tilde{X}_i$  are degree 0 coordinates and  $\chi_i, \zeta^i$  are degree 1 coordinates. The symplectic structure only differs in a sign from the first construction:

$$\omega_{\text{B2}} = dX^i \wedge d\chi_i - d\tilde{X}_i \wedge d\tilde{\chi}^i. \quad (3.26)$$

The AKSZ action is given by

$$\mathcal{S}_{\text{B2}}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} (\chi_i D X^i - \tilde{X}_i D \tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j), \quad (3.27)$$

where  $J^i_j$  is a constant complex structure on the target manifold. The first construction is just a special case of this: If we take  $J^i_j = i\delta^i_j$ , and rescale the fields  $\tilde{X}_i$  and  $\tilde{\chi}^i$  by  $i$ , then the action (3.27) reduces to  $\mathcal{S}_{\text{B1}}^{(2)}$ .

The case of non-constant complex structure  $J$  was also studied in [43], and an AKSZ sigma-model was proposed, of which the master equation gives the integrability condition

$$J^l_{[i} \partial_l J^k_{j]} - J^k_l \partial_{[i} J^l_{j]} = 0, \quad (3.28)$$

and the condition

$$J^i_k J^k_j = -\delta^i_j \quad (3.29)$$

is added by hand. The field content is the same as that of the constant case, and the action constructed by the AKSZ formalism is given by

$$\mathcal{S}_J^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} (\chi_i D X^i - \tilde{X}_i D \tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j + \partial_j J^i_k \tilde{X}_i \tilde{\chi}^j \tilde{\chi}^k). \quad (3.30)$$

## 4 Courant sigma-models

In this section we review the three-dimensional Courant sigma-model, and its specific examples which are relevant for us: the standard and the contravariant Courant sigma-models. We also study their boundary reductions in the exact gauge.

### 4.1 Courant algebroids

In dimension  $d = 3$ , the corresponding AKSZ sigma-model with source dg-manifold  $\mathcal{W} = T[1]\Sigma_3$  is defined on membrane worldvolume superfields with target space a QP-manifold of degree 2, which corresponds to a Courant algebroid [16]. In this paper we work only with the QP2-manifold  $\mathcal{M} = T^*[2]T[1]M$ . We choose local Darboux coordinates  $(X^i, \psi^i, \chi_i, F_i)$  with degrees  $(0, 1, 1, 2)$  in which the graded symplectic structure is given by

$$\omega_3 = dX^i \wedge dF_i + d\chi_i \wedge d\psi^i. \quad (4.1)$$

The graded Poisson brackets of the coordinates are canonical in the sense that  $\{X^i, F_j\} = \delta^i_j$  and  $\{\chi_i, \psi^j\} = \delta_i^j$ . For the Liouville potential we choose  $\vartheta_3 = F_i dX^i - \chi_i d\psi^i$ . Its zero locus is  $\mathcal{L}_3 = \{F_i = 0, \psi^i = 0\}$ . The most general form of the degree 3 Hamiltonian function is given by

$$\gamma_{\rho, T} = \rho^i{}_I(X) F_i \zeta^I + \frac{1}{3!} T_{IJK}(X) \zeta^I \zeta^J \zeta^K, \quad (4.2)$$

for degree 0 functions  $\rho^i_I$  and  $T_{IJK}$  on  $M$ , where we introduced a doubled index notation for the local degree 1 coordinates  $\zeta^I = (\psi^i, \chi_i)$ . The three-form  $T_{IJK}$  encodes the allowed geometric and non-geometric supergravity fluxes for given  $\rho^i_I$ .

We now define three operations given by taking derived brackets defined by  $\gamma_{\rho,T}$  and the graded Poisson bracket through

$$[e_1, e_2]_D = \{\{e_1, \gamma_{\rho,T}\}, e_2\}, \quad \langle e_1, e_2 \rangle = \{e_1, e_2\} \quad \text{and} \quad \rho(e) = \{e, \{\gamma_{\rho,T}, \cdot\}\}. \quad (4.3)$$

These operations are defined on degree 1 functions  $e$  with local expression  $e = f_I(X) \zeta^I$ , where  $f_I$  is a degree 0 function on the body  $M = \mathcal{M}_0$  of  $\mathcal{M}$ , which are identified as local sections of the generalized tangent bundle

$$E = TM \oplus T^*M \quad (4.4)$$

over  $M$ ; symbolically

$$A^i \chi_i + \alpha_i \psi^i \longleftrightarrow A^i \frac{\partial}{\partial X^i} + \alpha_i dX^i. \quad (4.5)$$

The classical master equation  $\{\gamma_{\rho,T}, \gamma_{\rho,T}\} = 0$  then implies that they endow  $E$  with the structure of a Courant algebroid: A *Courant algebroid* on a manifold  $M$  is a vector bundle  $E$  over  $M$  equipped with a symmetric non-degenerate bilinear form  $\langle \cdot, \cdot \rangle$  on its fibers, an anchor map  $\rho : E \rightarrow TM$ , and a binary bracket of sections  $[\cdot, \cdot]_D$ , called the Dorfman bracket, which together satisfy

$$\begin{aligned} [e_1, [e_2, e_3]_D]_D &= [[e_1, e_2]_D, e_3]_D + [e_2, [e_1, e_3]_D]_D, \\ \rho(e_1)\langle e_2, e_3 \rangle &= \langle [e_1, e_2]_D, e_3 \rangle + \langle e_2, [e_1, e_3]_D \rangle, \\ \rho(e_1)\langle e_2, e_3 \rangle &= \langle e_1, [e_2, e_3]_D + [e_3, e_2]_D \rangle, \end{aligned} \quad (4.6)$$

where  $e_1, e_2, e_3$  are sections of  $E$ . In this paper we consider two particular examples of Courant algebroids.

*Standard Courant algebroid.* The simplest Hamiltonian function with  $\rho^i_I = \delta^i_I$  and  $T_{IJK} = 0$  is given by

$$\gamma_0 = F_i \psi^i. \quad (4.7)$$

Its derived brackets on degree 1 functions (4.5) gives the *standard Courant algebroid* on the generalized tangent bundle  $E = TM \oplus T^*M$ , which features in generalized geometry [55, 56]. It is an extension of the Lie algebroid of tangent vectors by cotangent vectors with the three operations

$$\begin{aligned} \langle A + \alpha, B + \beta \rangle &= \iota_A \beta + \iota_B \alpha, \\ \rho(A + \alpha) &= A, \\ [A + \alpha, B + \beta]_{D,0} &= [A, B] + \mathcal{L}_A \beta - \iota_B d\alpha, \end{aligned} \quad (4.8)$$

where the sections of  $E = TM \oplus T^*M$  are composed of vector fields  $A, B$  and one-forms  $\alpha, \beta$ . The antisymmetrization of the standard Dorfman bracket in (4.8), called the Courant bracket, is the natural bracket in generalized geometry which is compatible with the commutator algebra of generalized Lie derivatives [55, 56].

Only the simplest case of pure NS–NS flux  $T_{IJK} = H_{ijk}$  is consistent with the choice of anchor map  $\rho^i_I = \delta^i_I$  of the standard Courant algebroid, which is necessarily closed by the classical master equation. Given a Kalb-Ramond two-form field  $B$  on  $M$ , with  $H = dB$ , canonical transformation of the Hamiltonian function (4.7) by the degree 2 function  $B = \frac{1}{2} B_{ij}(X) \psi^i \psi^j$  on  $\mathcal{M}$  yields the twisted Hamiltonian function

$$\gamma_H := e^{\delta_B} \gamma_0 = F_i \psi^i + \frac{1}{3!} H_{ijk} \psi^i \psi^j \psi^k. \quad (4.9)$$



The NS–NS  $H$ -flux thus appears as a twisting of the standard Courant algebroid, which gives rise to a deformation of the Dorfman bracket through an extra term as

$$[A + \alpha, B + \beta]_{D,H} = [A, B] + \mathcal{L}_A \beta - \iota_B d\alpha + \iota_A \iota_B H . \quad (4.10)$$

Poisson Courant algebroid. Consider the Hamiltonian defined through a bivector  $\pi$  and a three-vector  $R$  on  $M$  by setting  $\rho^i{}_I = \pi^{ij}$ ,  $T_{IJK} = (\partial_i \pi^{jk}, R^{ijk})$  to give

$$\gamma_{\pi,R} = \pi^{ij} F_i \chi_j - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \frac{1}{3!} R^{ijk}(X) \chi_i \chi_j \chi_k . \quad (4.11)$$

The master equation  $\{\gamma_{\pi,R}, \gamma_{\pi,R}\} = 0$  gives the constraints

$$[\pi, \pi]_S = 0 \quad \text{and} \quad [\pi, R]_S = 0 . \quad (4.12)$$

Note that the  $R$ -flux also enters here as a twist: The Hamiltonian (4.11) can be regarded as a canonical transformation  $\gamma_{\pi,R} = e^{\delta\beta} \gamma_{\pi,0}$  by a degree 2 function  $\beta = \frac{1}{2} \beta^{ij}(X) \chi_i \chi_j$  with  $R = d_\beta \beta$  where  $d_\beta = [\beta, \cdot]_S$ , regarded as a bivector  $\beta$  on  $M$  which is T-dual to the  $B$ -field of the  $H$ -flux frame. The corresponding Courant algebroid is the Poisson Courant algebroid [57], for which the identities are equivalent to the Poisson condition for  $\pi$  if  $R = 0$ : The *Poisson Courant algebroid* is the Courant algebroid on the generalized tangent bundle  $E = TM \oplus T^*M$  over a Poisson manifold  $(M, \pi)$  with the operations

$$\begin{aligned} \langle A + \alpha, B + \beta \rangle &= \iota_A \beta + \iota_B \alpha , \\ \rho(A + \alpha) &= \iota_\alpha \pi , \end{aligned} \quad (4.13)$$

$$[A + \alpha, B + \beta]_{D;\pi,R} = [\alpha, \beta]_\pi + \mathcal{L}_\alpha^\pi Y - \iota_\beta d_\pi X - \iota_\alpha \iota_\beta R ,$$

where  $\mathcal{L}_\alpha^\pi = \iota_\alpha d_\pi + d_\pi \iota_\alpha$  and  $[\cdot, \cdot]_\pi$  is the Koszul bracket on one-forms given by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\iota_\alpha \pi} \beta - \mathcal{L}_{\iota_\beta \pi} \alpha - d(\pi(\alpha \wedge \beta)) . \quad (4.14)$$

#### 4.2 Standard Courant sigma-model

It is evident from the general construction that Courant algebroids are uniquely encoded (up to isomorphism) in the corresponding AKSZ topological membrane theories, which are called Courant sigma-models [19]. In the particular example of the standard Courant algebroid on  $E = TM \oplus T^*M$  twisted by a closed NS–NS three-form flux  $H$ , the mapping space  $\mathcal{M}$  of superfields supports the canonical BV symplectic structure

$$\omega_3 = \int_{T[1]\Sigma_3} d^3 \hat{z} (\delta X^i \delta F_i + \delta \psi^i \delta \chi_i) , \quad (4.15)$$

where the ghost number 2 superfields  $F_i$  and ghost number 0 superfields  $X^i$ , as well as the conjugate pairs of ghost number 1 superfields  $\chi_i$  and  $\psi^i$ , contain each other's antifields respectively. The AKSZ construction leads to the action

$$\mathcal{S}_H^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( F_i D X^i - \chi_i D \psi^i + F_i \psi^i + \frac{1}{3!} H_{ijk} \psi^i \psi^j \psi^k \right) , \quad (4.16)$$

which solves the classical master equation  $(\mathcal{S}_H^{(3)}, \mathcal{S}_H^{(3)})_{\text{BV}} = 0$ . Integrating over  $\theta^\mu$  and restricting to degree 0 fields in (4.16) yields the first order membrane sigma-model action

$$\int_{\Sigma_3} \left( F_i \wedge (dX^i - \psi^i) - \chi_i \wedge d\psi^i + \frac{1}{3!} H_{ijk} \psi^i \wedge \psi^j \wedge \psi^k \right) , \quad (4.17)$$

which is classically equivalent to the topological bosonic membrane Wess-Zumino coupling

$$\int_{\Sigma_3} X^*(H) = \frac{1}{3!} \int_{\Sigma_3} H_{ijk} dX^i \wedge dX^j \wedge dX^k . \quad (4.18)$$

The standard Courant sigma-model on an open worldvolume is well-defined if, as usual, one specifies its boundary conditions. Instead we consider it in exact gauge as an illustration. The exact gauge defined in §2.2 reads here as

$$\mathbf{F}_i = \mathbf{D}\chi_i \quad \text{and} \quad \psi^i = -\mathbf{D}X^i . \quad (4.19)$$

It gives the gauge fixed BV symplectic structure

$$\omega_{3,\text{gf}} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \delta X^i \delta \chi_i , \quad (4.20)$$

and reduces the AKSZ action (4.16) without  $H$ -flux to zero. With  $H$ -flux, the AKSZ action leads to a pure Wess-Zumino coupling

$$\mathcal{S}_{H,\text{gf}}^{(3)} = -\frac{1}{3!} \int_{T[1]\Sigma_3} d^3\hat{z} H_{ijk} \mathbf{D}X^i \mathbf{D}X^j \mathbf{D}X^k \quad (4.21)$$

which is no longer an AKSZ action, as there are no BV gauge degrees of freedom in the bulk. This is reminiscent of the fact that the equation of motion for  $\mathbf{F}_i$  also gives the same action (4.21) up to a sign. If  $H_{ijk} = \partial_{[i} B_{jk]}$  is exact, then we obtain the boundary AKSZ action

$$-\frac{1}{2} \oint_{T[1]\partial\Sigma_3} d^2\hat{z} B_{ij} \mathbf{D}X^i \mathbf{D}X^j , \quad (4.22)$$

which is the quantization of the NS–NS  $B$ -field coupling. Hence the exact gauge is nicely applicable for boundary reductions of topological membranes describing flux deformations of string sigma-models.

Relation to the B-model. We will now show that the standard Courant sigma-model is related to the B-model on its boundary via the exact gauge. Although we have found that the standard Courant sigma-model has a trivial boundary reduction in the exact gauge, we can obtain a non-trivial boundary theory if we extend its field content and then set the extra fields to zero with gauge fixing.

The standard Courant sigma-model without  $H$ -flux is given by the Hamiltonian  $\gamma_0$  in (4.7) and the symplectic form  $\omega_3$  in (4.1). The AKSZ action is

$$\mathcal{S}_0^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} (\mathbf{F}_i \mathbf{D}X^i - \chi_i \mathbf{D}\psi^i + \mathbf{F}_i \psi^i) . \quad (4.23)$$

We double its fields with the introduction of degree 0 coordinates  $\tilde{X}_i$ , degree 1 coordinates  $\tilde{\chi}^i$ ,  $\tilde{\psi}_i$  and degree 2 coordinates  $\tilde{F}^i$  on the target QP2-manifold. The extra term in the symplectic structure is

$$d\tilde{X}_i \wedge d\tilde{F}^i + d\tilde{\chi}^i \wedge d\tilde{\psi}_i , \quad (4.24)$$

which together with the extended Hamiltonian

$$\gamma_0 + \tilde{\gamma}_0 = F_i \psi^i - \tilde{\psi}_i \tilde{F}^i \quad (4.25)$$

defines the AKSZ action

$$\tilde{\mathcal{S}}_0^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i - \chi_i \mathbf{D}\psi^i + \mathbf{F}_i \psi^i - \tilde{\psi}_i \tilde{\mathbf{F}}^i \right), \quad (4.26)$$

where we did not introduce all the possible kinetic terms. This extended standard Courant sigma-model is comparable to the membrane sigma-model in [43], which was introduced in order to uplift the AKSZ construction of the B-model in (3.27) to an AKSZ membrane theory with generalized complex structure. Our construction arrives at a different B-model construction and uses less fields, but does not include the generalized complex structure.

The last term in (4.26) decouples from the original standard Courant sigma-model. To see this we can choose a different gauge than that we will choose for the boundary reduction, but we use the same field-antifield decomposition. For example, if we set  $\tilde{\mathbf{F}}^i = 0$  and  $\tilde{\chi}^i = 0$  as a partial gauge fixing, we can trivially integrate out the fields  $\tilde{\mathbf{X}}_i$  and  $\tilde{\psi}_i$ , which gives the action of the standard Courant sigma-model in (4.23). Alternatively, we can arrive at the same conclusion if we rescale the fields by a real parameter  $\lambda$  in a way which leaves the symplectic structure invariant:

$$\tilde{\chi}^i \longrightarrow \lambda \tilde{\chi}^i \quad \text{and} \quad \tilde{\psi}_i \longrightarrow \frac{1}{\lambda} \tilde{\psi}_i, \quad (4.27)$$

which is a duality transformation given by a symplectomorphism at the BV level. Then we take the  $\lambda \rightarrow \infty$  limit: the term  $\tilde{\psi}_i \tilde{\mathbf{F}}^i$  in the action tends to zero and we get the standard Courant sigma-model in this way as well. Later on we will employ a similar rescaling technique.

Now we reduce the extended standard Courant sigma-model to its boundary with the previously defined exact gauge from §2.2. In this case it means the specific gauge choice

$$\tilde{\chi}^i = -\mathbf{D}\mathbf{X}^i, \quad \mathbf{F}_i = \mathbf{D}\tilde{\psi}_i, \quad \chi_i = -\mathbf{D}\tilde{\mathbf{X}}_i \quad \text{and} \quad \tilde{\mathbf{F}}^i = \mathbf{D}\psi^i. \quad (4.28)$$

The BV symplectic form becomes

$$\oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( \delta\mathbf{X}^i \delta\tilde{\psi}_i + \delta\tilde{\mathbf{X}}_i \delta\tilde{\chi}^i \right), \quad (4.29)$$

which is the same BV symplectic form induced by (3.22) and (3.24) with the relabelling  $\tilde{\psi}_i \rightarrow \chi_i$ . Our gauge fixing also reduces the AKSZ action in (4.26) to the boundary action

$$\tilde{\mathcal{S}}_{0,\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( \tilde{\psi}_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{X}}_i \mathbf{D}\tilde{\chi}^i + \tilde{\psi}_i \tilde{\chi}^i \right) = \mathcal{S}_{\text{B1}}^{(2)}, \quad (4.30)$$

which is the same action as that of the B-model in (3.25) with the same relabelling  $\tilde{\psi}_i \rightarrow \chi_i$  as before.

### 4.3 Contravariant Courant sigma-model

The contravariant Courant sigma-model was introduced in [45] as the Courant sigma-model corresponding to a Poisson Courant algebroid. It is defined by the AKSZ action

$$\mathcal{S}_{\pi,R}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i - \chi_i \mathbf{D}\psi^i + \pi^{ij} \mathbf{F}_i \chi_j - \frac{1}{2} \partial_i \pi^{jk} \psi^i \chi_j \chi_k + \frac{1}{3!} \mathbf{R}^{ijk} \chi_i \chi_j \chi_k \right). \quad (4.31)$$

In the absence of  $R$ -flux the master equation gives the Poisson condition for the bivector  $\pi$ , so we can expect that the contravariant Courant sigma-model is closely related to the Poisson sigma-model. This relation turns out to be the exact gauge boundary reduction. We use the same gauge

fixing as we used for the standard Courant sigma-model in (4.19) which gives the boundary BV symplectic form (4.20). The resulting boundary AKSZ action is that of the Poisson sigma-model in (3.5):

$$\mathcal{S}_{\pi,0;\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( \chi_i \mathbf{D}X^i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right) = \mathcal{S}_{\pi}^{(2)}. \quad (4.32)$$

Relation to the sigma-model for non-geometric R-flux. In the degenerate limit where the anchor of the contravariant Courant sigma-model is set to zero, we show that, in the exact gauge, it coincides precisely with the membrane sigma-model of [21] which quantizes the nonassociative phase space and geometry of the  $R$ -flux background [58–60]. This clarifies more precisely the geometrical meaning of the model of [21] in terms of a Courant algebroid structure. An alternative geometric description as a certain reduction of the standard Courant sigma-model for the target space of double field theory is discussed in [35], which we study in §5. A vanishing anchor map  $\rho$  with non-zero  $R$ -flux means that the bivector field  $\pi$  is identically zero, and the Dorfman bracket is given solely by the three-vector  $R$  in the simple form

$$[X + \alpha, Y + \beta]_{\text{D};0,R} = -\iota_{\alpha}\iota_{\beta}R, \quad (4.33)$$

so that the tangent bundle  $TM$  decouples completely from this structure.

We choose the exact gauge (4.19). In this case our gauge choice is not compatible with boundary conditions, because the BV master equation forces the flux term to be zero on the boundary, which means  $\chi_i = 0$  on  $T[1]\partial\Sigma_3$  if  $R \neq 0$ . This can be circumvented by adding a non-topological boundary term to the action as in [21]. We introduce this as a strictly classical term after the full gauge fixing, and for brevity avoid here issues concerning its quantization. Hence the AKSZ action (4.31) with  $\pi = 0$  reduces to

$$\mathcal{S}_{0,R;\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \chi_i \mathbf{D}X^i + \frac{1}{3!} \int_{T[1]\Sigma_3} d^3\hat{z} R^{ijk} \chi_i \chi_j \chi_k. \quad (4.34)$$

There is still a gauge degree of freedom on the boundary fields remaining, therefore we choose  $\chi_i^{(0)} = \chi_i^{(2)} = X^{i(1)} = 0$ . The bulk fields are not antifields in the BV sense, so we cannot set any of them to zero. Instead we eliminate the non-zero parts in the bulk by their equations of motion. The gauge fixed action for constant  $R$  in the superfield expansion (2.13) is

$$\mathcal{S}_{0,R;\text{gf}}^{(3)} = \oint_{\partial\Sigma_3} d^2\sigma \chi_i \wedge dX^i + \frac{1}{3!} \int_{\Sigma_3} d^3\sigma (R^{ijk} \chi_i \wedge \chi_j \wedge \chi_k - R^{ijk} f_i \chi_j \wedge \phi_k + R^{ijk} f_i f_j C_k) \quad (4.35)$$

where  $\chi_i^{(1)} = \chi_i$  is a degree 0 one-form,  $\chi_i^{(0)} = f_i$  is a degree 1 function,  $\chi_i^{(2)} = \phi_i$  is a degree  $-1$  two-form and  $\chi_i^{(3)} = C_i$  is a degree  $-2$  three-form. Both  $f_i$  and  $\phi_i$  vanish on the boundary, due to the boundary gauge fixing. The equations of motion of the three non-zero degree fields sets the last two bulk terms to zero, and they are consistent with each other. Now we introduce a boundary term given by the inverse of a target space metric  $g^{ij}$  since we need  $\chi_i$  to be non-zero on the boundary. Finally we arrive at the action containing only degree 0 fields:

$$\mathcal{S}_{0,R;\text{gf}}^{(3)} = \oint_{\partial\Sigma_3} d^2\sigma \chi_i \wedge dX^i + \frac{1}{3!} \int_{\Sigma_3} d^3\sigma R^{ijk} \chi_i \wedge \chi_j \wedge \chi_k + \oint_{\partial\Sigma_3} d^2\sigma g^{ij} \chi_i \wedge * \chi_j, \quad (4.36)$$

where  $*$  is the Hodge duality operator corresponding to a chosen metric on the membrane worldvolume  $\Sigma_3$ . This is precisely the string sigma-model derived in [21] which quantizes the non-geometric  $R$ -flux background.

## 5 DFT membrane sigma-models

Double field theory is a manifestly T-duality invariant theory, in which non-geometric backgrounds can be described naturally. It has its spacetime doubled, where the original coordinates conjugate to momentum modes are supplemented by dual coordinates conjugate to winding modes of closed strings. On the other hand, flux backgrounds and generalized geometry in closed string theory can be studied within the context of open topological membrane sigma-models, which uplift string theory on their boundary to bulk membrane theories. These membrane sigma-models fit nicely into the framework of the AKSZ construction of membrane sigma-models: the Courant bracket structure of generalized diffeomorphisms corresponds to the derived bracket of Courant sigma-models. The introduction of an analogous correspondence in double field theory appears in [35], where DFT algebroids are defined as the appropriate double field theory analogues of Courant algebroids in generalized geometry; they have the C-bracket as their bracket operation, and reduce to a Courant algebroid after imposing the strong section constraints. DFT algebroids can be implemented within topological membrane sigma-models, which can be obtained by reducing (or projecting) larger AKSZ sigma-models. In this section we describe the construction of DFT membrane sigma-models in AKSZ theory.

### 5.1 DFT algebroids

The definition of DFT algebroid starts with the large Courant algebroid which is a straightforward doubled version of a general Courant algebroid. We formulate the definition from the graded symplectic geometry viewpoint, since this is explicitly relevant in AKSZ constructions. For simplicity we consider the case when the large Courant algebroid is a Courant algebroid corresponding to the QP2-manifold  $T^*[2]T[1]T^*M$ , where the doubling of the original base manifold  $M$  appears as the total space of the cotangent bundle  $T^*M$ . We use the doubled index  $I = 1, \dots, 2d$  to label coordinates on the base space  $T^*M$ , which can be split into the first  $d$  indices  $I = 1, \dots, d$ , which are the original indices labelling coordinates on  $M$ , and the second  $d$  dual indices  $I = d+1, \dots, 2d$  labelling the covectors of  $T^*M$ ; both sets of indices are labeled by  $i = 1, \dots, d$ .

The symplectic form coming from (4.1) is

$$\begin{aligned}\omega_{\text{DFT}} &= dX^I \wedge dF_I + d\chi_I \wedge d\psi^I \\ &= dX^i \wedge dF_i + d\tilde{X}_i \wedge d\tilde{F}^i + d\chi_i \wedge d\psi^i + d\tilde{\chi}^i \wedge d\tilde{\psi}_i,\end{aligned}\tag{5.1}$$

where the splitting of a general doubled coordinate  $\phi^I$  has been used:  $\phi^I = (\phi^i, \tilde{\phi}_i)$ . As we know from (4.5) a degree one function  $A^I \chi_I + \alpha_I \psi^I$  corresponds to a section of  $T(T^*M) \oplus T^*(T^*M)$  symbolically as

$$A^i \chi_i + \tilde{A}_i \tilde{\chi}^i + \alpha_i \psi^i + \tilde{\alpha}^i \tilde{\psi}_i \quad \longleftrightarrow \quad A^i \frac{\partial}{\partial X^i} + \tilde{A}_i \frac{\partial}{\partial \tilde{X}_i} + \alpha_i dX^i + \tilde{\alpha}^i d\tilde{X}_i,\tag{5.2}$$

and the derived brackets (4.3) with a given general Hamiltonian (4.2):

$$\begin{aligned}\gamma_{\text{DFT}} &= \rho^I{}_J F_I \psi^J + \beta^{IJ} F_I \chi_J + \frac{1}{3!} T^{(0)}{}_{IJK} \psi^I \psi^J \psi^K \\ &\quad + \frac{1}{2} T^{(1)}{}_{IJ}{}^K \psi^I \psi^J \chi_K + \frac{1}{2} T^{(2)}{}_I{}^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} T^{(3)}{}^{IJK} \chi_I \chi_J \chi_K,\end{aligned}\tag{5.3}$$

define a Courant algebroid on  $T(T^*M) \oplus T^*(T^*M)$ .

The DFT algebroid is based on the projection to DFT vectors, which halves the number of degree 1 coordinates. We introduce a new basis for the subspace of degree 1 fields spanned by  $\chi_I$  and  $\psi^I$  given as

$$\tau_{\pm}^I = \frac{1}{2} (\psi^I \pm \eta^{IJ} \chi_J) , \quad (5.4)$$

where  $\eta_{IJ}$  is the  $O(d, d)$ -invariant constant metric

$$\eta_{IJ} = \begin{pmatrix} 0 & \delta^i_j \\ \delta_i^j & 0 \end{pmatrix} . \quad (5.5)$$

The projection  $\mathfrak{p}_+$  to the subspace  $L_+$  spanned by  $\tau_+^I$  yields the projection to DFT vectors, which are vectors under  $O(d, d)$ . The corresponding sub-bundle of  $T(T^*M) \oplus T^*(T^*M)$  is also denoted  $L_+$ .

For the symplectic structure  $\omega_{\text{DFT}}$  the projection means

$$d\chi_I \wedge d\psi^I = \eta_{IJ} d\tau_+^I \wedge d\tau_+^J - \eta_{IJ} d\tau_-^I \wedge d\tau_-^J \xrightarrow{\mathfrak{p}_+} \eta_{IJ} d\tau_+^I \wedge d\tau_+^J . \quad (5.6)$$

The coordinates  $\tau_+^I$  are counted twice in the symplectic structure compared to the original (5.1), so we halve their contribution in the symplectic structure solely:

$$\omega_{\text{DFT},+} = dX^I \wedge dF_I + \frac{1}{2} \eta_{IJ} d\tau_+^I \wedge d\tau_+^J . \quad (5.7)$$

For the Liouville potential we take  $\vartheta_{\text{DFT},+} = F_I dX^I - \frac{1}{2} \eta_{IJ} \tau_+^I d\tau_+^J$ . To specify its zero locus  $\mathcal{L}_{\text{DFT},+}$  as a Lagrangian submanifold, we choose a polarisation which is defined by a projector  $\mathcal{P}$  on  $L_+$  of rank  $d$  that is maximally isotropic with respect to the  $O(d, d)$  metric (5.5):

$$\mathcal{P}^i_K \eta^{KL} \mathcal{P}^j_L = 0 . \quad (5.8)$$

It acts on the basis  $\tau_+^I$  to give degree 1 coordinates

$$\tau^i = \mathcal{P}^i_J \tau_+^J \quad (5.9)$$

which span a  $d$ -dimensional subspace of  $L_+$ ; then  $\mathcal{L}_{\text{DFT},+} := \{F_I = 0, \tau^i = 0\}$ . Different polarizations define different Lagrangian submanifolds which are all related by  $O(d, d)$  transformations: Acting with  $\mathcal{O} \in O(d, d)$  changes the polarization as

$$\left( \frac{\mathcal{P}}{\tilde{\mathcal{P}}} \right) \mapsto \left( \frac{\mathcal{P}'}{\tilde{\mathcal{P}}'} \right) = \left( \frac{\mathcal{P}}{\tilde{\mathcal{P}}} \right) \mathcal{O} , \quad (5.10)$$

where  $\tilde{\mathcal{P}} = 1 - \mathcal{P}$  is the complementary projector.

The Hamiltonian is projected to the subspace  $L_+$  as

$$\gamma_{\text{DFT},+} = (\rho_+)^I_J F_I \tau_+^J + \frac{1}{3!} (T_+)_{IJK} \tau_+^I \tau_+^J \tau_+^K , \quad (5.11)$$

where the new structure functions are defined by

$$(\rho_+)^I_J = \rho^I_J + \beta^{IK} \eta_{KJ} , \quad (5.12)$$

and

$$\begin{aligned} (T_+)_{IJK} &= T^{(0)}_{IJK} + 3T^{(1)}_{[IJ}{}^{K'} \eta_{K]K'} + 3T^{(2)}_{[I}{}^{J'K'} \eta_{J']J} \eta_{K'|K]} \\ &\quad + T^{(3)I'J'K'} \eta_{I'[I} \eta_{J']J} \eta_{K'|K]} . \end{aligned} \quad (5.13)$$

The Hamiltonian  $\gamma_{\text{DFT},+}$  defined by these functions does not necessarily satisfy the master equation, despite the fact that the original Hamiltonian  $\gamma_{\text{DFT}}$  of the large Courant algebroid defined by  $\rho^I{}_J$ ,  $\beta^{IJ}$ ,  $T^{(0)}{}_{IJK}$ ,  $T^{(1)}{}_{IJ}{}^K$ ,  $T^{(2)}{}_I{}^{JK}$  and  $T^{(3)}{}^{IJK}$  does.

The C-bracket is defined on DFT vectors of  $L_+$ , which correspond to the degree 1 functions  $A$  in the subspace spanned by  $e_I = \eta_{IJ} \tau_+^J$ :

$$A = A^I e_I = \frac{1}{2} A^I (\chi_I + \eta_{IJ} \psi^J) . \quad (5.14)$$

It can be obtained from derived brackets of the QP2-manifold together with the symmetric pairing and anchor map as

$$\begin{aligned} \llbracket A_1, A_2 \rrbracket_{L_+} &= -\frac{1}{2} (\{\{A_1, \gamma_{\text{DFT},+}\}, A_2\} - \{\{A_2, \gamma_{\text{DFT},+}\}, A_1\}) , \\ \langle A_1, A_2 \rangle_{L_+} &= \{A_1, A_2\} , \\ \rho_+(A) &= \{A, \{\gamma_{\text{DFT},+}, \cdot\}\} , \end{aligned} \quad (5.15)$$

for DFT vectors  $A$ ,  $A_1$  and  $A_2$ . Since the master equation does not hold for  $\gamma_{\text{DFT},+}$ , the algebraic structure is not that of a Courant algebroid, but called a DFT algebroid in [35]: A *DFT algebroid* on  $T^*M$  is a vector bundle  $L_+$  of rank  $2d$  over  $T^*M$  equipped with a non-degenerate symmetric form  $\langle \cdot, \cdot \rangle_{L_+}$  on its fibres, an anchor map  $\rho_+ : L_+ \rightarrow T(T^*M)$ , and a skew-symmetric bracket of sections  $\llbracket \cdot, \cdot \rrbracket_{L_+}$ , called the C-bracket, which together satisfy

$$\begin{aligned} \langle \mathcal{D}_+ f, \mathcal{D}_+ g \rangle_{L_+} &= \frac{1}{4} \langle df, dg \rangle_{L_+} , \\ \llbracket A, fB \rrbracket_{L_+} &= f \llbracket A, B \rrbracket_{L_+} + (\rho_+(A)f) B - \langle A, B \rangle_{L_+} \mathcal{D}_+ f , \\ \rho_+(C) \langle A, B \rangle_{L_+} &= \langle \llbracket C, A \rrbracket_{L_+} + \mathcal{D}_+ \langle C, A \rangle_{L_+}, B \rangle_{L_+} + \langle A, \llbracket C, B \rrbracket_{L_+} + \mathcal{D}_+ \langle C, B \rangle_{L_+} \rangle_{L_+} , \end{aligned} \quad (5.16)$$

for all sections  $A, B, C \in C^\infty(T^*M, L_+)$  and functions  $f, g \in C^\infty(T^*M)$ , where  $\mathcal{D}_+ : C^\infty(T^*M) \rightarrow C^\infty(T^*M, L_+)$  is the derivative defined through  $\langle \mathcal{D}_+ f, A \rangle_{L_+} = \frac{1}{2} \rho_+(A)f$ .

## 5.2 AKSZ construction of DFT membrane sigma-models

The large Courant sigma-model constructed by AKSZ theory corresponds to the large Courant algebroid in the spirit of §4. Then we execute the projection by  $\mathfrak{p}_+$  on the level of AKSZ fields, which means selecting a special submanifold (by projection to DFT vectors) containing half of the ghost number 1 fields. This method is quite natural, because fields with identical properties appear twice, and we keep only one field of each identical pair. Note that there are infinitely many possibilities to perform the reduction on ghost number 1 fields, but only this projection to DFT vectors gives the right C-bracket structure of double field theory. One can think of the other reductions as a class of duality transformations, which leads out of the realm of the original physical double field theory.

The AKSZ action of the large Courant sigma-model corresponding to (5.3) is defined by

$$\begin{aligned} \mathcal{S}_{\text{DFT}}^{(3)} &= \int_{T[1]\Sigma_3} d^3 \hat{z} \left( F_I D X^I - \chi_I D \psi^I + \rho^I{}_J F_I \psi^J + \beta^{IJ} F_I \chi_J + \frac{1}{3!} T^{(0)}{}_{IJK} \psi^I \psi^J \psi^K \right. \\ &\quad \left. + \frac{1}{2} T^{(1)}{}_{IJ}{}^K \psi^I \psi^J \chi^K + \frac{1}{2} T^{(2)}{}_I{}^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} T^{(3)}{}^{IJK} \chi_I \chi_J \chi_K \right) . \end{aligned} \quad (5.17)$$

The BV symplectic structure coming from (5.1) is given by

$$\begin{aligned}\omega_{\text{DFT}} &= \int_{T[1]\Sigma_3} d^3\hat{z} \left( \delta\mathbf{X}^I \delta\mathbf{F}_I + \delta\chi_I \delta\psi^I \right) \\ &= \int_{T[1]\Sigma_3} d^3\hat{z} \left( \delta\mathbf{X}^I \delta\mathbf{F}_I + \eta_{IJ} \delta\tau_+^I \delta\tau_+^J - \eta_{IJ} \delta\tau_-^I \delta\tau_-^J \right),\end{aligned}\tag{5.18}$$

where we performed a duality transformation originating from (5.4):

$$\tau_{\pm}^I = \frac{1}{2} (\psi_I \pm \eta_{IJ} \chi^J).\tag{5.19}$$

Now we restrict the superfields to the submanifold  $\tau_-^I = 0$ . This is not a partial gauge fixing, since both fields and antifields are set to zero. The corresponding action is given by

$$\mathcal{S}_{\text{DFT},+}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_I \mathbf{D}\mathbf{X}^I - \eta_{IJ} \tau_+^I \mathbf{D}\tau_+^J + (\rho_+)^I{}_J \mathbf{F}_I \tau_+^J + \frac{1}{3!} (\mathbf{T}_+)_{IJK} \tau_+^I \tau_+^J \tau_+^K \right).\tag{5.20}$$

The reason we do not define this action directly from a DFT algebroid is that the action  $\mathcal{S}_{\text{DFT},+}^{(3)}$  does not satisfy the BV master equation, so it cannot be constructed by AKSZ theory, nor does it define a BV quantized sigma-model. In order for it to define a BV quantized theory or an AKSZ theory we have to impose additional conditions on the structure functions  $(\rho_+)^I{}_J$  and  $(\mathbf{T}_+)_{IJK}$  coming from the BV master equation for the reduced action. We will use this method below in our study of the topological A- and B-models within the framework of double field theory.

## 6 Double field theory for the A- and B-models

There have been several attempts to find an AKSZ sigma-model for both the topological A- and B-models, which unifies them within the framework of generalized geometry. These were based on AKSZ constructions for generalized complex geometry, which describes the Kähler structure of A-model and the complex structure of the B-model within one generalized complex structure. In the remainder of this paper we continue their study within the context of double field theory.

### 6.1 Doubled Poisson sigma-model

We shall start by proposing an AKSZ Poisson sigma-model with doubled target space coordinates, which gives the A- and B-models separately. Let us consider an AKSZ Poisson sigma-model with target QP1-manifold  $T^*[1]T^*M$  with degree 0 and degree 1 coordinates

$$X^I = \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} \quad \text{and} \quad \chi_I = \begin{pmatrix} \chi_i \\ \tilde{\chi}^i \end{pmatrix}\tag{6.1}$$

respectively. The doubled Poisson structure is denoted by  $\Omega^{IJ}$ , and it depends on both degree 0 coordinates  $X^i$  and  $\tilde{X}_i$ . The symplectic form and AKSZ action are as introduced in §3.1:

$$\omega_{\text{D2}} = d\chi_I \wedge dX^I,\tag{6.2}$$

and

$$\mathcal{S}_{\Omega}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_I \mathbf{D}\mathbf{X}^I + \frac{1}{2} \Omega^{IJ} \chi_I \chi_J \right).\tag{6.3}$$

The master equation imposes the Poisson condition for  $\Omega^{IJ}$ :

$$\Omega^{[I} \partial_L \Omega^{JK]} = 0,\tag{6.4}$$



where the doubled derivative  $\partial_I$  is defined by

$$\partial_I = \begin{pmatrix} \partial/\partial X^i \\ \partial/\partial \tilde{X}_i \end{pmatrix}. \quad (6.5)$$

We will now show that particular choices of  $\Omega^{IJ}$  give the A- or B-models.

A-model. The A-model is obtained by using the doubled Poisson structure

$$\Omega_A^{IJ} = \begin{pmatrix} \pi^{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.6)$$

where the bivector  $\pi^{ij}$  only depends on  $X^i$ . The AKSZ action  $\mathcal{S}_{\Omega_A}^{(2)}$  defined by  $\Omega_A^{IJ}$  is given by

$$\mathcal{S}_{\Omega_A}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i \mathbf{D}X^i + \tilde{\chi}^i \mathbf{D}\tilde{X}_i + \frac{1}{2} \pi^{ij} \chi_i \chi_j \right), \quad (6.7)$$

where the additional term  $\tilde{\chi}^i \mathbf{D}\tilde{X}_i$  can be removed with a partial gauge fixing. Thus it yields the original Poisson sigma-model, which is the AKSZ construction of the A-model (3.9). Then the constraint (6.4) reduces to the original constraint that  $\pi$  defines a Poisson structure on  $M$ .

B-model. The B-model cannot be obtained simultaneously with the A-model, as it arises from a different doubled Poisson structure. The AKSZ construction given by a constant complex structure in (3.27) can be obtained using

$$\Omega_B^{IJ} = \begin{pmatrix} 0 & J^i_j \\ -J^j_i & 0 \end{pmatrix}. \quad (6.8)$$

It gives the AKSZ construction of the B-model after the sign flip  $\tilde{X}_i \rightarrow -\tilde{X}_i$ . The AKSZ construction of the B-model in (3.25) can be derived directly with the choice  $J^i_j = i\delta^i_j$  from the doubled Poisson sigma-model using

$$\Omega_B'^{IJ} = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}. \quad (6.9)$$

The AKSZ formulation of the B-model for an arbitrary complex structure  $J^i_j$ , which only depends on  $X^i$ , is given in (3.30). Our doubled Poisson sigma-model also includes this construction and it is given by choosing

$$\Omega_J^{IJ} = \begin{pmatrix} 0 & J^i_j \\ -J^j_i & -2\partial_{[i}J^k_{j]} \tilde{X}_k \end{pmatrix} \quad (6.10)$$

after the sign flip  $\tilde{X}_i \rightarrow -\tilde{X}_i$ . The constraint (6.4) gives the same constraint as in the original construction, which is the integrability condition (3.28).<sup>3</sup>

General doubled Poisson structure. We write a general doubled Poisson structure in the block matrix form

$$\Omega_G^{IJ} = \begin{pmatrix} \mathbf{P}^{ij} & \mathbf{J}^i_j \\ -\mathbf{J}^j_i & \mathbf{Q}_{ij} \end{pmatrix}, \quad (6.11)$$

where the blocks  $\mathbf{P}^{ij}$ ,  $\mathbf{J}^i_j$  and  $\mathbf{Q}_{ij}$  are constrained by (6.4), and they depend on both  $X^i$  and  $\tilde{X}_i$ . The corresponding AKSZ action is

$$\mathcal{S}_{\Omega_G}^{(2)} = \int_{T[1]\Sigma_2} d^2\hat{z} \left( \chi_i \mathbf{D}X^i + \tilde{\chi}^i \mathbf{D}\tilde{X}_i + \frac{1}{2} \mathbf{P}^{ij} \chi_i \chi_j + \mathbf{J}^i_j \chi_i \tilde{\chi}^j + \frac{1}{2} \mathbf{Q}_{ij} \tilde{\chi}^i \tilde{\chi}^j \right). \quad (6.12)$$

---

<sup>3</sup>Since the complex structure appears as a Poisson structure on doubled space, it can be quantized using the Cattaneo-Felder approach [53]. In particular, the case of a constant complex structure can be quantized in closed form analogously to the Moyal star-product.

It is tempting to try to relate  $\Omega_G^{IJ}$  to the general complex structure  $\mathbb{J}^I{}_J$  in (3.21), but the identities (3.20) are not equivalent to (6.4). We will return to this problem later. It is also interesting to note that the action  $\mathcal{S}_{\Omega_G}^{(2)}$  gives the Zucchini model (3.18) if we replace  $\tilde{\chi}^i$  with  $\mathbf{D}\mathbf{X}^i$ , and nothing depends on  $\tilde{X}_i$ . But they are different BV theories with different constraints on their block structures.

## 6.2 Large contravariant Courant sigma-model

We have seen that the A- and B-models on the AKSZ level appear to be two different particular cases of the same two-dimensional AKSZ theory on a doubled target space. These Poisson sigma-models can be uplifted to the membrane level as a contravariant Courant sigma-model from §4.3, which gives them back on the boundary in the exact gauge. The novelty of the membrane description is that one can introduce flux terms in the bulk. We shall now study the doubled contravariant Courant sigma-model with doubled Poisson structures introduced in §6.1. These AKSZ constructions will play a similar role in the study of the A- and B-models within double field theory as the large Courant sigma-model in §5.

The BV symplectic form of the contravariant Courant sigma-model in doubled space with QP2-manifold  $T^*[2]T[1]T^*M$  is given by (5.18). The AKSZ action comes from (4.31) and is given by

$$\begin{aligned} \mathcal{S}_{\Omega, \mathcal{R}}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_I \mathbf{D}\mathbf{X}^I - \chi_I \mathbf{D}\psi^I + \Omega^{IJ} \mathbf{F}_I \chi_J \right. \\ \left. - \frac{1}{2} \partial_I \Omega^{JK} \psi^I \chi_J \chi_K + \frac{1}{3!} \mathcal{R}^{IJK} \chi_I \chi_J \chi_K \right), \end{aligned} \quad (6.13)$$

with the definition of a general three-vector flux  $\mathcal{R}^{IJK}$  on  $T^*M$ , which is allowed in the contravariant Courant sigma-model.

A-model. The doubled Poisson structure  $\Omega_A^{IJ}$  in (6.6) gives the original contravariant Courant sigma-model action (4.31) on  $M$  after the gauge fixing  $\tilde{\mathbf{F}}^i = 0$  and  $\tilde{\chi}^i = 0$ , which leaves only the R-flux  $\mathcal{R}^{ijk} = R^{ijk}$ . One needs to assume that  $R^{ijk}$  only depends on  $X^i$  in order to reduce the action purely to  $M$ . We have already seen that the contravariant Courant sigma-model in the exact gauge further reduces to the Poisson sigma-model formulation of the A-model in §4.3.

B-model. The doubled Poisson structure  $\Omega_B^{IJ}$  defined in (6.9) with vanishing flux  $\mathcal{R} = 0$  gives the standard Courant sigma-model and a ‘dual’ standard Courant sigma-model with action

$$\mathcal{S}_{\Omega'_B, 0}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{F}}^i \mathbf{D}\tilde{\mathbf{X}}_i - \chi_i \mathbf{D}\psi^i - \tilde{\chi}^i \mathbf{D}\tilde{\psi}_i + \mathbf{F}_i \tilde{\chi}^i - \tilde{\mathbf{F}}^i \chi_i \right). \quad (6.14)$$

It can be seen that the standard and the dual standard Courant sigma-models are decoupled from each other (in both the action and the symplectic form), so they can be separately gauge fixed. For example, the gauge  $\tilde{\mathbf{F}}^i = 0$  and  $\psi^i = 0$  yields the standard Courant sigma-model, which is related to the B-model as described in §4.2. The exact gauge defined in §2.2 reads here as

$$\mathbf{F}_i = \mathbf{D}\chi_i, \quad \psi^i = -\mathbf{D}\mathbf{X}^i, \quad \tilde{\mathbf{F}}^i = \mathbf{D}\tilde{\chi}^i \quad \text{and} \quad \tilde{\psi}_i = -\mathbf{D}\tilde{\mathbf{X}}_i, \quad (6.15)$$

and it gives the action of the B-model (3.25) on the boundary:

$$\mathcal{S}_{\Omega'_B, 0; \text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( \chi_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{X}}_i \mathbf{D}\tilde{\chi}^i + \chi_i \tilde{\chi}^i \right) = \mathcal{S}_{\text{B1}}^{(2)}. \quad (6.16)$$

The B-model construction with general constant complex structure is similar: the doubled Poisson structure  $\Omega_B^{IJ}$  in (6.8) leads to the AKSZ action

$$\mathcal{S}_{\Omega_B,0}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{F}}^i \mathbf{D}\tilde{\mathbf{X}}_i - \chi_i \mathbf{D}\psi^i - \tilde{\chi}^i \mathbf{D}\tilde{\psi}_i + J^i_j \mathbf{F}_i \tilde{\chi}^j - J^j_i \tilde{\mathbf{F}}^i \chi_j \right). \quad (6.17)$$

The exact gauge (6.15) gives the B-model construction (3.27) on the boundary:

$$\mathcal{S}_{\Omega_B,0;\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_2} d^2\hat{z} \left( \chi_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{X}}_i \mathbf{D}\tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j \right) = \mathcal{S}_{\text{B2}}^{(2)}. \quad (6.18)$$

Finally, we consider the choice of doubled Poisson structure  $\Omega_J$  from (6.10) which is associated to a non-constant complex structure. The corresponding AKSZ action is given by

$$\begin{aligned} \mathcal{S}_{\Omega_J,0}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} & \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{F}}^i \mathbf{D}\tilde{\mathbf{X}}_i - \chi_i \mathbf{D}\psi^i - \tilde{\chi}^i \mathbf{D}\tilde{\psi}_i + J^i_j \mathbf{F}_i \tilde{\chi}^j - J^j_i \tilde{\mathbf{F}}^i \chi_j \right. \\ & - 2 \partial_{[i} \mathbf{J}^k_{j]} \tilde{\mathbf{X}}_k \tilde{\mathbf{F}}^i \tilde{\chi}^j - \partial_i \mathbf{J}^j_k \psi^i \chi_j \tilde{\chi}^k + \partial_{[i} \mathbf{J}^k_{j]} \tilde{\psi}_k \tilde{\chi}^i \tilde{\chi}^j \\ & \left. + \partial_i \partial_j \mathbf{J}^l_k \tilde{\mathbf{X}}_l \psi^i \tilde{\chi}^j \tilde{\chi}^k \right). \end{aligned} \quad (6.19)$$

The master equation does not give the integrability condition (3.28) for  $J^i_j$  this time. We will use the DFT projection later to obtain the right Courant algebroid whose relations give the integrability condition. In the exact gauge (6.15), the action  $\mathcal{S}_{\Omega_J,0}^{(3)}$  reduces to the boundary action for the non-constant complex structure defined in (3.30) after a sign flip:

$$\mathcal{S}_{\Omega_J,0;\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( \chi_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{X}}_i \mathbf{D}\tilde{\chi}^i + J^i_j \chi_i \tilde{\chi}^j - \partial_j \mathbf{J}^i_k \tilde{\mathbf{X}}_i \tilde{\chi}^j \tilde{\chi}^k \right) = \mathcal{S}_J^{(2)}. \quad (6.20)$$

General doubled Poisson structure. The general AKSZ action (6.13) can be expanded in block form using the general doubled Poisson structure  $\Omega_G^{IJ}$  from (6.11) as

$$\begin{aligned} \mathcal{S}_{\Omega_G,0}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} & \left( \mathbf{F}_i \mathbf{D}\mathbf{X}^i + \tilde{\mathbf{F}}^i \mathbf{D}\tilde{\mathbf{X}}_i - \chi_i \mathbf{D}\psi^i - \tilde{\chi}^i \mathbf{D}\tilde{\psi}_i + \mathbf{P}^{ij} \mathbf{F}_i \chi_j + \mathbf{Q}_{ij} \tilde{\mathbf{F}}^i \tilde{\chi}^j \right. \\ & + \mathbf{J}^i_j \mathbf{F}_i \tilde{\chi}^j - \mathbf{J}^j_i \tilde{\mathbf{F}}^i \chi_j - \frac{1}{2} \partial_i \mathbf{P}^{jk} \psi^i \chi_j \chi_k - \frac{1}{2} \partial_i \mathbf{Q}_{jk} \psi^i \tilde{\chi}^j \tilde{\chi}^k \\ & \left. - \partial_i \mathbf{J}^j_k \psi^i \chi_j \tilde{\chi}^k - \frac{1}{2} \tilde{\partial}^i \mathbf{P}^{jk} \tilde{\psi}_i \chi_j \chi_k - \frac{1}{2} \tilde{\partial}^i \mathbf{Q}_{jk} \tilde{\psi}_i \tilde{\chi}^j \tilde{\chi}^k - \tilde{\partial}^i \mathbf{J}^j_k \tilde{\psi}_i \chi_j \tilde{\chi}^k \right), \end{aligned} \quad (6.21)$$

where the dual derivative is defined by  $\tilde{\partial}^i = \partial/\partial\tilde{\mathbf{X}}_i$ . As expected it reduces in the exact gauge (6.15) on the boundary to the action  $\mathcal{S}_{\Omega_G}^{(2)}$  given by (6.12).

Fluxes in the A- and B-models. Introducing  $\mathcal{R}$ -flux in (6.13) gives four different terms

$$\frac{1}{3!} \mathcal{R}^{IJK} \chi_I \chi_J \chi_K = \frac{1}{3!} \mathbf{R}^{ijk} \chi_i \chi_j \chi_k + \frac{1}{2} \mathbf{Q}_i{}^{jk} \tilde{\chi}^i \chi_j \chi_k + \frac{1}{2} \mathbf{F}_{ij}{}^k \tilde{\chi}^i \tilde{\chi}^j \chi_k + \frac{1}{3!} \mathbf{H}_{ijk} \tilde{\chi}^i \tilde{\chi}^j \tilde{\chi}^k. \quad (6.22)$$

Both geometric and non-geometric fluxes can appear in the membrane formulations of the A- and B-models, but the gauge fixings leave only  $\mathbf{R}^{ijk}$  in the case of the original contravariant Courant sigma-model and  $\mathbf{H}_{ijk}$  in the case of the standard Courant sigma-model. One of the main features of our new construction for the A- and B-models is that it allows for the introduction of four different fluxes. The compatibility condition  $[\Omega, \mathcal{R}]_S = 0$  between the fluxes and the doubled Poisson bivector  $\Omega$  can be derived from the master equation (4.12). The same fluxes (6.22) can be defined in the AKSZ theories (6.21) as well.

### 6.3 Courant sigma-model for generalized complex geometry

So far we have introduced a contravariant Courant sigma-model on doubled space, which reduces to the topological A- and B-models in the exact gauge. We shall now treat it as a large Courant sigma-model and use the projection to DFT vectors from §5, which halves the number of degree 1 coordinates. Explicitly, the degree 1 coordinates  $\chi_I$  and  $\psi^I$  are transformed to  $\tau_{\pm}^I$  in (5.4), which in components can be written as

$$\tau_{\pm}^I = \frac{1}{2} \begin{pmatrix} \psi^i \pm \tilde{\chi}^i \\ \tilde{\psi}_i \pm \chi_i \end{pmatrix}. \quad (6.23)$$

The coordinates  $\tau_-^I$  are projected out by  $\frac{1}{2} \mathbf{p}_+$  in the same way they were in (5.7), hence in Darboux coordinates

$$\tau_+^I = \begin{pmatrix} q^i \\ p_i \end{pmatrix} \quad (6.24)$$

the symplectic structure  $\omega_{\text{DFT}}$  from (5.1) becomes

$$\omega_{\text{DFT}} \xrightarrow{\frac{1}{2} \mathbf{p}_+} \omega_{\text{DFT},+} = dX^i \wedge dF_i + d\tilde{X}_i \wedge d\tilde{F}^i + dq^i \wedge dp_i. \quad (6.25)$$

In the Hamiltonian (5.3) we substitute

$$\chi_i \longrightarrow p_i, \quad \psi^i \longrightarrow q^i, \quad \tilde{\chi}^i \longrightarrow q^i \quad \text{and} \quad \tilde{\psi}_i \longrightarrow p_i, \quad (6.26)$$

which together with the symplectic form  $\omega_{\text{DFT},+}$  reduces the AKSZ action  $\mathcal{S}_{\Omega_{\text{G}},0}^{(3)}$  from (6.21) to the action

$$\begin{aligned} \mathcal{S}_{\Omega_{\text{G}},+}^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( & \mathbf{F}_i D X^i + \tilde{\mathbf{F}}^i D \tilde{X}_i - \mathbf{p}_i D q^i + \mathbf{P}^{ij} \mathbf{F}_i \mathbf{p}_j + \mathbf{Q}_{ij} \tilde{\mathbf{F}}^i q^j \right. \\ & + \mathbf{J}^i_j \mathbf{F}_i q^j - \mathbf{J}^j_i \tilde{\mathbf{F}}^i \mathbf{p}_j - \frac{1}{2} \partial_i \mathbf{P}^{jk} q^i \mathbf{p}_j \mathbf{p}_k - \frac{1}{2} \partial_i \mathbf{Q}_{jk} q^i q^j q^k + \partial_i \mathbf{J}^k_j q^i q^j \mathbf{p}_k \\ & \left. - \frac{1}{2} \tilde{\partial}^i \mathbf{P}^{jk} \mathbf{p}_i \mathbf{p}_j \mathbf{p}_k - \frac{1}{2} \tilde{\partial}^k \mathbf{Q}_{ij} q^i q^j \mathbf{p}_k - \tilde{\partial}^j \mathbf{J}^k_i q^i \mathbf{p}_j \mathbf{p}_k \right), \end{aligned} \quad (6.27)$$

where for simplicity we imposed the  $O(d, d)$ -invariant boundary condition  $(\mathbf{p}_i \mathbf{q}^i)|_{T[1]\partial\Sigma_3} = 0$ .<sup>4</sup>

We reduce the dual coordinates in (6.27) with a gauge fixing  $\tilde{\mathbf{F}}^i = 0$  and assume that none of the blocks  $\mathbf{P}^{ij}$ ,  $\mathbf{J}^i_j$  or  $\mathbf{Q}_{ij}$  depend on  $\tilde{X}_i$ . The resulting action is not necessarily an AKSZ action as it does not satisfy the master equation. Instead we impose the master equation as a further constraint on the blocks in order to satisfy the quantization condition, and we define the reduced action with the constrained blocks, which we write symbolically as

$$\mathbf{P}^{ij} \xrightarrow{\text{master}} \pi^{ij}, \quad \mathbf{J}^i_j \xrightarrow{\text{master}} J^i_j \quad \text{and} \quad \mathbf{Q}_{ij} \xrightarrow{\text{master}} \omega_{ij}. \quad (6.28)$$

The reduced AKSZ action is given by

$$\begin{aligned} \mathcal{S}_{\text{Z}}^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( & \mathbf{F}_i D X^i - \mathbf{p}_i D q^i + \pi^{ij} \mathbf{F}_i \mathbf{p}_j + J^i_j \mathbf{F}_i q^j \right. \\ & \left. - \frac{1}{2} \partial_i \pi^{jk} q^i \mathbf{p}_j \mathbf{p}_k - \frac{1}{2} \partial_i \omega_{jk} q^i q^j q^k + \partial_i J^k_j q^i q^j \mathbf{p}_k \right). \end{aligned} \quad (6.29)$$

<sup>4</sup>This boundary condition will be compatible with our further reduction to the B-model, but not to the A-model. To be compatible with the latter reduction we need to start with a different kinetic term for the large Courant algebroid, in order to obtain the right kinetic term of the Courant sigma-model for the generalized complex structure.

The special property of this AKSZ action is that in the exact gauge

$$F_i = Dp_i \quad \text{and} \quad q^i = -DX^i \quad (6.30)$$

it gives the Zucchini action (3.18):

$$\mathcal{S}_{Z,\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \left( p_i DX^i + \frac{1}{2} \pi^{ij} p_i p_j + \frac{1}{2} \omega_{ij} DX^i DX^j - J^i_j p_i DX^j \right) = \mathcal{S}_Z^{(2)}, \quad (6.31)$$

on the boundary after the sign flip  $J^i_j \rightarrow -J^i_j$ .

One may naturally expect that the master equation for  $\mathcal{S}_Z^{(3)}$  will give the constraints of a generalized complex structure (3.20) as the Zucchini model does, but this is not precisely true. If  $\omega_{ij}$  vanishes then we get the same identities as those of a generalized complex structure with  $\omega = 0$ , or if we set  $\omega = \pi^{-1}$  then  $d\omega = 0$  and the term involving  $\omega$  vanishes, thus we arrive at the same AKSZ action. Otherwise the  $\omega$  term generally prevents the constraints from being the identities of a generalized complex structure.

Thus we propose a Courant sigma-model

$$\begin{aligned} \mathcal{S}_{\pi,J}^{(3)} = \int_{T[1]\Sigma_3} d^3\hat{z} \left( F_i DX^i - p_i Dq^i + \pi^{ij} F_i p_j + J^i_j F_i q^j \right. \\ \left. - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^k_j q^i q^j p_k \right) \end{aligned} \quad (6.32)$$

for the generalized complex structure

$$\mathbb{J}^I_J = \begin{pmatrix} J^i_j & \pi^{ij} \\ 0 & -J^j_i \end{pmatrix}. \quad (6.33)$$

In the language of symplectic dg-geometry this means that the master equation for the Hamiltonian

$$\gamma_{\pi,J} = \pi^{ij} F_i p_j + J^i_j F_i q^j - \frac{1}{2} \partial_i \pi^{jk} q^i p_j p_k + \partial_i J^k_j q^i q^j p_k \quad (6.34)$$

with the symplectic form

$$\omega_3 = dX^i \wedge dF_i + dq^i \wedge dp_i \quad (6.35)$$

gives the conditions

$$\begin{aligned} \pi^{[i|l} \partial_l \pi^{jk]} &= 0, \\ J^l_i \partial_l \pi^{jk} + 2\pi^{jl} \partial_{[i} J^k_{l]} + \pi^{kl} \partial_l J^j_i - J^j_l \partial_i \pi^{lk} &= 0, \\ J^l_{[i} \partial_l J^k_{j]} - J^k_l \partial_{[i} J^l_{j]} &= 0. \end{aligned} \quad (6.36)$$

The first identity says that  $\pi^{ij}$  satisfies the Poisson condition, the third says  $J^i_j$  satisfies the integrability condition of the original complex structure, and the second identity is an additional compatibility condition needed to combine them into a generalized complex structure.

The Hamiltonian  $\gamma_{\pi,J}$  defines a Courant algebroid over the target space  $M$  for the generalized complex structures (6.33) with the Dorfman bracket and anchor

$$[e_1, e_2]_{D;\pi,J} = \{\{e_1, \gamma_{\pi,J}\}, e_2\} \quad \text{and} \quad \rho(e) = \{e, \{\gamma_{\pi,J}, \cdot\}\} \quad (6.37)$$

where the functions  $e$ ,  $e_1$  and  $e_2$  have degree 1. It would be interesting in its own right to study further this new Courant algebroid structure.

#### 6.4 Dimensional reductions to the A- and B-models

The relation of the Courant sigma-model (6.32) to the A-model is quite straightforward. If we set  $J$  to zero, and only keep  $\pi$  non-zero, the remaining identity from (6.36) is the Poisson condition. The resulting AKSZ action is just that of the contravariant Courant sigma-model, which reduces to the Poisson sigma-model on its boundary in the exact gauge, and thus to the A-model as well.

The relation to the B-model is not immediately apparent. Let  $\pi$  be zero, and  $J$  non-zero. The remaining identity from (6.36) is the integrability condition for the original complex structure  $J$ . The Hamiltonian associated to the resulting AKSZ action is

$$\gamma_{0,J} = J^i_j F_i q^j + \partial_i J^k_j q^i q^j p_k, \quad (6.38)$$

from which a Courant algebroid for a generic complex structure can be derived with the Dorfman bracket and anchor

$$[e_1, e_2]_{D;0,J} = \{\{e_1, \gamma_{0,J}\}, e_2\} \quad \text{and} \quad \rho(e) = \{e, \{\gamma_{0,J}, \cdot\}\} \quad (6.39)$$

respectively, where again  $e$ ,  $e_1$  and  $e_2$  are degree 1 functions. This structure is similar to that of the Poisson Courant algebroid, which is the derived Courant algebroid for a generic Poisson structure.

We apply the dimensional reduction method introduced in §2.3 on the AKSZ action with the original complex structure solely:

$$\mathcal{S}_{0,J}^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} (F_i D X^i - p_i D q^i + J^i_j F_i q^j + \partial_i J^k_j q^i q^j p_k). \quad (6.40)$$

The reduction method requires that the membrane worldvolume  $\Sigma_3$  be a product manifold, hence we apply it in a neighbourhood of the boundary. For this, choose an open subset  $U$  of  $\Sigma_3$  which includes  $\partial\Sigma_3$ :

$$\Sigma_3|_U = \partial\Sigma_3 \times \mathbb{R}^+, \quad (6.41)$$

where  $\mathbb{R}^+$  is the half-line parameterized with coordinate  $t$ , for which the  $t = 0$  points belong to the boundary. Then the worldvolume  $\Sigma_3$  is covered by open sets as

$$\Sigma_3 = U \cup U', \quad (6.42)$$

where the open set  $U'$  does not include the boundary, i.e. it is contained in the bulk interior  $\Sigma_3 \setminus \partial\Sigma_3$ . Then the BV symplectic form is given by the sum of two integrals over the covering sets  $U$  and  $U'$  as<sup>5</sup>

$$\begin{aligned} \omega_3 &= \omega_{3|U} + \omega_{3|U'} \\ &:= \int_{T[1]U} d^3 \hat{z} (\delta X^i \delta F_i + \delta q^i \delta p_i) + \int_{T[1]U'} d^3 \hat{z} (\delta X'^i \delta F'_i + \delta q'^i \delta p'_i), \end{aligned} \quad (6.43)$$

where we have rescaled the fields  $F_i$ ,  $p_i$ ,  $F'_i$  and  $p'_i$  with a suitable partition of unity subordinate to the covering (6.42). These fields are chosen so that the decomposition of the AKSZ action

$$\begin{aligned} \mathcal{S}_{0,J}^{(3)} &= \mathcal{S}_{0,J|U}^{(3)} + \mathcal{S}_{0,J|U'}^{(3)} \\ &:= \int_{T[1]U} d^3 \hat{z} (F_i D X^i - p_i D q^i + J^i_j F_i q^j + \partial_i J^k_j q^i q^j p_k) \\ &\quad + \int_{T[1]U'} d^3 \hat{z} (F'_i D X'^i - p'_i D q'^i + J^i_j F'_i q'^j + \partial_i J^k_j q'^i q'^j p'_k) \end{aligned} \quad (6.44)$$

---

<sup>5</sup>We use the same notation for the boundary fields as well for brevity, but they are not identical to those used earlier.

is independent of the choice of partition of unity.

First we deal with the boundary contributions. They are defined on a product manifold  $\partial\Sigma_3 \times \mathbb{R}^+$ , so we can apply the method of §2.3. We use a uniform notation for an arbitrary superfield  $\phi$ :

$$\phi = \widehat{\phi} + \phi_t \theta^t, \quad (6.45)$$

where the component superfields  $\widehat{\phi}$  and  $\phi_t$  do not depend on the odd coordinate  $\theta^t$  of  $T[1]\mathbb{R}^+$ . The integrals over  $U$  factorize and we get the BV symplectic form

$$\omega_{3|U;\text{gf}} = - \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \int_{\mathbb{R}^+} dt (\delta\widehat{\mathbf{X}}^i \delta(\mathbf{F}_t)_i + \delta\widehat{\mathbf{q}}^i \delta(\mathbf{p}_t)_i), \quad (6.46)$$

where we have used a different gauge fixing and also different antifields than those which were used for the reduction to the A-model: we have set  $\mathbf{X}_t^i$  and  $\mathbf{q}_t^i$  to zero. The gauge fixed boundary action is

$$\begin{aligned} \mathcal{S}_{0,J|U;\text{gf}}^{(3)} = & \oint_{T[1]\partial\Sigma_3} d^2\hat{z} \int_{\mathbb{R}^+} dt (\widehat{\mathbf{F}}_i \partial_t \widehat{\mathbf{X}}^i + \widehat{\mathbf{p}}_i \partial_t \widehat{\mathbf{q}}^i - (\mathbf{F}_t)_i \widehat{\mathbf{D}}\widehat{\mathbf{X}}^i - (\mathbf{p}_t)_i \widehat{\mathbf{D}}\widehat{\mathbf{q}}^i \\ & - \mathbf{J}^i_j (\mathbf{F}_t)_i \widehat{\mathbf{q}}^j + \partial_i \mathbf{J}^k_j \widehat{\mathbf{q}}^i \widehat{\mathbf{q}}^j (\mathbf{p}_t)_k). \end{aligned} \quad (6.47)$$

The first two terms determine the boundary conditions. Integrating out the fields  $\widehat{\mathbf{F}}_i$  and  $\widehat{\mathbf{p}}_i$  imposes the condition that the fields  $\widehat{\mathbf{X}}^i$  and  $\widehat{\mathbf{q}}^i$  are independent of  $t$ , while the zero modes of  $\widehat{\mathbf{F}}_i$  and  $\widehat{\mathbf{p}}_i$  on  $\mathbb{R}^+$  lead to the condition that  $\widehat{\mathbf{X}}^i$  and  $\widehat{\mathbf{q}}^i$  vanish at  $t = 0$  which means they vanish on the boundary.

We introduce the new notations

$$\chi_i = - \int_{\mathbb{R}^+} dt (\mathbf{F}_t)_i, \quad \mathbf{X}^i = \widehat{\mathbf{X}}^i, \quad \widetilde{\mathbf{X}}_i = - \int_{\mathbb{R}^+} dt (\mathbf{p}_t)_i \quad \text{and} \quad \widetilde{\chi}^i = \widehat{\mathbf{q}}^i, \quad (6.48)$$

and rewrite the BV symplectic form and the boundary AKSZ action with them as

$$\omega_{3|U;\text{gf}} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} (\delta\mathbf{X}^i \delta\chi_i + \delta\widetilde{\mathbf{X}}_i \delta\widetilde{\chi}^i), \quad (6.49)$$

and

$$\mathcal{S}_{0,J|U;\text{gf}}^{(3)} = \oint_{T[1]\partial\Sigma_3} d^2\hat{z} (\chi_i \mathbf{D}\mathbf{X}^i + \widetilde{\mathbf{X}}_i \mathbf{D}\widetilde{\chi}^i + \mathbf{J}^i_j \chi_i \widetilde{\chi}^j - \partial_j \mathbf{J}^i_k \widetilde{\mathbf{X}}_i \widetilde{\chi}^j \widetilde{\chi}^k) = \mathcal{S}_J^{(2)}, \quad (6.50)$$

which give the BV symplectic form corresponding to (3.26) and the AKSZ action (3.30) for the B-model after the sign flip  $\widetilde{\mathbf{X}}_i \rightarrow -\widetilde{\mathbf{X}}_i$ . Hence the action  $\mathcal{S}_{0,J}^{(3)}$  defined in (6.40) reduces to the B-model action in a neighbourhood of the boundary  $\partial\Sigma_3$ .

For the bulk contributions, one can gauge fix the bulk fields independently from the boundary fields using the same fields and antifields that were used for the reduction to the A-model. The exact gauge was a gauge fixing on the bulk as well, and not only on the boundary. Hence if we set  $\mathbf{F}'_i$  and  $\mathbf{q}'^i$  to zero, we get a vanishing bulk action  $\mathcal{S}_{0,J|U';\text{gf}}^{(3)}$ , and thus the action  $\mathcal{S}_{0,J}^{(3)}$  in (6.40) can be reduced entirely to the B-model action on the boundary.

We recall that, in all schemes presented in this paper, the reductions to the A- and B-models differ significantly: not only are the gauge choices different, but the antifields are also assigned differently, and the boundary conditions differ as well. But they both appear as boundary AKSZ sigma-models while the bulk fields are gauge fixed completely.

## 7 Topological S-duality in generalized complex geometry

Topological S-duality arises from the physical S-duality of type IIB superstring theory [46], and it relates the A- and B-models on the same Calabi-Yau manifold: D-instanton contributions of one model correspond to perturbative amplitudes of the other model. The string couplings  $g_A$  and  $g_B$  of the A- and B-models are related to each other by

$$g_A = \frac{1}{g_B} \quad (7.1)$$

and the Kähler forms  $k_A$  and  $k_B$  of the two theories are also related by

$$k_A = \frac{k_B}{g_B} . \quad (7.2)$$

In other words, S-duality exchanges the A- and B-models as a weak/strong coupling duality. As an application of the formalism developed in this paper, in this section we shall demonstrate how this duality is realised geometrically in generalized complex geometry using our Courant algebroids and AKSZ sigma-models.

### 7.1 Duality between Poisson and complex structure Courant algebroids

A duality transformation in the language of QP-manifolds is a transformation of supercoordinates which leaves the symplectic structure invariant. One of the simplest non-trivial cases is the renormalization of the fields by a scale transformation: a coordinate is scaled inversely with respect to its dual coordinate. In the following we study the Courant algebroid for generalized complex structures in this context.

The symplectic form  $\omega_3$  given by (6.35) is left invariant under the scale transformation

$$p_i \longmapsto \lambda p_i \quad \text{and} \quad q^i \longmapsto \frac{1}{\lambda} q^i , \quad (7.3)$$

with a constant parameter  $\lambda \in \mathbb{R}$ , which transforms the Hamiltonian  $\gamma_{\pi,J}$  from (6.34) to

$$\gamma_{\pi,J}^\lambda = \lambda \pi^{ij} F_i p_j - \frac{\lambda}{2} \partial_i \pi^{jk} q^i p_j p_k + \frac{1}{\lambda} J^i_j F_i q^j + \frac{1}{\lambda} \partial_i J^k_j q^i q^j p_k . \quad (7.4)$$

The scale transformation has no effect on the identities (6.36), and  $\gamma_{\pi,J}^\lambda$  satisfies the master equation as well.

Now we can take both the large or small  $\lambda$  limit. They give different Courant algebroids, namely the Poisson and the complex structure Courant algebroid respectively:

$$\frac{1}{\lambda} \gamma_{0,J} \xleftarrow{\lambda \ll 1} \gamma_{\pi,J}^\lambda \xrightarrow{\lambda \gg 1} \lambda \gamma_{\pi,0} , \quad (7.5)$$

where the Hamiltonian  $\gamma_{0,J}$  is defined in (6.38) while  $\gamma_{\pi,0}$  is defined in (4.11). After the limits are taken the parameter  $\lambda$  can be scaled back to obtain the original Hamiltonians which are independent of  $\lambda$ .

Thus scaling with  $\lambda$  introduces a type of weak/strong duality, which interpolates continuously between Poisson and complex structure Courant algebroids within the Courant algebroid for generalized complex geometry, and it exchanges them between the two limits. In the following we relate this duality to the topological S-duality between the A- and B-models based on our AKSZ constructions and boundary reductions from §6.



## 7.2 Topological S-duality

In the following we promote our duality to the level of AKSZ constructions. We start with the AKSZ action given by the Hamiltonian  $\lambda \gamma_{\pi, J}$  defined in (7.4):

$$\begin{aligned} \mathcal{S}_{A/B}^{(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( \frac{1}{\lambda} \mathbf{F}_i \mathbf{D} \mathbf{X}^i - \frac{1}{\lambda} \mathbf{p}_i \mathbf{D} \mathbf{q}^i + \pi^{ij} \mathbf{F}_i \mathbf{p}_j - \frac{1}{2} \partial_i \pi^{jk} \mathbf{q}^i \mathbf{p}_j \mathbf{p}_k \right. \\ \left. + \mathbf{J}^i_j \mathbf{F}_i \mathbf{q}^j + \partial_i \mathbf{J}^k_j \mathbf{q}^i \mathbf{q}^j \mathbf{p}_k \right), \end{aligned} \quad (7.6)$$

where we explicitly introduced an overall constant  $1/\lambda$  as a membrane tension in the definition of the action, which does not affect the BV quantization of the sigma-model. Now we perform the scaling duality (7.3). Since it leaves the BV symplectic form invariant, the kinetic terms do not change, only the interaction terms. The scale transformed AKSZ action is given by

$$\begin{aligned} \mathcal{S}_{A/B}^{\lambda(3)} = \int_{T[1]\Sigma_3} d^3 \hat{z} \left( \frac{1}{\lambda} \mathbf{F}_i \mathbf{D} \mathbf{X}^i - \frac{1}{\lambda} \mathbf{p}_i \mathbf{D} \mathbf{q}^i + \lambda \pi^{ij} \mathbf{F}_i \mathbf{p}_j - \frac{\lambda}{2} \partial_i \pi^{jk} \mathbf{q}^i \mathbf{p}_j \mathbf{p}_k \right. \\ \left. + \frac{1}{\lambda} \mathbf{J}^i_j \mathbf{F}_i \mathbf{q}^j + \frac{1}{\lambda} \partial_i \mathbf{J}^k_j \mathbf{q}^i \mathbf{q}^j \mathbf{p}_k \right). \end{aligned} \quad (7.7)$$

The large  $\lambda$  limit gives the contravariant Courant sigma-model without kinetic terms, which reduces to the A-model action given by (3.8) in the exact gauge in the same way that it reduced in §4.3:

$$\mathcal{S}_{A/B}^{\lambda(3)} \xrightarrow{\lambda \gg 1} \frac{\lambda}{2} \oint_{T[1]\partial\Sigma_3} d^2 \hat{z} \pi^{ij} \mathbf{p}_i \mathbf{p}_j. \quad (7.8)$$

Here  $\lambda$  appears as the inverse of the A-model string coupling:

$$\lambda = \frac{1}{g_A}. \quad (7.9)$$

On the other hand, if we take  $\lambda$  to be small, we get the AKSZ action of the complex structure Courant algebroid with an overall membrane tension  $1/\lambda$ , which can be reduced to the B-model on its boundary as in §6.4:

$$\mathcal{S}_{A/B}^{\lambda(3)} \xrightarrow{\lambda \ll 1} \frac{1}{\lambda} \oint_{T[1]\partial\Sigma_3} d^2 \hat{z} \left( \chi_i \mathbf{D} \mathbf{X}^i + \widetilde{\mathbf{X}}_i \mathbf{D} \widetilde{\chi}^i + \mathbf{J}^i_j \chi_i \widetilde{\chi}^j - \partial_j \mathbf{J}^i_k \widetilde{\mathbf{X}}_i \widetilde{\chi}^j \widetilde{\chi}^k \right). \quad (7.10)$$

Here  $\lambda$  appears as the B-model string coupling this time:

$$\lambda = g_B, \quad (7.11)$$

which together with (7.9) gives the relation (7.1). However, it says nothing about the scaling relation of Kähler forms in (7.2). This is due to the fact that the scalings of  $\pi$  and  $J$  are not fixed to each other by the constraints (6.36).

## 8 Conclusions and outlook

In this paper we studied AKSZ formulations of the topological A- and B-models within the framework of double field theory. We presented an observation that the Poisson sigma-model on doubled space describes both the A- and B-models simultaneously, which are given by different choices for the doubled Poisson structures. We uplifted the doubled Poisson sigma-model to the membrane level as a large contravariant Courant sigma-model, which is the three-dimensional AKSZ sigma-model description of a doubled Poisson sigma-model. We showed that it reduces to the doubled

Poisson sigma-model on the boundary if we use the exact gauge, and we have expounded some particular cases given by the reduction which are relevant in AKSZ theory.

We applied the projection to DFT vectors on the large contravariant Courant sigma-model of double field theory, which halves the number of ghost number 1 fields, and we reduced the dual coordinates, which led to the introduction of the Courant sigma-model of a particular class of generalized complex structures, and also its corresponding Courant algebroid. We pointed out that it gives the Zucchini model on the boundary in the exact gauge. We also studied two marginal cases, the purely Poisson and purely complex structure cases, which were reduced to the A- and B-models on their boundaries respectively.

We also proposed an S-duality between Poisson and complex structure Courant algebroids, which originates back to the generalized complex structure, in which they lie as different marginal limits. We promoted the duality to the AKSZ level, and related it to topological S-duality of the A- and B-models.

It would be interesting to study further the appearance of generalized geometry and double field theory in the context of the A- and B-models as they are defined originally in standard (not generalized) Calabi-Yau manifolds. The double field theory formulation of the A- and B-models also allows for the introduction of both geometric and non-geometric fluxes, which would be a further development in the study of its physical relevance, particularly in the context of topological string theory. The fluxes correspond to twist deformations of the proposed Courant algebroids which lead to the introduction of twists of the generalized and original complex structures, which is another avenue for further investigation. Another direction would be to find a Courant algebroid which gives the identities of a general version of generalized complex structure, and to relate it to the double field theory formulations of the A- and B-models. Our S-duality gives a continuous mapping between the A- and B-models, so it would be interesting to investigate whether the intermediate membrane theory has a clearer physical relevance. A surprising observation is that our S-duality arises from the T-duality inspired generalized complex geometry, thus it raises the question as to whether there is a physical origin behind this relation or whether it is just a coincidence found in the topological field theories.

## Acknowledgments

This work was completed while R.J.S. was visiting the Institut Montpellierain Alexander Grothendieck in Montpellier, France during May/June 2018; he warmly thanks the staff there for hospitality and for providing a stimulating environment, and in particular Damien Calaque for the invitation and numerous discussions. This work was supported by the Action MP1405 QSPACE, funded by the European Cooperation in Science and Technology (COST). The work of Z.K. was supported by the Hungarian Research Fund (OTKA). The work of R.J.S. was supported by the Consolidated Grant ST/P000363/1 from the UK Science and Technology Facilities Council, and by the Institut Universitaire de France.

## A Differential calculus of graded functionals

A nice and elaborate summary of formulas from differential calculus on graded manifolds can be found in [8]. In the following we rely on this treatment and only review formulas with regard to graded functionals.

The space of superfields is once again the mapping space

$$\mathcal{M} = \text{Map}(T[1]\Sigma_d, \mathcal{M}) , \tag{A.1}$$

and the superfield coordinates on  $\mathcal{M}$  are defined via the coordinates  $\hat{z}^{\hat{\mu}} \in T[1]\Sigma_d$ ,  $\hat{X}^{\hat{i}} \in \mathcal{M}$  as

$$\hat{\mathbf{X}}^{\hat{i}}(\hat{z}) = \phi^*(\hat{X}^{\hat{i}})(\hat{z}) , \quad (\text{A.2})$$

where  $\phi \in \mathcal{M}$  is an arbitrary superfield. We use the notation  $|\hat{i}|$  for the degree of  $\hat{X}^{\hat{i}}$ , and also for the ghost number of  $\hat{\mathbf{X}}^{\hat{i}}$ .

A graded functional<sup>6</sup> of the superfields  $\hat{\mathbf{X}}^{\hat{i}}(\hat{z})$  is defined by a function  $F(\hat{X})$  on  $\mathcal{M}$  as

$$\mathbf{F} = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(F) = \int_{T[1]\Sigma_d} d^d \hat{z} F(\hat{\mathbf{X}}(\hat{z})) , \quad (\text{A.3})$$

and it has ghost number  $|F| - d$ , where  $|F|$  denotes the ghost number of  $F(\hat{X}(\hat{z}))$ . The definition of a graded functional  $n$ -form  $\alpha$  with ghost number  $|\alpha| - d$  is analogous and given by an  $n$ -form  $\alpha$  on  $\mathcal{M}$  as

$$\begin{aligned} \alpha &= \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(\alpha) \\ &= \int_{T[1]\Sigma_d} d^d \hat{z} \delta \hat{\mathbf{X}}^{\hat{i}_1}(\hat{z}) \cdots \delta \hat{\mathbf{X}}^{\hat{i}_n}(\hat{z}) \alpha_{\hat{i}_1 \dots \hat{i}_n}(\hat{\mathbf{X}}(\hat{z})) \\ &:= \int_{T[1]\Sigma_d} d^d \hat{z} \alpha(\hat{\mathbf{X}}(\hat{z})) . \end{aligned} \quad (\text{A.4})$$

The exterior product of two graded functional forms  $\alpha$  and  $\beta$  also gives a graded functional form with ghost number  $|\alpha| + |\beta| - d$  which reads as

$$\alpha \wedge \beta = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(\alpha \wedge \beta) = \int_{T[1]\Sigma_d} d^d \hat{z} \alpha(\hat{\mathbf{X}}(\hat{z})) \beta(\hat{\mathbf{X}}(\hat{z})) . \quad (\text{A.5})$$

As can be seen, it depends on the integration measure. The one-form local basis elements have ghost number  $|\hat{i}| + 1 - d$  and they are given by

$$\delta \hat{\mathbf{X}}^{\hat{i}} = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(d\hat{X}^{\hat{i}}) = \int_{T[1]\Sigma_d} d^d \hat{z} \delta \hat{\mathbf{X}}^{\hat{i}}(\hat{z}) . \quad (\text{A.6})$$

We can write the  $n$ -form  $\alpha$  with the exterior product as

$$\alpha = \delta \hat{\mathbf{X}}^{\hat{i}_1} \wedge \cdots \wedge \delta \hat{\mathbf{X}}^{\hat{i}_n} \wedge \alpha_{\hat{i}_1 \dots \hat{i}_n} , \quad (\text{A.7})$$

where the scalar functional is given by

$$\alpha_{\hat{i}_1 \dots \hat{i}_n} = \int_{T[1]\Sigma_d} d^d \hat{z} \alpha_{\hat{i}_1 \dots \hat{i}_n}(\hat{\mathbf{X}}(\hat{z})) . \quad (\text{A.8})$$

A graded vector functional  $\mathbf{V}$  with ghost number  $|\mathbf{V}| - d$  can be written in the form

$$\mathbf{V} = \int_{T[1]\Sigma_d} d^d \hat{z} V^{\hat{i}}(\hat{\mathbf{X}}(\hat{z})) \frac{\delta}{\delta \hat{\mathbf{X}}^{\hat{i}}(\hat{z})} , \quad (\text{A.9})$$

which acts as a left functional derivative

$$\vec{\mathbf{V}} \mathbf{F} = \int_{T[1]\Sigma_d} d^d \hat{z} V^{\hat{i}}(\hat{\mathbf{X}}(\hat{z})) \frac{\vec{\delta} \mathbf{F}}{\delta \hat{\mathbf{X}}^{\hat{i}}(\hat{z})} \quad (\text{A.10})$$

---

<sup>6</sup>We only consider non-local graded functionals, hence the kernel function  $F(\hat{X})$  can be taken to be an ordinary graded function of  $\hat{X}$ .

on graded functionals  $\mathbf{F}$ , and is defined as

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbf{F}[\hat{\mathbf{X}} + \epsilon \hat{\boldsymbol{\eta}}] - \mathbf{F}[\hat{\mathbf{X}}]}{\epsilon} =: \int_{T[1]\Sigma_d} d^d \hat{z} \hat{\boldsymbol{\eta}}^i(\hat{z}) \frac{\vec{\delta} \mathbf{F}}{\delta \hat{\mathbf{X}}^i(\hat{z})}, \quad (\text{A.11})$$

where  $\hat{\boldsymbol{\eta}}^i$  is an arbitrary superfield with the same degree as  $\hat{\mathbf{X}}^i$ . The functional derivative with respect to  $\hat{\mathbf{X}}^i(\hat{z})$  has ghost number  $-|\hat{i}| + d$ . As graded functionals are non-local, the functional derivatives are given by ordinary derivatives of the kernel function as

$$\frac{\vec{\delta} \mathbf{F}}{\delta \hat{\mathbf{X}}^i(\hat{z})} = \left. \frac{\partial \mathbf{F}}{\partial \hat{\mathbf{X}}^i} \right|_{\hat{\mathbf{X}}^i(\hat{z})}. \quad (\text{A.12})$$

The interior product is given by contraction with the graded vector functional  $\mathbf{V}$ :

$$\iota_{\mathbf{V}} = \int_{T[1]\Sigma_d} d^d \hat{z} V^i(\hat{\mathbf{X}}(\hat{z})) \frac{\vec{\delta}}{\delta (\delta \hat{\mathbf{X}}^i(\hat{z}))}, \quad (\text{A.13})$$

which acts on exterior products as

$$\iota_{\mathbf{V}}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = \iota_{\mathbf{V}} \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^{(|V|+1)|\boldsymbol{\alpha}|} \boldsymbol{\alpha} \wedge \iota_{\mathbf{V}} \boldsymbol{\beta}. \quad (\text{A.14})$$

The de Rham differential can be written in the form

$$\boldsymbol{\delta} = \int_{T[1]\Sigma_d} d^d \hat{z} \boldsymbol{\delta} \hat{\mathbf{X}}^i(\hat{z}) \frac{\vec{\delta}}{\delta \hat{\mathbf{X}}^i(\hat{z})}. \quad (\text{A.15})$$

It has the following properties:

$$\begin{aligned} \boldsymbol{\delta} \mathbf{F} &= \lim_{\epsilon \rightarrow 0} \frac{\mathbf{F}[\hat{\mathbf{X}} + \epsilon \boldsymbol{\delta} \hat{\mathbf{X}}] - \mathbf{F}[\hat{\mathbf{X}}]}{\epsilon}, \\ \boldsymbol{\delta}^2 &= 0, \\ \boldsymbol{\delta}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) &= \boldsymbol{\delta} \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + (-1)^{|\boldsymbol{\alpha}|} \boldsymbol{\alpha} \wedge \boldsymbol{\delta} \boldsymbol{\beta}, \\ \boldsymbol{\delta} \boldsymbol{\alpha} &= \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(d\boldsymbol{\alpha}). \end{aligned} \quad (\text{A.16})$$

The symplectic structure on the target dg-manifold  $\mathcal{M}$  has degree  $d + 1$  and it can be written in the form

$$\boldsymbol{\omega} = (-1)^{(d+1)|a|} dq^a \wedge dp_a, \quad (\text{A.17})$$

where  $q^a$  and  $p_a$  are local Darboux coordinates such that  $|q^a| + |p_a| = d - 1$ . The BV symplectic form  $\boldsymbol{\omega}$  with ghost number 1 is defined by

$$\boldsymbol{\omega} = \int_{T[1]\Sigma_d} d^d \hat{z} \text{ev}^*(\boldsymbol{\omega}) = \int_{T[1]\Sigma_d} d^d \hat{z} (-1)^{(d+1)|q^a|} \boldsymbol{\delta} q^a(\hat{z}) \wedge \boldsymbol{\delta} p_a(\hat{z}). \quad (\text{A.18})$$

The definition of the Hamiltonian vector field of a graded functional  $\mathbf{F}$  is given by the expression

$$\iota_{\mathbf{X}_{\mathbf{F}}} \boldsymbol{\omega} := \boldsymbol{\delta} \mathbf{F}, \quad (\text{A.19})$$

and it has ghost number  $|F| - d + 1$ . The solution to this equation

$$\mathbf{X}_F = (-1)^{|F|+d} \int_{T[1]\Sigma_d} d^d \hat{z} \left( \frac{\overleftarrow{F} \delta}{\delta q^a(\hat{z})} \frac{\delta}{\delta p_a(\hat{z})} - (-1)^{|q^a| |p_a|} \frac{\overleftarrow{F} \delta}{\delta p_a(\hat{z})} \frac{\delta}{\delta q^a(\hat{z})} \right) \quad (\text{A.20})$$

is used to define the BV bracket of graded functionals  $\mathbf{F}$  and  $\mathbf{G}$  as

$$\begin{aligned} (\mathbf{F}, \mathbf{G})_{\text{BV}} &= (-1)^{|F|+d} \overrightarrow{\mathbf{X}}_F \mathbf{G} \\ &= (-1)^{|F|+d} \iota_{\mathbf{X}_F} \delta \mathbf{G} \\ &= (-1)^{|F|+d+1} \iota_{\mathbf{X}_F} \iota_{\mathbf{X}_G} \omega \\ &= \int_{T[1]\Sigma_d} d^d \hat{z} \left( \frac{\overleftarrow{F} \delta}{\delta q^a(\hat{z})} \frac{\overrightarrow{\delta} \mathbf{G}}{\delta p_a(\hat{z})} - (-1)^{|q^a| |p_a|} \frac{\overleftarrow{F} \delta}{\delta p_a(\hat{z})} \frac{\overrightarrow{\delta} \mathbf{G}}{\delta q^a(\hat{z})} \right). \end{aligned} \quad (\text{A.21})$$

The BV bracket has the following properties:

$$\begin{aligned} (\mathbf{F}, \mathbf{G})_{\text{BV}} &= -(-1)^{(|F|+d+1)(|G|+d+1)} (\mathbf{G}, \mathbf{F})_{\text{BV}}, \\ (\mathbf{F}, \mathbf{G} \mathbf{H})_{\text{BV}} &= (\mathbf{F}, \mathbf{G})_{\text{BV}} \mathbf{H} + (-1)^{(|F|+d+1)|G|} \mathbf{G} (\mathbf{F}, \mathbf{H})_{\text{BV}}, \\ (\mathbf{F}, (\mathbf{G}, \mathbf{H})_{\text{BV}})_{\text{BV}} &= ((\mathbf{F}, \mathbf{G})_{\text{BV}}, \mathbf{H})_{\text{BV}} + (-1)^{(|F|+d+1)(|G|+d+1)} (\mathbf{G}, (\mathbf{F}, \mathbf{H})_{\text{BV}})_{\text{BV}}. \end{aligned} \quad (\text{A.22})$$

## References

- [1] E. Witten, “Topological quantum field theory,” *Commun. Math. Phys.* **117** (1988) 353–386.
- [2] E. Witten, “Topological sigma-models,” *Commun. Math. Phys.* **118** (1988) 411–449.
- [3] E. Witten, “Mirror manifolds and topological field theory,” *AMS/IP Stud. Adv. Math.* **9** (1998) 121–160 [arXiv:hep-th/9112056].
- [4] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun. Math. Phys.* **165** (1994) 311–428 [arXiv:hep-th/9309140].
- [5] M. Alexandrov, M. Kontsevich, A. Schwarz and O. Zaboronky, “The geometry of the master equation and topological quantum field theory,” *Int. J. Mod. Phys. A* **12** (1997) 1405–1429 [arXiv:hep-th/9502010].
- [6] A. S. Cattaneo and G. Felder, “On the AKSZ formulation of the Poisson sigma-model,” *Lett. Math. Phys.* **56** (2001) 163–179 [arXiv:math.QA/0102108].
- [7] P. Bouwknegt and B. Jurčo, “AKSZ construction of topological open  $p$ -brane action and Nambu brackets,” *Rev. Math. Phys.* **25** (2013) 1330004 [arXiv:1110.0134 [math-ph]].
- [8] N. Ikeda, “Lectures on AKSZ sigma-models for physicists,” in: *Noncommutative Geometry and Physics 4*, eds. Y. Maeda, H. Moriyoshi, M. Kotani and S. Watamura (World Scientific, 2017) 79–170 [arXiv:1204.3714 [hep-th]].
- [9] C. M. Hull, “A geometry for non-geometric string backgrounds,” *JHEP* **0510** (2005) 065 [arXiv:hep-th/0406102].
- [10] J. Shelton, W. Taylor and B. Wecht, “Non-geometric flux compactifications,” *JHEP* **0510** (2005) 085 [arXiv:hep-th/0508133].

- [11] R. Blumenhagen, A. Deser, E. Plauschinn and F. Rennecke, “Bianchi identities for non-geometric fluxes: From quasi-Poisson structures to Courant algebroids,” *Fortsch. Phys.* **60** (2012) 1217–1228 [arXiv:1205.1522 [hep-th]].
- [12] M. A. Heller, N. Ikeda and S. Watamura, “Unified picture of non-geometric fluxes and T-duality in double field theory via graded symplectic manifolds,” *JHEP* **1702** (2017) 078 [arXiv:1611.08346 [hep-th]].
- [13] D. Roytenberg, “Courant algebroids, derived brackets and even symplectic supermanifolds,” PhD Thesis, University of California at Berkeley [arXiv:math.DG/9910078].
- [14] J.-S. Park, “Topological open  $p$ -branes,” in: *Symplectic Geometry and Mirror Symmetry*, eds. K. Fukaya, Y.-G. Oh, K. Ono and G. Tian (World Scientific, 2001) 311–384 [arXiv:hep-th/0012141].
- [15] N. Ikeda, “Chern-Simons gauge theory coupled with BF-theory,” *Int. J. Mod. Phys. A* **18** (2003) 2689–2702 [arXiv:hep-th/0203043].
- [16] D. Roytenberg, “On the structure of graded symplectic supermanifolds and Courant algebroids,” *Contemp. Math.* **315** (2002) 169–186 [arXiv:math.SG/0203110].
- [17] C. Hofman and J.-S. Park, “Topological open membranes,” arXiv:hep-th/0209148.
- [18] C. Hofman and J.-S. Park, “BV quantization of topological open membranes,” *Commun. Math. Phys.* **249** (2004) 249–271 [arXiv:hep-th/0209214].
- [19] D. Roytenberg, “AKSZ–BV formalism and Courant algebroid-induced topological field theories,” *Lett. Math. Phys.* **79** (2007) 143–159 [arXiv:hep-th/0608150].
- [20] N. Halmagyi, “Non-geometric backgrounds and the first order string sigma-model,” arXiv:0906.2891 [hep-th].
- [21] D. Mylonas, P. Schupp and R. J. Szabo, “Membrane sigma-models and quantization of non-geometric flux backgrounds,” *JHEP* **1209** (2012) 012 [arXiv:1207.0926 [hep-th]].
- [22] A. Chatzistavrakidis, L. Jonke and O. Lechtenfeld, “Sigma-models for genuinely non-geometric backgrounds,” *JHEP* **1511** (2015) 182 [arXiv:1505.05457 [hep-th]].
- [23] R. J. Szabo, “Higher quantum geometry and non-geometric string theory,” arXiv:1803.08861 [hep-th].
- [24] W. Siegel, “Two vierbein formalism for string inspired axionic gravity,” *Phys. Rev. D* **47** (1993) 5453–5459 [arXiv:hep-th/9302036].
- [25] W. Siegel, “Superspace duality in low-energy superstrings,” *Phys. Rev. D* **48** (1993) 2826–2837 [arXiv:hep-th/9305073].
- [26] C. M. Hull and B. Zwiebach, “Double field theory,” *JHEP* **0909** (2009) 099 [arXiv:0904.4664 [hep-th]].
- [27] G. Aldazabal, D. Marqués and C. Núñez, “Double field theory: A pedagogical review,” *Class. Quant. Grav.* **30** (2013) 163001 [arXiv:1305.1907 [hep-th]].
- [28] D. S. Berman and D. C. Thompson, “Duality symmetric string and M-theory,” *Phys. Rept.* **566** (2014) 1–60 [arXiv:1306.2643 [hep-th]].
- [29] O. Hohm, D. Lüst and B. Zwiebach, “The spacetime of double field theory: Review, remarks, and outlook,” *Fortsch. Phys.* **61** (2013) 926–966 [arXiv:1309.2977 [hep-th]].
- [30] C. M. Hull and B. Zwiebach, “The gauge algebra of double field theory and Courant brackets,” *JHEP* **0909** (2009) 090 [arXiv:0908.1792 [hep-th]].

- [31] I. Vaisman, “On the geometry of double field theory,” *J. Math. Phys.* **53** (2012) 033509 [arXiv:1203.0836 [math.DG]].
- [32] A. Deser and J. Stasheff, “Even symplectic supermanifolds and double field theory,” *Commun. Math. Phys.* **339** (2015) 1003–1020 [arXiv:1406.3601 [math-ph]].
- [33] A. Deser and C. Saemann, “Extended Riemannian geometry I: Local double field theory,” arXiv:1611.02772 [hep-th].
- [34] L. Freidel, F. J. Rudolph and D. Svoboda, “Generalised kinematics for double field theory,” *JHEP* **1711** (2017) 175 [arXiv:1706.07089 [hep-th]].
- [35] A. Chatzistavrakidis, L. Jonke, F. S. Khoo and R. J. Szabo, “Double field theory and membrane sigma-models,” arXiv:1802.07003 [hep-th].
- [36] D. Svoboda, “Algebroid structures on para-Hermitian manifolds,” arXiv:1802.08180 [math.DG].
- [37] A. Kapustin, “Topological strings on noncommutative manifolds,” *Int. J. Geom. Meth. Mod. Phys.* **1** (2004) 49–81 [arXiv:hep-th/0310057].
- [38] A. Kapustin and Y. Li, “Topological sigma-models with  $H$ -flux and twisted generalized complex manifolds,” *Adv. Theor. Math. Phys.* **11** (2007) 269–290 [arXiv:hep-th/0407249].
- [39] V. Pestun and E. Witten, “The Hitchin functionals and the topological B-model at one loop,” *Lett. Math. Phys.* **74** (2005) 21–51 [arXiv:hep-th/0503083].
- [40] R. Zucchini, “A sigma-model field theoretic realization of Hitchin’s generalized complex geometry,” *JHEP* **0411** (2004) 045 [arXiv:hep-th/0409181].
- [41] R. Zucchini, “Generalized complex geometry, generalized branes and the Hitchin sigma-model,” *JHEP* **0503** (2005) 022 [arXiv:hep-th/0501062].
- [42] V. Pestun, “Topological strings in generalized complex space,” *Adv. Theor. Math. Phys.* **11** (2007) 399–450 [arXiv:hep-th/0603145].
- [43] N. Ikeda and T. Tokunaga, “An alternative topological field theory of generalized complex geometry,” *JHEP* **0709** (2007) 009 [arXiv:0704.1015 [hep-th]].
- [44] V. Stojevic, “Topological A-type models with flux,” *JHEP* **0805** (2008) 023 [arXiv:0801.1160 [hep-th]].
- [45] T. Bessho, M. A. Heller, N. Ikeda and S. Watamura, “Topological membranes, current algebras and  $H$ -flux– $R$ -flux duality based on Courant algebroids,” *JHEP* **1604** (2016) 170 [arXiv:1511.03425 [hep-th]].
- [46] N. Nekrasov, H. Ooguri and C. Vafa, “S-duality and topological strings,” *JHEP* **0410** (2004) 009 [arXiv:hep-th/0403167].
- [47] A. S. Cattaneo, J. Qiu and M. Zabzine, “ $2D$  and  $3D$  topological field theories for generalized complex geometry,” *Adv. Theor. Math. Phys.* **14** (2010) 695–725 [arXiv:0911.0993 [hep-th]].
- [48] Z. Kökényesi, A. Sinkovics and R. J. Szabo, “AKSZ constructions for topological membranes on  $G_2$ -manifolds,” *Fortsch. Phys.* **66** (2018) 1800018 [arXiv:1802.04581 [hep-th]].
- [49] I. A. Batalin and G. A. Vilkovisky, “Gauge algebra and quantization,” *Phys. Lett. B* **102** (1981) 27–31.
- [50] P. Mnev, “Discrete BF-theory,” arXiv:0809.1160 [hep-th].
- [51] N. Ikeda, “Two-dimensional gravity and nonlinear gauge theory,” *Ann. Phys.* **235** (1994) 435–464 [arXiv:hep-th/9312059].

- [52] P. Schaller and T. Strobl, “Poisson structure induced (topological) field theories,” *Mod. Phys. Lett. A* **9** (1994) 3129–3136 [arXiv:hep-th/9405110].
- [53] A. S. Cattaneo and G. Felder, “A path integral approach to the Kontsevich quantization formula,” *Commun. Math. Phys.* **212** (2000) 591–611 [arXiv:math.QA/9902090].
- [54] C. Hofman, “On the open-closed B-model,” *JHEP* **0311** (2003) 069 [arXiv:hep-th/0204157].
- [55] N. Hitchin, “Generalized Calabi-Yau manifolds,” *Quart. J. Math.* **54** (2003) 281–308 [arXiv:math.DG/0209099].
- [56] M. Gualtieri, “Generalized complex geometry,” *Ann. Math.* **174** (2011) 75–123 [math.DG/0401221].
- [57] T. Asakawa, H. Muraki, S. Sasa and S. Watamura, “Poisson-generalized geometry and  $R$ -flux,” *Int. J. Mod. Phys. A* **30** (2015) 1550097 [arXiv:1408.2649 [hep-th]].
- [58] R. Blumenhagen and E. Plauschinn, “Nonassociative gravity in string theory?,” *J. Phys. A* **44** (2011) 015401 [arXiv:1010.1263 [hep-th]].
- [59] D. Lüster, “T-duality and closed string noncommutative (doubled) geometry,” *JHEP* **1012** (2010) 084 [arXiv:1010.1361 [hep-th]].
- [60] R. Blumenhagen, A. Deser, D. Lüster, E. Plauschinn and F. Rennecke, “Non-geometric fluxes, asymmetric strings and nonassociative geometry,” *J. Phys. A* **44** (2011) 385401 [arXiv:1106.0316 [hep-th]].