Uncertainty Quantification in Astro-Imaging by Optimisation

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***Abstract*—In our recent work [1], we proposed a Bayesian uncertainty quantification method for large scale imaging inverse problems. It aims to analyse the confidence in specific structures observed in maximum *a posteriori* (MAP) estimates (e.g., lesions in medical imaging, celestial sources in astronomical imaging), assuming a log-concave Bayesian model. This information can subsequently be used as evidence to inform decisions and conclusions. We propose to perform a hypothesis test on the structures of interest, to assert their uncertainty. The test is formulated as a convex minimisation problem, enabling the use of advanced optimisation algorithms. In this abstract we summarise the proposed Bayesian Uncertainty Quantification by Optimisation (BUQO) method and we provide results obtained on real data in the context of radio-astronomical imaging.**

I. Imaging inverse problems

In many imaging inverse problems, the objective is to find an estimate $\mathbf{x}^\dagger \in \mathbb{R}^N$ of an original unknown image $\overline{\mathbf{x}} \in \mathbb{R}^N$ from a degraded observation $\mathbf{y} = \mathbf{\Phi}\overline{\mathbf{x}} + \mathbf{w}$, where $\mathbf{\Phi} \in \mathbb{C}^{M \times N}$ is the measurement linear operator and $\mathbf{w} \in \mathbb{C}^M$ is a realization of a random noise. In this work, we assume that the noise has a bounded energy per block, i.e. there exist $(\varepsilon_d)_{1 \le d \le D} \in]0, +\infty[^D$ such that $\|\mathbf{w}_d\|_2 \le \varepsilon_d$, $\mathbf{w}_d \in \mathbb{C}^{M_d}$, $M = M_1 + \ldots + M_d$, and $\mathbf{w} = (\mathbf{w}_d)_{1 \le d \le D}$. The forward model can then be rewritten as $\mathbf{y}_d = \mathbf{\Phi}_d \overline{\mathbf{x}} + \mathbf{w}_d$, where $\mathbf{\Phi}_d \in \mathbb{C}^{M_d \times N}$ is a subpart of $\mathbf{\Phi}$ mapping $\overline{\mathbf{x}}$ to $\mathbf{y}_d \in \mathbb{C}^{M_d}$. A common Bayesian approach consists in modelling $\overline{\mathbf{x}}$ as a random vector with prior distribution $p(\mathbf{x})$ that is log-concave, and using a MAP estimation given by $\mathbf{x}^\dagger \in \operatorname{Argmin} f_{\mathbf{y}} + g$, where $f_{\mathbf{y}}(\mathbf{x}) = -\log p(\mathbf{y}|\mathbf{x})$ is associated with the forward model and $g(\mathbf{x}) = -\log p(\mathbf{x})$. In particular, we use $f_{\mathbf{y}}(\mathbf{x}) = \sum_{d=1}^D \iota_{\mathcal{B}_2(\mathbf{y}_d, \varepsilon_d)}(\mathbf{\Phi}_d \mathbf{x})$. and $g(\mathbf{x}) = \iota_{[0,+\infty[^N}(\mathbf{x}) + \lambda \|\mathbf{\Psi}\mathbf{x}\|_1$, where $\mathbf{\Psi} \in \mathbb{R}^{L \times N}$is a sparsity basis and $\lambda > 0$.

The MAP approach provides a single point estimate that can be easily visually analysed. Nevertheless, in many applications it is necessary to analyse as well the uncertainty in the delivered solutions. We propose to address this point by leveraging probability concentration phenomena and the model's underlying convex geometry to formulate Bayesian hypothesis tests as convex minimisation problems.

II. Hypothesis test

The proposed BUQO method takes the form of a Bayesian hypothesis test for specific structures appearing in the MAP estimate. To define this test we postulate the following null hypothesis:

H_0: The structure of interest is **absent** in the true image.

The null Hypothesis is rejected with significance $\alpha \in]0, 1[$ if $P(H_0|\mathbf{y}) \le \alpha$. To assess this probability, we introduce two sets:

- $\mathcal{C}_\alpha$: Conservative credible region, i.e. $P(\overline{\mathbf{x}} \in \mathcal{C}_\alpha|\mathbf{y}) \ge 1 - \alpha$ [2],
- $\mathcal{S}$: Set of all the images of $\mathbb{R}^N$ without the structure of interest.

These two sets are convex. Their formal definitions can be found in [2] and [1], respectively. Noticing that $P(H_0|\mathbf{y}) = P(\overline{\mathbf{x}} \in \mathcal{S}|\mathbf{y})$ and $P(\overline{\mathbf{x}} \in \mathbb{R}^N \setminus \mathcal{C}_\alpha|\mathbf{y}) \ge 1 - \alpha$, intuitively we can observe that, if $\mathcal{S} \subset \mathbb{R}^N \setminus \mathcal{C}_\alpha$ then $P(\overline{\mathbf{x}} \in \mathcal{S}|\mathbf{y}) \le \alpha$. We propose to determine if $\mathcal{S} \subset \mathbb{R}^N \setminus \mathcal{C}_\alpha$ by finding $(\mathbf{x}^\ddagger_{\mathcal{S}}, \mathbf{x}^\ddagger_{\mathcal{C}_\alpha}) \in \mathcal{S} \times \mathcal{C}_\alpha$ such that $\|\mathbf{x}^\ddagger_{\mathcal{S}} - \mathbf{x}^\ddagger_{\mathcal{C}_\alpha}\|^2$ is minimal. If $\|\mathbf{x}^\ddagger_{\mathcal{S}} - \mathbf{x}^\ddagger_{\mathcal{C}_\alpha}\|^2 > 0$, we can conclude that $\mathcal{S} \subset \mathbb{R}^N \setminus \mathcal{C}_\alpha$ and H_0 is rejected. In practice the sets $\mathcal{S}$ and $\mathcal{C}_\alpha$ are complicated sets which can be expressed as an intersection of simple convex sets (see [1], [3]). As proposed in [3], this problem can be solved leveraging a primal-dual proximal algorithm [4].

III. Results

The main contribution of this work is to provide preliminary results on real data in radio-astronomy. Fig. 1(top left) shows the MAP estimate of Cyg A, obtained from observations at X band with the VLA telescope [5]. In the central highlighted part of the image, a second black hole can be observed close to the supermassive primary black hole. Fig. 1(top right) shows a zoomed image in this area. We define a hypothesis test to quantify the uncertainty of the secondary black hole with $\alpha = 1\%$ (H_0: *the secondary black hole is absent in the true sky*). The images $\mathbf{x}^\ddagger_{\mathcal{S}}$ and $\mathbf{x}^\ddagger_{\mathcal{C}_\alpha}$ obtained using the BUQO method are shown in Fig. 1(bottom left and right), respectively. We observe that the secondary black hole is absent in $\mathbf{x}^\ddagger_{\mathcal{S}}$, but present in $\mathbf{x}^\ddagger_{\mathcal{C}_\alpha}$. Thus, we have $\|\mathbf{x}^\ddagger_{\mathcal{S}} - \mathbf{x}^\ddagger_{\mathcal{C}_\alpha}\| > 0$, and consequently H_0 is rejected and the existence of the second black hole is confirmed with 99% confidence. Note that a simpler approach [2] consisting in removing the black hole from the initial MAP estimate produces an image that does not belong to $\mathcal{C}_\alpha$, from which no conclusion can be inferred about the existence of the structure

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References

[1] A. Repetti, M. Pereyra, and Y. Wiaux, “Scalable Bayesian uncertainty quantification in imaging inverse problems via convex optimization,” Tech. Rep., 2018, https://arxiv.org/abs/1803.00889.

[2] M. Pereyra, “Maximum *a posteriori* estimation with Bayesian confidence regions,” *SIAM J. Imaging Sci.*, vol. 10, no. 1, pp. 285–302, 2017.

[3] A. Repetti, M. Pereyra, and Y. Wiaux, “Uncertainty quantification in imaging: When convex optimization meets Bayesian analysis,” in *Proc. of IEEE EUSIPCO*, 2018, pp. 2686–2690.

[4] N. Komodakis and J.-C. Pesquet, “Playing with duality: An overview of recent primal-dual approaches for solving large-scale optimization problems,” *IEEE Sign. Process. Mag.*, vol. 32, no. 6, pp. 31–54, 2015.

[5] A. Dabbech, A. Onose, A. Abdulaziz, R. A. Perley, O. M. Smirnov, and Y. Wiaux, “Cygnus a super-resolved via convex optimization from vla data,” *MNRAS*, vol. 476, no. 3, pp. 2853–2866, 2018.

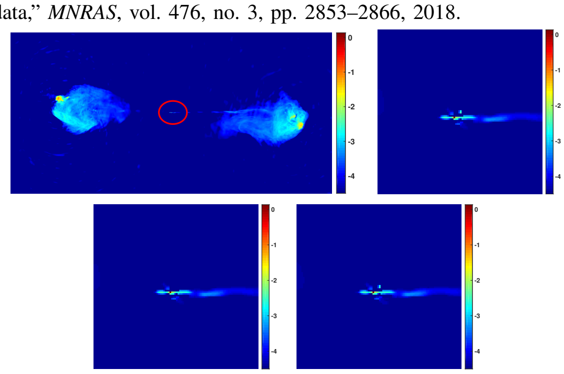

Fig. 1: Top row: MAP estimate (log scale) from [5] (left) and zoom into the highlighted red area (right). Bottom row: zooms on $\mathbf{x}^\ddagger_{\mathcal{S}}$ (left) and $\mathbf{x}^\ddagger_{\mathcal{C}_\alpha}$ (right).