Global asymptotic stability of bifurcating, positive equilibria of p-Laplacian boundary value problems with p-concave nonlinearities

GLOBAL ASYMPTOTIC STABILITY OF BIFURCATING, POSITIVE EQUILIBRIA OF p-LAPLACIAN BOUNDARY VALUE PROBLEMS WITH p-CONCAVE NONLINEARITIES

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ABSTRACT. We consider the initial value problem

$$\begin{aligned} v_t &= \Delta_p(v) + \lambda g(x,v)\phi_p(v), && \text{in } \Omega \times (0,\infty),\\ v &= 0, && \text{in } \partial\Omega \times (0,\infty), \\ v &= v_0 \geqslant 0, && \text{in } \Omega \times \{0\}, \end{aligned} \tag{IVP}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded domain with smooth boundary $\partial\Omega$, $\phi_p(s) := |s|^{p-1} \operatorname{sgn} s$, $s \in \mathbb{R}$, Δ_p denotes the p-Laplacian, with $p > \max\{2, N\}$, $v_0 \in C^0_0(\overline{\Omega})$, with $v_0 \geqslant 0$ on $\overline{\Omega}$, and $\lambda > 0$. The function $g : \overline{\Omega} \times \mathbb{R} \to (0,\infty)$ is C^0 and, for each $x \in \overline{\Omega}$, the function $g(x,\cdot) : [0,\infty) \to (0,\infty)$ is Lipschitz and decreasing. With these hypotheses, (IVP) has a unique, positive solution.

For each $\lambda > 0$, (IVP) has the trivial solution $v \equiv 0$. In addition, there exists $0 < \lambda_{\min}(g) < \lambda_{\max}(g)$ ($\lambda_{\max}(g)$ may be ∞) such that:

- if $\lambda \notin (\lambda_{\min}(g), \lambda_{\max}(g))$ then (IVP) has no non-trivial, positive equilibrium;
- if $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ then (IVP) has a unique, non-trivial, positive equilibrium $e_\lambda \in W^{1,p}_0(\Omega)$.

We prove the following stability results ('stability' means with respect to the set of non-trivial, positive solutions):

- if $0 < \lambda < \lambda_{\min}(g)$ then the trivial solution is globally asymptotically stable;
- if $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then e_λ is globally asymptotically stable;
- if $\lambda_{\max}(g) < \lambda$ then any non-trivial, positive solution blows up in finite time.

1. INTRODUCTION

We consider the degenerate, parabolic, initial-boundary value problem

$$\begin{aligned} v_t &= \Delta_p(v) + \lambda g(x,v)\phi_p(v), && \text{in } \Omega \times (0,\infty),\\ v &= 0, && \text{in } \partial\Omega \times (0,\infty), \\ v &= v_0 \geqslant 0, && \text{in } \Omega \times \{0\}, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded domain with smooth boundary $\partial\Omega$, $\phi_p(s) := |s|^{p-1} \operatorname{sgn} s$, $s \in \mathbb{R}$, and Δ_p denotes the p-Laplacian (see Section 2.1 for a precise definition), with $p > \max\{2, N\}$, $v_0 \in C^0_0(\overline{\Omega})$, with $v_0 \geqslant 0$ on $\overline{\Omega}$, and $\lambda > 0$. The function $g : \overline{\Omega} \times \mathbb{R} \to (0,\infty)$ is C^0 and we suppose that, for each $x \in \overline{\Omega}$,

$$g(x,\cdot) : [0,\infty) \to (0,\infty) \text{ is strictly decreasing,} \tag{1.2}$$

$$0 \leqslant g_\infty(x) := \lim_{\xi\to\infty} g(x,\xi) < g_0(x) := g(x,0), \tag{1.3}$$

and g is Lipschitz with respect to ξ, in the sense that for any $K > 0$ there exists L_K such that

$$|g(x,\xi_1) - g(x,\xi_2)| \leqslant L_K|\xi_1 - \xi_2|, \quad x \in \overline{\Omega},\ |\xi_1|, |\xi_2| \leqslant K. \tag{1.4}$$

We are only interested in positive (that is, $\geqslant 0$) solutions of (1.1), and in fact, under the above assumptions, (1.1) has a unique, positive solution on a maximal time-interval of existence, which may be finite, see Theorem 3.1 and Corollary 3.5 below (hence, the values of g on $\Omega \times (-\infty, 0)$ are irrelevant). In view of this, we will use the following notation: $C^0(\overline{\Omega})$ will denote the usual set of continuous functions on $\overline{\Omega}$, and $C^0_0(\overline{\Omega})$ will denote the set of $\omega \in C^0(\overline{\Omega})$ with $\omega = 0$ on $\partial\Omega$; $C^0_+(\overline{\Omega})$ (respectively $C^0_{0,+}(\overline{\Omega})$) will denote the set of functions $\omega \in C^0(\overline{\Omega})$ (respectively $\omega \in C^0_0(\overline{\Omega})$) with $\omega \geqslant 0$ in Ω. Similarly, $W^{1,p}_0(\Omega)$, and $W^{1,p}_{0,+}(\Omega)$ will denote the corresponding Sobolev spaces.

It is known that for any $v_0 \in C^0_{0,+}(\overline{\Omega})$ the problem (1.1) has a unique, positive solution $t \to v_{\lambda g, v_0}(t) \in W^{1,p}_{0,+}(\Omega)$, on some maximal interval $(0, T)$, and we will encounter both $T < \infty$ (finite-time blow-up) or $T = \infty$. What we mean by a solution of (1.1) will be made precise in Theorem 3.1 below. We are interested in the asymptotic (in time) behaviour of these solutions. This asymptotic behaviour is determined by the structure of the set of positive equilibria of (1.1), so we first describe this structure.

Naturally, for any $\lambda > 0$, a positive *equilibrium* is a time-independent solution $u \in W^{1,p}_{0,+}(\Omega)$ of (1.1), and it is clear that the function $v \equiv 0$ (or $(\lambda, v) = (\lambda, 0)$) is a (*trivial*) equilibrium. In addition, the complete structure of the set of non-trivial, positive equilibria of (1.1) has been obtained in [8], and is as follows (see Theorem 2.3 below for more details): there exists $0 < \lambda_{\min}(g) < \lambda_{\max}(g)$ ($\lambda_{\max}(g)$ may be ∞) such that:

- if $\lambda \notin (\lambda_{\min}(g), \lambda_{\max}(g))$ then (1.1) has no non-trivial equilibrium;
- if $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ then (1.1) has a unique, non-trivial equilibrium e_λ;
- the set of non-trivial equilibria $\mathcal{E}^+ := \{(\lambda, e_\lambda) : \lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))\}$ forms a continuum in $\mathbb{R} \times W^{1,p}_{0,+}(\Omega)$ bifurcating from the point $(\lambda_{\min}(g), 0)$.

We will prove the following results on the asymptotic behaviour of the positive solutions of (1.1). For any $0 \neq v_0 \in C^0_{0,+}(\overline{\Omega})$:

- $0 < \lambda < \lambda_{\min}(g) \implies \lim_{t \to \infty} \|v_{\lambda g, v_0}(t)\|_{0,p} = 0$;
- $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g) \implies \lim_{t \to \infty} \|v_{\lambda g, v_0}(t) - e_\lambda\|_{0,p} = 0$;
- $\lambda_{\max}(g) < \lambda < \infty \implies$ there exists $T < \infty$ such that $\lim_{t \nearrow T} |v_{\lambda g, v_0}(t)|_0 = \infty$.

Regarding (1.1) as a bifurcation problem, these results can be interpreted as global stability results for the trivial equilibria and the non-trivial equilibria e_λ (of course, 'stability' here means with respect to the set of non-trivial, positive solutions of (1.1)):

- if $0 < \lambda < \lambda_{\min}(g)$ then the trivial equilibrium is globally stable;
- as λ increases through $\lambda_{\min}(g)$, the trivial solution $(\lambda, 0)$ loses stability and a continuum, $\mathcal{E}^+$, of globally stable, positive equilibria bifurcates from $(\lambda_{\min}(g), 0)$
  (in a sense, there is a supercritical, transcritical bifurcation at $\lambda_{\min}(g)$, with exchange of stability between the equilibria);
- as λ increases through $\lambda_{\max}(g)$, the continuum $\mathcal{E}^+$ 'meets infinity' and then disappears, after which all non-trivial, positive solutions blow up in finite time.

These results are consistent with a bifurcation analysis of the corresponding semilinear ($p = 2$) problem, using the 'principle of linearised stability' to obtain local stability. Such problems have been extensively investigated, see [11] and the references therein for a summary of the main results. However, we do not use bifurcation theory to obtain our results, which usually yields local stability results. Instead, we use a mixture of comparison and compactness arguments to obtain the above results.

For the quasilinear problem involving the p-Laplacian with $p > 2$ considered here, these results are consistent with the results on 'linearised stability' in the 'p-concave' case in [12] (condition (1.2) is termed 'p-concavity' in [12]; this terminology has been used in other publications for very similar, but slightly different, conditions). However, the term 'linearised stability' in [12] refers to the sign of the principal eigenvalue of the linearisation of the problem at an equilibrium solution e_λ, not to the dynamic (time-dependent) stability that we consider. In the quasilinear case it is not clear that 'linearised stability', in this sense, implies stability in the usual dynamic sense. Even if such a result could be proved, it would give local rather than global stability.

The convergence results that we obtain say nothing about the rate of convergence. In particular, we do not obtain the exponential convergence that would be obtained from any sort of 'linearised stability' analysis, if such were possible. Convergence rates for quasilinear problems are discussed in [4], together with a broad survey of the literature relating to this. It is also noted in [4] that the known results are limited, and difficult to apply. In particular, the results discussed in [4] say nothing about the problem considered here.

2. Preliminaries

As mentioned above, $C^0(\overline{\Omega})$ will denote the standard space of real valued, continuous functions defined on $\overline{\Omega}$, with the standard sup-norm $|\cdot|_0$ (throughout, all function spaces will be real); $L^q(\Omega)$, $q > 1$, will denote the standard space of functions on Ω whose qth power is integrable, with norm $\|\cdot\|_q$; $W_0^{1,p}(\Omega)$ will denote the standard, first order Sobolev space of functions on $\overline{\Omega}$ which are zero on $\partial\Omega$, with norm $\|\cdot\|_{1,p}$. The dual space of $W_0^{1,p}(\Omega)$ will be denoted by $W^{-1,p'}(\Omega)$, where $p' := p/(p-1)$ is the conjugate exponent of p. By our assumption that $p > N$, the space $W_0^{1,p}(\Omega)$ is compactly embedded into $C_0^0(\overline{\Omega})$. We also define the corresponding sets of *positive* functions $C_+^0(\overline{\Omega})$, $C_{0,+}^0(\overline{\Omega})$, $W_{0,+}^{1,p}(\Omega)$, as above.

If $h : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is continuous then, for any $\omega \in C_+^0(\overline{\Omega})$, we define $h(\omega) \in C_+^0(\overline{\Omega})$ by

$$h(\omega)(x) := h(x, \omega(x)), \quad x \in \overline{\Omega}.$$

Clearly, the 'Nemytskii' mapping $\omega \to h(\omega) : C_+^0(\overline{\Omega}) \to C_+^0(\overline{\Omega})$ is continuous. In particular, we repeatedly use the Nemytskii mapping $\phi_p : \omega \to \phi_p(\omega) : C_+^0(\overline{\Omega}) \to C_+^0(\overline{\Omega})$.

2.1. The p-Laplacian.

Formally, the p-Laplacian is defined by

$$\Delta_p \omega := \nabla \cdot (|\nabla\omega|^{p-2}\nabla\omega),$$

for suitable ω, where $|\boldsymbol{v}| := (v_1^2 + \cdots + v_N^2)^{1/2}$ for $\boldsymbol{v} \in \mathbb{R}^N$. More precisely, for any $\omega \in W_0^{1,p}(\Omega)$, we define $\Delta_p(\omega) \in W^{-1,p'}(\Omega)$ by

$$\int_\Omega \Delta_p(\omega)\,\varphi := -\int_\Omega |\nabla\omega|^{p-2}\nabla\omega \cdot \nabla\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.1}$$

A precise definition of what is meant by a solution of (1.1) will be given in Section 3 below.

2.2. Principal eigenvalues of the p-Laplacian.

We briefly consider the weighted, nonlinear eigenvalue problem

$$-\Delta_p(\psi) = \mu\rho\phi_p(\psi), \quad \psi \in W_0^{1,p}(\Omega), \tag{2.2}$$

where $\mu \in \mathbb{R}$ and the weight function $\rho \in L^1(\Omega)$. We say that μ is an *eigenvalue* of (2.2), with *eigenfunction* $\psi \in W_0^{1,p}(\Omega) \setminus \{0\}$, if the following weak formulation of (2.2) holds

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi = \mu \int_\Omega \rho\phi_p(\psi)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.3}$$

A *principal eigenvalue* of (2.2) is an eigenvalue μ_0 which has a positive eigenfunction $\psi_0 \in W_{0,+}^{1,p}(\Omega)$ (which we will normalise by, say, $|\psi_0|_0 = 1$). The following result is well known — see, for example, [5, Sections 3-4].

Lemma 2.1. *Suppose that the weight function $\rho \in L^1(\Omega)$ satisfies $\rho \geqslant 0$ on Ω, with $\rho > 0$ on a set of positive Lebesgue measure. Then the eigenvalue problem (2.2) has a unique principal eigenvalue $\mu_0(\rho)$, with corresponding eigenfunction $\psi_0(\rho)$. This eigenvalue and eigenfunction have the following properties:*

(*a*) $\mu_0(\rho) > 0$, $\psi_0(\rho) > 0$ *on* Ω,

$$\int_\Omega |\nabla\omega|^p \geqslant \mu_0(\rho) \int_\Omega \rho|\omega|^p, \quad \forall\, \omega \in W_0^{1,p}(\Omega); \tag{2.4}$$

(*b*) *if ρ_0, ρ_1 are two such weight functions, with $\rho_1 \leqslant \rho_0$ on Ω and $\rho_1 < \rho_0$ on a set of positive Lebesgue measure, then $\mu_0(\rho_1) > \mu_0(\rho_0)$;*

(*c*) *if, for some $\epsilon > 0$ and each $\delta \in [0,\epsilon]$, $\rho_\delta \in C^0(\overline{\Omega})$ is a weight function, with $\rho_\delta \leqslant \rho_0$ for all $\delta \in (0,\epsilon]$ and $\lim_{\delta\to 0} |\rho_0 - \rho_\delta|_0 = 0$, then $\lim_{\delta\to 0^+} \mu_0(\rho_\delta) = \mu_0(\rho_0)$.*

Proof. The only result in the lemma that is not explicitly proved in [5] is part (c) – the proof is short, so for completeness we state it here. By part (b) and the minimisation characterisation of $\mu_0(\rho)$ in (1.3) of [5],

$$\mu_0(\rho_0) \leqslant \mu_0(\rho_\delta) \leqslant \frac{\int_\Omega |\nabla\psi_0(\rho_0)|^p}{\int_\Omega \rho_\delta\psi_0(\rho_0)^p} \to \frac{\int_\Omega |\nabla\psi_0(\rho_0)|^p}{\int_\Omega \rho_0\psi_0(\rho_0)^p} = \mu_0(\rho_0).$$

Of course, this result holds in considerably greater generality than stated here, but the above statement is what will be required below. □

By (1.3) and Lemma 2.1, we may now define

$$0 < \lambda_{\min}(g) := \mu_0(g_0) < \lambda_{\max}(g) := \begin{cases} \mu_0(g_\infty) < \infty, & \text{if } g_\infty \neq 0 \ \ (\text{in } L^\infty(\Omega)), \\ \infty, & \text{if } g_\infty = 0 \ \ (\text{in } L^\infty(\Omega)), \end{cases}$$

and we denote the corresponding normalised principal eigenfunctions by $\psi_{\min}(g)$, $\psi_{\max}(g)$.

2.3. **An energy functional.** We now define an 'energy' functional for (1.1) on $W^{1,p}_{0,+}(\Omega)$. Let

$$F(x,\xi) := \int_0^\xi g(x,s)s^{p-1}\,ds, \qquad (x,\xi) \in \Omega \times [0,\infty),$$

$$E_{\lambda g}(\omega) := \frac{1}{p}\int_\Omega |\nabla\omega|^p - \lambda\int_\Omega F(\omega), \quad \omega \in W^{1,p}_{0,+}(\Omega).$$

By the continuity of the embedding $W^{1,p}_{0,+}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, the *energy* functional $E_{\lambda g} : W^{1,p}_{0,+}(\Omega) \to \mathbb{R}$ is continuous.

Lemma 2.2. *For any* $\lambda \in (0, \lambda_{\max}(g))$ *there exists an increasing function* $M_\lambda : \mathbb{R} \to [0,\infty)$ *such that,*

$$\|\omega\|_{1,p} \leqslant M_\lambda(E_{\lambda g}(\omega)), \quad \omega \in W^{1,p}_{0,+}(\Omega).$$

Proof. Suppose that there exists $\lambda \in (0, \lambda_{\max}(g))$, $R \in \mathbb{R}$ and $\omega_n \in W^{1,p}_{0,+}(\Omega) \setminus \{0\}$, $n = 1,2,\dots$, such that $E_{\lambda g}(\omega_n) \leqslant R$ and $\lim_{n\to\infty} \|\omega_n\|_{1,p} = \infty$ Let $\widetilde{\omega}_n := \omega_n/\|\omega_n\|_{1,p}$, $n = 1,2,\dots$. By the compactness of the embedding $W^{1,p}_0(\Omega) \hookrightarrow C^0(\overline{\Omega})$ (since $p > N$), we may assume that $\widetilde{\omega}_n \to \widetilde{\omega}_\infty$ in $C^0_+(\overline{\Omega})$, for some $\widetilde{\omega}_\infty \in C^0_+(\overline{\Omega})$. We will show that this leads to a contradiction.

By definition,

$$E_{\lambda g}(\omega_n) = \frac{1}{p}\|\omega_n\|^p_{1,p}\left\{\int_\Omega |\nabla\widetilde{\omega}_n|^p - \lambda p\int_\Omega \frac{F(\omega_n)}{\|\omega_n\|^p_{1,p}}\right\}, \quad n \geqslant 1. \tag{2.5}$$

We now show that, as $n \to \infty$,

$$p\int_\Omega \frac{F(\omega_n)}{\|\omega_n\|^p_{1,p}} \to \int_\Omega g_\infty\widetilde{\omega}^p_\infty. \tag{2.6}$$

By (1.2) and (1.3) there exists $C > 0$ such that, for any $n \geqslant 1$,

$$p\frac{|F(\omega_n)|_0}{\|\omega_n\|^p_{1,p}} \leqslant \frac{|g_0|_0|\omega_n|^p_0}{\|\omega_n\|^p_{1,p}} \leqslant C, \tag{2.7}$$

and similarly, for any $x \in \Omega$ and $\epsilon > 0$, there exists $C(x,\epsilon) > 0$ such that, for any $n \geqslant 1$,

$$p\frac{F(\omega_n)(x)}{\|\omega_n\|^p_{1,p}} \leqslant \frac{C(x,\epsilon) + (g_\infty(x)+\epsilon)\,\omega_n(x)^p}{\|\omega_n\|^p_{1,p}} \to (g_\infty(x)+\epsilon)\,\widetilde{\omega}_\infty(x)^p.$$

Combining this with a similar lower bound shows that

$$p\frac{F(\omega_n)(x)}{\|\omega_n\|^p_{1,p}} \to g_\infty(x)\,\widetilde{\omega}_\infty(x)^p, \quad x \in \Omega, \tag{2.8}$$

so (2.6) follows from (2.7), (2.8) and the dominated convergence theorem.

Now suppose that $\int_\Omega g_\infty\widetilde{\omega}^p_\infty > 0$. Then, by Lemma 2.1, for $n \geqslant 1$,

$$\int_\Omega |\nabla\widetilde{\omega}_n|^p \geqslant \mu_0(g_\infty)\int_\Omega g_\infty\widetilde{\omega}^p_n \to \mu_0(g_\infty)\int_\Omega g_\infty\widetilde{\omega}^p_\infty > 0, \tag{2.9}$$

and combining (2.5), (2.6) and (2.9) shows that $E_{\lambda g}(\omega_n) \to \infty$ (since $\lambda < \mu_0(g_\infty)$). However, this contradicts the initial assumption that $E_{\lambda g}(\omega_n) \leqslant R$ for all $n \geqslant 1$.

Next, suppose that $\int_\Omega g_\infty \widetilde{\omega}_\infty^p = 0$, with $\|\widetilde{\omega}_\infty\|_p > 0$. Then, by Lemma 2.1, for $n \geqslant 1$,

$$\int_\Omega |\nabla \widetilde{\omega}_n|^p \geqslant \mu_0(\mathbf{1}) \|\widetilde{\omega}_n\|_p^p \to \mu_0(\mathbf{1}) > 0 \tag{2.10}$$

(where $\mathbf{1}$ denotes the weight function that is identically 1 on Ω), and combining (2.5), (2.6) and (2.10) again yields the contradiction $E_{\lambda g}(\omega_n) \to \infty$.

Finally, suppose that $\|\widetilde{\omega}_\infty\|_p = 0$. Since $\|\widetilde{\omega}_n\|_{1,p} = 1$, $n \geqslant 1$, this implies that $\int_\Omega |\nabla \widetilde{\omega}_n|^p \to 1$, so combining (2.5) and (2.6) again yields a contradiction.

These results show that for any $\lambda \in (0, \lambda_{\max}(g))$ and $R \in \mathbb{R}$ there exists $\widetilde{M}_\lambda(R) \geqslant 0$ such that

$$E_{\lambda g}(\omega) \leqslant R \implies \|\omega\|_{1,p} \leqslant \widetilde{M}_\lambda(R), \quad \omega \in W_{0,+}^{1,p}(\Omega).$$

We now obtain the desired increasing function M_λ by defining $M_\lambda(E) := \inf_{R \geqslant E}\{\widetilde{M}_\lambda(R)\} \geqslant 0$, $E \in \mathbb{R}$, which completes the proof of Lemma 2.2. □

2.4. **Existence and uniqueness of non-trivial, positive equilibria.** A *positive equilibrium* of (1.1) is a solution of the problem

$$-\Delta_p(u) = \lambda g(u)\phi_p(u), \quad u \in W_{0,+}^{1,p}(\Omega). \tag{2.11}$$

More precisely, a *solution* of (2.11) is defined to be a function $u \in W_{0,+}^{1,p}(\Omega)$ which satisfies the following weak formulation of (2.11),

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \lambda \int_\Omega g(u)\phi_p(u)\varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.12}$$

Clearly, for any $\lambda \in \mathbb{R}$, the function $u = 0$ is a (*trivial*) positive solution of (1.1) and (2.11).

We now describe the structure of the set of non-trivial, positive equilibria. Let

$$\Lambda := (\lambda_{\min}(g), \lambda_{\max}(g)).$$

Theorem 2.3. (*a*) *If $\lambda \notin \Lambda$ then* (2.11) *has no non-trivial solution $u \in W_{0,+}^{1,p}(\Omega)$.*

(*b*) *If $\lambda \in \Lambda$ then* (2.11) *has a unique, non-trivial solution $e_\lambda \in W_{0,+}^{1,p}(\Omega)$, and $e_\lambda > 0$ on Ω.*

(*c*) *The mapping $\lambda \to e_\lambda : \Lambda \to W_{0,+}^{1,p}(\Omega)$ is continuous, and*

$$\lim_{\lambda \searrow \lambda_{\min}(g)} \|e_\lambda\|_{1,p} = 0, \quad \lambda_{\max}(g) < \infty \implies \lim_{\lambda \nearrow \lambda_{\max}(g)} \|e_\lambda\|_{1,p} = \infty. \tag{2.13}$$

Proof. Parts (a) and (b) are proved in [8, Theorems 1, 2]. We observe that:

(i) the strict positivity of e_λ on Ω is not stated in [8, Theorem 2], but is derived in its proof; it also follows from Lemma 2.1;

(ii) part (*a*) also follows from Lemma 2.1 and the definitions of $\lambda_{\min}(g)$, $\lambda_{\max}(g)$.

To prove part (c), suppose firstly that $\lambda_n \in \Lambda$, $n = 1, 2, \dots$, is such that

$$\lim_{n\to\infty} \lambda_n = \lambda_\infty < \infty, \quad \lim_{n\to\infty} \|e_{\lambda_n}\|_{1,p} = \infty. \tag{2.14}$$

Defining $w_n := e_{\lambda_n}/\|e_{\lambda_n}\|_{1,p}$, $n = 1, 2, \dots$, it follows from the compactness and continuity properties described on p. 229 of [7] that we may suppose that $w_n \to w_\infty \in W_{0,+}^{1,p}(\Omega)$, with $w_\infty \neq 0$ and

$$\begin{aligned} -\Delta_p(w_\infty) &= \lambda_\infty \overline{g} \phi_p(w_\infty), \\ \overline{g}(x) &= \lim_{n\to\infty} g(e_{\lambda_n}(x)), \quad x \in \Omega. \end{aligned} \tag{2.15}$$

By (1.2) and (1.3), $0 \leqslant \overline{g} \leqslant g_0$ on $\overline{\Omega}$, so it follows from (2.15) and the invertibility of the operator Δ_p (see [7]) that we must have $\overline{g} > 0$ on a set of positive measure. Hence, by Lemma 2.1 and (2.15), for each $x \in \Omega$ we have $w_\infty(x) > 0$, and hence $e_{\lambda_n}(x) \to \infty$, and $\overline{g}(x) = g_\infty(x)$. Thus, by the definition of $\lambda_{\max}(g)$ and (2.15), $\lambda_\infty = \lambda_{\max}(g)$. We conclude that the mapping $\lambda \to e_\lambda : \Lambda \to W_{0,+}^{1,p}(\Omega)$ is bounded on any closed, bounded subinterval of Λ, and hence, again using the continuity properties in [7], this mapping is continuous on Λ.

Next, by similar arguments, it can be shown that if $\lambda_n \to \lambda_{\min}(g)$ then $\|e_{\lambda_n}\|_{1,p}$ cannot be bounded away from 0, and if $\lambda_n \to \lambda_{\max}(g)$ then $\|e_{\lambda_n}\|_{1,p}$ cannot be bounded, which proves (2.13), and so completes the proof of the theorem. □

Remark 2.4. (*a*) Theorem 2.3 shows that the set of non-trivial, positive equilibria $\mathcal{E}^+$ is a Rabinowitz-type global-continuum in $\Lambda \times W^{1,p}_{0,+}(\Omega)$, which bifurcates from $(\lambda_{\min}(g), 0)$ and 'meets infinity' at $\lambda_{\max}(g)$.

(*b*) It is shown in [9, 10] that if Ω is a ball, then $\mathcal{E}^+$ is in fact a smooth curve of radially symmetric solutions.

3. Time-dependent solutions

In Section 2.4 we discussed equilibrium (time-independent) solutions of equation (1.1). In this section we will discuss time-dependent solutions of (1.1). We first describe an existence and uniqueness result, and then a comparison result, which will be used to determine the long-time behaviour of the solutions.

3.1. **Existence and uniqueness of positive solutions.** In this section we will discuss the existence, uniqueness and properties of solutions of the time-dependent problem (1.1). To state precisely what we mean by a solution of (1.1) we define the spaces

$$\Sigma(T) := C([0,T), L^2(\Omega)) \cap C((0,T), W^{1,p}_0(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,T), L^2(\Omega)), \quad T > 0$$

(we allow $T = \infty$ here, and likewise for other such numbers below). The space $W^{1,2}((0,T), L^2(\Omega))$ is defined in [14, Example 10.2], using the notation $H^1((0,T), L^2(\Omega))$; the space $W^{1,2}_{\mathrm{loc}}((0,T), L^2(\Omega))$ can be defined by a simple adaptation of the definition in [14]. We will search for a solution of (1.1) in $\Sigma(T)$, for some $T > 0$. Thus, in this setting, a solution v will be regarded as a time-dependent mapping $t \to v(t) : (0,T) \to W^{1,p}_0(\Omega)$, with $\Delta_p(v(t)) \in W^{-1,p'}(\Omega)$ defined by (2.1), for each $t \in (0,T)$, and satisfying the initial condition at $t = 0$ as a limit in $L^2(\Omega)$. More (or less) regularity at $t = 0$ can be attained, depending on the regularity of v_0 (for example, if $v_0 \in W^{1,p}_{0,+}(\Omega)$ then the solution will belong to $C([0,T), W^{1,p}_0(\Omega)))$, but the above setting will suffice here.

In view of this, we will rewrite (1.1) in the form

$$\frac{dv}{dt} = \Delta_p(v) + \lambda g(v)\phi_p(v), \quad v(0) = v_0 \in C^0_{0,+}(\overline{\Omega}). \tag{3.1}$$

The following theorem describes the existence and uniqueness of solutions of (3.1), and various additional properties which will be required below. This theorem does not require g to be positive, nor to satisfy the conditions (1.2), (1.3). Once we have established the general existence of solutions we will then prove their positivity, and thereafter the values of $g(x,\xi)$, $\xi < 0$, will be irrelevant. If g is, initially, only defined on $\overline{\Omega} \times [0,\infty)$ then we may simply extend it to $\overline{\Omega} \times \mathbb{R}$ by setting $g(x,-\xi) = g(x,0)$, $\xi > 0$.

Theorem 3.1. *Suppose that $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition* (1.4) *on $\overline{\Omega} \times \mathbb{R}$, and $\lambda > 0$, $v_0 \in C^0_0(\overline{\Omega})$. Then* (3.1) *has a unique solution $v_{\lambda g,v_0} \in \Sigma(T_{\lambda g,v_0})$, defined on a maximal interval $[0, T_{\lambda g,v_0})$, for some $T_{\lambda g,v_0} > 0$, having the following properties.*

(*a*) $v_{\lambda g,v_0}(0) = v_0$.

(*b*) *The function $v_{\lambda g,v_0} : [0, T_{\lambda g,v_0}) \to L^2(\Omega)$ is differentiable at almost all $t \in [0, T_{\lambda g,v_0})$, and at such t,*

$$\frac{d}{dt}\, v_{\lambda g,v_0}(t)\,,\ \Delta_p(v_{\lambda g,v_0}(t)) \in L^2(\Omega),$$

and

$$\frac{d}{dt}\, v_{\lambda g,v_0}(t) = \Delta_p(v_{\lambda g,v_0}(t)) + \lambda g(v_{\lambda g,v_0}(t))\phi_p(v_{\lambda g,v_0}(t)), \quad \text{in } L^2(\Omega).$$

(*c*) *The function $E_{\lambda g}(v_{\lambda g,v_0}(\cdot)) : (0, T_{\lambda g,v_0}) \to \mathbb{R}$ is absolutely continuous, decreasing and*

$$\frac{d}{dt}\, E_{\lambda g}(v_{\lambda g,v_0}(t)) = -\Big\|\frac{d}{dt}\, v_{\lambda g,v_0}(t)\Big\|_2^2, \quad a.e. \ \ t \in (0, T_{\lambda g,v_0}). \tag{3.2}$$

(*d*) *The interval* $[0, T_{\lambda g, v_0})$ *on which the solution exists is maximal, in the sense that*

$$T_{\lambda g, v_0} < \infty \implies \lim_{t \nearrow T_{\lambda g, v_0}} |v_{\lambda g, v_0}(t)|_0 = \infty. \tag{3.3}$$

Proof. Let $\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ be a decreasing function with

$$\theta(s) = \begin{cases} 1, & s \leqslant 1, \\ 0, & s \geqslant 2, \end{cases}$$

and for any integer $n \geqslant 1$, define $\hat{f}_n : \overline{\Omega} \times \mathbb{R} \to (0, \infty)$ by

$$\hat{f}_n(x, \xi) := \theta(|\xi|/n)\, g(x, \xi)\, \phi_p(\xi), \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}.$$

Since $\hat{f}_n$ is bounded and Lipschitz, the results of [1, 2, 4] show that the problem

$$\hat{v}_t = \Delta_p(\hat{v}) + \lambda \hat{f}_n(\hat{v}), \quad \hat{v}(0) = v_0, \tag{3.4}$$

has a unique solution $\hat{v}_n \in \Sigma(\infty)$ having the properties (a)-(c) (we discuss this further in Remark 3.2 below). Clearly, $\hat{v}_n$ is a solution of (3.1) on the time interval $[0, T_n)$, where

$$T_n := \sup\{T : |\hat{v}_n(t)|_0 \leqslant n : t \in [0, T]\}, \quad n \geqslant 1.$$

The sequence (T_n) is increasing, so we may define $T_{\lambda g, v_0} := \lim_{n \to \infty} T_n$, and we see that (3.1) has a unique solution $v_{\lambda g, v_0} \in \Sigma(T_{\lambda g, v_0})$, having the properties (a)-(c), and also

$$T_{\lambda g, v_0} < \infty \implies \lim_{n \to \infty} |v_{\lambda g, v_0}(T_n)|_0 = \infty, \tag{3.5}$$

so that (3.3) holds, with 'limsup' instead of 'lim'. However, if the limit $\lim_{t \nearrow T_{\lambda g, v_0}} |v_{\lambda g, v_0}(t)|_0$ does not exist then there exists another sequence (s_n) such that $s_n \nearrow T_{\lambda g, v_0}$ and the sequence $(|v_{\lambda g, v_0}(s_n)|_0)$ is bounded. But then the argument in the proof of [15, Corollary 4.1] shows that this, together with (3.5), leads to a contradiction, so the limit in (3.3) must in fact exist, that is, property (d) must hold. This completes the proof of Theorem 3.1. □

Remark 3.2. (*a*) The existence and uniqueness of a solution $\hat{v}_n$ of (3.4), and the fact that $\hat{v}_n$ has properties (a)-(b), as asserted in the proof of Theorem 3.1, follows by combining various standard results on maximal monotone operators. Specifically, [2, Theorem 3.2], [1, Theorems 3.4, 3.11] and [1, Remark 3.6(5)]. How these results combine to give a solution with the desired properties is discussed in detail in [4, Remark 2.2]. It should be noted that, with the sign of f used here, the functions $\hat{f}_n$ are not monotone but, by assumption (1.4), they satisfy the Lipschitz condition imposed on the function f_2 in assumption (2.4) in [4]. Thus, to apply the discussion in [4] to the problem (3.4) above, we set (in the notation in [4]) $f_1 = 0$ and $f_2 = \hat{f}_n$.

(*b*) The fact that $\hat{v}_n$ has property (c) follows from [2, Lemma 3.3], and the argument in the proof of [4, Lemma 3.1].

(*c*) The existence and uniqueness of a local (in time) solution of (3.1), with weaker properties than those stated in Theorem 3.1, is proved in [15, Theorem 2.1], and global existence and uniqueness of such solutions of (3.4) (under similar Lipschitz conditions) is proved in [15, Theorem 3.1]. Hence, the solution $v_{\lambda g, v_0}$ given by Theorem 2.3 is unique in a considerably broader solution space than Σ^p.

3.2. **Comparison results.** We now consider the auxiliary problem

$$\frac{dw}{dt} = \Delta_p(w) + \lambda \gamma \phi_p(w), \quad w(0) = w_0 \in C^0_{0,+}(\overline{\Omega}), \tag{3.6}$$

where $\gamma \in L^\infty(\Omega)$ is independent of w, and $\gamma \geqslant 0$ on Ω. This is a special case of (3.1) (with $g(x, \cdot)$ having the form $\gamma(x)$) so, by Theorem 3.1, the problem (3.6) has a unique solution $w_{\lambda\gamma, w_0}$ defined on a maximal interval $[0, T_{\lambda\gamma, w_0})$.

Remark 3.3. Theorem 3.1 was stated, and proved, for continuous functions g depending on (x, ξ) (and Lipschitz with respect to ξ), but the results quoted from [4] (see Remark 3.2) in the proof of Theorem 3.1 apply equally to the problem (3.6), containing an x-dependent function $\gamma \in L^\infty(\Omega)$.

We now describe a 'comparison' result for solutions of (3.1) and (3.6). For any $T > 0$ and functions $\omega_1, \omega_2 \in \Sigma(T)$, we write $\omega_1 \geqslant \omega_2$ on $[0,T)$ if $\omega_1(t) \geqslant \omega_2(t)$, on $\overline{\Omega}$, for each $t \in [0,T)$.

Lemma 3.4. *Suppose that g satisfies the hypotheses of Theorem 2.3, and also g is positive, and satisfies* (1.2) *and* (1.3). *If $g_\infty, \gamma \in L^\infty(\Omega)$ satisfy $g_\infty \geqslant \gamma \geqslant 0$ and $v_0, w_0 \in C^0_{0,+}(\overline{\Omega})$ satisfy $v_0 \geqslant w_0 \geqslant 0$, then*

$$T_{\lambda g, v_0} \leqslant T_{\lambda\gamma, w_0} \quad \textit{and} \quad v_{\lambda g, v_0} \geqslant w_{\lambda\gamma, w_0} \quad \textit{on } [0, T_{\lambda g, v_0}).$$

Proof. The proof follows, with minor modifications, the proof of [13, Theorem 2.5], using our assumptions on g (in particular, the assumption $g_\infty \geqslant \gamma$ implies that $g(v) \geqslant \gamma$ for any $v \in W^{1,p}_{0,+}(\Omega)$). We omit the details. However, we note that [13, Theorem 2.5] considers equations of the form $v_t = \Delta_p(v) + \lambda\phi_p(v)$, but the proof can be adapted to give the above result; the argument in [13] is based on the proof of [6, Lemma 3.1, Ch. VI], which considered the equation $v_t = \Delta_p(v)$. □

If $\gamma = 0$ and $w_0 = 0$, then clearly $w_{\lambda\gamma, w_0} \equiv 0$, and since $g_\infty \geqslant 0$, Lemma 3.4 now yields the following positivity result for the solution $v_{\lambda g, v_0}$ of (3.1) found in Theorem 3.1.

Corollary 3.5. *If $v_0 \in C^0_{0,+}(\overline{\Omega})$ then $v_{\lambda g, v_0}(t) \in W^{1,p}_{0,+}(\Omega)$ for all $t \in (0, T_{\lambda g, v_0})$.*

In view of Corollary 3.5 we see that values of $g(x,\xi)$ when $\xi < 0$ are irrelevant to the solutions of the problem (1.1). From now on we resume supposing throughout that g satisfies our basic hypotheses, that is, g is positive and satisfies (1.2)-(1.4) on $\overline{\Omega} \times [0,\infty)$.

In the next section we will use the comparison result Lemma 3.4 to describe the behaviour of solutions of (3.1). The following criterion for finite time blow-up of solutions of (3.6) will be useful.

Lemma 3.6. *If $\lambda > \mu_0(\gamma)$ and $0 \neq w_0 \in C^0_{0,+}(\overline{\Omega})$, then $T_{\lambda\gamma, w_0} < \infty$.*

Proof. This result is proved in [13, Theorem 3.5] for the case $\gamma \equiv 1$. For the case of more general γ the proof is almost identical; for completeness we sketch the differences.

- The functional $\mathcal{E}$ defined in (3.2) in [13] coincides with the functional E_λ as defined here (with $g \equiv 1$); for general γ we simply replace $\mathcal{E} = E_\lambda$ with $\mathcal{E} = E_{\lambda\gamma}$.
- Following the proof of [13, Lemma 3.4], we move the number λ inside the integrals that are multiplied by λ, and then replace λ with $\lambda\gamma$ (we need the function γ inside the integrals). After this change the rest of the proof of the lemma holds verbatim.
- In the proof of [13, Theorem 3.5] the eigenfunctions used are those of the weighted problem (2.3) with $\rho = \gamma$ (rather than $\rho \equiv 1$ as in [13]). After this change the rest of the proof of the theorem holds verbatim (the technical positivity results in the proof are obtained in [13, Lemma 2.6] using solutions of the equation $u_t = \Delta_p(u)$, so are unaffected by the presence, or absence, of the function γ). □

The crux is the proof of [13, Lemma 3.4]. This can be proved in the general case by following, almost verbatim, the proof in [13, Lemma 3.4] except by replacing λ by $\lambda\gamma$ throughout the proof, and in the functional $\mathcal{E}$ defined in (3.2) in [13] (the functional $\mathcal{E}$ in [13] coincides with the functional E_λ as defined here, so we simply replace $\mathcal{E} = E_\lambda$ with $E_{\lambda\gamma}$, in our notation).

4. Global stability and instability of the equilibria

For any $\lambda > 0$ the time-dependent problem (3.1) has the trivial equilibrium solution $u = 0$, and also, by Theorem 2.3, for each $\lambda \in (\lambda_{\min}(g), \lambda_{\max}(g))$ there is a unique, non-trivial, positive equilibrium $e_\lambda \in W^{1,p}_{0,+}(\Omega)$. We will now consider the global stability, and instability, of these equilibria.

Theorem 4.1. *Suppose that $0 \neq v_0 \in C^0_{0,+}(\overline{\Omega})$.*

(*a*) *If $0 < \lambda \leqslant \lambda_{\min}(g)$ then $T_{\lambda g, v_0} = \infty$ and $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t)\|_{1,p} = 0$.*

(*b*) *If $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then $T_{\lambda g, v_0} = \infty$ and $\lim_{t\to\infty} \|v_{\lambda g, v_0}(t) - e_\lambda\|_{1,p} = 0$.*

(*c*) *If $\lambda_{\max}(g) < \lambda$ then $T_{\lambda g, v_0} < \infty$, that is, the solution $v_{\lambda g, v_0}$ blows up in finite time.*

Proof. The proof of part (c) is short, and the idea will also be used in the (much longer) proof of part (b), so we deal with it first. By hypothesis, $\lambda > \lambda_{\max}(g) = \mu_0(g_\infty)$, so by Lemmas 3.4 and 3.6 (with $\gamma = g_\infty$ and $w_0 = v_0$), $T_{\lambda g, v_0} \leqslant T_{\lambda g_\infty, v_0} < \infty$, which proves part (c).

We will now prove parts (a) and (b), so for the rest of the proof we will suppose that $0 < \lambda < \lambda_{\max}(g)$. Choose an arbitrary time $\bar{t} \in (0, T_{\lambda g, v_0})$ and let $\bar{v} = v_{\lambda g, v_0}(\bar{t}) \in W^{1,p}_{0,+}(\Omega)$. Then $E_{\lambda g}(\bar{v})$ is well-defined and, by Lemma 2.2 and Theorem 3.1 (c)-(d),

$$\begin{aligned} E_{\lambda g}(v_{\lambda g, v_0}(t)) \leqslant E_{\lambda g}(\bar{v}) &\implies \|v_{\lambda g, v_0}(t)\|_{1,p} \leqslant M_\lambda(E_{\lambda g}(\bar{v})), \quad \bar{t} \leqslant t < T_{\lambda g, v_0} \\ &\implies T_{\lambda g, v_0} = \infty \ \text{ and } \ E_{\lambda g}(v_{\lambda g, v_0}(\cdot)) \text{ is bounded on } (0, \infty) \\ &\implies \lim_{t \to \infty} E_{\lambda g}(v_{\lambda g, v_0}(t)) \text{ exists.} \end{aligned} \tag{4.1}$$

From now on, (t_n) will denote an increasing sequence in $(0, \infty)$ such that $t_n \to \infty$; we will choose various such sequences below, without continually relabelling them. For an arbitrary sequence (t_n) of this form it follows from (4.1) that the sequence $(v_{\lambda g, v_0}(t_n))$ is bounded in $W^{1,p}_0(\Omega)$, so we may also suppose (after taking a subsequence if necessary) that

$$v_{\lambda g, v_0}(t_n) \rightharpoonup v_\infty \text{ in } W^{1,p}_0(\Omega), \quad |v_{\lambda g, v_0}(t_n) - v_\infty|_0 \to 0, \tag{4.2}$$

for some $v_\infty \in W^{1,p}_{0,+}(\Omega)$ (where $\rightharpoonup$ denotes weak convergence in $W^{1,p}_0(\Omega)$). The argument in the proof of [4, Lemma 3.1 (3)] shows that

$$\lim_{t \to \infty} E_{\lambda g}(v_{\lambda g, v_0}(t)) \text{ exists and } |v_{\lambda g, v_0}(t_n) - v_\infty|_0 \to 0 \implies v_\infty \text{ is an equilibrium of (3.1),}$$

so by Theorem 2.3:

- if $0 < \lambda \leqslant \lambda_{\min}(g)$ then $v_\infty = 0$;
- if $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then either $v_\infty = 0$ or $v_\infty = e_\lambda$.

We will now show that these convergence results hold as $t \to \infty$, not just as $t_n \to \infty$.

Lemma 4.2. *(a) If $0 < \lambda \leqslant \lambda_{\min}(g)$ then $\lim_{t \to \infty} |v_{\lambda g, v_0}(t)|_0 = 0$.*

(b) If $\lambda_{\min}(g) < \lambda < \lambda_{\max}(g)$ then $\lim_{t \to \infty} |v_{\lambda g, v_0}(t) - e_\lambda|_0 = 0$.

Proof. (a) Suppose that the result is not true. Then there exists a sequence (t_n) (of the above form) and $\epsilon > 0$ such that $|v_{\lambda g, v_0}(t_n)|_0 \geqslant \epsilon > 0$. In addition, the paragraph preceding the lemma shows that we may suppose that (4.2) also holds (after taking a subsequence if necessary), so that $|v_{\lambda g, v_0}(t_n)|_0 \to 0$ (since $v_\infty = 0$ in this case), which contradicts the previous assertion, and so proves the result.

(b) We first suppose that there exists sequences (t^0_n), (t^1_n), such that

$$|v_{\lambda g, v_0}(t^0_n)|_0 \to 0, \quad |v_{\lambda g, v_0}(t^1_n) - e_\lambda|_0 \to 0.$$

Then, by continuity of the mapping $t \to |v_{\lambda g, v_0}(t)|_0$, there exists a sequence (t^2_n) such that

$$|v_{\lambda g, v_0}(t^2_n)|_0 = |e_\lambda|_0/2, \quad n \geqslant 1,$$

but this contradicts the results preceding the lemma. We conclude that in this case we must have $\lim_{t \to \infty} |v_{\lambda g, v_0}(t) - v_\infty|_0 = 0$, for either $v_\infty = 0$ or $v_\infty = e_\lambda$.

Now suppose that

$$\lim_{t \to \infty} |v_{\lambda g, v_0}(t)|_0 = 0. \tag{4.3}$$

For any $\delta > 0$, we define $\gamma_\delta \in C^0(\overline{\Omega})$ by $\gamma_\delta(x) := g(x, \delta)$, $x \in \overline{\Omega}$. By the properties of g, and the principal eigenvalue function $\mu_0(\cdot)$ (see Lemma 2.1), we have

$$\gamma_\delta \leqslant g_0 \ \text{ and } \ \lim_{\delta \searrow 0} |\gamma_\delta - g_0|_0 = 0, \quad \mu_0(\gamma_\delta) \geqslant \mu_0(g_0) \ \text{ and } \ \lim_{\delta \searrow 0} \mu_0(\gamma_\delta) = \mu_0(g_0).$$

Hence, since $\lambda > \lambda_{\min}(g) = \mu_0(g_0)$, we may choose δ sufficiently small that $\lambda > \mu_0(\gamma_\delta)$.

Having chosen δ, by (4.3) we may now choose $t_\delta \geqslant 0$ such that

$$0 < |v_{\lambda g, v_0}(t_\delta)|_0, \quad |v_{\lambda g, v_0}(t)|_0 \leqslant \tfrac{1}{2}\delta, \quad t \geqslant t_\delta, \tag{4.4}$$

and we define $v_\delta := v_{\lambda g, v_0}(t_\delta)$. We also define the function

$$\tilde{g}(x,\xi) := \begin{cases} g(x,\xi), & 0 \leqslant \xi \leqslant \delta, \\ \gamma_\delta(x), & \xi \geqslant \delta. \end{cases}$$

Clearly, $\tilde{g}_\infty = \gamma_\delta$ and, by (4.4) and the definition of $\tilde{g}$, $v_{\lambda g, v_0}(t + t_\delta) = v_{\lambda \tilde{g}, v_\delta}(t)$, $t \geqslant 0$. Also:

- by Lemma 3.4 (with $\gamma = \gamma_\delta$ and $v_0 = w_0 = v_\delta$), $v_{\lambda\tilde{g}, v_\delta}(t) \geqslant w_{\lambda\gamma_\delta, v_\delta}(t)$;
- by Lemma 3.6 (with $\lambda > \mu_0(\gamma_\delta)$ and $w_0 = v_\delta \neq 0$), $|v_{\lambda\gamma_\delta, v_\delta}(t)|_0 \to \infty$.

These results contradict (4.4), and so complete the proof of part (b) of Lemma 4.2. □

Next, to simplify the notation, and to combine the two cases (a) and (b), we define $\sigma_\lambda \in W^{1,p}_{0,+}(\Omega)$ by

$$\sigma_\lambda := \begin{cases} 0, & \text{if } 0 < \lambda \leqslant \lambda_{\min}(g), \\ e_\lambda, & \text{if } \lambda_{\min}(g) < \lambda < \lambda_{\max}(g), \end{cases}$$

and the preceding results show that

$$\lim_{t\to\infty} |v_{\lambda g, v_0}(t) - \sigma_\lambda|_0 = 0, \quad 0 < \lambda < \lambda_{\max}(g). \tag{4.5}$$

Thus, it only remains to prove that this convergence also holds with respect to the $W^{1,p}_0(\Omega)$ norm.

By integrating (3.2) with respect to t and using the existence of the limit $\lim_{t\to\infty} E_{\lambda g}(v_{\lambda g, v_0}(t))$ (by (4.1)), we see that the function on the right hand side of (3.2) lies in $L^1(0,\infty)$, so we may choose a sequence (t_n) such that

$$\|\Delta_p(v_{\lambda g, v_0}(t_n)) + \lambda g(v_{\lambda g, v_0}(t_n))\phi_p(v_{\lambda g, v_0}(t_n))\|_2 \to 0. \tag{4.6}$$

We may also suppose (after taking a subsequence if necessary) that (4.2) holds, with $v_\infty = \sigma_\lambda$. Hence, by (2.1), (2.12), (4.5) and (4.6),

$$\int_\Omega \big(\Delta_p(v_{\lambda g, v_0}(t_n)) + \lambda g(v_{\lambda g, v_0}(t_n))\phi_p(v_{\lambda g, v_0}(t_n))\big) v_{\lambda g, v_0}(t_n) \to 0, \quad \text{(by (4.1) and (4.6))}$$

$$\implies \int_\Omega |\nabla v_{\lambda g, v_0}(t_n)|^p \to \lambda \int_\Omega g(\sigma_\lambda)\sigma_\lambda^p = \int_\Omega |\nabla \sigma_\lambda|^p, \qquad \text{(by (2.1) and (2.12))}$$

and combining this with (4.2) yields

$$v_{\lambda g, v_0}(t_n) \rightharpoonup \sigma_\lambda, \quad \text{in } W^{1,p}_0(\Omega), \quad \|v_{\lambda g, v_0}(t_n)\|_{1,p} \to \|\sigma_\lambda\|_{1,p}. \tag{4.7}$$

Since $W^{1,p}_0(\Omega)$ is uniformly convex, it follows from (4.7) that $\|v_{\lambda g, v_0}(t_n) - \sigma_\lambda\|_{1,p} \to 0$, (see [3, Proposition 3.32]: in a uniformly convex Banach space, weak convergence of a sequence together with convergence of the norms of the elements of the sequence, implies strong convergence of the sequence). This now implies that $E_{\lambda g}(v_{\lambda g, v_0}(t_n)) \to E_{\lambda g}(\sigma_\lambda)$, and so

$$\lim_{t\to\infty} E_{\lambda g}(v_{\lambda g, v_0}(t)) = E_{\lambda g}(\sigma_\lambda) \tag{4.8}$$

(since this limit exists, by (4.1)).

Now suppose that there exists a sequence (t_n) and $\epsilon > 0$ such that $\|v_{\lambda g, v_0}(t_n) - \sigma_\lambda\|_{1,p} > \epsilon$, and also that (4.2) holds. Combining this with (4.8) (and the form of $E_{\lambda g}$) shows that (4.7) again holds, and so (by uniform convexity) $\|v_{\lambda g, v_0}(t_n) - \sigma_\lambda\|_{1,p} \to 0$, which contradicts this choice of sequence (t_n). Hence, we must have $\|v_{\lambda g, v_0}(t) - \sigma_\lambda\|_{1,p} \to 0$, which completes the proof of parts (a) and (b) of Theorem 4.1. □

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