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Quadratic risk minimization in a regime-switching model with portfolio constraints

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Abstract

We study a problem of stochastic control in mathematical finance, for which the asset prices are modeled by Itô processes. The market parameters exhibit "regime-switching" in the sense of being adapted to the joint filtration of the Brownian motion in the asset price models and a given finite-state Markov chain which models "regimes" of the market. The goal is to minimize a general quadratic loss function of the wealth at close of trade subject to the constraint that the vector of dollar amounts in each stock remains within a given closed convex set. We apply a conjugate duality approach, the essence of which is to establish existence of a solution to an associated dual problem and then use optimality relations to construct an optimal portfolio in terms of this solution. The optimality relations are also used to compute explicit optimal portfolios for various convex cone constraints when the market parameters are adapted specifically to the Markov chain.

Keywords: Convex analysis; duality synthesis; variational analysis; convex constraints; finite-state Markov chain; regime-switching

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1 Introduction

We study a problem of stochastic control in mathematical finance with the general goal of minimizing a quadratic loss function of the *terminal wealth* at close of trade. Appropriate choice of the loss function enables one to address more specific problems, such as minimizing the mean-square discrepancy between the terminal wealth and a specified square-integrable contingent claim (L_2-hedging), or minimizing the variance of the wealth at close of trade when its expected value is specified as a constraint (mean-variance portfolio selection). Problems of this general kind arise quite naturally in financial applications. For example, L_2-hedging is a useful tool in the management of a defined-benefit pension plan, in which one tries to minimize the mean-square discrepancy between the asset value and the liability at some future time T, so as to avoid either over-funding or under-funding the liability, both of which are undesirable from the viewpoint of managing the fund.

Two aspects of the problem of quadratic minimization addressed here deserve comment: the portfolio, which is specified by the vector of dollar amounts invested in each risky asset, must take values in a given closed and convex set (this amounts to a constraint on the portfolio), and the market model includes "regime-switching" among a finite number of "regimes" or "market states". To be more precise, we postulate a fairly classical continuous-time market model with finitely many risky assets and one risk-free asset; the asset prices are driven by a Brownian motion, and the model includes the additional element that the market parameters (risk-free interest rate, mean return rate on stocks, and volatility) undergo random "regime-switching" among a finite number of specified "market states". For example, a market model could include two market states, one of which corresponds to a "bull market" (with generally increasing prices) while the other regime corresponds to a "bear market" (with generally falling prices). Switching between states is modelled by means of a finite-state continuous-time Markov chain, the states of which effectively correspond to the market states we wish to include in the model. The market model is therefore "driven" by both a Brownian motion and a finite-state Markov chain, and the dependence of the market parameters on these driving processes is modelled by the stipulation that the market parameters be adapted to the *joint filtration* of the Brownian motion and the Markov chain. It is assumed that the Markov chain and the Brownian motion are independent, a condition which simplifies the analysis, and also has an economic justification: the Brownian motion models micro-economic effects on prices over short time-scales, while the Markov chain models macro-economic effects over long time-scales. The independence condition really amounts to the reasonable assumption that micro-economic and macro-economic effects on prices are independent.

A precursor to the present work is that of Zhou and Yin [2003], in which the problem of interest is similar to that summarized above, and which incorporates in particular a regime-switching market model. The portfolios in Zhou and Yin [2003] are unconstrained, and the market parameters are "Markov-modulated", in the sense that at any given instant the parameters are determined completely by the state of the Markov chain at that instant (in this regard we also draw attention to the recent results of Sotomayor and Cadenillas [2009] on the related problem of unconstrained utility maximization with Markov-modulated market parameters). Zhou and Yin [2003] adopt the approach of stochastic LQ control and completion of squares, to which their problem is ideally suited, and express the optimal portfolio in terms of affine feedback on the current wealth. When the portfolio is constrained and the market parameters are not specifically Markov-modulated, but depend at any instant on the joint history of the Brownian motion and the Markov chain up to that instant, as is the case in the present work, then it becomes rather difficult to follow the approach of stochastic LQ control used in Zhou and Yin [2003]. Indeed, portfolio constraints just by themselves, even without regime switching, constitute a definite challenge. A particularly effective approach for dealing with portfolio constraints is the method of “auxiliary markets”, introduced by Cvitanić and Karatzas [1992] for problems of constrained utility maximization. The essential idea is to formulate a complete “auxiliary” market model which has the property that *unconstrained* optimization in the auxiliary market amounts to *constrained* optimization in the given market. Despite the evident power of this method it is not apparent how to come up with an auxiliary market when the problem involves regime-switching in conjunction with portfolio constraints. For this reason, in the present work we shall follow an approach established by Labbé and Heunis [2007] for constrained portfolio optimization (either quadratic minimization or utility maximization), which allows for random market parameters (but in Labbé and Heunis [2007] did not allow for regime-switching in the market model) and which in particular does not require the formulation of an auxiliary market. The essence of the approach is to suppress the portfolio as the basic “free variable”, and write the given portfolio optimization problem as a type of *Bolza problem* involving the optimization of an objective functional over a vector space of Itô processes which includes all *wealth processes* arising from admissible portfolios. This re-formulated “primal” problem is ideally suited to direct application of the conjugate duality theory of Bismut [1973], which yields an appropriate dual functional, a weak duality relation between the primal and dual functionals, and optimality relations giving necessary and sufficient conditions for the primal problem and the dual problem (that is, minimization of the dual functional) to each have a solution with zero duality gap. Existence of a solution to the dual problem is established by the direct (Nagumo-Tonelli) method, and the optimality relations are then used to synthesize an optimal portfolio in terms of the solution of the dual problem. The goal of the present work is to generalize the approach of Labbé and Heunis [2007], briefly outlined above, to market models which include regime-switching as well as portfolio constraints.

In Sections 2 - 3 we set out the regime-switching market model, formulate the problem of interest, namely constrained quadratic loss minimization with random market parameters, and summarize some useful background. In Section 4 we construct the optimal portfolio. In Section 5 we specialize to the case where the market parameters are adapted only to the regime-state Markov chain, and use the optimality relations of Section 4 to construct explicit optimal portfolios in feedback form (on the current wealth) for problems which include portfolio constraints.

2 Market model and quadratic minimization

We assume investment in a continuous-time market model over a finite time horizon $[0, T]$ for a constant $T \in (0, \infty)$, with the following conditions in force:

Condition 2.1. The market is subject to regime-switching, as modelled by a continuous-time Markov chain $\{\alpha(t),\, t \in [0, T]\}$ which takes values in a finite state space $I = \{1, \dots, D\}$, with non-random initial state $\alpha(0) := i_0 \in I$. Associated with the Markov chain is a generator G, which is a $D \times D$ matrix $G = [g_{ij}]_{i,j=1}^{D}$ with the properties $g_{ij} \geq 0$, for all $i \neq j$ and $g_{ii} = -\sum_{j\neq i} g_{ij}$. The prices of the risky assets are driven by an N-dimensional, standard Brownian motion $\mathbf{W} = \{\mathbf{W}(t); t \in [0, T]\}$ with scalar entries $W_n(t)$, $n = 1, \dots, N$. The Markov chain and the Brownian motion are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are assumed to be independent. With $\mathcal{N}(\mathbb{P}) := \{A \in \mathcal{F} \ :\ \mathbb{P}(A) = 0\}$, the information available to investors is represented by the filtration

$$\mathcal{F}_t := \sigma\{(\alpha(s), \mathbf{W}(s)), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0, T]. \tag{2.1}$$

From Condition 2.1 the Markov processes α and $\mathbf{W}$ are independent Feller processes with values in I and $\mathbb{R}^N$ respectively. It then follows from Kallenberg [2002, Chapter 19, Exercise

10, page 389] that $(\alpha, \mathbf{W})$ is a Feller process with values in $I \times \mathbb{R}^N$, and therefore $\{\mathcal{F}_t\}$ is a right-continuous filtration (see Revuz and Yor [1994, Proposition III(2.10)]).

Remark 2.2. We use $\mathcal{P}^\star$ to denote the $\{\mathcal{F}_t\}$-predictable (or previsible) σ–algebra on $\Omega \times [0, T]$. For any mapping X on the set $\Omega \times [0, T]$ with values in some Euclidean space (the dimensionality of which is clear from the context), we write $X \in \mathcal{P}^\star$ to indicate that X is $\mathcal{P}^\star$-measurable. The measure space $(\Omega \times [0, T], \mathcal{P}^\star, \mathbb{P} \otimes Leb)$, where Leb stands for the Lebesgue measure on the Borel σ-algebra on $[0, T]$, is used throughout.

Condition 2.3. The market comprises a single risk-free asset with price $\{S_0(t); t \in [0, T]\}$ and several risky assets with prices $\{S_n(t); t \in [0, T]\}$, $n = 1, 2, \dots, N$, modeled by the relations

$$\mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, \qquad \mathrm{d}S_n(t) = S_n(t)\left(b_n(t)\,\mathrm{d}t + \sum_{m=1}^{N} \sigma_{nm}(t)\,\mathrm{d}W_m(t)\right), \tag{2.2}$$

with $S_0(0) = 1$ and $S_n(0)$ being a fixed, strictly positive constant, for each $n = 1, \dots, N$.

Condition 2.4. In (2.2), the risk-free rate of return $\{r(t)\}$ is a uniformly bounded, nonnegative, $\{\mathcal{F}_t\}$-predictable scalar stochastic process, and the entries $\{b_n(t)\}$ of the $\mathbb{R}^N$-valued mean rates of return $\{\mathbf{b}(t)\}$ and the entries $\{\sigma_{nm}(t)\}$ of the $N \times N$ matrix-valued volatility process $\{\boldsymbol{\sigma}(t)\}$ of the risky assets are uniformly bounded, $\{\mathcal{F}_t\}$-predictable scalar stochastic processes. Furthermore, using $\|\mathbf{z}\|$ for the Euclidean norm and $\mathbf{z}^\top$ for the transpose of a vector $\mathbf{z} \in \mathbb{R}^N$, there exists some constant $\kappa \in (0, \infty)$ such that $\mathbf{z}^\top \boldsymbol{\sigma}(\omega, t)\boldsymbol{\sigma}^\top(\omega, t)\mathbf{z} \geq \kappa\|\mathbf{z}\|^2$ for all $(\mathbf{z}, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$.

From (2.1) and the $\{\mathcal{F}_t\}$-predictability postulated at Condition 2.4 it follows that, at every instant $t \in [0, T]$, the market parameters are effectively determined by the paths $\{\alpha(s),\ s \in [0, t]\}$ and $\{\mathbf{W}(s),\ s \in [0, t]\}$. It is in this sense that"regime-switching" by the Markov chain α is included in the market model.

Remark 2.5. From Condition 2.4 and elementary linear algebra we have the useful upper bound $\max\{\|\boldsymbol{\sigma}^{-1}(\omega, t)\mathbf{z}\|, \|(\boldsymbol{\sigma}^\top)^{-1}(\omega, t)\mathbf{z}\|\} \leq (\kappa)^{-1/2}\|\mathbf{z}\|$ for all $(\mathbf{z}, \omega, t) \in \mathbb{R}^N \times \Omega \times [0, T]$.

Remark 2.6. Define the usual $\mathbb{R}^N$-valued *market price of risk* $\boldsymbol{\theta}(t) := \boldsymbol{\sigma}^{-1}(t)\,(\mathbf{b}(t) - r(t)\mathbf{1})$, $t \in [0, T]$ (where $\mathbf{1} \in \mathbb{R}^N$ has all unit entries). From Condition 2.4 it follows that the process $\boldsymbol{\theta}$ is $\{\mathcal{F}_t\}$-predictable and uniformly bounded, namely $\kappa_{\boldsymbol{\theta}} := \sup_{(\omega,t)}\|\boldsymbol{\theta}(\omega, t)\| < +\infty$.

We shall always suppose that an investor starts with a fixed non-random initial wealth $x_0 > 0$ and follows a self-financed strategy, investing at each instant $t \in [0, T]$ a *monetary* amount $\pi_n(t)$ in the n-th stock such that the $\mathbb{R}^N$-valued process $\boldsymbol{\pi} = \{\boldsymbol{\pi}(t); t \in [0, T]\}$ (for $\boldsymbol{\pi}(t) := (\pi_1(t), \dots, \pi_N(t))$) is a *square-integrable* portfolio process in the sense that $\boldsymbol{\pi} \in L^2(\mathbf{W})$ for

$$L^2(\mathbf{W}) := \left\{\boldsymbol{\Lambda} : \Omega \times [0, T] \to \mathbb{R}^N \,\middle|\, \boldsymbol{\Lambda} \in \mathcal{P}^\star \text{ and } \mathrm{E}\int_0^T \|\boldsymbol{\Lambda}(t)\|^2\,\mathrm{d}t < \infty\right\}. \tag{2.3}$$

The wealth process $X^{\boldsymbol{\pi}} = \{X^{\boldsymbol{\pi}}(t); t \in [0, T]\}$ corresponding to a portfolio process $\boldsymbol{\pi} \in L^2(\mathbf{W})$ is the continuous, $\{\mathcal{F}_t\}$-adapted, scalar-valued process given by the *wealth equation*

$$\mathrm{d}X^{\boldsymbol{\pi}}(t) = \big(r(t)X^{\boldsymbol{\pi}}(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\sigma}(t)\boldsymbol{\theta}(t)\big)\,\mathrm{d}t + \boldsymbol{\pi}^\top(t)\boldsymbol{\sigma}(t)\,\mathrm{d}\mathbf{W}(t), \quad X^{\boldsymbol{\pi}}(0) = x_0. \tag{2.4}$$

In order to define the problem of quadratic minimization addressed in this work we postulate:

Condition 2.7. We are given (i) the closed convex portfolio constraint set $K \subset \mathbb{R}^N$ with $\mathbf{0} \in K$; (ii) $\mathcal{F}_T$-measurable random variables A and B on $(\Omega, \mathcal{F}, \mathbb{P})$ such that B is square-integrable and $0 < \inf_{\omega\in\Omega}\{A(\omega)\} \leq \sup_{\omega\in\Omega}\{A(\omega)\} < \infty$.

From now on it will always be supposed without specific mention that Conditions 2.1, 2.3, 2.4 and 2.7 are in force. No further conditions besides these will be introduced until the examples of Section 5, when these conditions will be appropriately strengthened (see Conditions 5.3 and 5.12, which pertain to Examples 5.2 and 5.11 respectively). Now define the *risk measure* $J(\omega, x)$, set of *admissible portfolios* $\mathcal{A}$, and *primal value* $\mathcal{V}$ by

$$J(\omega, x) := \frac{1}{2}[A(\omega)x^2 + 2B(\omega)x], \quad \text{for all} \quad (\omega, x) \in \Omega \times \mathbb{R}, \tag{2.5}$$

$$\mathcal{A} := \{\boldsymbol{\pi} \in L^2(\mathbf{W}) \,\big|\, \boldsymbol{\pi}(\omega, t) \in K \quad \text{for } (\mathbb{P} \otimes Leb)\text{-almost all } (\omega, t) \in \Omega \times [0, T]\}, \tag{2.6}$$

$$\mathcal{V} := \inf_{\boldsymbol{\pi} \in \mathcal{A}} \{\mathrm{E}\left(J(X^{\boldsymbol{\pi}}(T))\right)\}. \tag{2.7}$$

Remark 2.8. It is clear from Condition 2.7 and Condition 2.4 that $-\infty < \mathcal{V} < \infty$. The problem of quadratic minimization is then to

$$\text{determine and characterize some } \bar{\boldsymbol{\pi}} \in \mathcal{A} \text{ such that } \mathcal{V} = \mathrm{E}\left(J(X^{\bar{\boldsymbol{\pi}}}(T))\right). \tag{2.8}$$

Our goal is therefore to establish existence of an "optimal portfolio" $\bar{\boldsymbol{\pi}}$ and characterize its dependence on the market coefficients $\{r(t)\}$, $\{\mathbf{b}(t)\}$ and $\{\boldsymbol{\sigma}(t)\}$, and the filtration $\{\mathcal{F}_t\}$.

The problems of *mean-square hedging* and *mean-variance portfolio selection* fit within the framework of problem (2.8). This is discussed further at Remark 4.14.

3 Canonical martingales and martingale representation

In this section we define the *canonical martingales* $\{M_{ij}(t)\}_{i \neq j}$ for the Markov chain α (recall Condition 2.1), and introduce some spaces of integrand processes which are needed for a martingale representation theorem for the filtration (2.1).

3.1 The canonical martingales of the Markov chain

Denoting by χ the zero-one indicator function, for each $i, j = 1, \dots, D$, $i \neq j$, define

$$[M_{ij}](t) := \sum_{0 < s \leq t} \chi[\alpha(s_-) = i]\, \chi[\alpha(s) = j], \quad \langle M_{ij} \rangle(t) := \int_0^t g_{ij} \chi[\alpha(s_-) = i]\, \mathrm{d}s, \tag{3.1}$$

$$M_{ij}(t) := [M_{ij}](t) - \langle M_{ij} \rangle(t), \quad \forall t \in [0, T]. \tag{3.2}$$

From Condition 2.1 and Rogers and Williams [2006, Lemma IV.21.12] it follows that M_{ij} is a square-integrable purely-discontinuous $\{\mathcal{F}_t\}$-martingale with optional and predictable quadratic variations given by $[M_{ij}]$ and $\langle M_{ij} \rangle$ respectively. Notice that $[M_{ij}](t)$ counts the number of jumps of α from states i to j over the interval $[0, t]$, from which it follows that $(\Delta M_{ij})(\Delta M_{pq}) = 0$ when $(i, j) \neq (p, q)$. We then get the following orthogonality relations from the definition of optional quadratic covariation (see Liptser and Shiryayev [1989, Section 1.8]):

$$\text{(i)}[W_n, W_k] = 0 \text{ when } k \neq n; \quad \text{(ii)}[M_{ij}, W_n] = 0; \quad \text{(iii)}[M_{ij}, M_{pq}] = 0 \text{ when } (i, j) \neq (p, q). \tag{3.3}$$

Remark 3.1. For notational convenience put $M_{ii} := 0$, for each $i = 1, \dots, D$. Then for any appropriately integrable process $\mathbf{f} = (f_{ij})_{i,j=1}^D$ we can (and always shall) write $\sum_{i,j=1}^D \int_0^t f_{ij}(s)\, \mathrm{d}M_{ij}(s)$ instead of the more cumbersome $\sum_{\substack{i,j=1 \\ i \neq j}}^D \int_0^t f_{ij}(s)\, \mathrm{d}M_{ij}(s)$.

For later use define the Doléans measure $\nu_{[M_{ij}]}$ on the measurable space $(\Omega\times[0,T],\mathcal{P}^\star)$ by

$$\nu_{[M_{ij}]}[A] := \mathrm{E}\int_0^T \chi_A(\omega,t)\,\mathrm{d}[M_{ij}](t),\quad \forall A\in\mathcal{P}^\star,\quad \forall i\neq j.$$

Notation 3.2. For $\mathbb{R}^{D\times D}$-valued processes $\mathbf{f} := (f_{ij})_{i,j=1}^D$, $\mathbf{h} := (h_{ij})_{i,j=1}^D$ on the set $\Omega\times[0,T]$, we mean by $\mathbf{f}=\mathbf{h}$, $\nu_{[M]}$-a.e., that $f_{ij}=h_{ij}$, $\nu_{[M_{ij}]}$-a.e. for each $i,j=1,2,\dots D$ with $i\neq j$.

3.2 Spaces of integrands and a martingale representation theorem

Recalling Remark 2.2 and $L^2(\mathbf{W})$ (see (2.3)) let

$$L_{21} := \left\{\Upsilon:\Omega\times[0,T]\to\mathbb{R}\,\middle|\,\Upsilon\in\mathcal{P}^\star \text{ and } \mathrm{E}\left(\int_0^T|\Upsilon(t)|\,\mathrm{d}t\right)^2<\infty\right\}, \tag{3.4}$$

$$\begin{aligned} L^2(\mathbf{M}) := \Big\{\mathbf{\Gamma} = \{\Gamma_{ij}\}_{i,j=1}^D : \Omega\times[0,T]\to\mathbb{R}^{D\times D}\,\big|\, &\Gamma_{ii}=0,\ (\mathbb{P}\otimes \mathit{Leb})\text{-a.e.},\ \Gamma_{ij}\in\mathcal{P}^\star,\\ &\text{and } \mathrm{E}\int_0^T|\Gamma_{ij}(t)|^2\,\mathrm{d}[M_{ij}](t)<\infty,\ \forall i,j\in I,\ i\neq j\Big\}, \end{aligned} \tag{3.5}$$

$$\mathbb{A} := \mathbb{R}\times L_{21}\times L^2(\mathbf{W}),\qquad \mathbb{B} := \mathbb{R}\times L_{21}\times L^2(\mathbf{W})\times L^2(\mathbf{M}). \tag{3.6}$$

Remark 3.3. We write $Y=(Y_0,\Upsilon^Y,\mathbf{\Lambda}^Y,\mathbf{\Gamma}^Y)\in\mathbb{B}$ (or $Y\in\mathbb{B}$ for short) to indicate that $Y=\{(Y(t),\mathcal{F}_t);t\in[0,T]\}$ is an $\mathbb{R}$-valued *cadlag* semimartingale of the form

$$Y(t) := Y_0+\int_0^t\Upsilon^Y(s)\,\mathrm{d}s+\sum_{n=1}^N\int_0^t\Lambda_n^Y(s)\,\mathrm{d}W_n(s)+\sum_{i,j=1}^D\int_0^t\Gamma_{ij}^Y(s)\,\mathrm{d}M_{ij}(s), \tag{3.7}$$

for some quadruple $(Y_0,\Upsilon^Y,\mathbf{\Lambda}^Y,\mathbf{\Gamma}^Y)\in\mathbb{R}\times L_{21}\times L^2(\mathbf{W})\times L^2(\mathbf{M})$. It follows from the orthogonality relations (3.3) that the integrands $\Upsilon^Y\in L_{21}$ and $\mathbf{\Lambda}^Y\in L^2(\mathbf{W})$ are uniquely determined $(\mathbb{P}\otimes \mathit{Leb})$-a.e. on $\Omega\times[0,T]$ and the integrand $\mathbf{\Gamma}^Y\in L^2(\mathbf{M})$ is uniquely determined $\nu_{[M]}$-a.e. on $\Omega\times[0,T]$ (see Notation 3.2). Furthermore, an easy application of Doob's L^2-inequality shows that $\mathrm{E}\left(\sup_{t\in[0,T]}|Y(t)|^2\right)<\infty$, for all $Y\in\mathbb{B}$. In exactly the same way the notation $X=(X_0,\Upsilon^X,\mathbf{\Lambda}^X)\in\mathbb{A}$ (or $X\in\mathbb{A}$ for short) indicates that $X=\{(X(t),\mathcal{F}_t);t\in[0,T]\}$ is an $\mathbb{R}$-valued Itô process of the form

$$X(t) := X_0+\int_0^t\Upsilon^X(s)\,\mathrm{d}s+\sum_{n=1}^N\int_0^t\Lambda_n^X(s)\,\mathrm{d}W_n(s), \tag{3.8}$$

for some $(\mathbb{P}\otimes \mathit{Leb})$-a.e. unique integrands $\Upsilon^X\in L_{21}$ and $\mathbf{\Lambda}^X\in L^2(\mathbf{W})$. Effectively, $\mathbb{A}$ is the vector subspace of continuous processes in $\mathbb{B}$.

The next proposition is immediate from Remark 3.3, (2.4) and (2.3):

Proposition 3.4. *One has* $X^{\boldsymbol\pi}\equiv(x_0,rX^{\boldsymbol\pi}+\boldsymbol\pi^\top\boldsymbol\sigma\boldsymbol\theta,\boldsymbol\sigma^\top\boldsymbol\pi)\in\mathbb{A}$ *for each* $\boldsymbol\pi\in L^2(\mathbf{W})$.

Notation 3.5. For a process H and $\{\mathcal{F}_t\}$-stopping time S, put $H[0,S](\omega,t):=H(\omega,t)$ when $t\in[0,S(\omega)]$, and $H[0,S](\omega,t):=0$ when $t>S(\omega)$.

Notation 3.6. $S^{(m)}\Uparrow T$ indicates that $(S^{(m)})_{m\in\mathbb{N}}$ is a sequence $0\le S^{(m)}\le S^{(m+1)}\le T$ of $\{\mathcal{F}_t\}$-stopping times and for each ω there is an integer $M(\omega)$ such that $S^{(m)}(\omega)=T$ for all $m\ge M(\omega)$.

Remark 3.7. The preceding notion of increasing stopping times ensures that the end-point T of the interval of trade $[0,T]$ is included in the localization, and rules out the possibility that the $S^{(m)}$ are all *strictly* less than T (i.e. $S^{(m)} < T$) for all $m = 1, 2, \ldots$ Increasing sequences of stopping times in the sense of Notation 3.6 occur quite naturally in later arguments.

Recalling (3.5) and (2.3), define the spaces of integrands

$$L^2_{\mathrm{loc}}(\mathbf{W}) := \Big\{\boldsymbol{\lambda} : \Omega \times [0,T] \to \mathbb{R}^N \,\Big|\, \text{there exists a sequence of } \{\mathcal{F}_t\}\text{-stopping times } (S^{(m)})_{m\in\mathbb{N}} \text{ such that } S^{(m)} \Uparrow T \text{ and } \boldsymbol{\lambda}[0, S^{(m)}] \in L^2(\mathbf{W}) \text{ for all } m \in \mathbb{N}\Big\},$$

$$L^2_{\mathrm{loc}}(\mathbf{M}) := \Big\{\boldsymbol{\gamma} = \{\gamma_{ij}\}_{i,j=1}^D : \Omega \times [0,T] \to \mathbb{R}^{D\times D} \,\Big|\, \text{there exists a sequence of } \{\mathcal{F}_t\}\text{-stopping times } (S^{(m)})_{m\in\mathbb{N}} \text{ such that } S^{(m)} \Uparrow T \text{ and } \boldsymbol{\gamma}[0, S^{(m)}] \in L^2(\mathbf{M}) \text{ for all } m \in \mathbb{N}\Big\}.$$

Definition 3.8. The $\mathbb{R}$-valued process $\{Z(t); t \in [0,T]\}$ is a locally-square integrable $\{\mathcal{F}_t\}$-martingale when there exists a sequence of $\{\mathcal{F}_t\}$-stopping times $(S^{(m)})_{m\in\mathbb{N}}$ such that $S^{(m)} \Uparrow T$ and $\{Z(t \wedge S^{(m)}); t \in [0,T]\}$ is a square integrable $\{\mathcal{F}_t\}$-martingale for each $m \in \mathbb{N}$.

To see that the stopping time convergence at Notation 3.6 and Definition 3.8 is reasonable suppose that $\{Z(t); t \in [0,\infty)\}$ is a locally-square integrable martingale; then there is a localizing sequence $(\sigma^{(m)})_{m\in\mathbb{N}}$ of $\{\mathcal{F}_t\}$-stopping times, and in particular $\sigma^{(m)}(\omega) \uparrow \infty$ for each ω. Now put $S^{(m)} := \sigma^{(m)} \wedge T$; then $S^{(m)} \Uparrow T$ (in the sense of Notation 3.6) and the sequence $(S^{(m)})_{m\in\mathbb{N}}$ localizes $\{Z(t); t \in [0,T]\}$ in the sense of Definition 3.8.

We shall need the following martingale representation theorem:

Proposition 3.9. *Suppose that $\{Z(t); t \in [0,T]\}$ is a locally-square integrable $\{\mathcal{F}_t\}$-martingale (see Definition 3.8) and null at the origin. Then there exist processes $\boldsymbol{\Lambda} = (\Lambda_1, \ldots, \Lambda_N)^\top \in L^2_{loc}(\mathbf{W})$ and $\boldsymbol{\Gamma} = (\Gamma_{ij})_{i,j=1}^D \in L^2_{loc}(\mathbf{M})$ such that Z has the stochastic integral representation*

$$Z(t) = \sum_{n=1}^N \int_0^t \Lambda_n(s)\, \mathrm{d}W_n(s) + \sum_{i,j=1}^D \int_0^t \Gamma_{ij}(s)\, \mathrm{d}M_{ij}(s), \quad \textit{for all } t \in [0,T], \quad \textit{a.s.} \tag{3.9}$$

In view of the orthogonality relations (3.3) the integrands $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ at (3.9) are $(\mathbb{P} \otimes \mathit{Leb})$-a.e. unique and $\nu_{[M]}$-a.e. unique respectively.

Proposition 3.9 follows from Elliott [1976, Theorem 5.1], but can also be easily obtained from Jacod and Shiryaev [1987, III(4.36)]. Indeed, put $E := \{(i,j) \in I \times I \,:\, i \neq j\}$ and let $\mathcal{E}$ denote the collection of all subsets of the finite set E. Define the integer-valued random measure $\mu(\omega, A \times B) := \sum_{(i,j)\in B} \int_A [M_{ij}](\omega, \mathrm{d}s)$ for all $\omega \in \Omega$, $A \in \mathcal{B}[0,T]$, and $B \in \mathcal{E}$. Then it is easily checked that μ is an E-valued multivariate point process (see Jacod and Shiryaev [1987, III(1.23)]) and the filtration $\mathcal{F}^\alpha_t := \sigma\{\alpha(s), s \in [0,t]\} \vee \mathcal{N}(\mathbb{P})$ satisfies Jacod and Shiryaev [1987, III(1.25)] (with $\mathcal{H} := \sigma\{\alpha(0)\} = \{\emptyset, \Omega\}$, where the equality follows from Condition 2.1). It then follows from Jacod and Shiryaev [1987, III(4.36)], together with (3.2), that each $\{\mathcal{F}^\alpha_t\}$-local martingale Z with $Z(0) = 0$, is given by

$$Z(t) = \sum_{(i,j)\in E} \int_0^t \Gamma_{ij}(s)\, \mathrm{d}M_{ij}(s), \quad \text{for all } t \in [0,T], \quad \text{a.s.} \tag{3.10}$$

for some $\{\mathcal{F}_t^\alpha\}$-predictable integrand processes Γ_{ij}. In the case where Z is a $\{\mathcal{F}_t\}$-local martingale (see (2.1)), we can use the main result of Xue [1993, Theorem on pages 226-227] to combine the martingale representation at (3.10) (for the filtration $\{\mathcal{F}_t^\alpha\}$) with the classical Itô martingale representation (for the filtration of the Brownian motion $\boldsymbol{W}$) to get the representation at (3.9) for $\{\mathcal{F}_t\}$-predictable integrands Λ_n and Γ_{ij} (recall Remark 3.1). Finally, when Z is also locally square-integrable, the localization of the integrands W_n and Γ_{ij} at (3.9) to the spaces $L^2_{\mathrm{loc}}(\mathbf{W})$ and $L^2_{\mathrm{loc}}(\mathbf{M})$ is immediate from the Itô isometry and the orthogonality at (3.3).

4 Application of convex duality

In this section we use convex duality on problem (2.8). The main steps are (a) re-write (2.8) as a "Bolza problem"; (b) synthesize a dual problem and optimality relations; and (c) use the optimality relations to construct an optimal portfolio. Each step is now dealt with in detail.

4.1 Re-write (2.8) in calculus-of-variations form (a "Bolza problem")

Motivated by Labbé and Heunis [2007] we re-formulate (2.8) as a "Bolza problem" which amounts to the minimization of a functional over the vector space $\mathbb{A}$ of Itô processes (see Remark 3.3). The advantage of this re-formulation is that it lends itself to the synthesis of a dual problem and optimality relations by elementary convex analysis. To this end, and recalling (2.6) and Remark 3.3, for each $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$ define

$$\begin{aligned} U(X) := \big\{ \boldsymbol{\pi} \in \mathcal{A} \,|\, \Upsilon^X(t) = r(t)X(t) + \boldsymbol{\pi}^\top(t)\boldsymbol{\sigma}(t)\boldsymbol{\theta}(t),\ (\mathbb{P}\otimes Leb)\text{-a.e.} \\ \text{and } \boldsymbol{\Lambda}^X(t) = \boldsymbol{\sigma}^\top(t)\boldsymbol{\pi}(t),\ (\mathbb{P}\otimes Leb)\text{-a.e.}\big\}. \end{aligned} \tag{4.1}$$

It follows from (2.4) that, for each $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$, one has $X = X^{\boldsymbol{\pi}}$, $(\mathbb{P}\otimes Leb)$-a.e. for some $\boldsymbol{\pi} \in \mathcal{A}$ if and only if $X_0 = x_0$ and $U(X) \neq \emptyset$. Then, from (2.7), we get

$$\mathcal{V} = \inf_{\substack{X\in\mathbb{A},\\ X_0=x_0,\\ U(X)\neq\emptyset}} \{\mathrm{E}\,(J(X(T)))\}. \tag{4.2}$$

We next define "infinite" penalty functions to remove the constraints under the infimum above, so that the minimization is over all of $\mathbb{A}$. For the initial wealth constraint $X_0 = x_0$ define

$$l_0(x) := \begin{cases} 0 & \text{if } x = x_0 \\ \infty & \text{otherwise,} \end{cases} \qquad \forall x \in \mathbb{R}. \tag{4.3}$$

For the path constraint $U(X) \neq \emptyset$ at (4.2), observe from Remark 2.5 and (4.1) that

$$U(X) \neq \emptyset \iff \Upsilon^X = rX + \left(\boldsymbol{\Lambda}^X\right)^\top \boldsymbol{\theta} \text{ and } (\boldsymbol{\sigma}^\top)^{-1}\boldsymbol{\Lambda}^X \in K,\ (\mathbb{P}\otimes Leb)\text{-a.e.} \tag{4.4}$$

for each $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$ (to get $\Longleftarrow$ at (4.4) observe that $(\boldsymbol{\sigma}^\top)^{-1}\boldsymbol{\Lambda}^X$ is necessarily a member of $L^2(\mathbf{W})$, as follows from Remark 2.5 and $\boldsymbol{\Lambda}^X \in L^2(\mathbf{W})$). Motivated by (4.4) define

$$L(\omega, t, x, \nu, \boldsymbol{\lambda}) := \begin{cases} 0 & \text{if } \nu = r(\omega,t)x + \boldsymbol{\lambda}^\top\boldsymbol{\theta}(\omega,t) \text{ and } (\boldsymbol{\sigma}^\top(\omega,t))^{-1}\boldsymbol{\lambda} \in K \\ \infty & \text{otherwise,} \end{cases} \tag{4.5}$$

for all $(\omega, t, x, \nu, \boldsymbol{\lambda}) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$. For $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$ it is clear that $L(t, X(t), \Upsilon^X(t), \boldsymbol{\Lambda}^X(t))$ is $\mathcal{P}^\star$-measurable in (ω, t), and hence from (4.5) and (4.4) we get

$$\mathrm{E}\int_0^T L(t, X(t), \Upsilon^X(t), \boldsymbol{\Lambda}^X(t))\,\mathrm{d}t = \begin{cases} 0 & \text{if } U(X) \neq \emptyset \\ \infty & \text{otherwise,} \end{cases} \tag{4.6}$$

for each $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$. Now define the functional $\Phi : \mathbb{A} \to (-\infty, +\infty]$ as

$$\Phi(X) := l_0(X_0) + \mathrm{E}\int_0^T L(t, X(t), \Upsilon^X(t), \boldsymbol{\Lambda}^X(t))\,\mathrm{d}t + \mathrm{E}\left(J(X(T))\right), \tag{4.7}$$

for all $X \equiv (X_0, \Upsilon^X, \boldsymbol{\Lambda}^X) \in \mathbb{A}$. Upon combining (4.2), (4.3), (4.6) and (4.7), we get that $\mathcal{V} = \inf_{X\in\mathbb{A}} \{\Phi(X)\}$ (recall (2.7)), and can introduce the following "Bolza problem":

$$\text{determine some } \bar{X} \equiv (\bar{X}_0, \Upsilon^{\bar{X}}, \boldsymbol{\Lambda}^{\bar{X}}) \in \mathbb{A} \text{ such that } \Phi(\bar{X}) = \inf_{X\in\mathbb{A}} \{\Phi(X)\} = \mathcal{V}. \tag{4.8}$$

Remark 4.1. Suppose that $\bar{X} \equiv (\bar{X}_0, \Upsilon^{\bar{X}}, \boldsymbol{\Lambda}^{\bar{X}}) \in \mathbb{A}$ is a solution of (4.8), i.e. (i) $\Phi(\bar{X}) = \mathcal{V} < +\infty$ (see Remark 2.8). Then the first and second terms on the right side of (4.7) must be zero (since these take values in the two-point set $\{0, \infty\}$), that is (ii) $\mathrm{E}(J(\bar{X}(T))) = \Phi(\bar{X})$ and (iii) $\bar{X}_0 = x_0$ (see (4.3)). Later (in Section 4.3) we shall construct an $\mathbb{R}^N$-valued process $\bar{\boldsymbol{\pi}} \in \mathcal{A}$ such that $\bar{X} = X^{\bar{\boldsymbol{\pi}}}$ (see (2.4)). Then, from (i) and (ii), we obtain (iv) $\mathrm{E}(J(X^{\bar{\boldsymbol{\pi}}}(T))) = \mathcal{V}$, that is $\bar{\boldsymbol{\pi}}$ is an optimal portfolio for (2.8). Our immediate goal is to characterize a solution $\bar{X} \in \mathbb{A}$ of (4.8) through conjugate duality. We address this in the following section.

4.2 Synthesis of a dual problem and optimality relations

We synthesize the cost functional of a problem which is dual to (4.8). Motivated by Bismut [1973, see especially eqns. (2.1) and (2.2)] we first calculate "pointwise" convex conjugates of the risk measure (see (2.5)) and the penalty functions (see (4.3) and (4.5)), namely

$$m_T(\omega, y) := J^\star(\omega, -y) := \sup_{x\in\mathbb{R}} \{x(-y) - J(\omega, x)\} = \frac{1}{2A(\omega)}\left(y + B(\omega)\right)^2, \tag{4.9}$$

$$m_0(y) := l_0^\star(y) := \sup_{x\in\mathbb{R}}\{xy - l_0(x)\} = x_0 y, \tag{4.10}$$

$$M(\omega, t, y, s, \boldsymbol{\xi}) := L^\star(\omega, t, s, y, \boldsymbol{\xi}) := \sup_{\substack{x,\nu\in\mathbb{R}\\ \boldsymbol{\lambda}\in\mathbb{R}^N}} \left\{xs + \nu y + \boldsymbol{\lambda}^\top\boldsymbol{\xi} - L(\omega, t, x, \nu, \boldsymbol{\lambda})\right\}, \tag{4.11}$$

for all $\omega \in \Omega$, $y, s \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^N$ (here we used (2.5) and (4.3) to calculate the explicit expressions on the right of (4.9) and (4.10) respectively). The conjugate at (4.11) is also easily calculated using (4.5) to yield

$$M(\omega, t, y, s, \boldsymbol{\xi}) = \begin{cases} \delta\left(-\boldsymbol{\sigma}(\omega, t)[\boldsymbol{\theta}(\omega, t)y + \boldsymbol{\xi}]\right) & \text{if } s + r(\omega, t)y = 0 \\ \infty & \text{otherwise,} \end{cases} \tag{4.12}$$

for all $\omega \in \Omega$, $y, s \in \mathbb{R}$ and $\boldsymbol{\xi} \in \mathbb{R}^N$, in which (recalling Condition 2.7(i))

$$\delta(\mathbf{z}) := \sup_{\boldsymbol{\pi}\in K} \left\{-\boldsymbol{\pi}^\top\mathbf{z}\right\}, \quad \forall \mathbf{z} \in \mathbb{R}^N. \tag{4.13}$$

For each semimartingale $Y \equiv (Y_0, \Upsilon^Y, \boldsymbol{\Lambda}^Y, \boldsymbol{\Gamma}^Y) \in \mathbb{B}$ (recall Remark 3.3) define

$$\Psi(Y) := m_0(Y_0) + \mathrm{E}\int_0^T M(t, Y(t), \Upsilon^Y(t), \boldsymbol{\Lambda}^Y(t))\,\mathrm{d}t + \mathrm{E}\left(m_T(Y(T))\right), \tag{4.14}$$

$$\boldsymbol{\Theta}_Y(t) := -\boldsymbol{\sigma}(t)\left(\boldsymbol{\theta}(t)Y(t) + \boldsymbol{\Lambda}^Y(t)\right), \quad \forall t \in [0, T]. \tag{4.15}$$

The functional $\delta(\cdot)$ is nonnegative (since $\mathbf{0} \in K$ from Condition 2.7(i)) and lower semi-continuous on $\mathbb{R}^N$; it then follows from (4.12) that $M(t, Y(t), \Upsilon^Y(t), \mathbf{\Lambda}^Y(t), \mathbf{\Gamma}^Y(t))$ is $\mathcal{P}^\star$-measurable for each $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B}$ and the second term on the right of (4.14) is defined. Notice that $\Psi(Y)$ exists in $(-\infty, \infty]$ for each $Y \in \mathbb{B}$. We can now introduce the following *dual problem*:

$$\text{determine some } \bar{Y} \in \mathbb{B} \text{ such that } \Psi(\bar{Y}) = \inf_{Y \in \mathbb{B}} \{\Psi(Y)\}, \tag{4.16}$$

in which $\Psi(\cdot)$ is called the *dual functional* and $\bar{Y}$ is dubbed a *dual solution*.

Observe that, contrary to what one might expect from calculus-of-variations, the second term on the right side of (4.14) (i.e. the "Lagrange term") does not depend explicitly on the $\mathrm{d}M_{ij}(s)$-integrands Γ_{ij}^Y in the integral representation at (3.7). This is because the primal variable $X \equiv (X_0, \Upsilon^X, \mathbf{\Lambda}^X) \in \mathbb{A}$ does not include $\mathrm{d}M_{ij}(s)$-integrands which can be paired with the Γ_{ij}^Y.

We next show that the functionals $\Phi(\cdot)$ and $\Psi(\cdot)$ are related by a weak duality principle and establish *optimality relations* which give necessary and sufficient conditions for a pair $(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$ to be solutions of the problems (4.8) and (4.16) with zero "duality gap". See Appendix A for the proof.

Proposition 4.2. *Recall (3.6), Remark 3.3, (4.1), (4.7), and (4.13) - (4.15). Then*

$$\Phi(X) + \Psi(Y) \geq 0, \qquad \textit{for all } (X, Y) \in \mathbb{A} \times \mathbb{B}. \tag{4.17}$$

Moreover, for each $\bar{X} \equiv (\bar{X}_0, \Upsilon^{\bar{X}}, \mathbf{\Lambda}^{\bar{X}}) \in \mathbb{A}$ *and* $\bar{Y} \equiv (\bar{Y}_0, \Upsilon^{\bar{Y}}, \mathbf{\Lambda}^{\bar{Y}}, \mathbf{\Gamma}^{\bar{Y}}) \in \mathbb{B}$, *we have*

$$\Phi(\bar{X}) + \Psi(\bar{Y}) = 0 \tag{4.18}$$

if and only if

$$\Phi(\bar{X}) = \inf_{X \in \mathbb{A}} \{\Phi(X)\} = \sup_{Y \in \mathbb{B}} \{-\Psi(Y)\} = -\Psi(\bar{Y}) \tag{4.19}$$

if and only if the following optimality relations (4.20) - (4.22) are satisfied:

$$\textit{(i) } \bar{X}_0 = x_0 \quad \textit{and} \quad \textit{(ii) } \bar{X}(T) = -\frac{1}{A}\left(\bar{Y}(T) + B\right), \ \textit{a.s.} \tag{4.20}$$

$$\Upsilon^{\bar{Y}}(t) + r(t)\bar{Y}(t) = 0, \qquad (\mathbb{P} \otimes \textit{Leb})\textit{-a.e.} \tag{4.21}$$

$$\begin{gathered}\textit{(i) } \delta(\mathbf{\Theta}_{\bar{Y}}(t)) + \bar{\boldsymbol{\pi}}^\top(t)\mathbf{\Theta}_{\bar{Y}}(t) = 0, \ (\mathbb{P} \otimes \textit{Leb})\textit{-a.e.} \quad \textit{and} \quad \textit{(ii) } \bar{\boldsymbol{\pi}} \in U(\bar{X}), \\ \textit{in which} \qquad \textit{(iii) } \bar{\boldsymbol{\pi}}(t) := (\boldsymbol{\sigma}^\top(t))^{-1}\mathbf{\Lambda}^{\bar{X}}(t).\end{gathered} \tag{4.22}$$

Remark 4.3. Proposition 4.2 states a logical equivalence between the assertions (4.18), (4.19), and (4.20) - (4.22), for *each and every* $(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$; the equivalence therefore holds even when $\bar{Y}$ is postulated *a priori* to be a member of $\mathbb{A} \subset \mathbb{B}$, that is we put $\mathbf{\Gamma}^{\bar{Y}} = \mathbf{0}$ (recall Remark 3.3). In this case the equivalence resembles that given by Labbé and Heunis [2007, Proposition 5.3]. One may reasonably question why the proposition is stated with the hypothesis $\bar{Y} \in \mathbb{B}$ instead of just $\bar{Y} \in \mathbb{A}$. The information filtration $\{\mathcal{F}_t\}$ at (2.1) is determined jointly by the Brownian motion $\{\boldsymbol{W}(t)\}$ and the Markov process $\{\alpha(t)\}$ (not just by $\{\boldsymbol{W}(t)\}$ alone, as in Labbé and Heunis [2007]) and consequently there is no guarantee that there exist pairs $(\bar{X}, \bar{Y})$ in the smaller space $\mathbb{A} \times \mathbb{A}$ which even satisfy (4.20) - (4.22). In fact, in the course of establishing the main result of the present section (see Theorem 4.13 to follow) we shall construct a pair $(\bar{X}, \bar{Y})$ in the larger space $\mathbb{A} \times \mathbb{B}$ which satisfies (4.20) - (4.22), for which the integrand $\mathbf{\Gamma}^{\bar{Y}} \in L^2(\mathbf{M})$ in the expansion $\bar{Y} \equiv (\bar{Y}_0, \Upsilon^{\bar{Y}}, \mathbf{\Lambda}^{\bar{Y}}, \mathbf{\Gamma}^{\bar{Y}}) \in \mathbb{B}$ (recall Remark 3.3) is necessarily non-trivial. Similarly, in the concrete examples of Section 5, non-trivial integrands $\mathbf{\Gamma}^{\bar{Y}} \in L^2(\mathbf{M})$ necessarily arise in the construction of pairs $(\bar{X}, \bar{Y})$ satisfying relations (4.20) - (4.22) (see Remark 5.17 which follows).

Remark 4.4. Problem (2.8) involves formally the same portfolio constraint and wealth-dynamics as are found in the problem addressed in Labbé and Heunis [2007], but incorporates the further element of regime-switching (see Condition 2.1). The basic dual variables in Labbé and Heunis [2007] are triplets $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y) \in \mathbb{A}$ (see (3.6)) of *Lagrange multipliers*, which collectively "enforce" the portfolio and wealth-dynamics constraints, whereas the basic dual variables in the present work are *quadruples* $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B}$ (recall Remark 3.3). If one regards the regime-state Markov process α as an "asset", then it follows from the wealth equation (2.4) that direct trade in this "asset" is prohibited (that is α is a "non-tradeable asset"), which constitutes a further portfolio constraint introduced by the regime-switching. The Lagrange multiplier for this additional constraint is precisely the fourth element $\mathbf{\Gamma}^Y \in L^2(\mathbf{M})$ (recall (3.5)) in the dual variables quadruple $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B}$, the remaining three elements $(Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y)$ serving as the Lagrange multipliers for the portfolio and wealth-dynamics constraints exactly as in Labbé and Heunis [2007]. See Remark 5.17 for further discussion on this.

Remark 4.5. The formulation of problem (2.8) in the calculus-of-variations form of (4.8) (i.e. to minimize the functional at (4.7) over the space of Itô-processes $\mathbb{A}$ at (3.6)), and the synthesis of the dual functional at (4.14) in terms of the convex conjugates (4.9) - (4.11), is motivated by Bismut [1973]. It remains an interesting and challenging problem to introduce convex portfolio constraints into the (unconstrained) quadratic minimization problems in incomplete semimartingale market models studied by Hou and Karatzas [2004] and Xia and Yan [2006], and extend the conjugate duality results in these works to include such constraints.

We next establish existence of a solution of the dual problem (4.16). From (4.21) of Proposition 4.2, one sees that a solution $\bar{Y} \in \mathbb{B}$ must satisfy $\Upsilon^{\bar{Y}}(t) = -r(t)\bar{Y}(t)$, $(\mathbb{P} \otimes Leb)$-a.e. In the search for dual solutions we therefore restrict attention to the space $\mathbb{B}_1 \subset \mathbb{B}$ given by

$$\mathbb{B}_1 := \left\{ Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B} \,|\, \Upsilon^Y(t) = -r(t)Y(t),\ (\mathbb{P} \otimes Leb)\text{-a.e.} \right\}. \tag{4.23}$$

Define the $\mathbb{R}$-valued processes $\{\beta(t); t \in [0,T]\}$ and $\{\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})(t); t \in [0,T]\}$ (for $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \mathbb{R} \times L^2(\mathbf{W}) \times L^2(\mathbf{M})$) as follows:

$$\beta(t) := \exp\left\{ -\int_0^t r(s)\,\mathrm{d}s \right\}, \tag{4.24}$$

$$\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})(t) := \beta(t) \left(y + \int_0^t \beta^{-1}(s) \boldsymbol{\lambda}^\top(s)\,\mathrm{d}\mathbf{W}(s) + \sum_{i,j=1}^D \int_0^t \beta^{-1}(s)\gamma_{ij}(s)\,\mathrm{d}M_{ij}(s) \right). \tag{4.25}$$

Elementary properties of the set $\mathbb{B}_1$ and the mapping Ξ are summarized in the next proposition. The proof is a routine application of Itô's formula and is omitted.

Proposition 4.6. *Put $\boldsymbol{S} := \mathbb{R} \times L^2(\mathbf{W}) \times L^2(\mathbf{M})$, and recall (2.3), (3.5) and Remark 3.3. Then*
(a) $\mathbb{B}_1$ is a real linear space;
(b) $\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \mathbb{B}_1$ for all $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$, and $\Xi : \boldsymbol{S} \to \mathbb{B}_1$ is a linear bijection;
(c) if $Y := \Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})$ for some $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$ then $Y_0 = y$, $\Upsilon^Y = -rY$, $\mathbf{\Lambda}^Y = \boldsymbol{\lambda}$, $\mathbf{\Gamma}^Y = \boldsymbol{\gamma}$.

Remark 4.7. When $Y \in \mathbb{B} \setminus \mathbb{B}_1$ then $(\mathbb{P} \otimes Leb)\{(\omega, t) :\ \Upsilon^Y(\omega, t) \neq -r(\omega, t)Y(\omega, t)\} > 0$, so that $\Psi(Y) = +\infty$ (recall (4.12) and (4.14)). From this observation, together with Proposition 4.6(b), we obtain $\inf_{Y \in \mathbb{B}} \{\Psi(Y)\} = \inf_{Y \in \mathbb{B}_1} \{\Psi(Y)\} = \inf_{(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}} \tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})$, in which we have defined (i) $\tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) := \Psi(\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}))$ for each $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$. For $Y := \Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})$ one sees from (4.12) and (4.15) that $M(t, Y(t), \Upsilon^Y(t), \mathbf{\Lambda}^Y(t)) = \delta(\mathbf{\Theta}_Y(t))$, thus, from (4.9), (4.10) and (4.14) we get

$$\tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = x_0 y + \mathrm{E}\int_0^T \delta(\mathbf{\Theta}_{\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})}(t))\,\mathrm{d}t + \mathrm{E}\left(\frac{1}{2A}[\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})(T) + B]^2 \right), \tag{4.26}$$

for each $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$, in which (see (4.15) and Proposition 4.6(c))

$$\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t) = -\boldsymbol{\sigma}(t)[\boldsymbol{\theta}(t)\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})(t) + \boldsymbol{\lambda}(t)], \quad \text{for all } (y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}. \tag{4.27}$$

The next result, the proof of which is given in Appendix A, establishes existence of a solution of the dual problem at (4.16):

Proposition 4.8. *There exists some* $(\bar{y}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\gamma}}) \in \boldsymbol{S}$ *such that* $\tilde{\Psi}(\bar{y}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\gamma}}) = \inf_{(y,\boldsymbol{\lambda},\boldsymbol{\gamma})\in\boldsymbol{S}}\{\tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})\}$ *(recall Remark 4.7(i)), and* $\bar{Y} := \Xi(\bar{y}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\gamma}}) \in \mathbb{B}_1$ *(see Proposition 4.6(b)) solves problem (4.16).*

4.3 Construction of an optimal portfolio

Remark 4.9. The goal of this section is to construct some $\bar{X} = (\bar{X}_0, \Upsilon^{\bar{X}}, \boldsymbol{\Lambda}^{\bar{X}}) \in \mathbb{A}$ (recall Remark 3.3) in terms of the solution $\bar{Y} = (\bar{Y}_0, \Upsilon^{\bar{Y}}, \boldsymbol{\Lambda}^{\bar{Y}}, \boldsymbol{\Gamma}^{\bar{Y}}) \in \mathbb{B}_1$ of the dual problem (4.16) given by Proposition 4.8 such that the pair $(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$ satisfies the relations (4.20) - (4.22). It then follows from Proposition 4.2 that (4.19) holds, in particular $\bar{X} \in \mathbb{A}$ is a solution of (4.8). Since $\bar{Y} \in \mathbb{B}_1$ (recall (4.23)) the condition (4.21) is already satisfied, so it really remains to construct $\bar{X} \in \mathbb{A}$ such that (4.20) and (4.22) hold. We shall also construct some $\bar{\boldsymbol{\pi}} \in \mathcal{A}$ such that $\bar{X} = X^{\bar{\boldsymbol{\pi}}}$; in view of Remark 4.1 this gives an optimal portfolio for the problem (2.8).

Recalling (4.24) and Remark 2.6, define the *state price density process*

$$H(t) := \beta(t)\mathcal{E}\left(-\boldsymbol{\theta} \bullet \mathbf{W}\right)(t), \quad \forall t \in [0, T], \tag{4.28}$$

in which the notation $\mathcal{E}\left(Z\right)(t) := \exp\{Z(t) - 1/2\langle Z\rangle(t)\}$ indicates the exponential of a continuous local martingale Z and '$\bullet$' denotes stochastic integration.

Remark 4.10. Since r and $\boldsymbol{\theta}$ are uniformly bounded (see Condition 2.4 and Remark 2.6) it easily follows from (4.28) that $\mathrm{E}[\sup_{t\in[0,T]}|H(t)|^p] < \infty$ for each $p \in \mathbb{R}$. Moreover, expanding (4.28) by Itô's formula gives $\mathrm{d}H(t) = -r(t)H(t)\,\mathrm{d}t - H(t)\boldsymbol{\theta}^\top(t)\,\mathrm{d}\mathbf{W}(t)$, so that (recalling Remark 3.3) one has $\Upsilon^H := -rH \in L_{21}$ and $\Lambda^H := -H\boldsymbol{\theta} \in L^2(\mathbf{W})$; in particular $H = (1, -rH, -H\boldsymbol{\theta}) \in \mathbb{A}$.

To get some idea of how to define a candidate $\bar{X}$ which attains the goals set forth in Remark 4.9 suppose that $\bar{X} \in \mathbb{A}$ actually satisfies (4.22)(ii). Then, from (4.1), it follows that $\bar{X} = (\bar{X}_0, r\bar{X} + \bar{\boldsymbol{\pi}}^\top\boldsymbol{\sigma}\boldsymbol{\theta}, \boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}}, \mathbf{0}) \in \mathbb{B}$, and of course $H = (1, -rH, -H\boldsymbol{\theta}, \mathbf{0}) \in \mathbb{B}$. From Proposition A.1 we then see that $\mathbb{M}(\bar{X}, H)(t) = \bar{X}(t)H(t) - \bar{X}(0)$ and $\{(\bar{X}(t)H(t), \mathcal{F}_t); t \in [0, T]\}$ is a martingale. In conjunction with (4.20)(ii), this *motivates* the following definition of $\bar{X}$:

$$\bar{X}(t) := -H^{-1}(t)\mathrm{E}\left(\frac{1}{A}\left(\bar{Y}(T) + B\right)H(T)\,\middle|\,\mathcal{F}_t\right), \quad \text{a.s.}, \quad \forall t \in [0, T]. \tag{4.29}$$

Since $\bar{Y}(T)$ is square-integrable (from $\bar{Y} \in \mathbb{B}$ and Remark 3.3), and A and B are subject to Condition 2.7(ii), it follows from the square-integrability of H (see Remark 4.10) that $(1/A)\left(\bar{Y}(T) + B\right)H(T)$ is integrable, so that the conditional expectation at (4.29) exists. The next proposition, the proof of which is given in Appendix A, establishes square-integrability properties of the process $\bar{X}$:

Proposition 4.11. *Recall (4.28) and (4.29). Then* $E[\sup_{t\in[0,T]}|\bar{X}(t)|^2] < \infty$ *and the process* $\{(\bar{X}(t)H(t), \mathcal{F}_t); t \in [0, T]\}$ *is a locally square-integrable martingale (see Definition 3.8).*

In view of Proposition 4.11 and Proposition 3.9, there are integrands $\boldsymbol{\Lambda}^{\bar{X}H} \in L^2_{\mathrm{loc}}(\mathbf{W})$ and $\boldsymbol{\Gamma}^{\bar{X}H} \in L^2_{\mathrm{loc}}(\mathbf{M})$ such that $\bar{X}H$ has the following representation for all $t \in [0, T]$:

$$\bar{X}(t)H(t) = \bar{X}(0)H(0) + \sum_{n=1}^{N}\int_0^t \Lambda_n^{\bar{X}H}(s)\;\mathrm{d}W_n(s) + \sum_{i,j=1}^{D}\int_0^t \Gamma_{ij}^{\bar{X}H}(s)\;\mathrm{d}M_{ij}(s), \quad \text{a.s.} \tag{4.30}$$

Setting $\xi(t) := \bar{X}(t)H(t)$ and using the integration-by-parts formula together with (4.30) and (4.28), we can expand $\bar{X}(t) = \xi(t)H^{-1}(t)$ to obtain

$$\begin{aligned}\bar{X}(t) = \bar{X}(0) &+ \int_0^t \left(r(s)\bar{X}(s_-) + \bar{\boldsymbol{\pi}}^\top(s)\boldsymbol{\sigma}(s)\boldsymbol{\theta}(s)\right)\,\mathrm{d}s + \int_0^t \bar{\boldsymbol{\pi}}^\top(s)\boldsymbol{\sigma}(s)\,\mathrm{d}\mathbf{W}(s) \\ &+ \sum_{i,j=1}^{D}\int_0^t H^{-1}(s)\,\Gamma_{ij}^{\bar{X}H}(s)\,\mathrm{d}M_{ij}(s),\end{aligned} \tag{4.31}$$

in which we define

$$\bar{\boldsymbol{\pi}}(t) := (\boldsymbol{\sigma}^\top(t))^{-1}\left(\boldsymbol{\Lambda}^{\bar{X}H}(t)H^{-1}(t) + \bar{X}(t_-)\boldsymbol{\theta}(t)\right). \tag{4.32}$$

From (4.32), one has $\bar{\boldsymbol{\pi}} \in \mathcal{P}^\star$. Moreover, from the uniform boundedness of $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ (recall Remark 2.6), the continuity and strict positivity of the state price density process H, the square-integrability of $\bar{X}$ (see Proposition 4.11), and $\boldsymbol{\Lambda}^{\bar{X}H} \in L^2_{\text{loc}}(\mathbf{W})$, one easily sees that $\int_0^T \|\bar{\boldsymbol{\pi}}(t)\|^2\,\mathrm{d}t < \infty$, a.s. so that the integrals on the right of (4.31) are defined. The next result, proved in Appendix A, establishes that the $\{\mathcal{F}_t\}$-semimartingale at (4.31) is a member of the space $\mathbb{B}$ (see (3.6) and Remark 3.3) and the process $\{\boldsymbol{\pi}(t)\}$ is square-integrable:

Proposition 4.12. *Recall (2.3), (4.29), (4.31), and (4.32). Then $\bar{\boldsymbol{\pi}} \in L^2(\mathbf{W})$ and*

$$\bar{X} \equiv (\bar{X}(0), r\bar{X}_- + \bar{\boldsymbol{\pi}}^\top\boldsymbol{\sigma}\boldsymbol{\theta}, \boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}}, \boldsymbol{\Gamma}^{\bar{X}H}H^{-1}) \in \mathbb{B}. \tag{4.33}$$

The main result of the present section follows next (see Appendix A for the proof):

Theorem 4.13. *Define the $\mathbb{R}$-valued process $\bar{X}$ as at (4.29) (in terms of the dual solution $\bar{Y}$ given by Proposition 4.8, the strictly positive state price density H at (4.28), the filtration $\{\mathcal{F}_t\}$ at (2.1), and the $\mathcal{F}_T$-measurable random variables A and B specified by Condition 2.7), and define the $\mathbb{R}^N$-valued process $\bar{\boldsymbol{\pi}}$ as at (4.32) (in terms of the process $\bar{X}$ and the $\mathrm{d}\mathbf{W}$-integrand $\boldsymbol{\Lambda}^{\bar{X}H}$ given by the martingale representation theorem at (4.30)). Then*

(a) $\bar{X} \in \mathbb{A}$ (in particular $\boldsymbol{\Gamma}^{\bar{X}H} = \mathbf{0}$, $\nu_{[M]}$-a.e. in the expansion (4.30) - recall Notation 3.2);
(b) the pair $(\bar{X}, \bar{Y}) \in \mathbb{A} \times \mathbb{B}$ satisfies (4.20) - (4.22);
(c) $\bar{\boldsymbol{\pi}} \in \mathcal{A}$ (see (2.6)), $\bar{X} = X^{\bar{\boldsymbol{\pi}}}$ (see (2.4)) and

$$E[J(X^{\bar{\boldsymbol{\pi}}}(T))] = \inf_{\boldsymbol{\pi}\in\mathcal{A}} E[J(X^{\boldsymbol{\pi}}(T))] = \sup_{Y\in\mathbb{B}}\{-\Psi(Y)\} = -\Psi(\bar{Y}). \tag{4.34}$$

In particular $\bar{\boldsymbol{\pi}}$ defined at (4.32) is optimal for the quadratic minimization problem (2.8).

Theorem 4.13 establishes the optimal portfolio $\bar{\boldsymbol{\pi}}$ for (2.8) in terms of the $\{\mathcal{F}_t\}$-predictable process $\boldsymbol{\Lambda}^{\bar{X}H}$ given by the $\{\mathcal{F}_t\}$-martingale representation theorem (see (4.30) and recall (2.1)). Notice that this optimal portfolio is similar in form to that given by Labbé and Heunis [2007, Proposition 5.6] except that in place of the integrand $\boldsymbol{\Lambda}^{\bar{X}H}$ at (4.32) one has an integrand $\boldsymbol{\psi}$ given by the classical Itô martingale representation theorem for the Brownian filtration of $\mathbf{W}$ only; this difference of course reflects the role of the regime-state Markov process α.

Remark 4.14. The quadratic minimization problem (2.8) includes the case of *mean-square hedging*, namely minimizing the mean-square discrepancy $\mathrm{E}\left[|X^\pi(T) - C|^2\right]$ of the wealth $X^\pi(T)$ at close of trade from a specified square-integrable contingent claim C. The problem of *mean-variance portfolio selection* also falls within the scope of problem (2.8). The goal is to minimize the *variance* $\mathrm{var}(X^\pi(T))$ over portfolios $\pi \in \mathcal{A}$ subject to the additional constraint $\mathrm{E}X^\pi(T) = d$ for some specified $d \in \mathbb{R}$, that is admissible portfolios which also attain the specified expected wealth d at close of trade. By introducing a scalar Lagrange multiplier for the constraint on the expected wealth at close of trade one reduces the problem to the form of (2.8) in which the coefficients A and B in the quadratic loss function (2.5) are constants. This was worked out in Labbé and Heunis [2007, Section 6]) for general convex portfolio constraints but without regime-switching. With the essential saddle-point relation (4.34) at hand, the calculation for the regime-switching case is completely identical and is not given here.

5 Examples

Remark 5.1. In this section, we study two special cases of problem (2.8) in which Conditions 2.4 and 2.7 are appropriately strengthened. In return for these stronger conditions, we can use the results on convex duality from Section 4 to get explicit portfolios in “feedback form” on the current (or instantaneous) wealth. With $\mathcal{N}(\mathbb{P})$ as at (2.1), define the filtration of the regime-state Markov process α, namely

$$\mathcal{F}_t^\alpha := \sigma\{\alpha(s), s \in [0,t]\} \vee \mathcal{N}(\mathbb{P}), \qquad t \in [0,T]. \tag{5.1}$$

Example 5.2. For this example we shall strengthen the basic conditions of Section 2 as follows:

Condition 5.3. Conditions 2.1, 2.3, 2.4 and 2.7 hold but are supplemented as follows: (i) the processes $\{r(t)\}$, $\{b_n(t)\}$ and $\{\sigma_{nm}(t)\}$ are specifically $\{\mathcal{F}_t^\alpha\}$-predictable (instead of just $\{\mathcal{F}_t\}$-predictable, as in Condition 2.4); (ii) the coefficients A and B in (2.5) are non-random constants with $A > 0$ (this strengthens Condition 2.7(ii)); and (iii) the portfolio constraint set K is specifically a vector subspace (this strengthens Condition 2.7(i)).

The vector subspace constraint set K can model trading restrictions such as prohibition of investment in some designated stocks and/or maintaining the investment in two or more designated stocks in a fixed ratio.

Remark 5.4. Zhou and Yin [2003] have addressed problem (2.8) with Condition 5.3 strengthened to the case of no portfolio constraints (i.e. $K := \mathbb{R}^N$), and with the $\{\mathcal{F}_t^\alpha\}$-predictability of the market parameters simplified to that of *Markov-modulation*, that is the market parameters are determined on $t \in [0,T]$ in terms of the Markov process $\{\alpha(t)\}$ by

$$r(t) := \tilde{r}(t, \alpha(t_-)), \qquad b_n(t) := \tilde{b}_n(t, \alpha(t_-)), \qquad \sigma_{nm}(t) := \tilde{\sigma}_{nm}(t, \alpha(t_-)), \tag{5.2}$$

in which $\tilde{r}(\cdot, i)$, $\tilde{b}_n(\cdot, i)$ and $\tilde{\sigma}_{nm}(\cdot, i)$ are given $\mathbb{R}$-valued uniformly bounded Borel-measurable deterministic functions on $[0,T]$ for all $i = 1, 2, \ldots, D$. The market parameters at instant t are thus determined completely by the instantaneous value $\alpha(t_-)$ of the Markov process, rather than through the more general predictable dependence on the paths of α allowed by Condition 5.3 (the Markov-modulated case is discussed further at Remark 5.9).

Define the $\mathbb{R}^N$-valued process (see Remark 2.6)

$$\boldsymbol{\xi}(t) := \boldsymbol{\theta}(t) - \mathrm{proj}\left[\boldsymbol{\theta}(t) \,\middle|\, \boldsymbol{\sigma}^{-1}(t)\tilde{K}\right], \qquad t \in [0,T], \tag{5.3}$$

in which $\mathrm{proj}[\mathbf{z}|C]$ is the uniquely defined projection of $\mathbf{z} \in \mathbb{R}^N$ onto a vector subspace $C \subset \mathbb{R}^N$, and $\tilde{K} := \{\mathbf{z} \in \mathbb{R}^N : \mathbf{z}^\top \boldsymbol{\eta} \geq 0 \text{ for all } \boldsymbol{\eta} \in K\}$ is the *polar subspace* of K.

Remark 5.5. From Condition 5.3 and Remark 2.6 one easily sees that the $\mathbb{R}^N$-valued process $\boldsymbol{\xi}$ at (5.3) is uniformly bounded and $\{\mathcal{F}_t^\alpha\}$-predictable. Moreover, from (5.3), $\boldsymbol{\xi}(t)$ and $\boldsymbol{\theta}(t) - \boldsymbol{\xi}(t)$ are orthogonal, thus (i) $\boldsymbol{\xi}^\top(t)\boldsymbol{\theta}(t) = \|\boldsymbol{\xi}(t)\|^2$. Since K is a vector subspace we also have (ii) $\tilde{K} = K^\perp$ (the orthogonal complement of K), as well as the elementary identity (iii) $(\boldsymbol{\sigma}^{-1}(t)\tilde{K})^\perp = \boldsymbol{\sigma}^\top(t)K$.

Problem (2.8) is addressed in Labbé and Heunis [2007, Example 6.2, page 94] in the case where the market parameters are uniformly bounded and *deterministic* (this strengthens Condition 5.3(i)), Condition 5.3(ii) holds, and the constraint set K is a closed convex cone. Under these conditions, the optimal portfolio is obtained in feedback form (see Labbé and Heunis [2007, eq. (6.25)]); in the special case where the constraint set K is not only a closed convex cone but a vector subspace, this optimal portfolio simplifies to the feedback form

$$\bar{\boldsymbol{\pi}}(t) := -\left(\bar{X}(t) + \frac{B}{A}\frac{\beta(T)}{\beta(t)}\right)\left(\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\xi}(t), \quad \text{for } \bar{X} := X^{\bar{\boldsymbol{\pi}}}, \tag{5.4}$$

(recall (2.4) and (4.24)).

Remark 5.6. Suppose that $B = 0$ in the loss function (2.5). The portfolio at (5.4) is then

$$\bar{\boldsymbol{\pi}}(t) := -\bar{X}(t)\left(\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\xi}(t), \quad \text{for } \bar{X} := X^{\bar{\boldsymbol{\pi}}}, \tag{5.5}$$

and (5.5) of course gives the optimal portfolio when the market parameters are deterministic. Moreover, if we discard the assumption that the market parameters are deterministic, and suppose that the market parameters satisfy Condition 5.3(i) (i.e. are $\{\mathcal{F}_t^\alpha\}$-predictable) then it is *intuitively plausible* that (5.5) still gives the optimal portfolio (when $B = 0$), that is one can effectively "ignore" the randomness arising from α in the market parameters. The intuition at work here is based on the notion of "totally unhedgeable coefficients" discussed by Karatzas and Shreve [1998, Example 6.7.4]: since the Brownian motion $\mathbf{W}$ in the price model (2.2) is *independent* of the Markov process α (recall Condition 2.1), to whose filtration the market parameters are adapted (by Condition 5.3(i)), the risk that is inherent in the market parameters is "undiversifiable" and should be "ignored" (see the discussion in Karatzas and Shreve [1998, page 306]). The practical significance of this is that the optimal portfolio at (5.5) for deterministic market parameters and $B = 0$ should then extend immediately to random market parameters which satisfy Condition 5.3(i). What happens when $B \neq 0$? In this case there is a decided technical obstacle in the way of applying the intuition of totally unhedgeable coefficients to the portfolio at (5.4), since, with a random interest rate r, the process $\{\beta(T)\beta^{-1}(t),\ t \in [0,T]\}$ cannot be $\{\mathcal{F}_t\}$-adapted (see (4.24)), and therefore (5.4) does not even define a valid portfolio process. Despite this, we shall nevertheless establish that the optimal portfolio for the problem (2.8) subject to Condition 5.3 has a structure very similar to that of (5.4), namely it is given by

$$\bar{\boldsymbol{\pi}}(t) := -\left(\bar{X}(t) + \frac{B}{A}\gamma(t_-)\right)\left(\boldsymbol{\sigma}^\top(t)\right)^{-1}\boldsymbol{\xi}(t), \quad \text{for } \bar{X} := X^{\bar{\boldsymbol{\pi}}}, \tag{5.6}$$

in which γ is an *appropriate* $\mathbb{R}$-valued uniformly bounded *cadlag* and $\{\mathcal{F}_t^\alpha\}$-adapted "offset" process which will be constructed from the convex duality results of Section 4 (the "left-continuous adjustment" in $\gamma(t_-)$ at (5.6) ensures the $\{\mathcal{F}_t\}$-predictability of $\bar{\boldsymbol{\pi}}$).

As a preliminary to the construction of an optimal portfolio in the form of (5.6), suppose we apply the feedback at (5.6) to the wealth equation (2.4) when γ is just some *arbitrary* $\mathbb{R}$-valued uniformly bounded *cadlag* $\{\mathcal{F}_t^\alpha\}$-adapted process. We obtain

$$\mathrm{d}\bar{X}(t) = r(t)\bar{X}(t)\,\mathrm{d}t - \left(\bar{X}(t) + \frac{B}{A}\gamma(t_-)\right)\boldsymbol{\xi}^\top(t)\boldsymbol{\theta}(t)\,\mathrm{d}t - \left(\bar{X}(t) + \frac{B}{A}\gamma(t_-)\right)\boldsymbol{\xi}^\top(t)\,\mathrm{d}\mathbf{W}(t). \tag{5.7}$$

Using Itô's formula it is easy to establish that the unique solution of (5.7) is given by

$$\bar{X}(t) = \beta^{-2}(t)\phi(t)\hat{H}(t)\left(x_0 - \frac{B}{A}\int_0^t (\beta\gamma)(s_-)\,\mathrm{d}G(s)\right), \qquad \text{for all} \quad t \in [0,T], \tag{5.8}$$

in which $(\beta\gamma)(t) := \beta(t)\gamma(t)$, $t \in [0,T]$, and (recalling Remark 5.5(i))

$$\phi(t) := \exp\left\{ -\int_0^t \boldsymbol{\xi}^\top(s)\boldsymbol{\theta}(s)\,\mathrm{d}s \right\} = \exp\left\{ -\int_0^t \|\boldsymbol{\xi}(s)\|^2\,\mathrm{d}s \right\}, \tag{5.9}$$

$$\hat{H}(t) := \beta(t)\mathcal{E}\left(-\boldsymbol{\xi} \bullet \mathbf{W}\right)(t), \qquad G(t) := \beta(t)\phi^{-1}(t)\hat{H}^{-1}(t). \tag{5.10}$$

Remark 5.7. From (5.8) we have (i) $\bar{X}(0) = x_0$, and in view of (5.7) and the uniform boundedness postulated for γ and the market parameters, it is easily established that

$$\mathrm{E}\left[\max_{t\in[0,T]} |\bar{X}(t)|^p\right] < \infty, \qquad \forall p \in [1,\infty), \tag{5.11}$$

(this is a standard calculation identical to that for Karatzas and Shreve [2005, Solution of 5.3.15]). In view of (5.11), (5.6), Remark 5.5, and Remark 2.5, it is clear that (ii) $\bar{\boldsymbol{\pi}} \in L^2(\mathbf{W})$ (see (2.3)). Now define (iii) $\Upsilon^{\bar{X}} := r\bar{X} + \bar{\boldsymbol{\pi}}^\top\boldsymbol{\sigma}\boldsymbol{\theta}$ and (iv) $\boldsymbol{\Lambda}^{\bar{X}} := \boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}}$; from (5.11) and (ii) we get (v) $\Upsilon^{\bar{X}} \in L_{21}$ (recall (3.4)) and (vi) $\boldsymbol{\Lambda}^{\bar{X}} \in L^2(\mathbf{W})$. Moreover, combining (5.7) and (5.6) gives the relation $\mathrm{d}\bar{X}(t) = \Upsilon^{\bar{X}}(t)\,\mathrm{d}t + (\boldsymbol{\Lambda}^{\bar{X}}(t))^\top\,\mathrm{d}\mathbf{W}(t)$, so that (vii) $\bar{X} = (x_0, \Upsilon^{\bar{X}}, \boldsymbol{\Lambda}^{X}) \in \mathbb{A}$ (from (v), (vi), and Remark 3.3). Now (5.3) and Remark 5.5(iii) give $\boldsymbol{\xi}(\omega,t) \in \boldsymbol{\sigma}^\top(\omega,t)K$ and thus (viii) $\bar{\boldsymbol{\pi}}(t) \in K$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. (see (5.6)). Then (ix) $\bar{\boldsymbol{\pi}} \in \mathcal{A}$ (from (viii), (ii), and (2.6)), hence (x) $\bar{\boldsymbol{\pi}} \in U(\bar{X})$ (from (ix), (iv), (iii) and (4.1)).

Remark 5.8. To summarize, we have shown that if $\bar{X}$ and $\bar{\boldsymbol{\pi}}$ are defined by (5.6) in terms of an *arbitrary* uniformly bounded cadlag and $\{\mathcal{F}_t^\alpha\}$-adapted process γ, then $\bar{X} \in \mathbb{A}$ (see Remark 5.7(vii)) and relations (4.20)(i) and (4.22)(ii)(iii) of Proposition 4.2 are satisfied (see Remark 5.7(i)(iv)(x)). It remains to construct some $\bar{Y} \equiv (\bar{Y}_0, \Upsilon^{\bar{Y}}, \boldsymbol{\Lambda}^{\bar{Y}}, \boldsymbol{\Gamma}^{\bar{Y}}) \in \mathbb{B}$, together with some uniformly bounded cadlag and $\{\mathcal{F}_t^\alpha\}$-adapted process γ, in such a way that (4.20)(ii), (4.21), and (4.22)(i) are satisfied when $\bar{\boldsymbol{\pi}}$ and $\bar{X}$ are defined by (5.6) in terms of this constructed γ. It then follows from Proposition 4.2 that $\bar{\boldsymbol{\pi}}$ given by (5.6) is the optimal portfolio in feedback form.

Before constructing $\bar{Y}$ and γ in accordance with Remark 5.8, let us recall the following: for *deterministic market parameters* we have seen that the optimal portfolio is given by (5.4), which is (5.6) with $\gamma(t) := \beta(T)\beta^{-1}(t)$, $t \in [0,T]$, thus in particular (a) γ is $\mathbb{R}$-valued uniformly bounded and nonrandom with $\gamma(T) = 1$; and (b) $(\beta\gamma)$ is a constant function, namely $(\beta\gamma)(t) = (\beta\gamma)(0)$ for all $t \in [0,T]$. In order to deal with $\{\mathcal{F}_t^\alpha\}$-predictable market parameters, we are going to suppose that γ at (5.6) has properties which are a "natural generalization" of properties (a) and (b) noted above for the deterministic case. In fact, motivated by (a) we shall suppose

$$\gamma \text{ is an } \mathbb{R}\text{-valued uniformly bounded cadlag } \{\mathcal{F}_t^\alpha\}\text{-adapted process with } \gamma(T) = 1 \text{ a.s.} \tag{5.12}$$

and, motivated by (b), we shall suppose that $(\beta\gamma)(t) := \beta(t)\gamma(t)$ is a cadlag $\{\mathcal{F}_t^\alpha\}$-adapted special semimartingale of the form

$$(\beta\gamma)(t) = (\beta\gamma)(0) + \sum_{i,j=1}^{D} \int_0^t \left\{ \zeta_{ij}^{(1)}(s)\,\mathrm{d}\langle M_{ij}\rangle(s) + \zeta_{ij}^{(2)}(s)\,\mathrm{d}M_{ij}(s) \right\}, \tag{5.13}$$

for some nonrandom $(\beta\gamma)(0) \in \mathbb{R}$ and some $\{\mathcal{F}_t^\alpha\}$-predictable integrands $\zeta^{(1)}$ and $\zeta^{(2)}$ for which the integrals are defined (recall the predictable quadratic variation $\langle M_{ij}\rangle$ at (3.1)).

It is not *a priori* evident that a process γ having properties (5.12) and (5.13) even exists. It will nevertheless be seen that, by *assuming* existence of such a γ, we shall be able to use the convex duality results of Section 4 to establish conditions which guide us in the *explicit* *construction* of a process γ which indeed satisfies (5.12) and (5.13), and which furthermore is such that (5.6) gives the optimal portfolio.

Since the canonical martingales $\{M_{ij}(t)\}$ are purely discontinuous (see (3.2)) it follows from (5.13) and (5.10) that $[G, \beta\gamma] = 0$, hence Itô's formula applied to the product of G and $\beta\gamma$ gives

$$\int_0^t (\beta\gamma)(s_-)\,\mathrm{d}G(s) = (\beta\gamma)(t)G(t) - (\beta\gamma)(0) - \sum_{i,j=1}^{D} \int_0^t G(s) \left\{ \zeta_{ij}^{(1)}(s)\,\mathrm{d}\langle M_{ij}\rangle(s) + \zeta_{ij}^{(2)}(s)\,\mathrm{d}M_{ij}(s) \right\}. \tag{5.14}$$

Upon substituting (5.14) into the right side of (5.8), taking $t = T$ and using $\gamma(T) = 1$ (recall (5.12)), we obtain

$$\begin{aligned} \bar{X}(T) + \frac{B}{A} = \beta^{-2}(T)\phi(T)\hat{H}(T)\Bigg(& \left(x_0 + \frac{B}{A}(\beta\gamma)(0) \right) \\ & + \frac{B}{A} \sum_{i,j=1}^{D} \int_0^T G(s) \left\{ \zeta_{ij}^{(1)}(s)\,\mathrm{d}\langle M_{ij}\rangle(s) + \zeta_{ij}^{(2)}(s)\,\mathrm{d}M_{ij}(s) \right\} \Bigg), \end{aligned} \tag{5.15}$$

(notice that (5.10) gives $\beta^{-1}(t)\phi(t)\hat{H}(t)G(t) = 1$). Motivated by the right side of (5.15) define

$$\eta(t) := \left(x_0 + \frac{B}{A}(\beta\gamma)(0) \right) + \frac{B}{A} \sum_{i,j=1}^{D} \int_0^t G(s) \left\{ \zeta_{ij}^{(1)}(s)\,\mathrm{d}\langle M_{ij}\rangle(s) + \zeta_{ij}^{(2)}(s)\,\mathrm{d}M_{ij}(s) \right\}, \tag{5.16}$$

$$R(t) := \mathrm{E}\left[\beta^{-2}(T)\phi(T) \,|\, \mathcal{F}_t^{\alpha} \right], \qquad \bar{Y}(t) := -A\hat{H}(t)\eta(t)R(t). \tag{5.17}$$

From (5.17), (5.16) and (5.15) we find

$$\bar{X}(T) + \frac{B}{A} = -\frac{\bar{Y}(T)}{A}. \tag{5.18}$$

We expand the triple product which defines $\bar{Y}$ at (5.17) by Itô's formula. To this end, first use the $\{\mathcal{F}_t^{\alpha}\}$-martingale representation theorem (see Elliott [1976, Lemma 3.3]) to represent the $\{\mathcal{F}_t^{\alpha}\}$-martingale R given by (5.17). Since R is uniformly lower-bounded by a strictly positive constant (the market parameters are uniformly bounded by Condition 2.4) we have

$$R(t) = R(0) + \sum_{i,j=1}^{D} \int_0^t R(s_-)\hat{\vartheta}_{ij}^{R}(s)\,\mathrm{d}M_{ij}(s), \tag{5.19}$$

for some $\{\mathcal{F}_t^{\alpha}\}$-predictable integrand $\{\hat{\boldsymbol{\vartheta}}^{R}\}$. Moreover, since R is uniformly bounded, from (5.19) and the Burkholder inequality (Liptser and Shiryayev [1989, Theorem 1.9.7, page 75]) one gets

$$\mathrm{E}\left[\left(\sum_{i,j=1}^{D} \int_0^T |\hat{\vartheta}_{ij}^{R}(s)|^2 \,\mathrm{d}[M_{ij}](s) \right)^{\frac{p}{2}} \right] < \infty, \qquad \forall p \in [1, \infty). \tag{5.20}$$

Next, use the Itô product formula to first expand $\hat{H}\eta$ (recall (5.16) and (5.10)) and then expand

$(\hat{H}\eta)R$ (using (5.19)). In view of (5.17) and (3.2) we obtain

$$\begin{aligned}\bar{Y}(t) &= -AR(0)\left(x_0 + \frac{B}{A}(\beta\gamma)(0)\right) - \int_0^t r(s)\bar{Y}(s_-)\,\mathrm{d}s - \int_0^t \bar{Y}(s_-)\boldsymbol{\xi}^\top(s)\,\mathrm{d}\mathbf{W}(s) \\ &\quad + \sum_{i,j=1}^{D}\int_0^t \bar{Y}(s_-)\hat{\vartheta}_{ij}^R(s)\,\mathrm{d}M_{ij}(s) - B\sum_{i,j=1}^{D}\int_0^t R(s_-)\beta(s)\phi^{-1}(s)\left(1+\hat{\vartheta}_{ij}^R(s)\right)\zeta_{ij}^{(2)}(s)\,\mathrm{d}M_{ij}(s) \\ &\quad - B\sum_{i,j=1}^{D}\int_0^t R(s_-)\beta(s)\phi^{-1}(s)\left(\zeta_{ij}^{(1)}(s) + \hat{\vartheta}_{ij}^R(s)\zeta_{ij}^{(2)}(s)\right)\,\mathrm{d}\langle M_{ij}\rangle(s).\end{aligned} \tag{5.21}$$

Now suppose the integrands $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$ (recall (5.13)) are related by

$$\zeta_{ij}^{(1)} + \hat{\vartheta}_{ij}^R\zeta_{ij}^{(2)} = 0. \tag{5.22}$$

Then it follows from (5.21) that $\bar{Y}$ has the "required form"

$$\bar{Y}(t) = \bar{Y}_0 + \int_0^t \Upsilon^{\bar{Y}}(s)\,\mathrm{d}s + \int_0^t (\boldsymbol{\Lambda}^{\bar{Y}}(s))^\top\mathbf{W}(s) + \sum_{i,j=1}^{D}\int_0^t \Gamma_{ij}^{\bar{Y}}(s)\,\mathrm{d}M_{ij}(s), \tag{5.23}$$

(compare with (3.7)) in which we have defined

$$\bar{Y}_0 := -R(0)\left(Ax_0 + B(\beta\gamma)(0)\right), \tag{5.24}$$

$$\Upsilon^{\bar{Y}}(t) := -r(t)\bar{Y}(t_-), \qquad \boldsymbol{\Lambda}^{\bar{Y}}(t) := -\bar{Y}(t_-)\boldsymbol{\xi}(t), \tag{5.25}$$

$$\Gamma_{ij}^{\bar{Y}}(t) := \bar{Y}(t_-)\hat{\vartheta}_{ij}^R(t) - BR(t_-)\beta(t)\phi^{-1}(t)\left(1+\hat{\vartheta}_{ij}^R(t)\right)\zeta_{ij}^{(2)}(t). \tag{5.26}$$

Moreover using (5.13), (5.19), (5.22), and the Itô product formula to expand $R(\beta\gamma)$, one obtains

$$R(t)(\beta\gamma)(t) = R(0)(\beta\gamma)(0) + \sum_{i,j=1}^{D}\int_0^t R(s_-)\left((\beta\gamma)(s_-)\hat{\vartheta}_{ij}^R(s) + \zeta_{ij}^{(2)}(s)\left(1+\hat{\vartheta}_{ij}^R(s)\right)\right)\,\mathrm{d}M_{ij}(s). \tag{5.27}$$

The integrand at (5.27) is $\{\mathcal{F}_t^\alpha\}$-predictable, thus $R(\beta\gamma)$ is a $\{\mathcal{F}_t^\alpha\}$-local martingale; but $R(\beta\gamma)$ is uniformly bounded (see (5.17) and (5.12)) so that $R(\beta\gamma)$ is actually a $\{\mathcal{F}_t^\alpha\}$-martingale. Moreover since $\gamma(T) = 1$ (see (5.12)), from (5.17) we have $R(T)(\beta\gamma)(T) = \beta^{-1}(T)\phi(T)$, that is

$$R(\beta\gamma) \text{ is a uniformly bounded } \{\mathcal{F}_t^\alpha\}\text{-martingale and } R(T)(\beta\gamma)(T) = \beta^{-1}(T)\phi(T). \tag{5.28}$$

To summarize: if there exists a process γ which satisfies (5.12) and (5.13), and if furthermore the integrands at (5.13) are related by (5.22), then (5.28) holds. This latter assertion is the essential clue for *constructing* the process γ, for it motivates the following definition: put

$$\gamma(t) := \frac{S(t)}{\beta(t)R(t)}, \qquad \text{for} \qquad S(t) := \mathrm{E}\left[\beta^{-1}(T)\phi(T)\,|\,\mathcal{F}_t^\alpha\right], \tag{5.29}$$

(recall (4.24) and $R(t)$ at (5.17)). Now it must be checked that γ defined by (5.29) indeed satisfies (5.12) and (5.13), and that the integrands $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$ at (5.13) also satisfy (5.22).

Clearly (5.12) is immediate from (5.29), (5.17), (5.9), (4.24) and the uniform boundedness of $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$ (see Remarks 5.5 and 2.6). As for verifying (5.13), we must expand the ratio $(\beta\gamma)(t) = S(t)/R(t)$ using Itô's formula. To this end, exactly as at (5.19) and (5.20), we have

$$S(t) = S(0) + \sum_{i,j=1}^{D} \int_0^t S(s_-)\hat{\vartheta}^S_{ij}(s)\,\mathrm{d}M_{ij}(s), \tag{5.30}$$

for some $\{\mathcal{F}^\alpha_t\}$-predictable integrand $\{\hat{\boldsymbol{\vartheta}}^S\}$ such that

$$\mathrm{E}\left[\left(\sum_{i,j=1}^{D}\int_0^T |\hat{\vartheta}^S_{ij}(s)|^2\,\mathrm{d}[M_{ij}](s)\right)^{\frac{p}{2}}\right] < \infty, \qquad \forall p \in [1,\infty). \tag{5.31}$$

Using (5.30), (5.19) and the general Itô formula (see Liptser and Shiryayev [1989, Theorem 2.3.1, page 118]) to expand $(\beta\gamma)(t) = S(t)/R(t)$, one finds that (5.13) indeed holds, with

$$\begin{aligned}(\beta\gamma)(0) &:= \frac{S(0)}{R(0)}, \qquad \zeta^{(1)}_{ij}(t) := \frac{S(t_-)}{R(t_-)}\frac{\hat{\vartheta}^R_{ij}(t)\left(\hat{\vartheta}^R_{ij}(t) - \hat{\vartheta}^S_{ij}(t)\right)}{1+\hat{\vartheta}^R_{ij}(t)},\\ \zeta^{(2)}_{ij}(t) &:= -\frac{S(t_-)}{R(t_-)}\frac{\hat{\vartheta}^R_{ij}(t) - \hat{\vartheta}^S_{ij}(t)}{1+\hat{\vartheta}^R_{ij}(t)},\end{aligned} \tag{5.32}$$

and it is clear that the integrands $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$ defined at (5.32) also satisfy the relation (5.22).

From now on the initial value $(\beta\gamma)(0)$ and integrands $\boldsymbol{\zeta}^{(1)}$ and $\boldsymbol{\zeta}^{(2)}$ appearing in (5.13) are defined by (5.32). With these parameters we have shown that $\bar{Y}$ defined by (5.16) and (5.17) satisfies (5.18), and has the form of (5.23), with $\bar{Y}_0$, $\Upsilon^{\bar{Y}}$, $\boldsymbol{\Lambda}^{\bar{Y}}$ and $\boldsymbol{\Gamma}^{\bar{Y}}$ given by (5.24) - (5.26). Inserting $(\beta\gamma)(0)$ and $\boldsymbol{\zeta}^{(2)}$ from (5.32) into (5.24) and (5.26) gives

$$\bar{Y}_0 = -[AR(0)x_0 + BS(0)], \tag{5.33}$$

$$\Gamma^{\bar{Y}}_{ij}(t) = \bar{Y}(t_-)\hat{\vartheta}^R_{ij}(t) + B\beta(t)\phi^{-1}(t)S(t_-)\left(\hat{\vartheta}^R_{ij}(t) - \hat{\vartheta}^S_{ij}(t)\right). \tag{5.34}$$

It remains to check that $\Upsilon^{\bar{Y}} \in L_{21}$, $\boldsymbol{\Lambda}^{\bar{Y}} \in L^2(\mathbf{W})$, and $\boldsymbol{\Gamma}^{\bar{Y}} \in L^2(\mathbf{M})$, as required to see that $\bar{Y} = (\bar{Y}_0, \Upsilon^{\bar{Y}}, \boldsymbol{\Lambda}^{\bar{Y}}, \Gamma^{\bar{Y}}) \in \mathbb{B}$ (see (5.23) and recall Remark 3.3). We address this next.

From (5.10) and Itô's formula we have $\mathrm{d}G(t) = G(t)\left(2\|\boldsymbol{\xi}(t)\|^2\,\mathrm{d}t + \boldsymbol{\xi}^\top(t)\,\mathrm{d}\mathbf{W}(t)\right)$; combining this with (5.16) and (5.14) then gives

$$\begin{aligned}\eta(t) = x_0 &+ \frac{B}{A}(\beta\gamma)(t)G(t) - \frac{2B}{A}\int_0^t (\beta\gamma)(s_-)G(s)\|\boldsymbol{\xi}(s)\|^2\,\mathrm{d}s\\ &- \frac{B}{A}\int_0^t (\beta\gamma)(s_-)G(s)\boldsymbol{\xi}^\top(s)\,\mathrm{d}\mathbf{W}(s).\end{aligned} \tag{5.35}$$

But, from (5.10), the uniform boundedness of $\boldsymbol{\xi}$ (see Remark 5.5), and (5.9), one also has

$$\mathrm{E}\left[\sup_{t\in[0,T]} |\hat{H}(t)|^q\right] < \infty \qquad \text{and} \qquad \mathrm{E}\left[\sup_{t\in[0,T]} |G(t)|^q\right] < \infty, \qquad \forall q \in \mathbb{R}. \tag{5.36}$$

From (5.36) and (5.35), and using the Burkholder inequality to bound the expected value of the supremum over $t \in [0,T]$ of the p-th order exponent of the magnitude of the $\mathrm{d}\mathbf{W}$-integral, one easily obtains $\mathrm{E}\left[\sup_{t\in[0,T]}|\eta(t)|^p\right] < \infty$ for all $p \in [1,\infty)$. From this, together with the bound on $\hat{H}$ given by (5.36), the uniform boundedness of R, and the definition of $\bar{Y}$ at (5.17), one finds

$$\mathrm{E}\left[\sup_{t\in[0,T]}|\bar{Y}(t)|^p\right] < \infty, \qquad \forall p \in [1,\infty). \tag{5.37}$$

In view of (5.37) and (5.25), it follows that $\Upsilon^{\bar{Y}} \in L_{21}$ and $\mathbf{\Lambda}^{\bar{Y}} \in L^2(\mathbf{W})$. It remains to show $\mathbf{\Gamma}^{\bar{Y}} \in L^2(\mathbf{M})$ (see (3.5)). We have

$$\mathrm{E}\left[\sum_{i,j=1}^{D}\int_0^T |\bar{Y}(t_-)\hat{\vartheta}_{ij}^R(t)|^2 \,\mathrm{d}[M_{ij}](t)\right] \le \mathrm{E}\left[\sup_{t\in[0,T]}|\bar{Y}(t)|^2\left(\sum_{i,j=1}^{D}\int_0^T |\hat{\vartheta}_{ij}^R(t)|^2\,\mathrm{d}[M_{ij}](t)\right)\right] < \infty, \tag{5.38}$$

(the final inequality follows from the Cauchy inequality with (5.37) and (5.20)), so that (5.38) gives $\hat{\boldsymbol{\vartheta}}^R\bar{Y}_- \in L^2(\mathbf{M})$. Since $\beta\phi^{-1}S_-$ is uniformly bounded, from (5.20) and (5.31) one obtains $\beta\phi^{-1}\hat{\boldsymbol{\vartheta}}^R S_- \in L^2(\mathbf{M})$ and $\beta\phi^{-1}\hat{\boldsymbol{\vartheta}}^S S_- \in L^2(\mathbf{M})$. Now $\mathbf{\Gamma}^{\bar{Y}} \in L^2(\mathbf{M})$ follows from (5.34).

We have shown that $\bar{Y}$, defined by (5.16), (5.17) and (5.32), satisfies (5.23), and that $\bar{Y} = (\bar{Y}_0, \Upsilon^{\bar{Y}}, \mathbf{\Lambda}^{\bar{Y}}, \mathbf{\Gamma}^{\bar{Y}}) \in \mathbb{B}$ (see Remark 3.3), with the integrands $\Upsilon^{\bar{Y}}$ and $\mathbf{\Lambda}^{\bar{Y}}$ given by (5.25), and the integrand $\mathbf{\Gamma}^{\bar{Y}}$ given by (5.34). Since $\bar{Y}(t_-) = \bar{Y}(t)$ for all except countably many values of t, the adjoint relation (4.21) follows from (5.25), while (5.18) gives (4.20)(ii). In view of this and Remark 5.8, it only remains to verify the complementary slackness relation (4.22)(i). From (5.3), (5.25) and (4.15), one has $\mathbf{\Theta}_{\bar{Y}}(t) = \bar{Y}(t)\boldsymbol{\sigma}(t)[\boldsymbol{\xi}(t) - \boldsymbol{\theta}(t)] \in \tilde{K}$ and thus $\delta(\mathbf{\Theta}_{\bar{Y}}(t)) = 0$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. But $\bar{\boldsymbol{\pi}}(t) \in K$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e., and $\tilde{K} = K^{\perp}$ (see Remark 5.7(viii) and Remark 5.5(ii)) thus $\bar{\boldsymbol{\pi}}^\top(t)\mathbf{\Theta}_{\bar{Y}}(t) = 0$ $(\mathbb{P}\otimes \mathit{Leb})$-a.e. which establishes (4.22)(i). Relations (4.20) - (4.22) are therefore satisfied by $(\bar{X}, \bar{Y}) \in \mathbb{A}\times\mathbb{B}$. From (5.29), (5.17), and Proposition 4.2, it follows that, with

$$\begin{aligned}\gamma(t) &:= \frac{\mathrm{E}\left[\beta^{-1}(T)\phi(T)\,|\,\mathcal{F}_t^\alpha\right]}{\beta(t)\mathrm{E}\left[\beta^{-2}(T)\phi(T)\,|\,\mathcal{F}_t^\alpha\right]}\\ &= \frac{\mathrm{E}\left[\exp\left\{\int_t^T\left[r(s) - \|\boldsymbol{\xi}(s)\|^2\right]\,\mathrm{d}s\right\}\,\middle|\,\mathcal{F}_t^\alpha\right]}{\mathrm{E}\left[\exp\left\{\int_t^T\left[2r(s) - \|\boldsymbol{\xi}(s)\|^2\right]\,\mathrm{d}s\right\}\,\middle|\,\mathcal{F}_t^\alpha\right]}, \qquad \forall\, t\in[0,T],\end{aligned} \tag{5.39}$$

the feedback portfolio $\bar{\boldsymbol{\pi}}$ defined by (5.6) is optimal for problem (2.8) when Condition 5.3 holds (the second equality at (5.39) follows from (5.9) and (4.24)).

Remark 5.9. Suppose the market parameters not only satisfy Condition 5.3 but are also Markov-modulated (recall Remark 5.4). Then, from Remark 2.6, (5.2) and (5.3), we have

$$r(t) = \tilde{r}(t,\alpha(t_-)), \qquad \boldsymbol{\theta}(t) = \tilde{\boldsymbol{\theta}}(t,\alpha(t_-)), \qquad \boldsymbol{\xi}(t) = \tilde{\boldsymbol{\xi}}(t,\alpha(t_-)), \tag{5.40}$$

where $\tilde{\boldsymbol{\theta}}(t,i) := \tilde{\boldsymbol{\sigma}}^{-1}(t,i)[\tilde{\mathbf{b}}(t,i) - \tilde{r}(t,i)\mathbf{1}]$ and $\tilde{\boldsymbol{\xi}}(t,i) := \tilde{\boldsymbol{\theta}}(t,i) - \mathrm{proj}[\tilde{\boldsymbol{\theta}}(t,i)|\tilde{\boldsymbol{\sigma}}^{-1}(t,i)\tilde{K}]$ for all $i = 1,2,\ldots,D$ and $t\in[0,T]$. With $\{r(t)\}$ and $\{\boldsymbol{\xi}(t)\}$ at (5.40) inserted into (5.39), the arguments in the conditional expectations of the numerator and denominator are $\sigma\{\alpha(u),\ u\in[t,T]\}$-measurable, and it follows from the Markov property of α (see e.g. Chung [1982, (iia) on page 3]) that γ simplifies to

$$\gamma(t) = \frac{\mathrm{E}\left[\exp\left\{\int_t^T\left[\tilde{r}(s,\alpha(s_-)) - \|\tilde{\boldsymbol{\xi}}(s,\alpha(s_-))\|^2\right]\,\mathrm{d}s\right\}\,\middle|\,\alpha(t)\right]}{\mathrm{E}\left[\exp\left\{\int_t^T\left[2\tilde{r}(s,\alpha(s_-)) - \|\tilde{\boldsymbol{\xi}}(s,\alpha(s_-))\|^2\right]\,\mathrm{d}s\right\}\,\middle|\,\alpha(t)\right]}, \qquad \forall\, t\in[0,T], \tag{5.41}$$

that is $\gamma(t)$ at (5.41) depends just on the instantaneous value $\alpha(t)$ of the regime-state Markov chain α.

Remark 5.10. The basic approach followed in Example 5.2 is as follows: motivated by the deterministic case and the intuition of totally unhedgeable coefficients we propose the candidate optimal portfolio at (5.6). Processes γ (see (5.39)) and $\bar{Y} \in \mathbb{B}$ (see (5.23), (5.25), (5.33), (5.34)) are then constructed such the pair $(\bar{X}, \bar{Y})$ satisfies the optimality relations of Proposition 4.2 (for $\bar{X}$ defined by (5.6) and (2.4)). It then follows that (5.6) is the optimal portfolio in feedback form, and $\bar{Y}$ solves the dual problem of minimizing the dual functional $\Psi(\cdot)$ (recall (4.14)) over the vector space $\mathbb{B}$.

Example 5.11. For this example we modify Condition 5.3 as follows:

Condition 5.12. Suppose Condition 5.3, except that the interest-rate process $\{r(t)\}$ is *non-random* (instead of $\{\mathcal{F}_t^{\alpha}\}$-predictable) and Borel-measurable on $[0, T]$, and the portfolio constraint set K is a non-empty closed convex cone in $\mathbb{R}^N$ (instead of specifically a vector subspace).

In Condition 5.12 we generalize the portfolio constraint set K from a vector subspace to a closed convex cone, but suppose in return that the interest-rate is non-random. The economic justification for this latter assumption is that the regime states of the market, given by the Markov chain $\{\alpha(t)\}$ (e.g. "bullish" or "bearish"), have a clear and direct influence on stock prices through the market parameters $\{b_n(t)\}$ and $\{\sigma_{nm}(t)\}$ (recall (2.2) and Condition 2.4). On the other hand the interest-rate $\{r(t)\}$ is (or should be) set by a central bank irrespective of the stock market and its regime states; this lack of dependence is captured by Condition 5.12 (a similar argument is given by Zhou and Yin [2003, Section 6]).

The goal is to construct an optimal portfolio $\bar{\boldsymbol{\pi}}$ in feedback form (see Remark 5.1) for the problem (2.8) when Condition 5.12 holds. To this end, define

$$\boldsymbol{\xi}_1(t) := \boldsymbol{\theta}(t) - \operatorname{proj}\left[\boldsymbol{\theta}(t) \,\middle|\, \boldsymbol{\sigma}^{-1}(t)\tilde{K}\right] \quad \text{and} \quad \boldsymbol{\xi}_2(t) := \boldsymbol{\theta}(t) + \operatorname{proj}\left[-\boldsymbol{\theta}(t) \,\middle|\, \boldsymbol{\sigma}^{-1}(t)\tilde{K}\right], \tag{5.42}$$

in which $\operatorname{proj}[\mathbf{z} \,|\, C] := \arg\min_{\boldsymbol{\eta} \in C} \|\mathbf{z} - \boldsymbol{\eta}\|$ is the (unique) projection of $\mathbf{z} \in \mathbb{R}^N$ onto a closed convex set $C \subset \mathbb{R}^N$ and $\tilde{K} := \{\mathbf{z} \in \mathbb{R}^N : \mathbf{z}^\top \boldsymbol{\eta} \geq 0 \ \text{ for all } \ \boldsymbol{\eta} \in K\}$ is the *polar cone* of $-K$.

Remark 5.13. It follows easily from Condition 5.12 and Remark 2.6 that the $\mathbb{R}^N$-valued processes $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ at (5.42) are uniformly bounded and $\{\mathcal{F}_t^{\alpha}\}$-predictable.

For $i = 1, 2$, define portfolio $\bar{\boldsymbol{\pi}}_i$ and corresponding wealth process $\bar{X}_i$ in the feedback form

$$\bar{\boldsymbol{\pi}}_i(t) := -\left(\bar{X}_i(t) + \frac{B}{A}\gamma(t)\right)\left(\boldsymbol{\sigma}^\top(t)\right)^{-1} \boldsymbol{\xi}_i(t), \quad \text{for } \bar{X}_i := X^{\bar{\boldsymbol{\pi}}_i}, \tag{5.43}$$

(recall (2.4)) in which the "wealth offset" process γ is given by (see (4.24))

$$\gamma(t) := \frac{\beta(T)}{\beta(t)} = \exp\left\{-\int_t^T r(s)\, \mathrm{d}s\right\}, \qquad \text{for all } \ t \in [0, T]. \tag{5.44}$$

When *all market parameters are non-random* then (see Labbé and Heunis [2007, Example 6.2]) the optimal portfolio is

$$\left.\begin{array}{l}\text{(a) given by } \bar{\boldsymbol{\pi}}_1 \text{ with corresponding wealth process } \bar{X}_1 \text{ when } Ax_0 + B\beta(T) \geq 0,\\ \text{(b) given by } \bar{\boldsymbol{\pi}}_2 \text{ with corresponding wealth process } \bar{X}_2 \text{ when } Ax_0 + B\beta(T) < 0.\end{array}\right\} \tag{5.45}$$

Notice that the portfolio given by (5.45) reduces to $\bar{\boldsymbol{\pi}}$ defined by (5.4) and (5.3) when the constraint set K is a vector subspace, for in this case $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 = \boldsymbol{\xi}$, thus $\bar{\boldsymbol{\pi}}_1 = \bar{\boldsymbol{\pi}}_2 = \bar{\boldsymbol{\pi}}$.

When Condition 5.12 holds, then of course γ at (5.44) is still non-random, and the intuition of "totally unhedgeable coefficients" articulated in Remark 5.6 suggests that (5.45) should again give the optimal portfolio. Our goal is to establish this on the basis of Proposition 4.2. To this end, observe from (5.43), (2.4) and Itô's formula (exactly as at (5.8)) that

$$\bar{X}_i(t) = \beta^{-2}(t)\phi_i(t)\hat{H}_i(t)\left(x_0 - \frac{B}{A}\int_0^t (\beta\gamma)(s_-)\,\mathrm{d}G_i(s)\right), \quad \text{for } i = 1,2, \tag{5.46}$$

in which (recall (4.24))

$$\phi_i(t) := \exp\left\{-\int_0^t \boldsymbol{\xi}_i^\top(s)\boldsymbol{\theta}(s)\,\mathrm{d}s\right\}, \quad \text{for } i = 1,2, \tag{5.47}$$

$$\hat{H}_i(t) := \beta(t)\mathcal{E}\left(-\boldsymbol{\xi}_i \bullet \mathbf{W}\right)(t), \qquad G_i(t) := \beta(t)\phi_i^{-1}(t)\hat{H}_i^{-1}(t), \quad \text{for } i = 1,2. \tag{5.48}$$

Remark 5.14. Define (i) $\Upsilon^{\bar{X}_i} := r\bar{X}_i + \bar{\boldsymbol{\pi}}_i^\top\boldsymbol{\sigma}\boldsymbol{\theta}$ and (ii) $\Lambda^{\bar{X}_i} := \boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}}_i$. From (5.43) and (2.4) we have the relation $\mathrm{d}\bar{X}_i(t) = \Upsilon^{\bar{X}_i}(t)\,\mathrm{d}t + (\Lambda^{\bar{X}_i}(t))^\top\,\mathrm{d}\mathbf{W}(t)$; we shall later establish (see Remark 5.15(b)) that $\Upsilon^{\bar{X}_i} \in L_{21}$ and $\Lambda^{\bar{X}_i} \in L^2(\mathbf{W})$, that is $\bar{X}_i = (x_0, \Upsilon^{\bar{X}_i}, \Lambda^{\bar{X}_i}) \in \mathbb{A}$ (see Remark 3.3).

Since $(\beta\gamma)(t) = \beta(T)$, $t \in [0,T]$ (recall (5.44)), we have $\int_0^t (\beta\gamma)(s_-)\,\mathrm{d}G_i(s) = \beta(T)[G_i(t) - 1]$. Substituting this into (5.46), and using $\beta^{-1}(t)\phi_i(t)\hat{H}_i(t)G_i(t) = 1$ (see (5.48)), gives

$$\bar{X}_i(t) + \frac{B}{A}\gamma(t) = \beta^{-2}(t)\phi_i(t)\hat{H}_i(t)\left(x_0 + \frac{B}{A}\beta(T)\right), \qquad \text{for all } t \in [0,T]. \tag{5.49}$$

From (5.48), (5.49), Remark 5.13, and uniform boundedness of the market parameters (see Condition 5.12), one has

$$\mathrm{E}\left[\max_{t\in[0,T]}|\hat{H}_i(t)|^p\right] < \infty \quad \text{and} \quad \mathrm{E}\left[\max_{t\in[0,T]}|\bar{X}_i(t)|^p\right] < \infty, \text{ for each } p \in [1,\infty). \tag{5.50}$$

Remark 5.15. (a) Given the closed convex cone $C \subset \mathbb{R}^N$ define the *polar cone* of $-C$, namely (i) $\tilde{C} := \{\mathbf{z} \in \mathbb{R}^N \ : \ \mathbf{z}^\top\boldsymbol{\eta} \geq 0, \ \ \forall\, \boldsymbol{\eta} \in C\}$ (this is a closed convex cone). From Hiriart-Urruty and Lemaréchal [2001, Theorem A.3.2.5] one has the identity (ii) $\mathbf{z} = \mathrm{proj}\,[\mathbf{z}\,|\,-C] + \mathrm{proj}[\mathbf{z}\,|\,\tilde{C}]$ for all $\mathbf{z} \in \mathbb{R}^N$. Now put (iii) $C := \boldsymbol{\sigma}^\top(\omega,t)K$ and $\mathbf{z} := \boldsymbol{\theta}(\omega,t)$ (for some fixed (ω,t)); then it is easily checked that (iv) $\tilde{C} = \boldsymbol{\sigma}^{-1}(\omega,t)\tilde{K}$, and it follows from (5.42) together with (ii), (iii), and (iv) that $\boldsymbol{\xi}_1(\omega,t) = \boldsymbol{\theta}(\omega,t) - \mathrm{proj}[\boldsymbol{\theta}(\omega,t)\,|\boldsymbol{\sigma}^{-1}(\omega,t)\tilde{K}] = \mathrm{proj}[\boldsymbol{\theta}(\omega,t)\,|\,-\boldsymbol{\sigma}^\top(\omega,t)K]$, that is (v) $\boldsymbol{\xi}_1(\omega,t) \in -\boldsymbol{\sigma}^\top(\omega,t)K$ for all (ω,t). In the same way, but using $\mathbf{z} := -\boldsymbol{\theta}(\omega,t)$ and C given by (iii) in (ii), one finds (vi) $\boldsymbol{\xi}_2(\omega,t) = -\,\mathrm{proj}[-\boldsymbol{\theta}(\omega,t)\,|\,-\boldsymbol{\sigma}^\top(\omega,t)K] \in \boldsymbol{\sigma}^\top(\omega,t)K$ for all (ω,t).

(b) When $Ax_0 + B\beta(T) \geq 0$, then $\bar{X}_1(t) + BA^{-1}\gamma(t) \geq 0$, $t \in [0,T]$ (see (5.49)), thus from (5.43), (v), and the fact that K is a cone, we get (vii) $\bar{\boldsymbol{\pi}}_1(t) \in K$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. In exactly the same way, when $Ax_0 + B\beta(T) < 0$, then $\bar{X}_2(t) + BA^{-1}\gamma(t) < 0$, $t \in [0,T]$, and it follows from (5.43) and (vi) that (viii) $\bar{\boldsymbol{\pi}}_2(t) \in K$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. But, from (5.50), (5.43), Remark 5.13, and Remark 2.5, it follows that (ix) $\bar{\boldsymbol{\pi}}_i \in L^2(\mathbf{W})$ (see (2.3)) and thus (x) $\bar{\boldsymbol{\pi}}_i \in \mathcal{A}$ (see (2.6), (vii), (viii)) for $i = 1,2$. Recalling Remark 5.14(i)(ii), it follows from (5.50) and (ix) that (xi) $\Upsilon^{\bar{X}_i} \in L_{21}$ (see (3.4)) and $\Lambda^{\bar{X}_i} \in L^2(\mathbf{W})$, and therefore (xii) $\bar{X}_i = (x_0, \Upsilon^{\bar{X}_i}, \Lambda^{\bar{X}_i}) \in \mathbb{A}$ (see Remark 3.3). Moreover, from (xii), (x), Remark 5.14(i)(ii), and (4.1) we have (xiii) $\bar{\boldsymbol{\pi}}_i \in U(\bar{X}_i)$.

(c) We therefore see that (4.20)(i) and (4.22)(ii) are satisfied (as follows from (xii) and (xiii)), and (4.22)(iii) also holds (see Remark 5.14(ii)). It therefore remains to construct dual processes

$\bar{Y}_i = (\bar{Y}_i(0), \Upsilon^{\bar{Y}_i}, \mathbf{\Lambda}^{\bar{Y}_i}, \mathbf{\Gamma}^{\bar{Y}_i}) \in \mathbb{B}$, for $i = 1, 2$, such that the pair $(\bar{X}_1, \bar{Y}_1)$ [respectively $(\bar{X}_2, \bar{Y}_2)$] satisfies the remaining relations (4.20)(ii), (4.21), and (4.22)(i) when $Ax_0 + B\beta(T) \geq 0$ [respectively $Ax_0 + B\beta(T) < 0$]. It then follows from Proposition 4.2 that the optimal portfolio is indeed given by (5.45). We construct the dual processes $\bar{Y}_i$ next.

Motivated by the right side of (5.49), for $i = 1, 2$, define (see (5.42), (5.47), (5.48))

$$R_i(t) := \mathrm{E}\left[\beta^{-2}(T)\phi_i(T) \,|\, \mathcal{F}_t^{\alpha}\right], \qquad \bar{Y}_i(t) := -\left(Ax_0 + B\beta(T)\right) R_i(t)\hat{H}_i(t), \tag{5.51}$$

($\beta^{-2}(T)$ is non-random, and thus can be factored out of the conditional expectation defining $R_i(t)$, but it is easier to leave it in place as shown). From (5.51) and (5.49) with $t = T$, we have

$$\bar{X}_i(T) + \frac{B}{A} = -\frac{\bar{Y}_i(T)}{A}, \qquad i = 1, 2. \tag{5.52}$$

We next calculate $\bar{Y}_i$ at (5.51) using Itô's formula. Just as at (5.19) and (5.20), we can expand the strictly positive and uniformly bounded $\{\mathcal{F}_t^{\alpha}\}$-martingale R_i defined at (5.51) as follows:

$$R_i(t) = R_i(0) + \sum_{j,k=1}^{D} \int_0^t R_i(s_-)\hat{\vartheta}_{jk}^{R_i}(s)\, \mathrm{d}M_{jk}(s), \tag{5.53}$$

for some $\{\mathcal{F}_t^{\alpha}\}$-predictable integrand $\{\hat{\boldsymbol{\vartheta}}^{R_i}(t)\}$ such that

$$\mathrm{E}\left[\left(\sum_{j,k=1}^{D} \int_0^T |\hat{\vartheta}_{jk}^{R_i}(s)|^2\, \mathrm{d}[M_{jk}](s)\right)^{\frac{p}{2}}\right] < \infty, \qquad \forall p \in [1, \infty). \tag{5.54}$$

Then from (5.53), (5.51), the definition of $\hat{H}_i$ at (5.48), and the Itô product formula, we obtain

$$\bar{Y}_i(t) = \bar{Y}_i(0) + \int_0^t \Upsilon^{\bar{Y}_i}(s)\, \mathrm{d}s + \int_0^t \left(\mathbf{\Lambda}^{\bar{Y}_i}(s)\right)^{\top} \mathrm{d}\mathbf{W}(s) + \sum_{j,k=1}^{D} \int_0^t \Gamma_{jk}^{\bar{Y}_i}(s)\, \mathrm{d}M_{jk}(s), \tag{5.55}$$

in which

$$\bar{Y}_i(0) = -R_i(0)\left(Ax_0 + B\beta(T)\right), \tag{5.56}$$

$$\Upsilon^{\bar{Y}_i}(t) := -r(t)\bar{Y}_i(t_-), \quad \mathbf{\Lambda}^{\bar{Y}_i}(t) := -\bar{Y}_i(t_-)\boldsymbol{\xi}_i(t), \quad \mathbf{\Gamma}^{\bar{Y}_i}(t) := \bar{Y}_i(t_-)\hat{\boldsymbol{\vartheta}}^{R_i}(t). \tag{5.57}$$

In view of (5.50), together with the uniform boundedness of R_i and (5.51), we find

$$\mathrm{E}\left[\sup_{t\in[0,T]} |\bar{Y}_i(t)|^p\right] < \infty, \quad \forall\, p \in [1, \infty). \tag{5.58}$$

Remark 5.16. In view of (5.58), (5.57), and Remark 5.13, we get $\Upsilon^{\bar{Y}_i} \in L_{21}$ and $\mathbf{\Lambda}^{\bar{Y}_i} \in L^2(\mathbf{W})$ (see (3.4) and (2.3)). Moreover, from (5.57), (5.54) and a calculation identical to that at (5.38), we obtain $\mathbf{\Gamma}^{\bar{Y}_i} \in L^2(\mathbf{M})$, so it follows that $\bar{Y}_i = (\bar{Y}_i(0), \Upsilon^{\bar{Y}_i}, \mathbf{\Lambda}^{\bar{Y}_i}, \mathbf{\Gamma}^{\bar{Y}_i}) \in \mathbb{B}$, $i = 1, 2$ (see (5.55) and Remark 3.3). From (5.52), the pair $(\bar{X}_i, \bar{Y}_i) \in \mathbb{A} \times \mathbb{B}$ satisfies (4.20)(ii), and, since $\bar{Y}_i(t) = \bar{Y}_i(t_-)$ for all except countably many values of t, it follows from (5.57) that each $\bar{Y}_i$ satisfies (4.21). In view of Remark 5.15(c), it remains only to verify the complementary slackness relation (4.22)(i); it is here that the sign of $Ax_0 + B\beta(T)$ plays a critical role. From (4.15), (5.42) and (5.57),

$$\mathbf{\Theta}_{\bar{Y}_i}(t) = (-1)^i \bar{Y}_i(t)\boldsymbol{\sigma}(t)\, \mathrm{proj}\left[(-1)^{i+1}\boldsymbol{\theta}(t) \,\middle|\, \boldsymbol{\sigma}^{-1}(t)\tilde{K}\right], \quad (\mathbb{P} \otimes \mathit{Leb})\text{-a.e.}, \tag{5.59}$$

and then, from (5.59) and (5.43),

$$\bar{\boldsymbol{\pi}}_i^\top(t)\boldsymbol{\Theta}_{\bar{Y}_i}(t) = (-1)^{i+1}\left[\bar{X}_i(t) + \frac{B}{A}\gamma(t)\right]\bar{Y}_i(t)\boldsymbol{\xi}_i^\top(t)\,\mathrm{proj}\left[(-1)^{i+1}\boldsymbol{\theta}(t)\,\middle|\,\boldsymbol{\sigma}^{-1}(t)\tilde{K}\right], \quad (\mathbb{P}\otimes \mathit{Leb})\text{-a.e.} \tag{5.60}$$

Now it follows from Hiriart-Urruty and Lemaréchal [2001, Theorem A.3.2.5] that $\boldsymbol{\xi}_i(t)$ and $\mathrm{proj}[(-1)^{i+1}\boldsymbol{\theta}(t)\,|\,\boldsymbol{\sigma}^{-1}(t)\tilde{K}]$ are orthogonal for $i = 1, 2$ (recall (5.42)), thus from (5.60)

$$\bar{\boldsymbol{\pi}}_i^\top(t)\boldsymbol{\Theta}_{\bar{Y}_i}(t) = 0, \quad (\mathbb{P}\otimes \mathit{Leb})\text{ -a.e.}, \tag{5.61}$$

and of course

$$\boldsymbol{\sigma}(\omega,t)\,\mathrm{proj}\left[(-1)^{i+1}\boldsymbol{\theta}(\omega,t)\,\middle|\,\boldsymbol{\sigma}^{-1}(\omega,t)\tilde{K}\right] \in \tilde{K} = \left\{\mathbf{z}\in\mathbb{R}^N \mid \delta(\mathbf{z}) = 0\right\}, \text{ for each } (\omega,t). \tag{5.62}$$

Now suppose that $Ax_0 + B\beta(T) \geq 0$: from (5.51) one obtains $\bar{Y}_1(t) \leq 0$, $t \in [0,T]$, and thus, since $\tilde{K}$ is a cone, it follows from (5.62) and (5.59) that $\boldsymbol{\Theta}_{\bar{Y}_1}(t) \in \tilde{K}$, that is $\delta(\boldsymbol{\Theta}_{\bar{Y}_1}(t)) = 0$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. Combining this with (5.61) shows that $(\bar{\boldsymbol{\pi}}_1, \bar{Y}_1)$ satisfies (4.22)(i). Next, suppose that $Ax_0 + B\beta(T) < 0$: then (5.51) gives $\bar{Y}_2(t) > 0$, $t \in [0,T]$, so that (5.62), (5.59), and the fact that $\tilde{K}$ is a cone, gives $\boldsymbol{\Theta}_{\bar{Y}_2}(t) \in \tilde{K}$, that is $\delta(\boldsymbol{\Theta}_{\bar{Y}_2}(t)) = 0$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. Combining this with (5.61) shows that $(\bar{\boldsymbol{\pi}}_2, \bar{Y}_2)$ also satisfies (4.22)(i). It now follows from Remark 5.15(c) that (5.45) is the optimal portfolio for the problem (2.8) when Condition 5.12 holds.

Exactly as at Remark 5.10, we see that $\bar{Y}_1$ (respectively $\bar{Y}_2$) is a solution of the dual problem of minimizing $\Psi(\cdot)$ when $Ax_0 + B\beta(T) \geq 0$ (respectively $Ax_0 + B\beta(T) < 0$).

Remark 5.17. In Example 5.2, the $\{\mathcal{F}_t^\alpha\}$-martingale representation theorem is used in the expansions at (5.19) and (5.30). Notice that the integrands $\hat{\boldsymbol{\vartheta}}^R$ and $\hat{\boldsymbol{\vartheta}}^S$ obtained from these expansions do not feature at all in the optimal portfolio, which is given only by (5.6) and (5.39). These integrands nevertheless play an absolutely essential role in constructing the integrand $\boldsymbol{\Gamma}^Y \in L^2(\mathbf{M})$ for the dual process $\bar{Y}$ in such a way that the pair $(\bar{X}, \bar{Y})$ satisfies the relations (4.20) - (4.22) of Proposition 4.2 (see the expression for $\boldsymbol{\Gamma}^{\bar{Y}}$ at (5.34)). This is consistent with the comments made in Remark 4.3. A similar remark applies to Example 5.11 (see the expansion at (5.53) and the expression for $\boldsymbol{\Gamma}^{\bar{Y}_i}$ at (5.57)). Finally, as noted at Remark 4.4, the integrand $\boldsymbol{\Gamma}^{\bar{Y}}$ is a Lagrange multiplier for the constraint of prohibited hedging in the regime-state α.

The optimal portfolio in Example 5.11 (see (5.45)) depends on the *sign* of the quantity $Ax_0 + B\beta(T)$. The reason for this is that projection onto a closed convex cone is generally not linear; the sign condition on $Ax_0 + B\beta(T)$ nevertheless ensures that the processes $\bar{X}_i(t) + BA^{-1}\gamma(t)$ (for $i = 1, 2$, see (5.43)) are also of fixed sign over the interval $t \in [0,T]$ in such a way that the portfolios $\bar{\boldsymbol{\pi}}_i$ at (5.43) take values in the closed convex cone K (see Remark 5.15(b)) and the complementary slackness condition (4.22)(i) is satisfied (see Remark 5.16). On the other hand, projection onto a vector subspace is linear, and therefore the optimal portfolio in Example 5.2 (in which the constraint set K is a vector subspace - recall Condition 5.3) does not depend on the sign of $Ax_0 + B\beta(T)$. It is for this reason that we can deal with a *random* interest rate process in Example 5.2; although the sign of the quantity $\bar{X}(t) + BA^{-1}\gamma(t_-)$ in the optimal portfolio (5.6) is not necessarily constant (with $\gamma(t)$ given by (5.39)), this does not matter since projection onto a vector subspace is linear. The following question arises from this discussion: what is the optimal portfolio in the case where all market parameters (including the interest rate r) are random as in Condition 5.3(i), and the constraint set K is a general closed convex cone (as in Condition 5.12)? This case is not clearly understood and requires further effort.

A Appendix: Proofs

We collect in this appendix the proofs of several results already stated.

We shall need the following technical result on semimartingales in $\mathbb{B}$ (recall Remark 3.3) which follows from Bismut [1973, Proposition I-1].

Proposition A.1. *For any* $X \equiv (X_0, \Upsilon^X, \mathbf{\Lambda}^X, \mathbf{\Gamma}^X) \in \mathbb{B}$ *and* $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B}$*, define*

$$\begin{aligned}\mathbb{M}(X,Y)(t) :=& X(t)Y(t) - X_0Y_0 - \int_0^t \left(\Upsilon^X(s)Y(s) + X(s)\Upsilon^Y(s)\right) \mathrm{d}s \\ & - \sum_{n=1}^N \int_0^t \Lambda_n^X(s)\Lambda_n^Y(s)\,\mathrm{d}s - \sum_{i,j=1}^D \int_0^t \Gamma_{ij}^X(s)\Gamma_{ij}^Y(s)\,\mathrm{d}[M_{ij}](s).\end{aligned} \tag{A.1}$$

Then $\{(\mathbb{M}(X,Y)(t), \mathcal{F}_t); t \in [0,T]\}$ *is a martingale, null at the origin.*

Proof of Proposition 4.2: Fix $X \equiv (X_0, \Upsilon^X, \mathbf{\Lambda}^X) \in \mathbb{A}$ and $Y \equiv (Y_0, \Upsilon^Y, \mathbf{\Lambda}^Y, \mathbf{\Gamma}^Y) \in \mathbb{B}$. From the convex conjugates defined at (4.9)-(4.11), we get the following inequalities (i), (ii) and (ii):
(i) $l_0(X_0) + m_0(Y_0) \geq X_0Y_0$, (ii) $J(X(T)) + m_T(Y(T)) \geq -X(T)Y(T)$,
(iii) $L(t, X(t), \Upsilon^X(t), \mathbf{\Lambda}^X(t)) + M(t, Y(t), \Upsilon^Y(t), \mathbf{\Lambda}^Y(t)) \geq X(t)\Upsilon^Y(t) + \Upsilon^X(t)Y(t) + (\mathbf{\Lambda}^X(t))^\top \mathbf{\Lambda}^Y(t)$,
for all $(\omega, t) \in [0,T] \times \Omega$. It follows from (i), (ii) and (iii), together with (4.7) and (4.14), that

$$\Phi(X) + \Psi(Y) \geq X_0Y_0 + \mathrm{E}\int_0^T \left(X(t)\Upsilon^Y(t) + \Upsilon^X(t)Y(t) + \left(\mathbf{\Lambda}^X(t)\right)^\top \mathbf{\Lambda}^Y(t)\right) \mathrm{d}t - \mathrm{E}\left(X(T)Y(T)\right).$$

From Proposition A.1 and the fact that $X \in \mathbb{A}$, hence $\mathbf{\Gamma}^X := \mathbf{0}$ (recall Remark 3.3), one sees that the right-hand side is $-\mathrm{E}(\mathbb{M}(X,Y)(T)) = 0$, as required for (4.17).

That (4.18) implies (4.19) follows from (4.17), while the converse is immediate. Now fix arbitrary $\bar{X} \equiv (\bar{X}_0, \Upsilon^{\bar{X}}, \mathbf{\Lambda}^{\bar{X}}) \in \mathbb{A}$ and $\bar{Y} \equiv (\bar{Y}_0, \Upsilon^{\bar{Y}}, \mathbf{\Lambda}^{\bar{Y}}, \mathbf{\Gamma}^{\bar{Y}}) \in \mathbb{B}$. An argument which is *absolutely identical* to that for the proof of Labbé and Heunis [2007, Proposition 5.2] establishes that (4.18) holds if and only if the following relations (iv), (v) and (vi) hold:
(iv) $l_0(\bar{X}_0) + m_0(\bar{Y}_0) = \bar{X}_0\bar{Y}_0$,
(v) $L(t, \bar{X}(t), \Upsilon^{\bar{X}}(t), \mathbf{\Lambda}^{\bar{X}}(t)) + M(t, \bar{Y}(t), \Upsilon^{\bar{Y}}(t), \mathbf{\Lambda}^{\bar{Y}}(t)) = \bar{X}(t)\Upsilon^{\bar{Y}}(t) + \Upsilon^{\bar{X}}(t)\bar{Y}(t) + \left(\mathbf{\Lambda}^{\bar{X}}\right)^\top(t)\mathbf{\Lambda}^{\bar{Y}}(t)$,
(vi) $J(\bar{X}(T)) + m_T(\bar{Y}(T)) = -\bar{X}(T)\bar{Y}(T)$.
Again, an argument which is *absolutely identical* to that in the proof of Labbé and Heunis [2007, Proposition 5.3] establishes that the preceding relations (iv), (v) and (vi) hold if and only if relations (4.20) - (4.22) hold. Thus (4.18) and (4.20) - (4.22) are equivalent. □

Remark A.2. Put $\|(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})\|_{\boldsymbol{S}} := \{y^2 + \Sigma_n \mathrm{E}\int_0^T |\lambda_n(t)|^2\,\mathrm{d}t + \Sigma_{i,j}\mathrm{E}\int_0^T |\gamma_{ij}(t)|^2\,\mathrm{d}[M_{ij}](t)\}^{1/2}$ for each $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$ (see the definition of $\boldsymbol{S}$ at Proposition 4.6). Then it is clear that $\|\cdot\|_{\boldsymbol{S}}$ is a norm on $\boldsymbol{S}$ with respect to which $\boldsymbol{S}$ is a reflexive Banach space (in fact a Hilbert space).

Proof of Proposition 4.8: From Condition 2.7(ii) it is easy (although tedious) to show that $\tilde{\Psi}(\cdot)$ is $\|\cdot\|_{\boldsymbol{S}}$-coercive (that is $\tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \to \infty$ as $\|(y, \boldsymbol{\lambda}, \boldsymbol{\gamma})\|_{\boldsymbol{S}} \to \infty$), and it is clear that $\tilde{\Psi}(\cdot)$ is convex on $\boldsymbol{S}$ and proper (since $\tilde{\Psi}(0) \leq \mathrm{E}[B^2/2A] < \infty$ and $\tilde{\Psi}(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \geq x_0 y > -\infty$ for all $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \boldsymbol{S}$). From Fatou's theorem and Condition 2.7(ii) one sees that $\tilde{\Psi}(\cdot)$ is $\|\cdot\|_{\boldsymbol{S}}$-lower semi-continuous. Existence of a minimizer $(\bar{y}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\gamma}}) \in \boldsymbol{S}$ of $\tilde{\Psi}(\cdot)$ follows from Remark A.2 and Ekeland and Témam [1976, Proposition II.1.2, p.35], and $\bar{Y} := \Xi(\bar{y}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\gamma}})$ solves (4.16) (see Remark 4.7). □

Proof of Proposition 4.11: Proof of the square-integrability of $\bar{X}$ is *completely identical* to the proof of Labbé and Heunis [2007, Lemma 5.1]. It remains to see that $\bar{X}H$ is a locally square-integrable martingale. For each $m = 1, 2, \ldots$ put $S^{(m)} := \inf\{t \geq 0 \,:\, H(t) \geq m\} \wedge T$. Then $S^{(m)}$ is a $\{\mathcal{F}_t\}$-stopping time, and, since H is continuous, one has $S^{(m)} \Uparrow T$. It follows from (4.29) that the stopped process $(\bar{X}H)^{S^{(m)}}$ is a $\{\mathcal{F}_t\}$-martingale, and $E|\bar{X}(t \wedge S^{(m)})H(t \wedge S^{(m)})|^2 \leq m^2 \mathrm{E}\,[\sup_{t\in[0,T]}|\bar{X}(t)|^2] < \infty$ for each $t \in [0,T]$ as required. □

Proof of Proposition 4.12: In view of (4.31) and (3.6) this is a matter of showing that $r\bar{X}_- + \bar{\boldsymbol{\pi}}^\top \boldsymbol{\sigma\theta} \in L_{21}$, $\boldsymbol{\sigma}^\top \bar{\boldsymbol{\pi}} \in L^2(\mathbf{W})$, and $H^{-1}\boldsymbol{\Gamma}^{\bar{X}H} \in L^2(\mathbf{M})$. For each $m \in \mathbb{N}$, define the $\{\mathcal{F}_t\}$–stopping time

$$S_1^{(m)} := \inf\left\{t > 0 \,:\, \int_0^t \|\bar{\boldsymbol{\pi}}(s)\|^2 \,\mathrm{d}s > m \quad \text{or} \quad |\bar{X}(t_-)|^2 > m \quad \text{or} \quad |H^{-1}(t)|^2 > m\right\} \wedge T.$$

Since $\boldsymbol{\Gamma}^{\bar{X}H} \in L^2_{\mathrm{loc}}(\mathbf{M})$, there exists a sequence $\{S_2^{(m)}\}_{m\in\mathbb{N}}$ of $\{\mathcal{F}_t\}$-stopping times such that $S_2^{(m)} \Uparrow T$, and $\boldsymbol{\Gamma}^{\bar{X}H}[0, S_2^{(m)}] \in L^2(\mathbf{M})$ for all $m \in \mathbb{N}$ (see Notation 3.5). Define the $\{\mathcal{F}_t\}$–stopping time $S^{(m)} := S_1^{(m)} \wedge S_2^{(m)}$, which is clearly such that $S^{(m)} \Uparrow T$.

We have (i) $-2\bar{X}(s_-)\boldsymbol{\theta}^\top(s)\boldsymbol{\sigma}(s)\bar{\boldsymbol{\pi}}(s) \leq (1/2)[4\bar{X}^2(s_-)\|\boldsymbol{\theta}(s)\|^2 + \|\boldsymbol{\sigma}^\top(s)\bar{\boldsymbol{\pi}}(s)\|^2]$ (exactly as in the proof of Labbé and Heunis [2007, Lemma 5.2]). Now expand $t \mapsto \bar{X}^2(t \wedge S^{(m)})$ by Itô's formula (recall (4.31)), take expectations and insert (i) to get

$$\kappa \geq \mathrm{E}\int_0^{S^{(m)}} \|\boldsymbol{\sigma}^\top(s)\bar{\boldsymbol{\pi}}(s)\|^2 \,\mathrm{d}s + \sum_{i,j=1}^{D} \mathrm{E}\int_0^{S^{(m)}} H^{-2}(s)|\Gamma_{ij}^{\bar{X}H}(s)|^2 \,\mathrm{d}[M_{ij}](s), \tag{A.2}$$

in which $\kappa := [2 + 4T\kappa_{\boldsymbol{\theta}}^2]\mathrm{E}[\sup_{t\in[0,T]} |\bar{X}(t)|^2] < \infty$ (see Proposition 4.11 and Remark 2.6). Since $S^{(m)} \Uparrow T$, we can take $m \to \infty$ at (A.2) to get $H^{-1}\boldsymbol{\Gamma}^{\bar{X}H} \in L^2(\mathbf{M})$ and $\boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}} \in L^2(\mathbf{W})$, hence $\bar{\boldsymbol{\pi}} \in L^2(\mathbf{W})$ (by Remark 2.5). That $r\bar{X}_- + \bar{\boldsymbol{\pi}}^\top\boldsymbol{\sigma\theta} \in L_{21}$ follows from Proposition 4.11 and the uniform boundedness of r, $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ (recall Condition 2.4 and Remark 2.6). □

In order to establish Theorem 4.13 we shall need the following Lemma A.3 and Lemma A.4:

Lemma A.3. *For each $\boldsymbol{\rho} \in L^2(\mathbf{W})$ and $\boldsymbol{\gamma} = (\gamma_{ij})_{i,j=1}^D \in L^2(\mathbf{M})$, there exists some $(\mathbb{P}\otimes Leb)$-a.e. unique $\boldsymbol{\lambda} \in L^2(\mathbf{W})$ such that $(\mathbb{P} \otimes Leb)$-a.e.*

$$\boldsymbol{\lambda}(t) + \boldsymbol{\theta}(t)\int_0^t \boldsymbol{\lambda}^\top(s)\,\mathrm{d}\mathbf{W}(s) = \boldsymbol{\rho}(t) - \boldsymbol{\theta}(t)\sum_{i,j=1}^{D}\int_0^t \gamma_{ij}(s)\,\mathrm{d}M_{ij}(s). \tag{A.3}$$

Proof. Define the norm $\|\cdot\|_{L^2(\mathbf{W})}$ on $L^2(\mathbf{W})$ by $\|\boldsymbol{\lambda}\|^2_{L^2(\mathbf{W})} := \mathrm{E}\int_0^T \|\boldsymbol{\lambda}(t)\|^2\,\mathrm{d}t$ (see (2.3) and recall that $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^N$); with this norm $L^2(\mathbf{W})$ is a Banach space. Put $\eta(t) := \sum_{i,j=1}^D \int_0^t \gamma_{ij}(s)\,\mathrm{d}M_{ij}(s)$ and $\boldsymbol{\xi}(t) := \boldsymbol{\rho}(t) - \boldsymbol{\theta}(t)\eta(t_-)$, $t \in [0,T]$, and for each $\boldsymbol{\lambda} \in L^2(\mathbf{W})$ put $\mathbb{G}\boldsymbol{\lambda}(t) := \boldsymbol{\xi}(t) - \boldsymbol{\theta}(t)\int_0^t \boldsymbol{\lambda}^\top(s)\,\mathrm{d}\mathbf{W}(s)$, $t \in [0,T]$. From the Doob L^2-inequality and the Itô isometry one easily checks that $\mathbb{G}\boldsymbol{\lambda} \in L^2(\mathbf{W})$ for each $\boldsymbol{\lambda} \in L^2(\mathbf{W})$. From the Itô isometry and induction it is easily seen that $\|\mathbb{G}^m\boldsymbol{\lambda}_1 - \mathbb{G}^m\boldsymbol{\lambda}_2\|^2_{L^2(\mathbf{W})} \leq \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|^2_{L^2(\mathbf{W})}\kappa_{\boldsymbol{\theta}}^{2m}T^m/m!$ for all $m = 1, 2, \ldots$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in L^2(\mathbf{W})$ (recall $\kappa_{\boldsymbol{\theta}}$ defined at Remark 2.6). Now fix some positive integer m such that $\kappa_{\boldsymbol{\theta}}^{2m}T^m/m! < 1$; then $\mathbb{G}^m$ is a contraction on the Banach space $L^2(\mathbf{W})$ and the generalized contraction mapping theorem (see Kolmogorov and Fomin [1975, Theorem 1′, page 70]) establishes that $\boldsymbol{\lambda}(t) = \mathbb{G}\boldsymbol{\lambda}(t)$, $(\mathbb{P}\otimes Leb)$-a.e. for some unique $\boldsymbol{\lambda} \in L^2(\mathbf{W})$. The result follows since $\eta(t) = \eta(t_-)$, $(\mathbb{P}\otimes Leb)$-a.e. □

Lemma A.4. *For each* $(y,\boldsymbol{\lambda},\boldsymbol{\gamma}) \in \boldsymbol{S} := \mathbb{R}\times L^2(\mathbf{W})\times L^2(\mathbf{M})$ *we have the inequality*

$$\begin{aligned} 0 \le y(x_0 - \bar{X}(0)) + \lim_{\epsilon\downarrow 0}\left\{E\int_0^T \frac{1}{\epsilon}\left[\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)+\epsilon\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right)-\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right)\right]\,\mathrm{d}t\right\} \\ + E\int_0^T \bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\,\mathrm{d}t - E\sum_{i,j=1}^{D}\int_0^T H^{-1}(t)\Gamma_{ij}^{\bar{X}H}(t)\gamma_{ij}(t)\,\mathrm{d}[M_{ij}](t); \end{aligned} \tag{A.4}$$

here $\bar{Y}\equiv\Xi(\bar{y},\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}})$ *is given by Proposition 4.8 and* $\boldsymbol{\Gamma}^{\bar{X}H}\in L^2_{loc}(\mathbf{M})$ *is the* $\mathrm{d}M_{ij}$*-integrand at (4.30) (recall also (4.13), (4.25), (4.27), (4.28), (4.29), and (4.32)).*

Proof. Fix $(y,\boldsymbol{\lambda},\boldsymbol{\gamma})\in\boldsymbol{S}$ and put (i) $R := \Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})$. For $\epsilon\in(0,\infty)$, define the perturbation $(y^\epsilon,\boldsymbol{\lambda}^\epsilon,\boldsymbol{\gamma}^\epsilon) := (\bar{y},\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}})+\epsilon(y,\boldsymbol{\lambda},\boldsymbol{\gamma})\in\boldsymbol{S}$. Then (ii) $\Xi(y^\epsilon,\boldsymbol{\lambda}^\epsilon,\boldsymbol{\gamma}^\epsilon)=\bar{Y}+\epsilon R$ (since Ξ is linear, by Proposition 4.6(b)) and (iii) $\boldsymbol{\Theta}_{\Xi(y^\epsilon,\boldsymbol{\lambda}^\epsilon,\boldsymbol{\gamma}^\epsilon)}=\boldsymbol{\Theta}_{\bar{Y}}+\epsilon\boldsymbol{\Theta}_R$ (see (4.27)). From Proposition 4.8 we have the optimality inequality (iv) $\epsilon^{-1}\left[\tilde{\Psi}(y^\epsilon,\boldsymbol{\lambda}^\epsilon,\boldsymbol{\gamma}^\epsilon)-\tilde{\Psi}(\bar{y},\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\gamma}})\right]\ge 0$. Upon taking $\epsilon\to 0$ in (iv) and using (i), (ii), (iii), (4.26), and $-\bar{X}(T)=A^{-1}[\bar{Y}(T)+B]$ (see (4.29)), we find

$$x_0 y + \lim_{\epsilon\downarrow 0}\left\{\mathrm{E}\int_0^T \frac{1}{\epsilon}\left[\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)+\epsilon\boldsymbol{\Theta}_R(t)\right)-\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right)\right]\,\mathrm{d}t\right\} - \mathrm{E}\left(\bar{X}(T)R(T)\right)\ge 0. \tag{A.5}$$

But $\bar{X}\in\mathbb{B}$, with $\Upsilon^{\bar{X}}=r\bar{X}_- + \bar{\boldsymbol{\pi}}^\top\boldsymbol{\sigma}\boldsymbol{\theta}$, $\boldsymbol{\Lambda}^{\bar{X}}=\boldsymbol{\sigma}^\top\bar{\boldsymbol{\pi}}$, $\boldsymbol{\Gamma}^{\bar{X}}=\boldsymbol{\Gamma}^{\bar{X}H}H^{-1}$ (see Proposition 4.12), and $R\in\mathbb{B}$, with $R(0)=y$, $\Upsilon^R=-rR$, $\boldsymbol{\Lambda}^R=\boldsymbol{\lambda}$, $\boldsymbol{\Gamma}^R=\boldsymbol{\gamma}$ (see Proposition 4.6(c)). Then, from (A.1),

$$\mathbb{M}(\bar{X},R)(T)=\bar{X}(T)R(T)-\bar{X}(0)y+\int_0^T \bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_R(t)\,\mathrm{d}t-\sum_{i,j=1}^{D}\int_0^T H^{-1}(t)\Gamma_{ij}^{\bar{X}H}(t)\gamma_{ij}(t)\,\mathrm{d}[M_{ij}](t),$$

(since $\boldsymbol{\Theta}_R(t)=-\boldsymbol{\sigma}(t)\left[\boldsymbol{\theta}(t)R(t)+\boldsymbol{\lambda}(t)\right]$ - see (i)). From Proposition A.1 we have $\mathrm{E}\mathbb{M}(\bar{X},R)(T)=0$; taking expectations in the preceding expression and combining with (A.5) then gives (A.4). ☐

Proof of Theorem 4.13: Fix an arbitrary $y\in\mathbb{R}$. From the uniform boundedness of $\boldsymbol{\theta}$ (see Remark 2.6) we have that $-y\boldsymbol{\theta}\in L^2(\mathbf{W})$. Applying Lemma A.3 to $\boldsymbol{\rho}:=-y\boldsymbol{\theta}$ and $\boldsymbol{\gamma}:=\mathbf{0}$, there exists $\boldsymbol{\lambda}_y\in L^2(\mathbf{W})$ such that (i) $\boldsymbol{\lambda}_y(t)+\boldsymbol{\theta}(t)\int_0^t\boldsymbol{\lambda}_y^\top(s)\,\mathrm{d}\mathbf{W}(s)=-y\boldsymbol{\theta}(t)$, $(\mathbb{P}\otimes Leb)$-a.e. Now put (ii) $R:=\Xi(y,\beta\boldsymbol{\lambda}_y,\mathbf{0})$; from (4.25) we find (iii) $\beta^{-1}(t)R(t)=y+\int_0^t\boldsymbol{\lambda}_y^\top(s)\,\mathrm{d}\mathbf{W}(s)$. Then, from (i) and (iii), we obtain (iv) $\beta^{-1}(t)R(t)\boldsymbol{\theta}(t)+\boldsymbol{\lambda}_y(t)=0$, $(\mathbb{P}\otimes Leb)$-a.e., and, from (ii), (iv) and (4.27), it follows that (v) $\boldsymbol{\Theta}_{\Xi(y,\beta\boldsymbol{\lambda}_y,\mathbf{0})}(t)=-\boldsymbol{\sigma}(t)[\boldsymbol{\theta}(t)R(t)+\beta(t)\boldsymbol{\lambda}_y(t)]=0$, $(\mathbb{P}\otimes Leb)$-a.e. Upon identifying $(y,\boldsymbol{\gamma},\boldsymbol{\lambda})$ in Lemma A.4 with $(y,\beta\boldsymbol{\lambda}_y,\mathbf{0})$ and using (v), we obtain $0\le y(x_0-\bar{X}(0))$. The arbitrary choice of $y\in\mathbb{R}$ then gives

$$\bar{X}(0)=x_0. \tag{A.6}$$

Since the support functional $\delta(\cdot)$ (recall (4.13)) is positively homogeneous and subadditive (see Karatzas and Shreve [1998, Section 5.4]), for each $\epsilon\in(0,\infty)$ we have

$$\frac{1}{\epsilon}\left[\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)+\epsilon\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right)-\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right)\right]\le\delta\left(\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right), \tag{A.7}$$

for all $(y,\boldsymbol{\lambda},\boldsymbol{\gamma})\in\mathbb{R}\times L^2(\mathbf{W})\times L^2(\mathbf{M})$. Substituting (A.7) into (A.4) and using (A.6), we get

$$\mathrm{E}\sum_{i,j=1}^{D}\int_0^T H^{-1}(t)\Gamma_{ij}^{\bar{X}H}(t)\gamma_{ij}(t)\,\mathrm{d}[M_{ij}](t)\le\mathrm{E}\int_0^T\left[\delta\left(\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right)+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right]\,\mathrm{d}t, \tag{A.8}$$

for all $(y,\boldsymbol{\lambda},\boldsymbol{\gamma})\in\mathbb{R}\times L^2(\mathbf{W})\times L^2(\mathbf{M})$. We next use (A.8) to establish

$$\bar{\boldsymbol{\pi}}\in\mathcal{A}, \tag{A.9}$$

(recall (4.32) and (2.6)). Put $y:=0$ and $\boldsymbol{\gamma}:=\mathbf{0}$ in (A.8) to obtain

$$\mathrm{E}\int_0^T\left[\delta\left(\boldsymbol{\Theta}_{\Xi(0,\boldsymbol{\lambda},\mathbf{0})}(t)\right)+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_{\Xi(0,\boldsymbol{\lambda},\mathbf{0})}(t)\right]\,\mathrm{d}t\geq 0,\quad\text{for all}\quad\boldsymbol{\lambda}\in L^2(\mathbf{W}), \tag{A.10}$$

and set $Q:=\{(\omega,t)\in\Omega\times[0,T]\,:\,\bar{\boldsymbol{\pi}}(\omega,t)\in K\}$. By a trivial adaptation of Karatzas and Shreve [1998, Lemma 5.4.2], corresponding to the $\mathbb{R}^N$-valued $\{\mathcal{F}_t\}$-predictable process $\bar{\boldsymbol{\pi}}$ there exists an $\{\mathcal{F}_t\}$-predictable mapping $\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}:\Omega\times[0,T]\to\mathbb{R}^N$ such that $\|\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)\|\leq 1$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e., with $|\,\delta(\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t))|\leq 1$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. and

$$\begin{cases}\delta(\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t))+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)=0, & (\mathbb{P}\otimes \mathit{Leb})\text{-a.e. on } Q\\ \delta(\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t))+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)<0, & (\mathbb{P}\otimes \mathit{Leb})\text{-a.e. on } \Omega\times[0,T]-Q.\end{cases} \tag{A.11}$$

Suppose $(\mathbb{P}\otimes \mathit{Leb})(\Omega\times[0,T]-Q)>0$; then it follows from (A.11) that

$$\mathrm{E}\int_0^T\left[\delta\left(\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)\right)+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)\right]\,\mathrm{d}t<0. \tag{A.12}$$

We shall next show that $\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)=\boldsymbol{\Theta}_{\Xi(0,\boldsymbol{\lambda},\mathbf{0})}(t)$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. for some $\boldsymbol{\lambda}\in L^2(\mathbf{W})$; it then follows that (A.12) contradicts (A.10), and consequently $(\mathbb{P}\otimes \mathit{Leb})(\Omega\times[0,T]-Q)=0$, that is $\bar{\boldsymbol{\pi}}(t)\in K$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e.; since $\bar{\boldsymbol{\pi}}\in L^2(\mathbf{W})$ (see Proposition 4.12) we get (A.9). To this end put $\boldsymbol{\rho}(t):=-\beta^{-1}(t)\boldsymbol{\sigma}^{-1}(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)$. From the boundedness of β and $\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}$, together with Remark 2.5, we get $\boldsymbol{\rho}\in L^2(\mathbf{W})$. From Lemma A.3 for this $\boldsymbol{\rho}$ and for $\boldsymbol{\gamma}:=0$, there exists $\xi\in L^2(\mathbf{W})$ such that

$$\xi(t)+\boldsymbol{\theta}(t)\int_0^t\xi^\top(s)\,\mathrm{d}\mathbf{W}(s)=-\beta^{-1}(t)\boldsymbol{\sigma}^{-1}(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t),\qquad(\mathbb{P}\otimes \mathit{Leb})\text{-a.e.} \tag{A.13}$$

Now (4.25) gives (vi) $\Xi(0,\beta\xi,\mathbf{0})(t)=\beta(t)\int_0^t\xi^\top(s)\,\mathrm{d}\mathbf{W}(s)$, and, from (4.27) we also have that (vii) $\boldsymbol{\Theta}_{\Xi(0,\beta\xi,\mathbf{0})}(t)=-\boldsymbol{\sigma}(t)\left[\boldsymbol{\theta}(t)\Xi(0,\beta\xi,\mathbf{0})(t)+\beta(t)\xi(t)\right]$. Upon combining (A.13), (vi) and (vii), we find that $\boldsymbol{\Theta}_{\Xi(0,\beta\xi,\mathbf{0})}(t)=\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e; since $\beta\xi\in L^2(\mathbf{W})$, this establishes (A.9). We next show that

$$\boldsymbol{\Gamma}^{\bar{X}H}=\mathbf{0},\quad\nu_{[M]}\text{-a.e. (recall Notation 3.2).} \tag{A.14}$$

Put $\boldsymbol{\rho}:=-\beta^{-1}\boldsymbol{\sigma}^{-1}\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}$ and $\boldsymbol{\gamma}:=\beta^{-1}H^{-1}\boldsymbol{\Gamma}^{\bar{X}H}$. Then $\boldsymbol{\rho}\in L^2(\mathbf{W})$ (exactly as before - see text preceding (A.13)) and also $\boldsymbol{\gamma}\in L^2(\mathbf{M})$ (see Proposition 4.12). We can therefore use Lemma A.3 for this pair $(\boldsymbol{\rho},\boldsymbol{\gamma})$ to see that there exists $\boldsymbol{\eta}\in L^2(\mathbf{W})$ such that

$$\boldsymbol{\eta}(t)+\boldsymbol{\theta}(t)\int_0^t\boldsymbol{\eta}^\top(s)\,\mathrm{d}\mathbf{W}(s)=-\beta^{-1}(t)\boldsymbol{\sigma}^{-1}(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)-\boldsymbol{\theta}(t)\sum_{i,j=1}^D\int_0^t\beta^{-1}(s)H^{-1}(s)\Gamma_{ij}^{\bar{X}H}(s)\,\mathrm{d}M_{ij}(s). \tag{A.15}$$

Set (viii) $R:=\Xi(0,\beta\boldsymbol{\eta},H^{-1}\boldsymbol{\Gamma}^{\bar{X}H})$. Then (ix) $R(t)\boldsymbol{\theta}(t)=-\beta(t)\boldsymbol{\eta}(t)-\boldsymbol{\sigma}^{-1}(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)$, as follows from (A.15) and (4.25), and thus (x) $\boldsymbol{\Theta}_R(t)=-\boldsymbol{\sigma}(t)\left[\boldsymbol{\theta}(t)R(t)+\beta(t)\boldsymbol{\eta}(t)\right]=\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)$, in which the first equality follows from (4.27) and (viii), and the second equality follows from (ix). From (A.9), (A.11) and (x), we obtain (xi) $\delta\left(\boldsymbol{\Theta}_R(t)\right)+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_R(t)=\delta(\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t))+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\nu}^{\bar{\boldsymbol{\pi}}}(t)=0$, $(\mathbb{P}\otimes \mathit{Leb})$-a.e. Upon taking $(y,\boldsymbol{\lambda},\boldsymbol{\gamma}):=(0,\beta\boldsymbol{\eta},H^{-1}\boldsymbol{\Gamma}^{\bar{X}H})$ in (A.8), and recalling (viii) and (xi), we find

$$\mathrm{E}\sum_{i,j=1}^D\int_0^T H^{-2}(t)|\Gamma_{ij}^{\bar{X}H}(t)|^2\,\mathrm{d}[M_{ij}](t)\leq\mathrm{E}\int_0^T\left[\delta\left(\boldsymbol{\Theta}_R(t)\right)+\bar{\boldsymbol{\pi}}^\top(t)\boldsymbol{\Theta}_R(t)\right]\,\mathrm{d}t=0. \tag{A.16}$$

Now (A.14) follows from (A.16) and the strict positivity of H (see (4.28)). We next establish

$$\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right) + \bar{\boldsymbol{\pi}}^{\top}(t)\boldsymbol{\Theta}_{\bar{Y}}(t) = 0, \quad (\mathbb{P}\otimes Leb)\text{-a.e.} \tag{A.17}$$

Set $(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) := (-\bar{y}, -\bar{\boldsymbol{\lambda}}, -\bar{\boldsymbol{\gamma}})$ (recall Proposition 4.8); from (4.25) we get (xii) $\Xi(y, \boldsymbol{\lambda}, \boldsymbol{\gamma}) = -\bar{Y}$, and from (4.27) we also have (xiii) $\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})} = -\boldsymbol{\Theta}_{\bar{Y}}$. Since $\delta(\cdot)$ is positively homogeneous, it follows from (xiii) that $\delta(\boldsymbol{\Theta}_{\bar{Y}}(t) + \epsilon\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)) = \delta((1-\epsilon)\boldsymbol{\Theta}_{\bar{Y}}(t)) = (1-\epsilon)\delta(\boldsymbol{\Theta}_{\bar{Y}}(t))$ for all $0 < \epsilon < 1$, that is

$$\frac{1}{\epsilon}\left[\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t) + \epsilon\boldsymbol{\Theta}_{\Xi(y,\boldsymbol{\lambda},\boldsymbol{\gamma})}(t)\right) - \delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right)\right] = -\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right), \qquad \text{for all } 0 < \epsilon < 1. \tag{A.18}$$

From (A.18), (xiii), (A.14), (A.6), (A.4), we get (xiv) $\mathrm{E}\int_0^T \left(\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right) + \bar{\boldsymbol{\pi}}^{\top}(t)\boldsymbol{\Theta}_{\bar{Y}}(t)\right)\, \mathrm{d}t \leq 0$; but, from (A.9) and (4.13), we also have (xv) $\delta\left(\boldsymbol{\Theta}_{\bar{Y}}(t)\right) = \sup_{\boldsymbol{\pi}\in K}\left\{-\boldsymbol{\pi}^{\top}\boldsymbol{\Theta}_{\bar{Y}}(t)\right\} \geq -\bar{\boldsymbol{\pi}}^{\top}(t)\boldsymbol{\Theta}_{\bar{Y}}(t)$, $(\mathbb{P}\otimes Leb)$-a.e., and (xv) and (xiv) together give (A.17).

We can now complete the proof of Theorem 4.13: part (a) is immediate from (A.14), Proposition 4.12 and Remark 3.3. As for part (b), (4.20)(i) is (A.6), (4.20)(ii) follows from (4.29) with $t := T$, (4.21) follows since $\bar{Y} \in \mathbb{B}_1$ (recall Proposition 4.8 and (4.23)), (4.22)(i) is just (A.17), and (4.22)(ii)(iii) are immediate from (A.14), (A.9), (4.33) and (4.1). For part (c), we have $\bar{\boldsymbol{\pi}} \in \mathcal{A}$ from (A.9). Moreover, upon inserting (A.14) and (A.6) into (4.31), the resulting equation for $\bar{X}$ is identical to the equation (2.4) for $X^{\bar{\boldsymbol{\pi}}}$ so that $\bar{X} = X^{\bar{\boldsymbol{\pi}}}$. In view of part (b) and Proposition 4.2 we obtain (4.19), that is $\bar{X}$ solves the Bolza problem (4.8). Now (4.34) is immediate from (4.19) and Remark 4.1. $\square$

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