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Classical r-matrices for the generalised Chern-Simons formulation of 3d gravity

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Abstract

We study the conditions for classical r-matrices to be compatible with the generalised Chern-Simons action for 3d gravity. Compatibility means solving the classical Yang-Baxter equations with a prescribed symmetric part for each of the real Lie algebras and bilinear pairings arising in the generalised Chern-Simons action. We give a new construction of r-matrices via a generalised complexification and derive a non-linear set of matrix equations determining the most general compatible r-matrix. We exhibit new families of solutions and show that they contain some known r-matrices for special parameter values.

1 Introduction and Motivation

The application of the combinatorial quantisation programme [1, 2, 3, 4, 5, 6, 7, 8] to the Chern-Simons formulation of 3d gravity [9, 10, 11] has provided a systematic way of studying the role of quantum groups and non-commutative geometry in 3d quantum gravity. What emerges in this framework is that quantisation deforms the classical phase space geometry into a non-commutative geometry in which the model spacetimes are replaced by non-commutative spaces and the local isometry groups (gauge groups for the Chern-Simons theory) by quantum isometry groups.

In the combinatorial quantisation procedure, one begins with the description of the Poisson structure on the classical phase space in terms of a classical r-matrix. In its original formulation for generic Chern-Simons theories this description is due to Fock and Rosly [12]. It requires that the r-matrix satisfies a certain compatibility condition with the inner product used in defining the Chern-Simons action. When the r-matrix is the classical limit of a quantum R-matrix associated to a certain quantum group, one can formulate the quantum theory in terms of the representation theory of that quantum group, see [3] for the general framework and also [13, 14] for survey accounts related to 3d gravity.

In the case of the Chern-Simons formulation of 3d gravity, r-matrices satisfying the compatibility condition are known for all signatures and values of the cosmological constant [15]. The associated quantum groups are deformations of the classical isometry groups and thus natural candidates for the 'quantum isometry groups' of 3d quantum gravity.

Unfortunately, Fock and Rosly's compatibility requirement does not specify the classical r-matrix uniquely. For a given classical theory, several compatible r-matrices are possible in general, and the associated quantum groups are equally valid contenders for the role of 'quantum isometry group'.

The family of classical r-matrices and corresponding quantum groups that can be associated to the most general Chern-Simons action for 3d gravity via the Fock-Rosly compatibility requirement has not yet been fully characterised. The quantum groups are known to include a family of Drinfeld quantum doubles and Maijd's bicrossproduct quantum groups with spacelike, timelike and null deformation parameters [15, 16]. However, recent work [17] shows that there are more and that, at the infinitesimal Lie bialgebra level, all the known solutions can in fact be viewed as classical doubles. There are also partial results on the relationship between different compatible r-matrices [16].

Our goal here is to formulate the general equations which determine compatible r-matrices in 3d gravity, and to exhibit some new solutions. It turns out to be mathematically natural to consider a generalised form of the Chern-Simons action for 3d gravity, based on the most general symmetric bilinear form for the relevant Lie algebra. This generalised action was previously considered in [10, 18, 19, 15]. In addition to the usual gravitational term it contains a non-standard term which has been interpreted as an analogue of the Immirzi term in 4d gravity [19], though this interpretation has recently been questioned [20].

The family of real Lie algebras used in the Chern-Simons formulation of 3d gravity depends on the speed of light and the value of the cosmological constant. It includes $so(p, q)$ with $p+q=4$ as well as the Poincaré and Euclidean group in, respectively, 2+1 and 3 dimensions. All of these are much studied in geometry and physics. Our results are therefore also of interest in the context of the bialgebra structures for these Lie algebras and of the Poisson-Lie algebra structures of the corresponding groups. They generalise previous studies of r-matrices for the Euclidean and Poincaré group in [21] and [22][1].

We should stress that, in general, the Chern-Simons formulation of 3d gravity is not equivalent to the Einstein or metric formulation of 3d gravity. The differences arise because the frame field is necessarily non-degenerate in the metric formulation but may degenerate in the Chern-Simons formulation, and because of different notions of gauge invariance in the two formulations. The resulting phase spaces may therefore, in general, be different. This issue has been discussed extensively in the literature, starting with the papers [23, 24] (showing that the resulting phases spaces may be the same under certain assumptions) and [25](exhibiting an example where they differ). This discussion has some parallels with that comparing gauge and metric formulations of gravity in 4d, see, e.g., [26], but the flatness of classical solutions in the Chern-Simons formulation leads to special features in 3d. This paper is not intended as a contribution to this discussion. We take the Chern-Simons formulation as the starting point of our treatment, and explore its consequences with a view to gaining a systematic understanding of its quantisation.

Our paper is organised as follows. We start in Sect. 2 with a review of the generalised Chern-Simons action for 3d gravity, the description of the relevant Lie algebras as generalised

[1] Readers referring to [22] should be aware of the 'dual' use of the parameter λ in that paper compared to the current paper.

complexifications of $so(3)$ and $so(2,1)$. We give a precise formulation of the compatibility requirement for an r-matrix in this context, and explain the reality condition we impose. In Sect. 3 we derive a particular class of solutions by a process of (generalised) complexification of the standard r-matrix for the Lie bialgebra of $sl(2,\mathbb{R}) \simeq so(2,1)$. Sect. 4 contains the main results of this paper. We derive a set of non-linear coupled equations which characterises the most general compatible r-matrix in 3d gravity. In Sect. 5 we derive two new families of solutions, and show how they relate to the complexified solutions of Sect. 3 and that they include some known solutions as special cases. Sect. 6 contains our conclusions.

2 Lie algebras, Chern-Simons actions and compatible r-matrices

2.1 Lie algebra conventions

We use the conventions of [22], so write $\mathfrak{g}$ for either $so(3)$ or $so(2,1)$, with generators J_a, $a = 0,1,2$. The metric $\eta_{ab} = \eta^{ab}$ is, respectively, the Euclidean metric diag$(1,1,1)$ or the Lorentzian metric diag$(1,-1,-1)$. The Lie brackets of $\mathfrak{g}$ are then

$$[J_a, J_b] = \epsilon_{abc} J^c, \tag{2.1}$$

where indices are raised with η^{ab} and we adopt the convention $\epsilon_{012} = \epsilon^{012} = 1$. We make frequent use of the Killing form as an invariant (possibly Lorentzian) inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and assume that it is normalised so that

$$\langle J_a, J_b\rangle = \eta_{ab}. \tag{2.2}$$

We write A^t for the transpose of a map $A : \mathfrak{g} \to \mathfrak{g}$ with respect to the inner product (2.2), i.e.,

$$\langle v, Aw\rangle = \langle A^t v, w\rangle \qquad \forall v, w \in \mathfrak{g}. \tag{2.3}$$

When we expand elements of $\mathfrak{g}$ via

$$p = p^a J_a, \quad q = q^a J_a, \tag{2.4}$$

we write $\boldsymbol{p} = (p^0, p^1, p^2)$ for the coordinate vector and

$$\langle p, q\rangle = \boldsymbol{p}\cdot\boldsymbol{q}. \tag{2.5}$$

We also use the following notation for the dual of a vector as a linear form:

$$v^t = \langle v, \cdot\rangle, \quad v \in \mathfrak{g}. \tag{2.6}$$

We write $\mathfrak{g}_\lambda$ for the family of Lie algebras which arise as isometry Lie algebras in 3d gravity. Depending on the real parameter λ (which is related to the cosmological constant, see [14] and our discussion in Sect. 2.2), $\mathfrak{g}_\lambda$ is the Lie algebra of the Poincaré, de Sitter or anti-de Sitter group or their Euclidean analogues in three dimensions, with brackets

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = \lambda\epsilon_{abc} J^c. \tag{2.7}$$

The Lie algebra $\mathfrak{g}_\lambda$ can be interpreted as the real form of a generalised complexification of $\mathfrak{g}$, and this viewpoint will prove useful in the following. We refer the reader to [27] and [28] for details but briefly summarise the main results. The idea is to introduce a formal parameter θ satisfying

$$\theta^2 = \lambda, \tag{2.8}$$

and to set

$$P_a = \theta J_a. \tag{2.9}$$

Then the commutators (2.7) follow from (2.1) by extending the Lie brackets linearly in θ. More formally, one defines the ring $R_\lambda = (\mathbb{R}^2, +, \cdot)$ as the commutative ring obtained from $\mathbb{R}^2$ with the usual addition and the λ-dependent multiplication law

$$(a + \theta b) \cdot (c + \theta d) = (ac + \lambda bd) + \theta(ad + bc) \quad \forall a, b, c, d \in \mathbb{R}, \tag{2.10}$$

and defines $\mathfrak{g}_\lambda$ as the realification of $\mathfrak{g} \otimes R_\lambda$ [27].

The Lie algebra $\mathfrak{g}_\lambda$ has a two-parameter family of non-degenerate symmetric Ad-invariant bilinear forms [10]. Defining

$$t(J_a, J_b) = 0, \qquad t(P_a, P_b) = 0, \qquad t(J_a, P_b) = \eta_{ab}, \tag{2.11}$$

$$s(J_a, J_b) = \eta_{ab}, \qquad s(J_a, P_b) = 0, \qquad s(P_a, P_b) = \lambda\eta_{ab}, \tag{2.12}$$

the most general such form is given by

$$(\cdot, \cdot)_\tau = \alpha t(\cdot, \cdot) + \beta s(\cdot, \cdot), \tag{2.13}$$

in terms of two real parameters α, β. It turns out [15] that the condition for the non-degeneracy for (2.13) can conveniently be written in terms of the complexified parameter $\tau = \alpha + \theta\beta \in R_\lambda$ as

$$\tau\bar{\tau} = \alpha^2 - \lambda\beta^2 \neq 0. \tag{2.14}$$

2.2 Chern-Simons theory and 3d gravity

The Chern-Simons theory on a 3d manifold depends on a gauge group and an invariant, non-degenerate symmetric bilinear form on the Lie algebra of that gauge group. One recovers the Einstein-Hilbert action for 3d gravity for any signature and value of the cosmological constant from the Chern-Simons action by choosing the appropriate local isometry group G_λ with Lie algebra $\mathfrak{g}_\lambda$ as gauge group and using the non-degenerate form (2.11), see [9, 10]. Here we are interested in the Chern-Simons action with the more general bilinear form (2.13). This was previously considered in [10, 18, 19] and, in our notation, in [15]

Consider a three-dimensional spacetime manifold M^3 of the product topology $\mathbb{R} \times S$, where S is an oriented two-dimensional manifold, possibly with handles and punctures. Physically, S represents 'space' and the punctures particles. The gauge field of the Chern-Simons theory is locally a one-form A on the spacetime with values in the Lie algebra $\mathfrak{g}_\lambda$. It can be expanded in terms of the generators J_a and P_a as

$$A = \omega_a J^a + e_a P^a, \tag{2.15}$$

where $\omega = \omega^a J_a$ is geometrically interpreted as the spin connection on the frame bundle and the one-form e_a as a dreibein. The curvature of this connection combines the Riemann curvature R, the torsion T and a cosmological term, see [15] for details.

The Chern-Simons action for the gauge field A is

$$I_\tau(A) = \int_M (A \wedge dA)_\tau + \frac{1}{3}(A \wedge [A, A])_\tau. \tag{2.16}$$

Integrating by parts and ignoring boundary terms, this can be expanded as

$$\begin{aligned} I_\tau(A) = \alpha &\int_M \left(2e^a \wedge R_a + \frac{\lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right) \\ + \beta &\int_M \left(\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c + \lambda e^a \wedge T_a \right). \end{aligned} \tag{2.17}$$

The first term is the usual Einstein-Hilbert action for 3d gravity with a cosmological constant. To make contact with the physical constants of 3d gravity identify

$$\alpha = \frac{1}{16\pi G}, \qquad \lambda = -c^2\Lambda, \tag{2.18}$$

where G is Newton's constant in 2+1 dimensions, c is the speed of light and taken to be imaginary in the case of Euclidean signature, and Λ is the cosmological constant.

The term proportional to β contains the Chern-Simons action for the spin connection and a cosmological contribution. There have been attempts to interpret this term as the analogue of the Immirzi term in $4d$ [19], but this is contentious [20].

A discussion of the classical equations of motion obtained when varying the action (2.17) with respect to e_a and ω_a, treated as independent variables, can be found in [15].

2.3 Bi-algebras and classical r-matrices

We refer the reader to [29, 30] for details on the background reviewed in this short section. A Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ equipped with a map $\delta : \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$ (the cocommutator, or cobracket) satisfying the following condition:

(i) $\delta : \mathfrak{g} \mapsto \mathfrak{g} \otimes \mathfrak{g}$ is a skew-symmetric linear map, i.e., $\delta : \mathfrak{g} \mapsto \wedge^2\mathfrak{g}$

(ii) δ satisfies the coJacobi identity $(\delta \otimes \mathrm{id}) \circ \delta(X) + \mathrm{cyclic} = 0, \quad \forall X \in \mathfrak{g}$.

(iii) For all $X, Y \in \mathfrak{g}$, $\quad \delta([X, Y]) = (\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X)\delta(Y) - (\mathrm{ad}_Y \otimes 1 + 1 \otimes \mathrm{ad}_Y)\delta(X)$.

An element $r \in \wedge^2\mathfrak{g}$ is said to be a coboundary structure of the Lie bialgebra $(\mathfrak{g}, [\ ,\ ], \delta)$ if $\delta(X) = \mathrm{ad}_X(r) = [X \otimes 1 + 1 \otimes X, r]$.

For any Lie algebra $\mathfrak{g}$, let $r = r^{ab} X_a \otimes Y_b \in \mathfrak{g} \otimes \mathfrak{g}$, $r_{21} = \sigma(r) = r^{ab} Y_b \otimes X_a$ and set

$$r_{12} = r^{ab} X_a \otimes Y_b \otimes 1,\ r_{13} = r^{ab} X_a \otimes 1 \otimes Y_b,\ r_{23} = r^{ab} 1 \otimes X_a \otimes Y_b. \tag{2.19}$$

The classical Yang-Baxter map is the map

$$\mathrm{CYB} : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}, \quad r \mapsto [[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \tag{2.20}$$

It is easy to check that CYB restricts to a map $\wedge^2\mathfrak{g} \to \wedge^3\mathfrak{g}$. The equation

$$[[r, r]] = 0, \tag{2.21}$$

is called the classical Yang-Baxter equation(CYBE). Any solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ of the CYBE is called a classical r-matrix. If $[[r, r]]$ is non-zero but an invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ then r is said to satisfy the modified classical Yang-Baxter equation(MCYBE). The triple $(\mathfrak{g}, [\ ,\ ], r)$ defines a coboundary Lie bialgebra if and only if

$$\mathrm{ad}_X([[r, r]]) = 0, \quad \mathrm{ad}_X(r + r_{21}) = 0, \quad \forall X \in \mathfrak{g}, \tag{2.22}$$

where ad is the usual Lie algebra adjoint action extended to products in the usual way (as a derivation).

2.4 Compatible r-matrices

As reviewed above, a Chern-Simons theory requires a gauge group and an invariant, non-degenerate, symmetric bilinear form on the Lie algebra of the gauge group. In the Fock-Rosly construction, a classical r-matrix is said to be compatible with a Chern-Simons action if it satisfies the CYBE (2.21) and its symmetric part is equal to the Casimir associated to the invariant, non-degenerate symmetric bilinear form used in the Chern-Simons action. Fock and Rosly went on to show how to describe the Poisson structure of an extended phase space for the Chern-Simons theory in terms of a compatible r-matrix.

Decomposing a compatible r-matrix into the symmetric Casimir part K and antisymmetric part r' via $r = K + r'$ we have the identity [29]

$$[[K + r', K + r']] = [[K, K]] + [[r', r']]. \tag{2.23}$$

Therefore, an r-matrix is compatible with a Chern-Simons action if its symmetric part equals the associated Casimir and its antisymmetric part satisfies the modified classical Yang-Baxter equation (MCYBE)

$$[[r', r']] = -[[K, K]]. \tag{2.24}$$

We can now apply this prescription to the Chern-Simons action (2.16). The Casimir in $\mathfrak{g}_\lambda$ associated to the bilinear invariant form (2.13) is

$$K_\tau = \frac{\alpha}{\tau\bar{\tau}}(J_a \otimes P^a + P_a \otimes J^a) - \frac{\beta}{\tau\bar{\tau}}(\lambda J_a \otimes J^a + P_a \otimes P^a). \tag{2.25}$$

It is shown in [15][2] that

$$\begin{aligned}\Omega_\tau &= [[K_\tau, K_\tau]] \\ &\mu\epsilon_{abc}(\lambda J^a \otimes J^b \otimes J^c + J^a \otimes P^b \otimes P^c + P^a \otimes J^b \otimes P^c + P^a \otimes P^b \otimes J^c) \\ &+ \nu\epsilon_{abc}(P^a \otimes P^b \otimes P^c + \lambda P^a \otimes J^b \otimes J^c + \lambda J^a \otimes J^b \otimes P^c + \lambda J^a \otimes P^b \otimes J^c),\end{aligned} \tag{2.26}$$

where we introduced

$$\mu = \frac{\alpha^2 + \lambda\beta^2}{(\tau\bar{\tau})^2}, \qquad \nu = -\frac{2\alpha\beta}{(\tau\bar{\tau})^2}, \tag{2.27}$$

so that

$$\mu + \theta\nu = \frac{1}{\tau^2}. \tag{2.28}$$

[2] In the corresponding expression in [15], there is a missing factor of 2 in the second term. This term was not considered further there, so the missing factor does not affect the conclusions of that paper.

In fact, the right-hand-side of (2.26) is the most general general invariant element in $(\mathfrak{g}_\lambda)^3$.

We sum up the discussion in the following definition.

Definition 2.1 *An r-matrix for any of the Lie algebras* $\mathfrak{g}_\lambda$ *is compatible with the Chern-Simons action* (2.16) *if* $r = r' + K_\tau$, *and* r' *satisfies*

$$[[r', r']] = -\Omega_\tau. \tag{2.29}$$

Our goal here is to find the most general *real* solution of equation (2.29). At the level of Lie algebras we need to keep track of the real structures in order to distinguish between the various physically distinct regimes of 3d gravity. The reality of the classical r-matrices is required in the Fock-Rosly construction in order to have a real Poisson structure on the extended phase. This allows for the usual interpretation of functions on the phase space as observables, and is expected in the classical limit of any quantisation of the theory where observables are Hermitian with respect to a given $*$-structure.

The combination of a real Lie algebra structure with a real r-matrix amounts, in the terminology of [30], to a real-real form of the Lie bialgebra structure defined by the r-matrix. It is mathematically possible to impose other conditions (for example the half-real structure defined in [30] where the anti-symmetric part of the r-matrix is imaginary) but the relevance of such structures to the generalised Chern-Simons formulation of 3d gravity is not clear. We therefore restrict our discussions in the following to real r-matrices; the interested reader should have no difficulty in obtaining solutions satisfying other reality conditions.

3 Compatible r-matrices via generalised complexification

In this section we show that a particular class of solutions of (2.29) can be obtained by a process of (generalised) complexification of the standard solution of the MCYBE for the Lie algebra $sl(2,\mathbb{R})$. As explained in the appendix of the paper [15], the well-known identity

$$[X, [Y, Z]] = \langle X, Z\rangle Y - \langle X, Y\rangle Z \tag{3.1}$$

for the three dimensional Lie algebras $\mathfrak{g}$ can be used, together with the Jacobi identity and the invariance of the Killing form, to establish that

$$r' = \epsilon^{abc} m_a J_b \otimes J_c \tag{3.2}$$

satisfies

$$[[r', r']] = m_a m^a \epsilon^{bcd} J_b \otimes J_c \otimes J_d. \tag{3.3}$$

The quadratic Casimir

$$K = J_a \otimes J^a \tag{3.4}$$

satisfies

$$[[K, K]] = \Omega, \tag{3.5}$$

where Ω is the cubic Casimir $\Omega = \epsilon_{abc} J^a \otimes J^b \otimes J^c$. Therefore, by (2.24),

$$r = K + r' \tag{3.6}$$

satisfies the CYBE (2.21) provided

$$m_a m^a = -1. \tag{3.7}$$

This has a non-trivial real solution only in the case $\mathfrak{g} = sl(2,\mathbb{R})$ and leads to the standard bialgebra structure in that case.

Consider now the (generalised) complexification $\mathfrak{g}_\lambda \otimes R_\lambda$ of the (real, 6-dimensional) Lie algebra $\mathfrak{g}_\lambda$. The generators

$$J_a^\pm = \frac{1}{2}(P_a \pm \theta J_a) \tag{3.8}$$

satisfy

$$[J_a^\pm, J_b^\pm] = \pm\theta\epsilon_{abc}(J^\pm)^c, \quad [J_a^+, J_b^-] = 0. \tag{3.9}$$

It follows that

$$K^\pm = J_a^\pm \otimes (J^\pm)^a \tag{3.10}$$

are both invariant element of the universal enveloping algebra $U(\mathfrak{g}_\lambda \otimes R_\lambda)$, and that

$$r^\pm(\tau, \boldsymbol{m}) = \frac{1}{\tau}K^\pm + \epsilon^{abc} m_a J_b^\pm \otimes J_c^\pm \tag{3.11}$$

both satisfy the CYBE over R_λ for any invertible $\tau \in R_\lambda$ and vector $\boldsymbol{m} \in R_\lambda^3$ satisfying the condition

$$\boldsymbol{m}^2 = -\frac{1}{\tau^2}. \tag{3.12}$$

Since the $+$ and the $-$ copy of the Lie algebra $\mathfrak{g}_\lambda \otimes R_\lambda$ commute, we also deduce that any linear combination

$$a^+ r^+(\tau^+, \boldsymbol{m}^+) + a^- r^-(\tau^-, \boldsymbol{m}^-) \tag{3.13}$$

satisfies the CYBE for any $a^\pm \in R_\lambda$, $\boldsymbol{m}^\pm \in R_\lambda^3$, provided the condition (3.12) holds for the parameters $\tau^\pm$ and $\boldsymbol{m}^\pm$. In particular, therefore, the combinations

$$r^+(\tau, \boldsymbol{m}) \pm r^-(\bar{\tau}, \bar{\boldsymbol{m}}) = r(\tau, \boldsymbol{m}) \pm \overline{r(\tau, \boldsymbol{m})} \tag{3.14}$$

satisfy the CYBE.

In order to obtain a real solution of the CYBE for $\mathfrak{g}_\lambda$ with the symmetric part agreeing with the general Casimir element (2.25), we note that

$$\frac{1}{\tau}K^+ - \frac{1}{\bar{\tau}}K^- = \frac{\theta}{2\tau\bar{\tau}}\left(\alpha(P_a \otimes J^a + J_a \otimes P^a) - \beta(P_a \otimes P^a + \lambda J_a \otimes J^a)\right). \tag{3.15}$$

Assuming for a moment that $\lambda \neq 0$ (so that θ is no a zero-divisor), we can take the negative sign in (3.14) and multiply the result by $2/\theta$ to obtain a solution with the required symmetric part:

$$r(\tau, \boldsymbol{m}) = \frac{2}{\theta}\left(r^+(\tau, \boldsymbol{m}) - r^-(\bar{\tau}, \bar{\boldsymbol{m}})\right). \tag{3.16}$$

Writing

$$\boldsymbol{m} = \boldsymbol{p} + \theta\boldsymbol{q}, \tag{3.17}$$

the solution takes the form

$$r(\tau, \boldsymbol{m}) = K_\tau + \epsilon^{abc}(p_a(P_b \otimes J_c + J_b \otimes P_c) + q_a(P_b \otimes P_c + \lambda J_b \otimes J_c)). \tag{3.18}$$

Spelling out the condition for (3.18) to be a solution of the CYBE, we find

$$\boldsymbol{p}^2 + \lambda \boldsymbol{q}^2 = -\mu, \quad 2\boldsymbol{p}\cdot\boldsymbol{q} = -\nu, \tag{3.19}$$

where we used the abbreviations (2.27).

The division by θ is potentially problematic in the case $\lambda = 0$. However, one can also check that (3.18) is a solution of the CYBE in the limit $\lambda \to 0$ by making careful use of the identity

$$q_a p^d \epsilon_{dbc} + q_b p^d \epsilon_{adc} + q_c p^d \epsilon_{abd} = q_d p^d \epsilon_{abc}. \tag{3.20}$$

One needs to check if (3.19) has solutions. To simplify the analysis we define

$$\boldsymbol{m}' = \tau \boldsymbol{m}, \tag{3.21}$$

and expand $\boldsymbol{m}' = \boldsymbol{p}' + \theta \boldsymbol{q}'$. The condition (3.12) is now simply

$$\boldsymbol{m}'^2 = -1 \tag{3.22}$$

or

$$(\boldsymbol{p}')^2 + \lambda(\boldsymbol{q}')^2 = -1, \qquad \boldsymbol{p}'\cdot\boldsymbol{q}' = 0. \tag{3.23}$$

In other words, solutions are determined by two orthogonal vectors whose squares satisfy one constraint. When $\lambda > 0$, this constraint is clearly impossible to satisfy (for real vectors!) in the Euclidean case, but has various types of solutions in the Lorentzian case, including $\boldsymbol{p}'$ and $\boldsymbol{q}'$ being orthogonal spacelike vectors, but also cases where only one of either $\boldsymbol{p}'$ or $\boldsymbol{q}'$ is spacelike and the other either lightlike or timelike.

When $\lambda < 0$, we can write the constraint as

$$(\boldsymbol{p}')^2 + 1 = (-\lambda)(\boldsymbol{q}')^2, \qquad \boldsymbol{p}'\cdot\boldsymbol{q}' = 0. \tag{3.24}$$

This clearly has solutions in the Euclidean case. In fact, assuming $\lambda = -1$ without loss of generality, any such solutions may be interpreted as determining an ellipse in $\mathbb{R}^3$ with minor axis $\boldsymbol{p}'$ (including the degenerate case $\boldsymbol{p}' = 0$) and major axis $\boldsymbol{q}'$. There are also solutions in the Lorentzian case, of all the types described for the $\lambda > 0$ case above.

4 Conditions for compatible r-matrices in 3d gravity

4.1 A Lie-algebraic equation

We first derive a Lie-algebraic condition for the most general solution of (2.29). In order to distinguish the various kinds of terms in that equation it is helpful to the use the notation of generalised complexification. Then the invariant element (2.26) takes the form

$$\begin{aligned}\Omega_\tau = (&\mu(\lambda \mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id} + \mathrm{id}\otimes\theta\otimes\theta + \theta\otimes\mathrm{id}\otimes\theta + \theta\otimes\theta\otimes\mathrm{id}) + \nu(\theta\otimes\theta\otimes\theta \\ &+ \lambda\theta\otimes\mathrm{id}\otimes\mathrm{id} + \lambda\mathrm{id}\otimes\mathrm{id}\otimes\theta + \lambda\mathrm{id}\otimes\theta\otimes\mathrm{id}))\epsilon_{abc}J^a\otimes J^b\otimes J^c.\end{aligned} \tag{4.1}$$

The antisymmetric part r' of the r-matrix can be written as

$$\begin{aligned} r' &= (\mathrm{id}\otimes A + \theta\otimes B - B\otimes\theta + \theta\otimes\theta\, C)J^a\otimes J_a \\ &= J^a\otimes A(J_a) + P^a\otimes B(J_a) - B(J_a)\otimes P^a + P^a\otimes C(P_a) \\ &= A_{ba}J^a\otimes J^b + B_{ba}P^a\otimes J^b - B_{ba}J^b\otimes P^a + C_{ba}P^a\otimes P^b, \end{aligned} \tag{4.2}$$

where we can assume that A and C are anti-symmetric, i.e., $A_{ab} = -A_{ba}$ and $C_{ab} = -C_{ba}$.

Note that all the matrices appearing in (4.2) should be thought of as matrices for linear maps

$$A, B, C : \mathfrak{g} \to \mathfrak{g} \tag{4.3}$$

with respect to the orthonormal basis $\{J_0, J_1, J_2\}$ of $\mathfrak{g}$. An antisymmetric map $A : \mathfrak{g} \to \mathfrak{g}$ can always and uniquely be expressed in terms of an element $u = u_a J^a \in \mathfrak{g}$, acting via commutator. Also note that

$$A = [u, \cdot] \quad \Rightarrow \quad A_{ab} = -\epsilon_{abc} u^c, \tag{4.4}$$

and that

$$A = [u, \cdot],\ C = [v, \cdot] \quad \Rightarrow \quad AC = vu^t - \langle u, v \rangle \,\mathrm{id}, \tag{4.5}$$

which we will use frequently later in this paper.

Inserting (4.2) into the left hand side of (2.29) generates 48 terms. Setting them equal to the right hand side of (2.29) yields eight equations, each having six terms. We have found it useful to contract the equations with an element $X \otimes Y \otimes Z \in \mathfrak{g}^3$, using the metric $\langle \cdot, \cdot \rangle$. That way, for example, the equation

$$[[J^a \otimes A(J_a), J^b \otimes A(J_b)]] = \epsilon_{abc} J^a \otimes J^b \otimes J^c \tag{4.6}$$

turns into

$$\langle X, [A^t(Y), A^t(Z)] \rangle + \langle Y, [A(X), A^t(Z)] \rangle + \langle Z, [A(X), A(Y)] \rangle = \langle X, [Y, Z] \rangle \quad \forall X, Y, Z \in \mathfrak{g} \tag{4.7}$$

which in turn is equivalent to

$$A([X, A^t(Y)]) + A([Y, A(X)]) + [A(X), A(Y)] = [X, Y] \quad \forall X, Y \in \mathfrak{g}. \tag{4.8}$$

This is the sort of equation studied and solved in [22]. In the case at hand we can simplify the condition by using that, because of the antisymmetry $A^t = -A$, A is an adjoint action with a suitable Lie algebra element and therefore obeys the Jacobi identity

$$A([X, Y]) = [A(X), Y] + [X, A(Y)]. \tag{4.9}$$

Then the condition (4.8) can be written as

$$[A(X), A(Y)] - A^2([X, Y]) = [X, Y] \quad \forall X, Y \in \mathfrak{g}. \tag{4.10}$$

Proceeding similarly with (2.29) but omitting the explicit statement of $\forall X, Y \in \mathfrak{g}$, we find, the following 'raw' terms, where we have not yet used the antisymmetry of both A and C.

From the $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id}$ term:

$$\begin{aligned} &A([X, A^t(Y)] + [Y, A(X)]) + [A(X), A(Y)] \\ &\qquad + \lambda(B([X, B^t(Y)] + [B^t(X), Y]) + [B^t(X), B^t(Y)]) = -\mu\lambda[X, Y]. \end{aligned} \tag{4.11}$$

From the $\theta \otimes \mathrm{id} \otimes \mathrm{id}$ term:

$$\begin{aligned} &A([Y, B(X)] + [X, B^t(Y)]) + B([X, A^t(Y)]) + [B(X), A(Y)] \\ &\qquad + \lambda(B([Y, C(X)]) + [B^t(Y), C(X)]) = -\lambda\nu[X, Y]. \end{aligned} \tag{4.12}$$

From the $\mathrm{id} \otimes \theta \otimes \mathrm{id}$ term:

$$\begin{aligned} B([Y, A(X)]) + [A(X), B(Y)] - A([X, B(Y)] + [Y, B^t(X)]) \\ + \lambda(B([X, C^t(Y)]) - [B^t(X), C(Y)]) = -\lambda\nu[X, Y]. \end{aligned} \quad (4.13)$$

From the $\mathrm{id} \otimes \mathrm{id} \otimes \theta$ term:

$$\begin{aligned} - B^t([X, A^t(Y)] + [Y, A(X)]) - [B^t(X), A(Y)]) - [A(X), B^t(Y)] \\ + \lambda(C([X, B^t(Y)]) - [Y, B^t(X)]) = -\lambda\nu[X, Y]. \end{aligned} \quad (4.14)$$

From the $\theta \otimes \theta \otimes \mathrm{id}$ term:

$$\begin{aligned} B([Y, B(X)] - [X, B(Y)]) + [B(X), B(Y)] + A([X, C^t(Y)] + [Y, C(X)]) \\ + \lambda[C(X), C(Y)]) = -\mu[X, Y]. \end{aligned} \quad (4.15)$$

From the $\theta \otimes \mathrm{id} \otimes \theta$ term:

$$\begin{aligned} - B^t([X, B^t(Y)] + [Y, B(X)]) - [B(X), B^t(Y)] + C([X, A^t(Y)] + [C(X), A(Y)]) \\ + \lambda C[Y, C(X)]) = -\mu[X, Y]. \end{aligned} \quad (4.16)$$

From the $\mathrm{id} \otimes \theta \otimes \theta$ term:

$$\begin{aligned} B^t([X, B(Y)]) + [Y, B^t(X)]) - [B^t(X), B(Y)] + C([Y, A(X)] + [A(X), C(Y)]) \\ + \lambda C[X, C^t(Y)]) = -\mu[X, Y]. \end{aligned} \quad (4.17)$$

Finally, from the $\theta \otimes \theta \otimes \theta$ term:

$$\begin{aligned} - B^t([X, C^t(Y)]) - B^t[Y, C(X)]) + [B(X), C(Y)] + [C(X), B(Y)] \\ + C([Y, B(X)] - [X, B(Y)]) = -\nu[X, Y]. \end{aligned} \quad (4.18)$$

We can simplify the equations (4.11) and (4.18) using the antisymmetry of A and C to find

$$[A(X), A(Y)] - A^2([X, Y]) + \lambda(B([X, B^t(Y)] + [B^t(X), Y]) + [B^t(X), B^t(Y)]) = -\mu\lambda[X, Y]. \quad (4.19)$$

and

$$B^t C([X, Y]) + [B(X), C(Y)] + [C(X), B(Y)] - C([B(X), Y] + [X, B(Y)]) = -\nu[X, Y]. \quad (4.20)$$

The equations (4.15)-(4.17) turn out to be mutually equivalent. The best way to see this is to contract again with a general vector Z and to use cyclic identities. Thus we can replace the three equations by the single equation obtained from (4.15) after using the antisymmetry of C and the Jacobi identity:

$$[B(X), B(Y)] - B([X, B(Y)] + [X, B(Y)]) - AC([X, Y]) + \lambda[C(X), C(Y)] = -\mu[X, Y]. \quad (4.21)$$

The equations (4.12)-(4.14) are also mutually equivalent. We can see this again by projecting onto Z, using cyclic identities and re-naming. Using also the antisymmetry of A and C, we obtain

$$B^tA([X,Y]) - [B^t(X), A(Y)]) - [A(X), B^t(Y)] + \lambda C([X, B^t(Y)] + [B^t(X), Y]) = -\lambda\nu[X,Y]. \tag{4.22}$$

We thus obtain a set of four coupled equations for the linear maps A, B, C. We combine them here for clarity:

$$\begin{aligned}
[A(X), A(Y)] - A^2([X,Y]) + \lambda(B([X, B^t(Y)] + [B^t(X), Y]) + [B^t(X), B^t(Y)]) &= -\mu\lambda[X,Y],\\
B^tC([X,Y]) + [B(X), C(Y)] + [C(X), B(Y)] - C([B(X), Y] + [X, B(Y)]) &= -\nu[X,Y],\\
[B(X), B(Y)] - B([X, B(Y)] + [X, B(Y)]) - AC([X,Y]) + \lambda[C(X), C(Y)] &= -\mu[X,Y],\\
B^tA([X,Y]) - [B^t(X), A(Y)]) - [A(X), B^t(Y)] + \lambda C([X, B^t(Y)] + [B^t(X), Y]) &= -\lambda\nu[X,Y].\\
\forall X, Y \in \mathfrak{g}
\end{aligned} \tag{4.23}$$

4.2 An equation involving three linear maps

Our main tool in this section is Lemma 3.1 of [22]. We recall it here for the convenience of the reader.

Lemma 4.1 *For every linear map $F : \mathfrak{g} \to \mathfrak{g}$, there is a uniquely determined linear map*

$$F^{\mathrm{adj}} : \mathfrak{g} \to \mathfrak{g}, \tag{4.24}$$

which satisfies

$$\langle F^{\mathrm{adj}}(Z), [X,Y]\rangle = \langle Z, [F(X), F(Y)]\rangle \quad \forall X, Y, Z \in \mathfrak{g}. \tag{4.25}$$

It is given by

$$F^{\mathrm{adj}} = F^2 - \mathrm{tr}(F)\,F + \frac{1}{2}\left(\mathrm{tr}(F)^2 - \mathrm{tr}(F^2)\right)\mathrm{id}, \tag{4.26}$$

which is the adjugate of F.

We need a corollary of this lemma, which is obtained by polarisation:

Lemma 4.2 *For linear maps $E, F : \mathfrak{g} \to \mathfrak{g}$, there is a uniquely determined linear map*

$$(E,F)^{\mathrm{adj}} : \mathfrak{g} \to \mathfrak{g}, \tag{4.27}$$

which satisfies

$$\langle (E,F)^{\mathrm{adj}}(Z), [X,Y]\rangle = \langle Z, [(E(X), F(Y)] + [F(X), E(Y)]\rangle \quad \forall X, Y, Z \in \mathfrak{g}. \tag{4.28}$$

It is given by

$$(E,F)^{\mathrm{adj}} = FE + EF - \mathrm{tr}(F)\,E - \mathrm{tr}(E)\,F + (\mathrm{tr}(F)\mathrm{tr}(E) - \mathrm{tr}(EF))\mathrm{id}. \tag{4.29}$$

It follows that

$$(\mathrm{id}, B)^{\mathrm{adj}} = -B + \mathrm{tr}(B)\,\mathrm{id} \tag{4.30}$$

Thus, taking adjugates of equation (4.19), we obtain, using that $\mathrm{tr}(A) = 0$,

$$\begin{aligned} -\frac{1}{2}\mathrm{tr}(A^2)\mathrm{id} + \lambda((-B^t + \mathrm{tr}(B)\,\mathrm{id})B^t \\ + (B^t)^2 - \mathrm{tr}(B)B^t + \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id} = -\mu\lambda\,\mathrm{id}, \end{aligned} \tag{4.31}$$

or

$$-\frac{1}{2}\mathrm{tr}(A^2)\mathrm{id} + \frac{\lambda}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id} = -\mu\lambda\,\mathrm{id}. \tag{4.32}$$

Similarly taking adjugates of equation (4.20) and using $\mathrm{tr}(C) = 0$, we find

$$-CB + (BC + CB - \mathrm{tr}(B)\,C - \mathrm{tr}(CB)\mathrm{id}) + (-B + \mathrm{tr}(B)\,\mathrm{id})C = -\nu\,\mathrm{id}, \tag{4.33}$$

or

$$-\mathrm{tr}(CB)\mathrm{id} = -\nu\mathrm{id}, \tag{4.34}$$

Proceeding with the equation (4.21) in a similar fashion gives

$$\begin{aligned} B^2 - \mathrm{tr}(B)\,B + \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id} - (-B + \mathrm{tr}(B)\,\mathrm{id})B^t - CA \\ + \lambda(C^2 - \frac{1}{2}\mathrm{tr}(C^2)\mathrm{id}) = -\mu\,\mathrm{id} \end{aligned} \tag{4.35}$$

or

$$(B - \mathrm{tr}(B))(B + B^t) + \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id} - CA + \lambda(C^2 - \frac{1}{2}\mathrm{tr}(C^2)\mathrm{id}) = -\mu\,\mathrm{id} \tag{4.36}$$

Finally, we obtain from equation (4.22) that

$$-AB - (AB^t + B^tA - \mathrm{tr}(B)\,A - \mathrm{tr}(AB^t)\mathrm{id}) - \lambda(-B^t + \mathrm{tr}(B)\,\mathrm{id})C = -\lambda\nu\,\mathrm{id} \tag{4.37}$$

or

$$-A(B + B^t) - (B^t - \mathrm{tr}(B))\,A - \mathrm{tr}(AB)\mathrm{id} + \lambda(B^tC - \mathrm{tr}(B)C) = -\lambda\nu\,\mathrm{id} \tag{4.38}$$

The conclusion from these calculations is one of the main results of this paper:

Theorem 4.3 *The r-matrix $r = r' + K_\tau$ with r' given in* (4.2) *is compatible with the Chern-Simons action* (2.16) *in the sense of definition 2.1 if the linear maps A, B and C satisfy the following coupled equations:*

$$\begin{aligned} \frac{1}{2}\mathrm{tr}(A^2) - \frac{\lambda}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right) &= \mu\lambda, \\ \mathrm{tr}(CB) &= \nu, \\ (B - \mathrm{tr}(B)\mathrm{id})(B + B^t) + \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id} - CA + \lambda(C^2 - \frac{1}{2}\mathrm{tr}(C^2)\mathrm{id}) &= -\mu\,\mathrm{id}, \\ -A(B + B^t) + (B^t - \mathrm{tr}(B)\mathrm{id})\,(\lambda C - A) - \mathrm{tr}(AB)\mathrm{id} &= -\lambda\nu\,\mathrm{id}. \end{aligned} \tag{4.39}$$

The coupled matrix equations (4.39) are non-linear but can be analysed with linear algebra of the sort used in [22] and [21]. We have not been able obtain a complete set of solutions. In the next section, we give two classes of solutions which contain some compatible r-matrices known in the literature, as well as new ones.

5 Two families of compatible r-matrices

5.1 Solutions for the case $A = \lambda C$

It is possible to determine a family of solutions by making the ansatz $A = \lambda C$. If $\lambda \neq 0$, the equations (4.39) reduce to

$$\begin{aligned} \frac{\lambda}{2}\mathrm{tr}(C^2) - \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right) &= \mu, & (5.1)\\ \mathrm{tr}(CB) &= \nu, & (5.2)\\ (B - \mathrm{tr}(B)\mathrm{id})(B + B^t) &= 0, & (5.3)\\ -\lambda C(B + B^t) &= 0. & (5.4) \end{aligned}$$

We can solve this system in terms of non-zero element $v \in \mathfrak{g}$, a further element $w \in \mathfrak{g}$ and real numbers x, y by parametrising (without loss of generality)

$$C = x[v, \cdot], \tag{5.5}$$

and making the further ansatz

$$B = y\, vv^t + [w, \cdot]. \tag{5.6}$$

Then $\mathrm{tr}(B) = y\langle v, v\rangle$ and

$$B + B^t = 2y\ vv^t \qquad B - \mathrm{tr}(B)\mathrm{id} = y(vv^t - \langle v, v\rangle \mathrm{id}) + [w, \cdot], \tag{5.7}$$

so that (5.3) and (5.4) are satisfied provided

$$y[v, w] = 0. \tag{5.8}$$

It follows that

$$B^2 = y^2\langle v, v\rangle vv^t + ww^t - \langle w, w\rangle \mathrm{id}, \tag{5.9}$$

Moreover, if $y \neq 0$, the condition (5.8) means

$$w = zv, \quad z \in \mathbb{R}. \tag{5.10}$$

Assuming this, one checks that

$$\frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right) = z^2\langle v, v\rangle. \tag{5.11}$$

so that (5.1) and (5.2) give us the two normalisation conditions

$$-(\lambda x^2 + z^2)\langle v, v\rangle = \mu, \qquad -2xz\langle v, v\rangle = \nu. \tag{5.12}$$

A particular solution in this case is the 'special' double solutions in [17]. In this case $\mu > 0$, $\lambda > 0$, the metric is necessarily Lorentzian and the solution is parametrised by a spacelike element $v \in \mathfrak{g}$ satisfying $\langle v, v\rangle = -1$. Comparing with equation 5.26 in [17], the parameter ρ used there is related to our parameters x, y, z via

$$x = \frac{1-\rho^2}{4\sqrt{\lambda}}, \qquad y = \frac{\rho}{2}, \qquad z = \frac{1+\rho^2}{4}, \tag{5.13}$$

so that

$$\lambda x^2 + z^2 = \frac{1+\rho^4}{8}, \qquad 2xz = \frac{1-\rho^4}{8\sqrt{\lambda}}. \tag{5.14}$$

From equation 5.25 in [17], our parameters μ, ν can be written in terms of the parameter ρ is as

$$\mu = \frac{1+\rho^4}{8}, \qquad \nu = \frac{1-\rho^4}{8\sqrt{\lambda}}, \tag{5.15}$$

so that, with $\langle v, v\rangle = -1$, our condition (5.12) is indeed satisfied.

To sum up, the most general solution with $\lambda \neq 0$, $A = \lambda C$ and B of the form (5.6) with $y \neq 0$ has an r-matrix with anti-symmetric part

$$r' = (yv^a v^b + zv^c \epsilon_{abc})(P^a \otimes J^b - J^b \otimes P^a) + xv^c \epsilon_{abc}(\lambda J^a \otimes J^b + P^a \otimes P^b), \tag{5.16}$$

and x, z and v satisfying (5.12). The real parameter y can take any value.

If $y = 0$, we need no longer require that $[v, w] = 0$ and then (5.16) is not the most general solution of the form (5.6). In this case, the family (5.16) coincides with the family of solutions obtained by complexification in Sect. 3. To make contact with the notation there, we expand

$$B_{ab} = -\epsilon_{abc} p^c, \qquad C_{ab} = -\epsilon_{abc} q^c, \qquad A_{ab} = \lambda C_{ab}. \tag{5.17}$$

Then

$$\mathrm{tr}(B) = 0, \quad \mathrm{tr}(B^2) = -2\boldsymbol{p}^2, \quad \mathrm{tr}(C^2) = -2\boldsymbol{q}^2, \quad \mathrm{tr}(BC) = -2\boldsymbol{p}\cdot\boldsymbol{q}, \tag{5.18}$$

and so (5.1) and (5.2) become the condition (3.19) derived via complexification.

Finally, we turn to the limiting case $\lambda = 0$. In that case, the ansatz $A = \lambda C$ means that $A = 0$, and the first and fourth equation in (4.39) are automatically satisfied. In order to retain the simplification in the third equation we need to impose

$$-\frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right) = \mu, \tag{5.19}$$

and solve (5.3). The matrix C can be chosen freely subject to the constraint (5.2). This reproduces the $\lambda = 0$ solutions discussed in Sect. 3, but also includes solutions of the form (5.16) with $y \neq 0$.

5.2 Solution for $\nu = 0$

It is important to classify solutions compatible with either the gravitational pairing (2.11) or the pairing (2.12). This case is characterised by $\alpha\beta = 0$ or $\nu = 0$. We obtain the complete family of solutions and highlight the new ones.

We first show that, if $\nu = 0$, and $\lambda \neq 0$, one can use the fourth equation in (4.39) to express B in terms of A and C. Inserting the ansatz

$$B = AC + x\,\mathrm{id} \tag{5.20}$$

into the fourth equation, and again writing the antisymmetric maps A and B in terms of Lie algebra elements u, v as

$$A = [u, \cdot], \qquad C = [v, \cdot], \tag{5.21}$$

we can use (4.5) to write

$$B = vu^t + (x - \langle u, v\rangle)\,\mathrm{id} \tag{5.22}$$

and hence to write the equations (4.39) in terms of u, v and x. In particular, one finds

$$\frac{1}{2}(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)) = (2x - \langle u, v\rangle)^2 - x^2. \tag{5.23}$$

Inserting(5.21) and (5.22) in the fourth equation of (4.39), and setting $\nu = 0$ leads to

$$-\langle u, w\rangle[u, v] + \langle u, v\rangle[u, w] - \langle v, [u, w]\rangle u + \lambda(\langle u, v\rangle - 2x)[v, w] = 0. \tag{5.24}$$

It follows from the identity

$$[w, [u, [u, v]]] = \langle u, v\rangle[w, u] - \langle u, u\rangle[w, v] = \langle w, [u, v]\rangle u - \langle w, u\rangle[u, v], \tag{5.25}$$

that

$$\langle u, v\rangle[w, u] + \langle w, u\rangle[u, v] + \langle w, [v, u]\rangle u = \langle u, u\rangle[w, v]. \tag{5.26}$$

Using this to simplify (5.24), one concludes that, for $\lambda \neq 0$, the fourth equation in (4.39) is equivalent to

$$x = \frac{1}{2}\left(\langle u, v\rangle + \frac{\langle u, u\rangle}{\lambda}\right). \tag{5.27}$$

Inserting (5.22) with this expression for x into the third equation in (4.39), yields

$$(\langle u, u, \rangle + \lambda)(vv^t - \frac{1}{\lambda}uv^t) - \frac{1}{4}\left(\langle u, v\rangle - \frac{\langle u, u\rangle}{\lambda}\right)^2 \mathrm{id} + \langle u, v\rangle\mathrm{id} = -\mu\,\mathrm{id}. \tag{5.28}$$

This is equivalent to

$$\langle u, u\rangle = -\lambda, \qquad (\langle u, v\rangle - 1)^2 = 4\mu. \tag{5.29}$$

The second equation in (4.39) is automatically satisfied for B of the form (5.20) since CAC is antisymmetric and so $\mathrm{tr}(CAC) = 0$. Finally, if $\lambda = -\langle u, u\rangle$ then $x = \frac{1}{2}(\langle u, v\rangle - 1)$ and so the first equation in (4.39) becomes

$$\lambda - \lambda(1 - x^2) = \lambda\mu, \tag{5.30}$$

which, for $\lambda \neq 0$, is equivalent to the second equation in (5.29).

The equation (5.29) has real solutions when $\mu \geq 0$. When $\lambda < 0$ there are solutions for both Euclidean and Lorentzian signatures, but when $\lambda > 0$ we require Lorentzian signature and a spacelike element u. Assuming these conditions are met, the only condition on C is that

$$\langle u, v\rangle = 1 \pm 2\sqrt{\mu}. \tag{5.31}$$

The map B then takes the form

$$B = vu^t - (1 \pm \sqrt{\mu})\,\mathrm{id}, \tag{5.32}$$

and the antisymmetric part of the r-matrix is

$$r' = \left(u^a v^b - (1 \pm \sqrt{\mu})\eta_{ab}\right)(P^a \otimes J^b - J^b \otimes P^a) + u^c\epsilon_{abc}J^a \otimes J^b + v^c\epsilon_{abc}P^a \otimes P^b, \tag{5.33}$$

with $\lambda \neq 0$ and $u, v \in \mathfrak{g}$ subject to the constraints (5.29).

6 Conclusion

The set of equations (4.39) determines the most general r-matrix which is compatible with the generalised Chern-Simons action for 3d gravity. In Table 1 we summarise the solutions found in this paper. Some of the solutions listed there, like the 'standard doubles' and 'generalised bicrossproducts' for some parameter values, have been known for some time, but the family of solutions obtained by generalised complexification and the family of solutions for $\nu = 0$ are new.

Type	Solution	Constraint
Standard doubles	$B = \pm\sqrt{\mu}\,\mathrm{id},\ A = [p, \cdot],\ C = 0$	$\mu \geq 0,\ \langle p, p\rangle = -\lambda$
Generalised bicrossproducts	$B = y\, vv^t + z[v, \cdot],$ $A = \lambda C,\ C = x\,[v, \cdot]$	$(\lambda x^2 + z^2)\langle v, v\rangle = -\mu,$ $2xz\langle v, v\rangle = -\nu$
Generalised complexifications	$A = \lambda C,\ B = [p, \cdot],\ C = [q, \cdot]$	$\langle p, p\rangle + \lambda\langle q, q\rangle = -\mu,$ $2\langle p, q\rangle = -\nu$
Solutions for $\nu = 0$	$A = [u, \cdot],\ B = [v, \cdot],$ $B = vu^t - (1 \pm \sqrt{\mu})\,\mathrm{id}$	$\mu \geq 0,\ \langle u, u\rangle = -\lambda,$ $(\langle u, v\rangle - 1)^2 = 4\mu$

Table 1: Types of compatible r-matrices discussed in this paper: the maps A, B, C parametrise r' via equation (4.2) so that $r' + K_\tau$, with K_τ given in (2.25) satisfies the classical Yang-Baxter equation (2.21). The names are not standard; note in particular that some of the 'generalised bicrossproducts' may also be viewed as doubles for certain parameter values, see the discussion in Sect. 5.1 and in [17].

It remains a challenge to determine all solutions systematically. Meeting this challenge would allow us to understand possible non-commutative geometries arising in the quantisation of (generalised) 3d gravity and the relation between them.

It would also be of interest in the context of gravitational scattering. When studying 3d gravity in a universe with compact spatial slices, one expects different r-matrices compatible with the same Chern-Simons action to give rise to equivalent quantum theories. However, when the spatial slices have boundaries, different r-matrices may encode different physics. In [31], for example, it was shown how quantum R-matrices determine scattering of massive particles in a universe with non-compact spatial slices. Corresponding results for other scattering processes, for example of massive particles with BTZ black holes in AdS_3, are not known, but would require quantum R-matrices corresponding the solutions of our set of equations (4.39).

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