The Filon construct for moving dislocations

Citation for published version:
Acharya, A & Knops, RJ 2018, 'The Filon construct for moving dislocations', *Journal of Engineering Mathematics*, vol. 109, no. 1, pp. 239-257. https://doi.org/10.1007/s10665-018-9951-0

Digital Object Identifier (DOI):
10.1007/s10665-018-9951-0

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Publisher's PDF, also known as Version of record

Published In:
Journal of Engineering Mathematics

J Eng Math (2018) 109:239–257
https://doi.org/10.1007/s10665-018-9951-0

The Filon construct for moving dislocations

A. Acharya · R. J. Knops

Received: 12 April 2017 / Accepted: 5 January 2018 / Published online: 31 January 2018

Abstract Filon's construct, originally developed for plane isotropic linear elastostatics, examines the difference between solutions to the same boundary value problem but for two different values of the elastic moduli, and relates the difference solution to one occurring in a corresponding stationary dislocation problem. This paper generalises the result to three-dimensional linear elastodynamics with particular reference to moving dislocations. Also studied is the inverse procedure which derives the pair of elastodynamical problems from a given distribution of dislocations. Body-forces and an auxiliary plastic distortion tensor are novel and essential features of the argument. Specialisation to the static and quasi-static theories is straightforward. The inverse Filon construct is illustrated by the stationary edge dislocation and uniformly moving screw dislocation in a homogeneous isotropic linear elastic whole space.

Keywords Linear elastodynamics · Moving dislocations · Variation of elastic moduli

1 Introduction

This paper continues the study of Filon's construct first announced at the Edinburgh meeting in 1921 of the British Association for the Advancement of Science. Filon [1] demonstrated for plane isotropic linear elasticity in a multiply connected plate that an analogy exists between dislocation theory and the difference between linear elastic solutions to the same boundary value problem but for two different values of Poisson's ratio. Muskhelishvili [2] and Boley and Weiner [3] established a similar relationship between dislocation and thermoelastic problems in two- and three-dimensions, respectively. The unifying concept depends upon the notion of residual or initial stress and ultimately upon the subcutaneous Riemann structure of linear elasticity.

The present aim is to explore the construct in two- and three-dimensional linear theories of anisotropic non-homogeneous elastostatics and elastodynamics. Certain aspects of the analogy in statics are discussed in [4,5]. Consequently, primary concern is with dynamic problems.

A. Acharya
Department of Civil and Environmental Engineering, Carnegie-Mellon University, Pittsburgh, PA 15213, USA

R. J. Knops (✉)
The Maxwell Institute of Mathematical Sciences and School of Mathematical and Computing Sciences, Heriot-Watt University, Edinburgh EH14 4AS, Scotland, UK
e-mail: R.J.Knops@hw.ac.uk

Requisite elements of plastic dislocation theory are presented in Sect. 2, while properties of linear elastodynamics are reviewed in Sect. 3. The difference between the solutions to the elastic initial boundary problem for two different sets of moduli and different body-forces, initial, and boundary data is shown in Sect. 4 to possess the same structure as the solution to the moving dislocation problem. A correspondence between the respective solutions is therefore possible which considerably extends Filon's original construct to arrays of moving dislocations in three-dimensional regions. The general process is described in Sect. 5. The inverse Filon construct, treated in Sect. 6, generates constituent elastodynamic solutions from a given dislocation problem. Complete generality requires the introduction of an auxiliary plastic distortion tensor and retention of body-forces. Section 7, for convenience, specialises the main formulae to isotropic elasticity. Filon's construct for equilibrium problems is illustrated in [4,5] by application to the stationary edge and screw dislocations, and these examples are not repeated here. Nor is Filon's construct applied to moving dislocations since the static treatment suggests the procedure is comparatively straightforward. Instead, attention is devoted to the inverse Filon construct which is considered in Sect. 8 for the stationary edge dislocation and uniformly moving screw dislocation both in a homogeneous isotropic linear elastic whole space. The edge dislocation leads to constituent plane elastic fields corresponding to a point force, while the moving screw dislocation has anti-plane shear motions as one possible constituent pair. A different pair is derived to demonstrate that such pairs are not unique. A few brief concluding remarks are contained in Sect. 9.

Vector and tensor quantities are represented by bold lower and upper case letters, respectively, although lower case bold letters are occasionally employed to also denote tensor quantities. The distinction should be obvious from the context. Both indicial and direct notations are used as convenient, with the standard conventions adopted of summation over repeated subscripts and superscripts, and a subscript comma to denote partial differentiation. Latin subscripts range over 1, 2, 3, while Greek indices assume the values 1, 2. The spatial argument of a function is abbreviated so that, for example, $\boldsymbol{h}(x_i)$ indicates $\boldsymbol{h}(x_1, x_2, x_3)$. In the direct notation, the gradient, divergence, and rotation operators as applied to vectors and tensors are denoted, respectively, by ∇, grad, Grad, div, Div, and $\nabla\times$, curl, Curl. Transposition is indicated by the superscript T, the identity tensor by $\boldsymbol{I}$, and the scalar and tensor products by their usual symbols. The indicial forms of these operations become

$$(\boldsymbol{u}\times\boldsymbol{v})_i = e_{ijk}u_j v_k, \qquad (\boldsymbol{u}\otimes\boldsymbol{v})_{ij} = u_i v_j, \tag{1.1}$$

$$(\mathrm{grad}\,\phi)_{\mathrm{i}} = \phi_{,i}, \qquad (\mathrm{curl}\,\boldsymbol{u})_{\mathrm{i}} = e_{ijk}u_{k,j}, \tag{1.2}$$

$$\mathrm{div}\,\boldsymbol{u} = u_{i,i}, \qquad (\boldsymbol{A}\times\boldsymbol{v})_{im} = e_{mjk}A_{ij}v_k, \tag{1.3}$$

$$(\boldsymbol{A}\times\boldsymbol{B})_i = e_{ijk}A_{jr}B_{rk}, \qquad (\mathrm{Grad}\,\boldsymbol{u})_{\mathrm{ij}} = u_{i,j}, \tag{1.4}$$

$$(\mathrm{Div}\boldsymbol{A})_{\mathrm{i}} = (\nabla\cdot\boldsymbol{A})_i = A_{ij,j}, \qquad (\mathrm{Curl}\boldsymbol{A})_{\mathrm{im}} = (\nabla\times\boldsymbol{A})_{im} = e_{ijk}A_{mk,j}, \tag{1.5}$$

where e_{ijk} denotes components of the alternating tensor.

We assume the existence of a solution suitable to our needs.

2 Moving and stationary dislocations

Elements of the theory of moving dislocations required subsequently are briefly recalled. A more complete account is presented, for example, in [6,7]. Specialisation recovers the stationary and quasi-static theories.

Consider an unbounded or bounded, simply or multiply connected region $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ that when bounded possesses a Lipschitz smooth boundary $\partial\Omega$ on which the outward unit vector normal is $\boldsymbol{n}$. The region Ω is occupied by an inhomogeneous anisotropic compressible linear elastic material subject to prescribed time-dependent external body-force, mixed boundary conditions, and Cauchy initial data. The body, however, is self-stressed due to an array of moving dislocations that can be approximated by a continuous distribution of dislocations of prescribed density specified by the second-order non-symmetric tensor $\boldsymbol{\alpha}(x_i, t)$, $(\boldsymbol{x}, t) \in \Omega\times[0, T)$. Here, and in what follows, x_i, $i = 1, 2, 3$, are the Cartesian components of the position vector $\boldsymbol{x}$, and $[0, T)$ is the maximal interval of existence in the initial boundary problem of moving dislocations. It is convenient to suppose that this is also the interval of existence for the elastic problems discussed in Sect. 3.

Let $\Sigma \subset \Omega$ be any (time-varying) open surface bounded by the closed curve $\partial\Sigma$ described in a right-handed sense. The Burgers vector $\boldsymbol{b}$ corresponding to the dislocation distribution is given by (c.p., [8,9])

$$\boldsymbol{b} = \int_{\Sigma} \boldsymbol{\alpha} \cdot \mathbf{d}\boldsymbol{S}, \tag{2.1}$$

where $\mathbf{d}\boldsymbol{S}$ denotes the vector surface area element. The sign convention is opposite to that usually adopted.

Kröner [9], amongst others, postulates the second-order non-symmetric *elastic distortion* tensor $\boldsymbol{U}^{(|E)}(x_i, t)$ as a second state variable and relates it to Burgers vector by

$$\boldsymbol{b} = \int_{\partial\Sigma} \boldsymbol{U}^{(E)} \cdot \mathbf{d}\boldsymbol{s} \tag{2.2}$$

$$= \int_{\Sigma} \nabla \times \boldsymbol{U}^{(E)} \cdot \mathbf{d}\boldsymbol{S}, \tag{2.3}$$

where Stokes' theorem is employed to derive the last equation, and $\mathbf{d}\boldsymbol{s}$ denotes the curvilinear vector line element of $\partial\Sigma$. Elimination of $\boldsymbol{b}$ between (2.1) and (2.3) yields the fundamental field equation

$$\boldsymbol{\alpha}(x_i, t) = \nabla \times \boldsymbol{U}^{(\boldsymbol{E})}(x_i, t), \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{2.4}$$

from which is deduced the condition

$$\mathrm{Div}\,\boldsymbol{\alpha} = \nabla \cdot \boldsymbol{\alpha} = 0, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, \mathrm{T}). \tag{2.5}$$

A *compatible* (non-symmetric) continuously differentiable tensor field $\boldsymbol{U}(x_i)$ is defined to satisfy

$$\nabla \times \boldsymbol{U} = 0, \qquad \boldsymbol{x} \in \Omega. \tag{2.6}$$

When $\boldsymbol{U}$ is not compatible, the tensor is *incompatible*. Poincaré's lemma states that when Ω is simply connected, a necessary and sufficient condition for compatibility is the existence of a twice continuously differentiable vector field $\boldsymbol{v}(x_i)$ such that

$$\boldsymbol{U}(x_i) = \nabla \boldsymbol{v}(x_i), \qquad \boldsymbol{x} \in \Omega. \tag{2.7}$$

A different compatibility condition for a symmetric second-order tensor is stated in (2.17).

For non-vanishing dislocation density $\boldsymbol{\alpha}$, (2.4) implies that $\boldsymbol{U}^{(E)}$ is an incompatible tensor. Nevertheless, the stress $\boldsymbol{\sigma}(x_i, t)$ in Ω is linearly related to $\boldsymbol{U}^{(E)}(x_i, t)$ by

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{U}^{(E)}, \tag{2.8}$$

where $\boldsymbol{C}$ denotes the fourth-order elastic moduli tensor whose Cartesian components possess the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}, \tag{2.9}$$

which imply the fourth symmetry $C_{ijkl} = C_{ijlk}$. We also suppose that the tensor $\boldsymbol{C}$ is convex in the sense that there exists a constant C_0 such that

$$C_0 \psi_{ij} \psi_{ij} \le C_{ijkl} \psi_{ij} \psi_{kl}, \tag{2.10}$$

for all symmetric second-order tensors $\boldsymbol{\psi} = \boldsymbol{\psi}^{\mathrm{T}}$.

The symmetries (2.9) enable the constitutive relation (2.8) to be rewritten as

$$\boldsymbol{\sigma}(x_i, t) = \boldsymbol{C}\,\boldsymbol{E}^{(E)}(x_i, t), \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{2.11}$$

where

$$\boldsymbol{E}^{(E)} = \frac{1}{2}\left(\boldsymbol{U}^{(E)} + \left(\boldsymbol{U}^{(E)}\right)^{\mathrm{T}}\right), \tag{2.12}$$

$$\boldsymbol{W}^{(E)} = \frac{1}{2}\left(\boldsymbol{U}^{(E)} - \left(\boldsymbol{U}^{(E)}\right)^{\mathrm{T}}\right), \tag{2.13}$$

so that

$$\boldsymbol{U}^{(E)} = \boldsymbol{E}^{(E)} + \boldsymbol{W}^{(E)}. \tag{2.14}$$

A similar notation is adopted for other second-order non-symmetric tensors.

The time evolution of the dislocation density together with time-varying boundary conditions causes the body to change shape with time. The change is measured by the compatible *total displacement* $\boldsymbol{u}(x_i, t)$, in terms of which the second-order incompatible *plastic distortion* tensor $\boldsymbol{U}^{(P)}(x_i, t)$ is defined to be

$$\boldsymbol{U}^{(P)} = \nabla \boldsymbol{u} - \boldsymbol{U}^{(E)}. \tag{2.15}$$

The strain tensor $\boldsymbol{e}$ corresponding to $\boldsymbol{u}$ is given by the symmetric expression

$$\boldsymbol{e} = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right), \tag{2.16}$$

which for a spatially continuously differentiable displacement field $\boldsymbol{u}(x_i, t)$ implies

$$\nabla \times \nabla \times \boldsymbol{e} = 0. \tag{2.17}$$

Conversely, for a given symmetric second-order tensor field $\boldsymbol{e}(x_i, t)$, there exists a single-valued continuously differentiable vector field $\boldsymbol{u}(x_i, t)$ satisfying (2.16) if the region Ω is simply connected and the symmetric tensor $\boldsymbol{e}$ satisfies the Saint-Venant compatibility relations (2.17)

Note that the representations (2.7) and (2.16) are associated with different compatibility conditions (2.6) and (2.17), respectively.

It follows from (2.4) that $\boldsymbol{U}^{(E)}$ is incompatible and indeed (2.4) and (2.15) yield

$$\begin{aligned}
\boldsymbol{\alpha}(x_i, t) &= \nabla \times \boldsymbol{U}^{(E)}(x_i, t) && (2.18)\\
&= \nabla \times \left(\nabla \boldsymbol{u}(x_i, t) - \boldsymbol{U}^{(P)}(x_i, t)\right) && (2.19)\\
&= -\nabla \times \boldsymbol{U}^{(P)}(x_i, t), \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), && (2.20)\\
&= -\nabla \times \boldsymbol{E}^{(P)} - \nabla \times \boldsymbol{W}^{(P)}, && (2.21)
\end{aligned}$$

from which we conclude that $\boldsymbol{U}^{(P)}$ is incompatible. Moreover, (2.18) and (2.20) imply that

$$\nabla \times \nabla \times \boldsymbol{U}^{(P)} = -\nabla \times \nabla \times \nabla \boldsymbol{U}^{(E)} = -\nabla \times \boldsymbol{\alpha} \neq 0. \tag{2.22}$$

Elimination of the antisymmetric tensor $\boldsymbol{W}^{(P)}$ from (2.21) leads to the relation

$$\nabla \times \boldsymbol{\alpha} + (\nabla \times \boldsymbol{\alpha})^{\mathrm{T}} = -2\nabla \times \nabla \times \boldsymbol{E}^{(P)} \equiv 2\boldsymbol{\eta}(x_i, t), \tag{2.23}$$

which may be solved for $\boldsymbol{E}^{(P)}$ in terms of $\boldsymbol{\eta}$ (Eshelby [10]). Furthermore, the relation (2.23) implies that $\boldsymbol{\eta}$ vanishes with $\boldsymbol{\alpha}$ but the converse is false. The symmetric part $\boldsymbol{E}^{(P)}$ of $\boldsymbol{U}^{(P)}$ may vanish or be the gradient of some differentiable vector field, while the non-symmetric component $\boldsymbol{W}^{(P)}$ satisfies $\nabla \times \boldsymbol{W}^{(P)} \neq 0$. In consequence, it may be concluded from (2.21) that $\boldsymbol{\alpha}$ does not vanish with $\boldsymbol{\eta}$.

In the quasi-static theory, inertial effects due to the total displacement are discarded, but in the dynamic theory, the corresponding equations of motion are given by

$$\mathrm{Div}\,\boldsymbol{\sigma} + \rho\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, \mathrm{T}), \tag{2.24}$$

where $\rho(x_i)$ represents the mass density of the body, and $\boldsymbol{f}(x_i, t)$ is the body-force per unit mass. The importance of including body-forces will become apparent later.

The inertial term necessitates initial Cauchy data for the total displacement vector $\boldsymbol{u}(x_i, t)$ in terms of which the dislocation initial boundary value problem is given by

$$\boldsymbol{\alpha} = -\nabla \times \boldsymbol{U}^{(P)}, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{2.25}$$

$$\boldsymbol{\sigma}(x_i, t) = \boldsymbol{C}\left(\boldsymbol{e} - \boldsymbol{E}^{(P)}\right), \quad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{2.26}$$

$$\mathrm{Div}\,\boldsymbol{C}\left(\boldsymbol{e} - \boldsymbol{E}^{(\mathrm{P})}\right) + \rho\boldsymbol{f} = \rho\ddot{\boldsymbol{u}}, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{2.27}$$

to which are adjoined initial conditions,

$$\boldsymbol{u}(x_i, 0) = \boldsymbol{u}_0(x_i), \qquad \dot{\boldsymbol{u}}(x_i, 0) = \boldsymbol{u}_1(x_i), \qquad \boldsymbol{x} \in \Omega, \tag{2.28}$$

displacement boundary conditions,

$$\boldsymbol{u}(x_i, t) = \boldsymbol{h}(x_i, t), \qquad (\boldsymbol{x}, t) \in \partial\Omega_1 \times [0, T), \tag{2.29}$$

and traction boundary conditions

$$\boldsymbol{C}\left(\boldsymbol{e} - \boldsymbol{E}^{(P)}\right) \cdot \boldsymbol{n} = \boldsymbol{g}(x_i, t), \qquad (\boldsymbol{x}, t) \in \partial\Omega_2 \times [0, T), \tag{2.30}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and $\boldsymbol{h}(x_i, t)$, $\boldsymbol{g}(x_i, t)$, $\boldsymbol{u}_0(x_i)$, $\boldsymbol{u}_1(x_i)$ are prescribed functions on their respective domains of definition.

Detailed discussion of the uniqueness of the solution to this initial boundary value problem is presented in [11], which also considers the relation between the total displacement gradient plasticity theory just formulated and the Volterra theory of dislocations (see [11, §3]).

We repeat that the corresponding quasi-static problem suppresses the inertial term in (2.27), while for the corresponding equilibrium boundary value problem all terms are independent of time, and initial conditions (2.28) are irrelevant.

3 The three-dimensional elastic boundary and initial boundary value problems on multiply connected regions

We consider a non-homogeneous anisotropic compressible linear elastic body that occupies a bounded or unbounded region $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ which may be simply or multiply connected. When bounded, Ω possesses the smooth boundary $\partial\Omega$ with unit outward vector normal $\boldsymbol{n}$.

It is well known that multiply connected bodies can be self-stressed (see, e.g., [12–14]) and the standard Kirchhoff and Neumann uniqueness theorems of linear elasticity no longer apply.

Two different sets $\boldsymbol{C}^{(\gamma)}$, $\gamma = 1, 2$, of the elastic modulus tensor are considered whose components with respect to a Cartesian system of orthogonal coordinates with origin in Ω satisfy the same symmetries specified in (2.9); that is,

$$C^{(\gamma)}_{ijkl}(x_i) = C^{(\gamma)}_{klij}(x_i) = C^{(\gamma)}_{jikl}(x_i), \qquad \boldsymbol{x} \in \Omega, \quad \gamma = 1, 2. \tag{3.1}$$

It is further supposed that the respective elastic modulus tensors are positive-definite in the sense that for all non-zero second- order symmetric tensors $\boldsymbol{\psi}$, the following inequality holds for $\gamma = 1, 2$:

$$0 < C^{(\gamma)}_{ijkl}(x_i)\psi_{ij}\psi_{kl}, \qquad \boldsymbol{x} \in \Omega, \qquad \forall\, \psi_{ij} = \psi_{ji}. \tag{3.2}$$

When the elastic modulus tensors $\boldsymbol{C}^{(\gamma)}(x_i)$ are continuously differentiable on Ω, the positive-definite condition (3.2) by continuity implies a convexity condition similar to (2.10).

The elastic compliance tensor $\boldsymbol{C}^{(-1)(\gamma)}$ is the inverse of the elastic modulus tensor and consequently for each γ the components satisfy the relation

$$C^{(\gamma)}_{ijpq}C^{(-1)(\gamma)}_{pqkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \equiv I_{ijkl}, \tag{3.3}$$

where there is no sum over γ, I_{ijkl} are the Cartesian components of the fourth-order unit tensor $\boldsymbol{I}$ and δ_{ij} denotes the Kronecker delta function. The elastic compliance tensor enjoys the same minor symmetries as the elastic modulus tensor.

For each γ, components of the elastic modulus and compliance tensors for an isotropic elastic material, respectively, become

$$C^{(\gamma)}_{ijkl} = \lambda^{(\gamma)}\delta_{ij}\delta_{kl} + \mu^{(\gamma)}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{3.4}$$

$$C^{(-1)(\gamma)}_{ijkl} = \frac{1}{4\mu^{(\gamma)}}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{\lambda^{(\gamma)}}{2\mu^{(\gamma)}\left(3\lambda^{(\gamma)} + 2\mu^{(\gamma)}\right)}\delta_{ij}\delta_{kl}, \tag{3.5}$$

where $\lambda^{(\gamma)}$, $\mu^{(\gamma)}$ are the Lamé moduli related to Poisson's ratio $\nu^{(\gamma)}$ by

$$\lambda^{(\gamma)} = \frac{2\mu^{(\gamma)}\nu^{(\gamma)}}{\left(1 - 2\nu^{(\gamma)}\right)}. \tag{3.6}$$

In plane strain linear isotropic elasticity, the corresponding expressions are

$$C^{(\gamma)}_{\alpha\beta\kappa\delta} = \lambda^{(\gamma)}\delta_{\alpha\beta}\delta_{\kappa\delta} + \mu^{(\gamma)}(\delta_{\alpha\kappa}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\kappa}), \tag{3.7}$$

$$C^{(-1)(\gamma)}_{\alpha\beta\kappa\delta} = \frac{1}{4\mu^{(\gamma)}}(\delta_{\alpha\kappa}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\kappa}) - \frac{\lambda^{(\gamma)}}{4\mu^{(\gamma)}(\lambda^{(\gamma)} + \mu^{(\gamma)})}\delta_{\alpha\beta}\delta_{\kappa\delta} \tag{3.8}$$

$$= \frac{1}{4\mu^{(\gamma)}}(\delta_{\alpha\kappa}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\kappa}) - \frac{\nu^{(\gamma)}}{2\mu^{(\gamma)}}\delta_{\alpha\beta}\delta_{\kappa\delta}. \tag{3.9}$$

Cartesian components of the symmetric linear strain tensor $\boldsymbol{e}^{(\gamma)}$ are derived as usual from the differentiable displacement vector field $\boldsymbol{u}^{(\gamma)}$ according to

$$e^{(\gamma)}_{ij} = \frac{1}{2}\left(u^{(\gamma)}_{i,j} + u^{(\gamma)}_{j,i}\right), \tag{3.10}$$

and as described in Sect. 2, are compatible in the sense that for $\boldsymbol{u}^{(\gamma)} \in C^3(\Omega, \mathbb{R}^3)$ they satisfy the Saint–Venant compatibility relations analogous to (2.17):

$$\nabla \times \nabla \times \boldsymbol{e}^{(\gamma)} = 0, \tag{3.11}$$

or equivalently

$$e_{irs}e_{jpk}e^{(\gamma)}_{ks,pr} = 0, \tag{3.12}$$

where, as already stated, e_{ijk} are the components of the usual alternating tensor. We note that the compatibility condition (3.11), necessary for the existence of a continuously differentiable single-valued displacement vector field, is also sufficient provided that the region Ω is simply connected. (See, for example, Gurtin [15, §.14.2].) It should be remarked, however, that subsequent discussion assumes certain solutions contain point singularities for which (3.12) is satisfied only in regions external to the neighbourhood of the singularity.

Introduction of suitable cuts converts multiply connected regions into simply connected regions. Assume that (3.12) holds so that the strains $\boldsymbol{e}^{(\gamma)}$ are compatible on the cut region $\widehat{\Omega}$. Let $\boldsymbol{u}^{(\gamma)}$ be the corresponding continuously differentiable displacement, and let $\mathcal{C}$ be any continuous curve in $\widehat{\Omega}$ connecting the points with position vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}$. The Volterra–Cesaro integration procedure (see, for example, [16] [15, §14]) then gives

$$\boldsymbol{u}^{(\gamma)}(x_i, t) = \boldsymbol{u}^{(\gamma)}(x_{0\,i},\ t) - \int_{\mathcal{C}} \boldsymbol{V}^{(\gamma)} \cdot \mathrm{d}\boldsymbol{x}, \tag{3.13}$$

where

$$\boldsymbol{V}^{(\gamma)}(x_i, t) = \boldsymbol{e}^{(\gamma)}(x_i, t) + (\boldsymbol{x} - \boldsymbol{x}_0) \times \left(\nabla \times \boldsymbol{e}^{(\gamma)}(x_i, t)\right), \tag{3.14}$$

which is independent of $\mathcal{C}$ provided a cut is not traversed.

When the endpoints $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are on contiguous sides of a cut, the last expression determines the jump in $\boldsymbol{u}^{(\gamma)}$ across the cut as $\boldsymbol{x} \to \boldsymbol{x}_0$, and consequently the corresponding Burgers vector.

Let $\boldsymbol{\sigma}^{(\gamma)}$ be the stress tensor which for $\gamma = 1, 2$ is related to the strain $\boldsymbol{e}^{(\gamma)}$ by the constitutive assumption

$$\boldsymbol{\sigma}^{(\gamma)} = \boldsymbol{C}^{(\gamma)}\boldsymbol{e}^{(\gamma)}, \qquad \boldsymbol{x} \in \Omega. \tag{3.15}$$

The equations of motion, boundary, and initial conditions satisfied by the respective elastodynamic fields are

$$\operatorname{Div} \boldsymbol{\sigma}^{(\gamma)} + \rho \boldsymbol{f}^{(\gamma)} = \rho \ddot{\boldsymbol{u}}^{(\gamma)}, \qquad (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{3.16}$$

$$\boldsymbol{u}^{(\gamma)} = \boldsymbol{h}^{(\gamma)}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_1 \times [0, T), \tag{3.17}$$

$$\boldsymbol{\sigma}^{(\gamma)} \cdot \boldsymbol{n} = \boldsymbol{g}^{(\gamma)}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_2 \times [0, T), \tag{3.18}$$

$$\boldsymbol{u}^{(\gamma)}(x_i, 0) = \boldsymbol{u}_0^{(\gamma)}(x_i), \qquad \dot{\boldsymbol{u}}^{(\gamma)}(x_i, 0) = \boldsymbol{u}_1^{(\gamma)}(x_i), \qquad \boldsymbol{x} \in \Omega, \tag{3.19}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, and the mass density $\rho(x_i)$ (assumed the same for both sets of elastic moduli), body-force vector $\boldsymbol{f}^{(\gamma)}(x_i, t)$ per unit mass, surface traction $\boldsymbol{g}^{(\gamma)}(x_i, t)$, surface displacement vector $\boldsymbol{h}^{(\gamma)}(x_i, t)$, and initial data $\boldsymbol{u}_0^{(\gamma)}(x_i)$, $\boldsymbol{u}_1^{(\gamma)}(x_i)$ are prescribed. Of later separate interest are problems for which the quantities $\boldsymbol{h}^{(\gamma)}$and $\boldsymbol{g}^{(\gamma)}$ are, respectively, the same, e.g., $\boldsymbol{h}^{(1)} = \boldsymbol{h}^{(2)}$. The interval of existence of a solution is given by $[0, T)$, where T is either finite or infinite. As already indicated, this interval is assumed to coincide with that for the dislocation problem.

The elastostatic and quasi-static boundary value problems may be derived by taking, for example, the inertial term to be zero in (3.16) and ignoring the initial values (3.19).

4 Variation of elastic moduli

Michell [12] (see also Gurtin [15, §§46, 47]) has proved that stresses in the traction boundary value problem of plane isotropic homogeneous linear elasticity are independent of the elastic moduli provided body-forces vanish and the region is simply connected. The result also holds for multiply connected regions when the resultant force, but not necessarily the couple, vanishes over each separate boundary. Earlier discussion is given by Maxwell [17, p. 201], and Lévy [18], while subsequent consideration is due to Timpe [19], who remarked that contravention of the result is to be expected in plane multiply connected regions since these can be self-stressed by notional cutting and welding operations. The multi-valued displacement, stress, and strain depend upon the elastic moduli. Volterra developed a general theory of such multi-valued displacement which is fundamental to his theory of dislocations.

In other two- and three-dimensional boundary value problems of linear elastostatics, and initial boundary value problems of linear elastodynamics the stress, strain, and displacement in general depend upon the elastic moduli in both simply and multiply connected regions. Conditions for independence in isotropic elasticity are discussed by several authors including Carlson [20–22], Dundurs [23], Knops [5], and Markenscoff [24]. In particular, Carlson [22] notes that for elastodynamics there are only a few problems for which it can be expected that the solution is independent of elastic moduli. Continuous dependence is explored by Bramble and Payne [25], amongst others.

We derive equations for the differences between the quantities introduced in Sect. 3 when the moduli are varied, but when the mass density remains unaltered. Accordingly, we define the *difference* fields to be

$$\boldsymbol{u} = \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}, \qquad (\boldsymbol{x}, t) \in \bar{\Omega} \times [0, T], \tag{4.1}$$

$$\boldsymbol{e} = \boldsymbol{e}^{(1)} - \boldsymbol{e}^{(2)}, \qquad (\boldsymbol{x}, t) \in \bar{\Omega} \times [0, T], \tag{4.2}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}, \qquad (\boldsymbol{x}, t) \in \bar{\Omega} \times [0, T], \tag{4.3}$$

$$\boldsymbol{f} = \boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}, \qquad (\boldsymbol{x}, t) \in \bar{\Omega} \times [0, T], \tag{4.4}$$
$$\boldsymbol{h} = \boldsymbol{h}^{(1)} - \boldsymbol{h}^{(2)}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_1 \times [0, T), \tag{4.5}$$
$$\boldsymbol{g} = \boldsymbol{g}^{(1)} - \boldsymbol{g}^{(2)}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_2 \times [0, T), \tag{4.6}$$
$$\boldsymbol{u}_0 = \boldsymbol{u}_0^{(1)} - \boldsymbol{u}_0^{(2)}, \qquad \boldsymbol{u}_1 = \boldsymbol{u}_1^{(1} - \boldsymbol{u}_1^{(2)}, \qquad \boldsymbol{x} \in \bar{\Omega}, \tag{4.7}$$

where $\bar{\Omega}$ denotes the closure of Ω.

Subtraction of the constitutive relations (3.15) leads to

$$\boldsymbol{\sigma} = \boldsymbol{C}^{(1)} \left\{ \boldsymbol{e} + (\boldsymbol{I} - \boldsymbol{D})\boldsymbol{e}^{(2)} \right\}, \tag{4.8}$$

where we recall that $\boldsymbol{I}$ is the identity tensor, whose Cartesian components are here repeated for convenience:

$$I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{4.9}$$

the tensor $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \boldsymbol{C}^{(-1)(1)}\boldsymbol{C}^{(2)}, \tag{4.10}$$

and $\boldsymbol{C}^{(-1)(1)}$ is the corresponding elastic compliance tensor satisfying (3.3). Definition (4.10) implies that $\boldsymbol{D}$ possesses minor symmetries so that its Cartesian components satisfy

$$D_{ijkl} = D_{jikl} = D_{ijlk}. \tag{4.11}$$

Subtraction of the respective elastodynamics equations, initial, and boundary conditions (3.16)–(3.19), together with the constitutive relations (4.8) shows that the difference fields satisfy the initial boundary value problem

$$\rho\ddot{\boldsymbol{u}} = \mathrm{Div}\, \boldsymbol{C}^{(1)} \left[\boldsymbol{e} + (\boldsymbol{I} - \boldsymbol{D})\, \boldsymbol{e}^{(2)} \right] + \rho \boldsymbol{f}, \quad (\boldsymbol{x}, t) \in \Omega \times [0, \mathrm{T}), \tag{4.12}$$
$$\boldsymbol{u} = \boldsymbol{h}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_1 \times [0, T), \tag{4.13}$$
$$\boldsymbol{C}^{(1)}\boldsymbol{e} \cdot \boldsymbol{n} = \boldsymbol{C}^{(1)}\,(\boldsymbol{D} - \boldsymbol{I})\, \boldsymbol{e}^{(2)} \cdot \boldsymbol{n} + \boldsymbol{g}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_2 \times [0, T), \tag{4.14}$$
$$\boldsymbol{u} = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}} = \boldsymbol{u}_1, \qquad (\boldsymbol{x}, t) \in \Omega \times \{0\}. \tag{4.15}$$

This system corresponds to the standard initial (mixed) boundary value problem with non-zero body-force, non-zero initial data, and non-zero surface traction and displacement.

The "strain" $\boldsymbol{De}^{(2)}$ is incompatible in the sense that in general

$$\nabla \times \nabla \times \boldsymbol{De}^{(2)} \neq 0. \tag{4.16}$$

5 The Filon construct

The dislocation initial boundary value problem specified by (2.26)–(2.30) becomes identical to the variation system (4.12)–(4.15) on setting

$$\boldsymbol{C} = \boldsymbol{C}^{(1)}, \tag{5.1}$$
$$\boldsymbol{E}^{(P)} = -\,(\boldsymbol{I} - \boldsymbol{D})\, \boldsymbol{e}^{(2)}, \tag{5.2}$$

and letting $\boldsymbol{u}$, $\boldsymbol{e}$, $\boldsymbol{f}$, $\boldsymbol{h}$, $\boldsymbol{g}$ retain their respective definitions assigned in Sects. 2 and 4.

The analogy may be used to derive solutions to a dislocation problem from known constituent elastodynamics solutions defined in Sect. 4. Their difference uniquely specifies the total displacement $\boldsymbol{u}$, the stress $\boldsymbol{\sigma}$, and therefore the symmetric part $\boldsymbol{E}^{(E)}$ of the elastic distortion $\boldsymbol{U}^{(E)}$ for the dislocation problem in which the symmetric part of the plastic distortion $\boldsymbol{U}^{(P)}$ is prescribed by (5.2)

The dislocation solution, however, when derived from the construct is restricted. Only the symmetric part of the plastic distortion tensor $\boldsymbol{U}^{(P)}$ is determined by (5.2), and moreover must satisfy the condition

$$\nabla \times \nabla \times (\boldsymbol{I}-\boldsymbol{D})^{-1}\boldsymbol{E}^{(P)} = 0, \tag{5.3}$$

since the strain $\boldsymbol{e}^{(2)}$ is supposed compatible in the sense that the Saint-Venant relation (2.17) is satisfied. Relation (5.2) allows condition (5.3) to be alternatively expressed as

$$\nabla \times \nabla \times \boldsymbol{E}^{(P)} = \nabla \times \nabla \times \boldsymbol{D}\boldsymbol{e}^{(2)}. \tag{5.4}$$

Restrictions on $\boldsymbol{E}^{(P)}$ may be partially relaxed by introduction of an auxiliary second-order symmetric tensor field $\boldsymbol{E}^{*}(x_i, t)$, which in general is not derivable from a displacement vector field. Therefore, $\boldsymbol{E}^{*}$ is incompatible and (2.17) is not satisfied. For each time instant t, we take $\boldsymbol{E}^{*}$ to be a particular solution to the boundary value problem

$$\mathrm{Div}\, \boldsymbol{C}^{(1)}\boldsymbol{E}^{*} = \rho\left(\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)} - \boldsymbol{f}\right), \qquad \boldsymbol{x} \in \Omega, \tag{5.5}$$

$$\boldsymbol{C}^{(1)}\boldsymbol{E}^{*}\cdot\boldsymbol{n} = \boldsymbol{g}^{(2)} - \boldsymbol{g}^{(1)} - \boldsymbol{g}, \qquad \boldsymbol{x} \in \partial\Omega_2, \tag{5.6}$$

where, for the moment, the vector functions $\boldsymbol{f}$, $\boldsymbol{g}$ are arbitrary. Alternatively, the tensor $\boldsymbol{E}^{*}$ can be arbitrarily chosen but still incompatible with $\boldsymbol{f}$ and $\boldsymbol{g}$ determined from (5.5) and (5.6).

When $\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)} = \boldsymbol{f}$, $\boldsymbol{g}^{(2)} - \boldsymbol{g}^{(1)} = \boldsymbol{g}$, and $\partial\Omega_1 = \emptyset$, a particular solution is the trivial solution $\boldsymbol{E}^{*} \equiv 0$.

Once $\boldsymbol{E}^{*}$ is determined, the symmetric part of the plastic distortion is no longer derived from relation (5.2), but instead is obtained from the formula

$$\boldsymbol{E}^{(P)} + \boldsymbol{E}^{*} = -\,(\boldsymbol{I}-\boldsymbol{D})\,\boldsymbol{e}^{(2)}. \tag{5.7}$$

The Filon construct by virtue of (5.5) and (5.6) then leads to the equations of motion for the dislocation problem. We have

$$\begin{aligned}
\rho\ddot{\boldsymbol{u}} &= \rho\ddot{\boldsymbol{u}}^{(1)} - \rho\ddot{\boldsymbol{u}}^{(2)} && (5.8)\\
&= \mathrm{Div}\,\boldsymbol{C}^{(1)}\left(\boldsymbol{e} + (\boldsymbol{I}-\boldsymbol{D})\boldsymbol{e}^{(2)}\right) + \rho\left(\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}\right) && (5.9)\\
&= \mathrm{Div}\,\boldsymbol{C}^{(1)}\boldsymbol{e} - \mathrm{Div}\,\boldsymbol{C}^{(1)}\left(\boldsymbol{E}^{(\mathrm{P})} + \boldsymbol{E}^{*}\right) + \rho\left(\boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}\right) && (5.10)\\
&= \mathrm{Div}\,\boldsymbol{C}^{(1)}\left(\boldsymbol{e} - \boldsymbol{E}^{(\mathrm{P})}\right) + \rho\boldsymbol{f}, && (5.11)
\end{aligned}$$

subject to the boundary and initial conditions

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{h}^{(1)} - \boldsymbol{h}^{(2)} = \boldsymbol{h}(x_i), \qquad (\boldsymbol{x}, t) \in \partial\Omega_1 \times [0, T), && (5.12)\\
\boldsymbol{C}^{(1)}\boldsymbol{E}^{(P)}\cdot\boldsymbol{n} &= \boldsymbol{C}^{(1)}\boldsymbol{e}\cdot\boldsymbol{n} + \boldsymbol{g}, \qquad (\boldsymbol{x}, t) \in \partial\Omega_2, && (5.13)\\
\boldsymbol{u}(x_i, 0) &= \boldsymbol{u}_0^{(1)}(x_i) - \boldsymbol{u}_0^{(2)}(x_i), \qquad \boldsymbol{x} \in \Omega, && (5.14)\\
\boldsymbol{u}_1(x_i, 0) &= \boldsymbol{u}_1^{(1)}(x_i) - \boldsymbol{u}_1^{(2)}(x_i), \qquad x \in \Omega, && (5.15)
\end{aligned}$$

where $\boldsymbol{f}, \boldsymbol{g}$ are the arbitrary vector functions appearing in (5.5) and (5.6)

The total displacement $\boldsymbol{u} = \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$ is the solution to the dislocation initial boundary value problem (5.11)–(5.15) for which the plastic distortion strain tensor $\boldsymbol{E}^{(P)}$ is determined by (5.7) subject to body-force $\boldsymbol{f}$ and traction boundary conditions (5.13). The elastic distortion strain tensor $\boldsymbol{E}^{(E)}$ remains given by $\nabla\boldsymbol{u} - \boldsymbol{E}^{(P)}$, but the dislocation stress $\boldsymbol{\sigma} = \boldsymbol{C}^{(1)}\boldsymbol{E}^{(E)} = \boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} + \boldsymbol{C}^{(1)}\boldsymbol{E}^{*}$ is no longer the difference stress $\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}$.

These calculations demonstrate the importance of body- and surface forces upon relaxation of the restriction on $\boldsymbol{E}^{(P)}$ imposed by condition (5.2). In the absence of $\boldsymbol{E}^{*}$, the body-force $\boldsymbol{f}$ and traction $\boldsymbol{g}$ in (5.11) and (5.13) are not arbitrary, but are given by $\boldsymbol{f} = \boldsymbol{f}^{(1)} - \boldsymbol{f}^{(2)}$ and $\boldsymbol{g} = \boldsymbol{g}^{(2)} - \boldsymbol{g}^{(1)}$. The stationary single edge and screw dislocation discussed in [4] require only (5.2) and are examples of when the auxiliary tensor $\boldsymbol{E}^{*}$ and body-forces are not necessary.

The Filon construct in which relation (5.7) determines the symmetric part $\boldsymbol{E}^{(P)}$ of $\boldsymbol{U}^{(P)}$ fails to uniquely determine the dislocation density from (2.23). The Volterra theory of dislocations suggests that results involving the Filon construct will be of interest when the region Ω is multiply connected or when the constituent solutions become singular in a certain sense, e.g., point or line singularities. The discussion in [4] of the stationary edge dislocation also shows that in the multiply connected or punctured region Ω the symmetric strain tensor $\boldsymbol{E}^{(E)}$ satisfies the Saint-Venant relation (2.17) and is derivable from an associated vector field $\boldsymbol{u}^{(E)}$. Suitable cuts convert Ω into a singly connected region and enable the corresponding Burgers vector to be calculated from the Volterra–Cesaro formula (3.13) in which $\boldsymbol{e}^{(\gamma)}$ is replaced by $\boldsymbol{E}^{(E)}$. The reader is referred to [4] for further details.

6 The inverse Filon construct

Previous sections concern the derivation of certain dislocation solutions from linear elastic solutions whose strains are compatible. This section studies the inverse process, namely, the recovery of the elastic solutions from a uniquely specified dislocation solution. As already mentioned, body- forces are often essential to achieve complete generality. Non-uniqueness is inherent since one constituent initial boundary problem can be arbitrarily selected for the elastic field $(\boldsymbol{u}^{(2)}, \boldsymbol{e}^{(2)}, \boldsymbol{\sigma}^{(2)})$ provided the strain $\boldsymbol{e}^{(2)}$ is compatible. The solution is not required to be unique. In consequence, however, the second constituent elastic solution $(\boldsymbol{u}^{(1)}, \boldsymbol{e}^{(1)}, \boldsymbol{\sigma}^{(1)})$, which must also be compatible, is unambiguously determined such that the difference between it and the first constituent solution corresponds to the given dislocation solution.

We examine further implications of the inverse procedure. Assume as before, that the mass density is unaltered throughout the discussion. The compatible strain $\boldsymbol{e}^{(2)}(x_i, t)$ is derived from the displacement $\boldsymbol{u}^{(2)}$ and belongs to the solution of the elastodynamic problem subject to body-force $\boldsymbol{f}^{(2)}(x_i, t)$, boundary conditions $\boldsymbol{h}^{(2)}(x_i, t)$, $\boldsymbol{g}^{(2)}(x_i, t)$, and initial conditions $\boldsymbol{u}_0^{(2)}(x_i)$, $\boldsymbol{u}_1^{(2)}(x_i)$.

The Filon relation (5.2) formally leads to

$$\boldsymbol{e}^{(2)} = -(\boldsymbol{I}-\boldsymbol{D})^{-1}\boldsymbol{E}^{(P)}, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T). \tag{6.1}$$

But in general for given $\boldsymbol{E}^{(P)}$, we have

$$\nabla \times \nabla \times (\boldsymbol{I}-\boldsymbol{D})^{-1}\boldsymbol{E}^{(P)} \neq 0, \tag{6.2}$$

and consequently (6.1) is not necessarily consistent with the assumed compatibility of the strain $\boldsymbol{e}^{(2)}$ and satisfaction of the Saint–Venant relation (2.17). The difficulty is again resolved by introduction of an auxiliary second-order tensor field $\boldsymbol{E}^{**}$ and replacement of (6.1) by the relation

$$\boldsymbol{e}^{(2)} = -(\boldsymbol{I}-\boldsymbol{D})^{-1}\left[\boldsymbol{E}^{(P)} + \boldsymbol{E}^{**}\right], \tag{6.3}$$

where $\boldsymbol{E}^{**}(x_i, t)$ is chosen to satisfy

$$\nabla \times \nabla \times (\boldsymbol{I}-\boldsymbol{D})^{-1}\boldsymbol{E}^{**} = -\nabla \times \nabla \times (\boldsymbol{I}-\boldsymbol{D})^{-1}\boldsymbol{E}^{(P)}. \tag{6.4}$$

An obvious choice is

$$\boldsymbol{E}^{**} = -(\boldsymbol{I}-\boldsymbol{D})\,\boldsymbol{e}^{(2)} - E^{(P)}. \tag{6.5}$$

Although the last relation is formally identical to (5.7), and the tensors $\boldsymbol{E}^{*}$ and $\boldsymbol{E}^{**}$ are superficially the same, we prefer to retain a separate notation for purposes of conceptual clarity.

Another method of selection is to arbitrarily choose the vector $\boldsymbol{u}^{(2)}(x_i, t)$ to ensure compatibility of the strain $\boldsymbol{e}^{(2)}$, and also to arbitrarily select the elastic modulus tensor $\boldsymbol{C}^{(2)}$. The corresponding body-force is then determined from the equations of motion

$$\rho\boldsymbol{f}^{(2)} = \rho\ddot{\boldsymbol{u}}^{(2)} - \mathrm{Div}\,\boldsymbol{C}^{(2)}\boldsymbol{e}^{(2)}, \qquad (\boldsymbol{x}, t) \in \Omega \times [0, \mathrm{T}). \tag{6.6}$$

Boundary and initial conditions are derived by a similar semi-inverse method. Provided the chosen elastic modulus tensor $\boldsymbol{C}^{(2)}$ satisfies the major symmetry in (2.9), the corresponding elastodynamic solution is unique. (c.p., [26]).

Remark 6.1 (*Alternative procedure*) Instead of arbitrarily selecting $\boldsymbol{u}^{(2)}$, we may proceed as follows. After the introduction of the tensor $\boldsymbol{E}^{**}$ defined to satisfy (6.4), Kröner's method (see, e.g., [9]) is used to obtain a particular solution to (6.4), at least for an isotropic elastic material, which upon substitution in (6.3) generates a compatible strain $\boldsymbol{e}^{(2)}$. The Volterra–Cesaro method of integration, outlined in Sect. 3, applied to $\boldsymbol{e}^{(2)}$ yields the displacement $\boldsymbol{u}^{(2)}$ and enables the corresponding body-force $\boldsymbol{f}^{(2)}$ to be derived from the equations of motion (6.6).

Accordingly, $\boldsymbol{u}^{(2)}(x_i, t)$, $\boldsymbol{e}^{(2)}(x_i, t)$, body-force $\boldsymbol{f}^{(2)}(x_i, t)$, boundary conditions $\boldsymbol{h}^{(2)}(x_i, t)$, $\boldsymbol{g}^{(2)}(x_i, t)$, and initial conditions $\boldsymbol{u}_0^{(2)}(x_i)$, $\boldsymbol{u}_1^{(2)}(x_i)$ can be calculated by any of the procedures just described. In order to determine the second constituent displacement $\boldsymbol{u}^{(1)}$, body-force $\boldsymbol{f}^{(1)}$, boundary conditions $\boldsymbol{h}^{(1)}$, $\boldsymbol{g}^{(1)}$, and initial conditions $\boldsymbol{u}_0^{(1)}$, $\boldsymbol{u}_1^{(1)}$, we select the elastic tensor $\boldsymbol{C}^{(1)}$ to be the same as for the dislocation problem, i.e., $\boldsymbol{C}^{(1)} = \boldsymbol{C}$. The strain $\boldsymbol{e}$ is compatible and derived from the dislocation total displacement $\boldsymbol{u}$ which is assumed known along with the dislocation body-force $\boldsymbol{f}$, boundary conditions $\boldsymbol{h}$, $\boldsymbol{g}$, and initial conditions $\boldsymbol{u}_0$, $\boldsymbol{u}_1$. In consequence, define $\boldsymbol{u}^{(1)}$ by

$$\boldsymbol{u}^{(1)}(x_i, t) = \boldsymbol{u}(x_i, t) + \boldsymbol{u}^{(2)}(x_i, t), \qquad (\boldsymbol{x}, t) \in \Omega \times [0, T), \tag{6.7}$$

and use this expression to derive the compatible strain $\boldsymbol{e}^{(1)}(x_i, t)$ and corresponding stress $\boldsymbol{\sigma}^{(1)} = \boldsymbol{C}^{(1)}\boldsymbol{e}^{(1)} = \boldsymbol{C}\boldsymbol{e}^{(1)}$. The body-force required to maintain these fields is obtained from the respective equations of motion (6.6) and (2.27) which, together with (6.5) and the identity $\boldsymbol{e} = \boldsymbol{e}^{(1)} - \boldsymbol{e}^{(2)}$, gives

$$\rho\ddot{\boldsymbol{u}}^{(1)} = \rho\ddot{\boldsymbol{u}} + \rho\ddot{\boldsymbol{u}}^{(2)} \tag{6.8}$$

$$= \operatorname{Div} \boldsymbol{C}^{(1)}\left(\boldsymbol{e} - \boldsymbol{E}^{(\mathrm{P})}\right) + \rho\boldsymbol{f} + \operatorname{Div} \boldsymbol{C}^{(2)}\boldsymbol{e}^{(2)} + \rho\boldsymbol{f}^{(2)} \tag{6.9}$$

$$= \operatorname{Div} \boldsymbol{C}^{(1)}\boldsymbol{e}^{(1)} - \operatorname{Div} \boldsymbol{C}^{(1)}\left[(\boldsymbol{I} - \boldsymbol{D})\boldsymbol{e}^{(2)} + \boldsymbol{E}^{(\mathrm{P})}\right] + \rho\left(\boldsymbol{f} + \boldsymbol{f}^{(2)}\right) \tag{6.10}$$

$$= \operatorname{Div} \boldsymbol{C}^{(1)}\boldsymbol{e}^{(1)} + \operatorname{Div} \boldsymbol{C}^{(1)}\boldsymbol{E}^{**} + \rho\left(\boldsymbol{f} + \boldsymbol{f}^{(2)}\right) \tag{6.11}$$

$$= \operatorname{Div} \boldsymbol{C}^{(1)}\boldsymbol{e}^{(1)} + \rho\boldsymbol{f}^{(1)}, \tag{6.12}$$

where

$$\rho\boldsymbol{f}^{(1)} = \rho\left(\boldsymbol{f} + \boldsymbol{f}^{(2)}\right) + \operatorname{Div} \boldsymbol{C}^{(1)}\boldsymbol{E}^{**}. \tag{6.13}$$

Boundary and initial conditions are obtained by semi-inverse methods.

Remark 6.2 The above arguments confirm that non-trivial body-forces can be crucial for the inverse Filon construct to be effective.

Notwithstanding Remark 6.2, in Sect. 8.1, we present an example in which body-forces are zero.

7 Isotropic elasticity

It is convenient to list explicit expressions for the Cartesian components of the fourth-order tensors $\boldsymbol{D}$, $(\boldsymbol{I} - \boldsymbol{D})$ and $(\boldsymbol{I} - \boldsymbol{D})^{-1}$ for isotropic linear elasticity. Corresponding expressions for the elastic modulus and its inverse, or compliance, are given by (3.4)–(3.9).

From Definition (4.10), we obtain

$$D_{ijkl} = D_1\delta_{ij}\delta_{kl} + D_2 I_{ijkl}, \tag{7.1}$$

where components I_{ijkl} of the identity matrix are given by (4.9), and

$$D_1 = \frac{\mu^{(2)}\left(\nu^{(1)} - \nu^{(2)}\right)}{\mu^{(1)}\left(1 + \nu^{(1)}\right)\left(1 - 2\nu^{(2)}\right)}, \tag{7.2}$$

$$D_2 = \frac{\mu^{(2)}}{\mu^{(1)}}. \tag{7.3}$$

Consequently, we have

$$[\boldsymbol{I}-\boldsymbol{D}]_{ijkl} = -D_1\delta_{ij}\delta_{kl} + D_3 I_{ijkl}, \tag{7.4}$$

where

$$D_3 = \frac{\left(\mu^{(1)}-\mu^{(2)}\right)}{\mu^{(1)}}. \tag{7.5}$$

Now suppose

$$[\boldsymbol{I}-\boldsymbol{D}]^{-1}_{ijkl} = A_1\delta_{ij}\delta_{kl} + A_2 I_{ijkl}, \tag{7.6}$$

where the scalar quantities $A_1,\ A_2$ are chosen to ensure that

$$[\boldsymbol{I}-\boldsymbol{D}]\,[\boldsymbol{I}-\boldsymbol{D}]^{-1} = \boldsymbol{I}. \tag{7.7}$$

We conclude that

$$\begin{aligned} A_1 &= \frac{D_1}{D_3\,(D_3-3D_1)} \\ &= \frac{\mu^{(1)}\mu^{(2)}\left(\nu^{(1)}-\nu^{(2)}\right)}{\left(\mu^{(1)}-\mu^{(2)}\right)\left[\mu^{(1)}\left(1+\nu^{(1)}\right)\left(1-2\nu^{(2)}\right)-\mu^{(2)}\left(1+\nu^{(2)}\right)\left(1-2\nu^{(1)}\right)\right]}, \end{aligned} \tag{7.8}$$

$$A_2 = \frac{!}{D_3} = \frac{\mu^{(1)}}{\left(\mu^{(1)}-\mu^{(2)}\right)}. \tag{7.9}$$

When $\mu^{(1)} \neq \mu^{(2)}$ but $\nu^{(1)} = \nu^{(2)}$, expressions (7.2) and (7.8) give $D_1 = A_1 = 0$. On the other hand, when $\mu^{(1)} = \mu^{(2)}$ but $\nu^{(1)} \neq \nu^{(2)}$ both A_1 and A_2 become singular and the respective inverses require alternative derivation. In this case, $D_3 = 0$, and

$$[\boldsymbol{I}-\boldsymbol{D}]_{ijkl} = -D_1^*\delta_{ij}\delta_{kl}, \tag{7.10}$$

where

$$D_1^* = \frac{(\nu^{(2)}-\nu^{(1)})}{(1+\nu^{(1)})(1-2\nu^{(2)})}, \tag{7.11}$$

and

$$[\boldsymbol{I}-\boldsymbol{D}]^{-1}_{ijkl} = -\frac{1}{9D_1^*}\delta_{ij}\delta_{kl}. \tag{7.12}$$

It follows from (5.7) that

$$E^*_{ij} + E^{(P)}_{ij} = D_1^* e^{(2)}_{kk}\delta_{ij}, \tag{7.13}$$

which gives $e^{(2)}_{kk} = (E^*_{kk} + E^{(P)}_{kk})/3D_1^*$. Only the polar parts of the respective tensors are immediately involved so that an appropriate representation is

$$\begin{aligned} e^{(2)}_{ij} &= \frac{1}{3}e^{(2)}_{kk}\delta_{ij} + \varepsilon^{(2)}_{ij} && (7.14)\\ &= \left(E^*_{kk} + E^{(P)}_{kk}\right)\delta_{ij}/9D_1^* + \varepsilon^{(2)}_{ij}, && (7.15) \end{aligned}$$

where the deviatoric part $\boldsymbol{\varepsilon}^{(2)}$ of $\boldsymbol{e}^{(2)}$ satisfies $\varepsilon^{(2)}_{kk} = 0$. Both $\boldsymbol{E}^*$ and $\boldsymbol{\epsilon}^{(2)}$ are chosen such that

$$\nabla\times\nabla\times\boldsymbol{e}^{(2)} = 0. \tag{7.16}$$

On using (3.7) and (3.8), we may derive analogous tensors for plane strain isotropic elasticity. When $\nu^{(1)} \neq \nu^{(2)},\ \mu^{(1)} \neq \mu^{(2)}$, we have

$$[\boldsymbol{I}-\boldsymbol{D}]_{\alpha\beta\kappa\delta} = -D_1\delta_{\alpha\beta}\delta_{\kappa\delta} + D_3 I_{\alpha\beta\kappa\delta}, \tag{7.17}$$

where

$$D_1 = \frac{\mu^{(2)}}{\mu^{(1)}}\left(\frac{(\nu^{(2)} - \nu^{(1)})}{(1 - 2\nu^{(2)})}\right), \qquad D_3 = \frac{(\mu^{(1)} - \mu^{(2)})}{\mu^{(1)}}, \tag{7.18}$$

while the inverse matrix has components

$$[\boldsymbol{I} - \boldsymbol{D}]^{-1}_{\alpha\beta\kappa\delta} = A_1\delta_{\alpha\beta}\delta_{\kappa\delta} + A_2 I_{\alpha\beta\kappa\delta}, \tag{7.19}$$

where

$$A_1 = \frac{\mu^{(1)}\mu^{(2)}(\nu^{(2)} - \nu^{(1)})}{(\mu^{(1)} - \mu^{(2)})[\mu^{(1)}(1 - 2\nu^{(2)}) - \mu^{(2)}(1 - 2\nu^{(1)})]}, \tag{7.20}$$

$$A_2 = \frac{\mu^{(1)}}{(\mu^{(1)} - \mu^{(2)})}. \tag{7.21}$$

When $\nu^{(1)} = \nu^{(2)}$, $\mu^{(1)} \neq \mu^{(2)}$, we easily conclude that $D_1 = A_1 = 0$. When $\nu^{(1)} \neq \nu^{(2)}$, $\mu^{(1)} = \mu^{(2)}$, we have $D_3 = 0$, and $D_1 = D_1^{**}$ where

$$D_1^{**} = \frac{(\nu^{(2)} - \nu^{(1)})}{(1 - 2\nu^2)}, \tag{7.22}$$

but as before $A_1,\ A_2$ become singular. Similar arguments, however, to those previously employed lead to the inverse relations

$$e^{(2)}_{\alpha\beta} = \frac{1}{2D_1^{**}}\left(E^{(P)}_{\alpha\beta} + E^{**}_{\alpha\beta}\right)\delta_{\alpha\beta} + \varepsilon_{\alpha\beta}, \tag{7.23}$$

where $\varepsilon_{\kappa\kappa} = 0$, and $\varepsilon_{\alpha\beta}$ and $E^{**}_{\alpha\beta}$ are selected to ensure that the strain tensor $\boldsymbol{e}^{(2)}$ is compatible.

8 Examples

Both the inclusion problem and the Filon construct for stationary edge and screw dislocations are discussed in [4,5] for isotropic linear elasticity. Therefore they are not considered here.

Instead, we apply the inverse Filon construct to the stationary edge dislocation and to the uniformly moving screw dislocation both in a homogeneous isotropic linear elastic whole space from which the positive Cartesian x_3-axis is deleted. The aim is to determine a pair of elastic states from the respective dislocation fields.

8.1 Inverse construct for the stationary edge dislocation

The homogeneous isotropic linear elastic material in the whole space has Poisson's ratio ν and shear modulus μ. The stationary edge dislocation is distributed so that its endpoints lie along the x_3-axis, and its Burgers vector, given by $\boldsymbol{b} = (0, B, 0)$ for constant scalar B, has jumps across the x_1-axis. Let

$$r^2 = x_1^2 + x_2^2, \tag{8.1}$$

and scale quantities so that $r = 1$ corresponds to a typical length. Suppose the plastic strain tensor and total displacement have components given by (see, for example, [4])

$$E^{(P)}_{\alpha\beta} = B\frac{x_1}{2\pi r^2}\delta_{\alpha\beta}, \tag{8.2}$$

$$u_1(x_1, x_2) = \frac{B}{4\pi(1 - \nu)}\left[\ln r - \frac{x_2^2}{r^2}\right], \tag{8.3}$$

$$u_2(x_1, x_2) = \frac{B}{4\pi(1-\nu)} \frac{x_1 x_2}{r^2}. \tag{8.4}$$

It is easily verified that for $r \neq 0$ the corresponding stress (2.26) is in equilibrium under zero body-force so that in the previous notation we have $\boldsymbol{f} = 0$.

We seek to determine a pair of equilibrium elastic fields whose difference between components of their vector displacements equals (8.3) and (8.4). Let $\nu^{(2)}$ be Poisson's ratio belonging to one component solution. In this example, the shear modulus is kept fixed at the value μ of the dislocation problem so that by (5.1) we have $\mu^{(2)} = \mu = \mu^{(1)}$. Only Poisson's ratio is varied. From (7.23), components of the corresponding strain tensor $\boldsymbol{e}^{(2)}$ are obtained from

$$e^{(2)}_{\alpha\beta} = \frac{1}{2D_1^{**}} \left[E^{(P)}_{\alpha\beta} + E^{**}_{\alpha\beta} \right] + \varepsilon_{\alpha\beta} \tag{8.5}$$

$$= \frac{1}{4D_1^{**}} E^{(P)}_{\kappa\kappa} \delta_{\alpha\beta} + G_{\alpha\beta}, \tag{8.6}$$

where

$$G_{\alpha\beta} = \frac{1}{2D_1^{**}} E^{**}_{\alpha\beta} + \varepsilon_{\alpha\beta}, \qquad D_1^{**} = \frac{(\nu^{(2)} - \nu)}{(1-2\nu^{(2)})}. \tag{8.7}$$

The tensors $\boldsymbol{E}^{**}$, $\boldsymbol{\varepsilon}$ must be chosen to ensure that the strain tensor $\boldsymbol{e}^{(2)}$ is compatible in the whole space for which $r \neq 0$. We set $\boldsymbol{E}^{**} = 0$ and put

$$G_{11} = -G_{22} = \frac{B}{2\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_1}{r^2} \left(\frac{x_1^2}{r^2} - \nu^{(2)} \right), \tag{8.8}$$

$$G_{12} = G_{21} = \frac{(1-\nu^{(2)})B}{4\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_2}{r^2} \left(1 - 2\nu^{(2)} + \frac{2x_1^2}{r^2} \right). \tag{8.9}$$

Substitution of (8.8) and (8.9) in (8.6) and use of (8.2) yields the following expressions for components of the compatible strain tensor $\boldsymbol{e}^{(2)}$:

$$e^{(2)}_{11} = \frac{B}{4\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_1}{r^2} \left[(3-4\nu^{(2)}) - \frac{2x_2^2}{r^2} \right], \tag{8.10}$$

$$e^{(2)}_{22} = \frac{B}{4\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_1(x_2^2 - x_1^2)}{r^4}, \tag{8.11}$$

$$e^{(2)}_{12} = \frac{(1-\nu^{(2)})B}{4\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_2}{r^2} \left[(1-2\nu^{(2)}) + \frac{2x_1^2}{r^2} \right]. \tag{8.12}$$

The displacement $\boldsymbol{u}^{(2)}$ from which the strain tensor $\boldsymbol{e}^{(2)}$ is derived has components

$$u^{(2)}_1 = \frac{B}{4\pi D_1^{(**}(1-2\nu^{(2)})} \left[(3-4\nu^{(2)}) \ln r + \frac{x_2^2}{r^2} \right], \tag{8.13}$$

$$u^{(2)}_2 = -\frac{B}{4\pi D_1^{**}(1-2\nu^{(2)})} \frac{x_1 x_2}{r^2}. \tag{8.14}$$

On setting

$$\frac{2\mu B(1-\nu^{(2)})}{D_1^{**}(1-2\nu^{(2)})} = -X_1, \tag{8.15}$$

we recognise these components as due to a point force of magnitude X_1 uniformly distributed along the x_3-axis and directed along the positive x_1-axis. (See, for example, [16].)

The second component elastic field occurring in the inverse Filon construct, in view of (5.1), has Poisson's ratio and shear modulus given by $\nu^{(1)} = \nu$, $\mu^{(1)} = \mu$. The corresponding displacement vector $\boldsymbol{u}^{(1)}$ is obtained from $\boldsymbol{u}^{(1)} = \boldsymbol{u} + \boldsymbol{u}^{(2)}$. Direct substitution from (8.3), (8.4), (8.13), (8.14), and use of (8.15) gives

$$u_1^{(1)} = -\frac{X_1}{8\pi\mu(1-\nu)}\left[(3-4\nu)\ln r + \frac{x_2^2}{r^2}\right], \tag{8.16}$$

$$u_2^{(1)} = \frac{X_1}{8\pi\mu(1-\nu)}\frac{x_1x_2}{r^2}, \tag{8.17}$$

which, as for the first constituent solution, are the components of the displacement for a static point force $(X_1,\ 0,\ 0)$ distributed along the x_3-axis.

The vanishing of the body-force $\boldsymbol{f}^{(1)}$ everywhere except on the x_3−axis is confirmed from relation (6.13) upon recalling that in the present example it is assumed that $E^{**} = 0$ and that $\boldsymbol{f} = \boldsymbol{f}^{(2)} = 0$ apart from the x_3−axis. These conclusions demonstrate that non-zero body-force and non-zero tensor $\boldsymbol{E}^{**}$ are not always required in applications of the inverse Filon construct. Examples are presented in the next section when these conditions are required.

8.2 Uniformly moving screw dislocation

We continue to assume that the isotropic elastic material occupies the whole space apart from the x_3-axis. We further suppose that the screw dislocation is parallel to the x_3-axis, and moves with uniform speed v in the x_1-direction. Subject to appropriate initial conditions, let the total displacement in the dislocation problem have components $(0, 0, u_3(x_1, x_2, t))$, where

$$u_3(x_\alpha, t) = \frac{b}{2\pi}\tan^{-1}\frac{\gamma x_2}{(x_1 - vt)} - \frac{b}{2}\mathrm{H}(x_1 - vt)\mathrm{sign}(x_2), \quad \gamma > 0, \tag{8.18}$$

$$= \frac{b}{2\pi}\left[\frac{\pi}{2}\mathrm{sign}(x_1 - vt)\mathrm{sign}(\gamma x_2) - \tan^{-1}\frac{(x_1 - vt)}{\gamma x_2}\right] \tag{8.19}$$

$$-\frac{b}{2}\mathrm{H}(x_1 - vt)\mathrm{sign}(x_2)$$

$$= -\frac{b}{4}\mathrm{sign}(x_2) - \frac{\mathrm{b}}{2\pi}\tan^{-1}\frac{(x_1 - vt)}{\gamma x_2}. \tag{8.20}$$

In these formulae, b is the constant magnitude of the Burgers vector $(0, 0, b)$, H(.) and sign(.) are the Heaviside and sign functions, $0 \le v < c$, and

$$\gamma^2 = (1 - v^2/c^2), \qquad c^2 = \mu/\rho, \qquad \mathrm{sign}(\gamma x) = \mathrm{sign}(x), \tag{8.21}$$

$$\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}\mathrm{sign}(x). \tag{8.22}$$

As indicated, it is supposed that the constant γ satisfies $\gamma > 0$.

Let ν and μ denote the Poisson ratio and shear modulus of the elastic medium containing the moving screw dislocation. It is known [27, p. 185] that the non-zero components of the dislocation stress are

$$\sigma_{13} = -\frac{\mu b}{2\pi}\frac{\gamma x_2}{(x_1 - vt)^2 + \gamma^2 x_2^2}, \tag{8.23}$$

$$\sigma_{23} = \frac{\mu b}{2\pi}\frac{\gamma(x_1 - vt)}{(x_1 - vt)^2 + \gamma^2 x_2^2}. \tag{8.24}$$

We deduce from the formula $\boldsymbol{\sigma} = \boldsymbol{C}\left(\nabla \boldsymbol{u} - \boldsymbol{U}^{(P)}\right)$ that the only non-zero component of the plastic distortion tensor is

$$U_{32}^{(P)} = -b\mathrm{H}(x_1 - vt)\delta(x_2), \tag{8.25}$$

where $\delta(.)$ denotes the delta function. The dislocation vector body-force $\boldsymbol{f}$ is zero.

The first constituent displacement $\boldsymbol{u}^{(2)}$ may be arbitrarily selected (for possible choices, see, for example, [28–31]). Then $\boldsymbol{E}^{**}$ is defined by (6.5). In particular, fix Poisson's ratio at $\nu^{(2)} = \nu$, but vary the shear modulus to the value $\mu^{(2)}$ and let $\boldsymbol{u}^{(2)}$ be any anti-plane shear motion $\boldsymbol{u}^{(2)}(x_\alpha, t) = (0, 0, w(x_1, x_2, t))$, with corresponding body-force $\boldsymbol{f}^{(2)} = (0, 0, f_3^{(2)})$. We consider the explicit choice (see [32, p. 413])

$$\begin{aligned} w(x_\alpha, t) = \rho \int_{\mathbb{R}^2} &\left\{ v_{33} f_3^{(2)} + w_1 V_{33} \right\} \mathrm{d}y_1 \mathrm{d}y_2 \\ &+ \rho \frac{\partial}{\partial t} \int_{\mathbb{R}^2} w_0 V_{33} \, \mathrm{d}y_1 \mathrm{d}y_2, \end{aligned} \tag{8.26}$$

where $w_0,\ w_1$ denote prescribed initial values of w and $\dot{w}$,

$$v_{33}(x_\alpha, y_\alpha) = \frac{\mathrm{H}(t - \widetilde{r}/c^{(2)})}{2\pi\mu^{(2)}} \int_r^{c^{(2)}t} f\left(t - \chi c^{(2)}\right) \frac{\mathrm{d}\chi}{(\chi^2 - \widetilde{r}^2)^{1/2}}, \tag{8.27}$$

$$V_{33}(x_\alpha, y_\alpha) = \frac{1}{2\pi\mu^{(2)}} \frac{\mathrm{H}(t - \widetilde{r}/c^{(2)})}{[t^2 - \rho\widetilde{r}^2/\mu^{(2)}]^{1/2}}, \tag{8.28}$$

and the twice differentiable function $f(t)$ is the amplitude of the body-force for the fundamental singular elastodynamic solution. We also have

$$\widetilde{r}^2 = (x_\alpha - y_\alpha)(x_\alpha - y_\alpha), \tag{8.29}$$

$$c^{(2)} = \left(\mu^{(2)}/\rho\right)^{1/2}. \tag{8.30}$$

Substitution in (6.5) enables non-zero components of $\boldsymbol{E}^{**}$ to be obtained as

$$E_{13}^{**} = -\frac{1}{2A_2} w_{,1}, \tag{8.31}$$

$$E_{23}^{**} = -\frac{1}{2A_2} w_{,2} + \frac{b}{2}\mathrm{H}(x_1 - vt)\delta(x_2). \tag{8.32}$$

By construction, Poisson's ratio $\nu^{(1)}$ and the shear modulus $\mu^{(1)}$ for the second constituent elastic state in the Filon inverse procedure have the same values $\nu,\ \mu$ as those for the dislocation problem. The displacement $\boldsymbol{u}^{(1)}$ according to (6.7) has Cartesian components $(0, 0, u_3^{(1)}(x_\alpha, t)) = (0, 0, w + u_3)$, where w and u_3 are specified in (8.26) and (8.20). The corresponding body-force from (6.13), (8.31), and (8.32) becomes $\boldsymbol{f}^{(1)} = (0, 0, f_3^{(1)}(x_\alpha, t))$ where

$$\rho f_3^{(1)}(x_\beta, t) = \rho f_3^{(2)}(x_\beta, t) - \left(\mu - \mu^{(2)}\right) w_{,\gamma\gamma} + \mu b\mathrm{H}(x_1 - vt)\delta'(x_2). \tag{8.33}$$

This completes the solution.

The simple example of the uniformly moving screw dislocation may be used also to compare the alternative calculation based upon the relation (6.3), which, since $\nu^{(2)} = \nu = \nu^{(1)}$, for the present problem reduces to

$$e_{ij}^{(2)} = -\frac{A_2}{2}\left(U_{ij}^{(P)} + U_{ji}^{(P)} + 2E_{ij}^{**}\right), \tag{8.34}$$

where $\boldsymbol{E}^{**}$ is chosen to ensure that $\boldsymbol{e}^{(2)}$ is compatible. Accordingly, we select the non-zero components of $\boldsymbol{E}^{**}$ to be

$$E_{31}^{**} = \frac{1}{2}p_{,1}, \qquad E_{32}^{**} = \frac{1}{2}p_{,2}, \tag{8.35}$$

in which the function $p(x_\alpha, t)$ is to be chosen. For this purpose, put

$$\gamma_2^2 = \left(1 - \rho v^2/\mu^{(2)}\right), \tag{8.36}$$

and for constant k appeal to the relation (8.22) in deriving the following equivalent expressions for p:

$$p = k\left[\tan^{-1}\frac{(x_1 - vt)}{\gamma_2 x_2} + \frac{\pi}{2}\mathrm{sign}(x_2)\right] \tag{8.37}$$

$$= k\left[\frac{\pi}{2}\mathrm{sign}(x_1 - vt)\mathrm{sign}(x_2) - \tan^{-1}\frac{\gamma_2 x_2}{(x_1 - vt)} + \frac{\pi}{2}\mathrm{sign}(\gamma_2 x_2)\right] \tag{8.38}$$

$$= k\left[\frac{\pi}{2}\left(1 + \mathrm{sign}(x_1 - vt)\right)\mathrm{sign}(x_2) - \tan^{-1}\frac{\gamma_2 x_2}{(x_1 - vt)}\right] \tag{8.39}$$

$$= k\left[\pi \mathrm{H}(x_1 - vt)\mathrm{sign}(x_2) - \tan^{-1}\frac{\gamma_2 x_2}{(x_1 - vt)}\right]. \tag{8.40}$$

It is now easily verified by substitution from (8.40) and (8.25) that the relation (6.4) is satisfied on selecting

$$2k\pi = b = \frac{2\pi}{A_2}, \qquad A_2 = \frac{\mu}{\mu - \mu^{(2)}}, \tag{8.41}$$

which determines k in terms of the specified b, $\mu^{(2)}$, and μ.

Insertion of (8.41), (8.37), and (8.40) into (8.35), yields

$$E^{**}_{31} = \frac{b\gamma_2}{4\pi}\frac{x_2}{R^2}, \tag{8.42}$$

$$E^{**}_{32} = \frac{b\gamma_2}{4\pi}\left[2\pi \mathrm{H}(x_1 - vt)\delta(x_2) - \frac{(x_1 - vt)}{R^2}\right], \tag{8.43}$$

where

$$R^2 = (x_1 - vt)^2 + \gamma_2^2 x_2^2. \tag{8.44}$$

Other components of $\boldsymbol{E}^{**}$ vanish.

The non-zero components of the compatible strain $\boldsymbol{e}^{(2)}$, calculated from (6.3), are given by

$$e^{(2)}_{13} = -\frac{1}{2}\frac{\gamma_2 x_2}{R^2}, \tag{8.45}$$

$$e^{(2)}_{23} = \frac{1}{2}\frac{\gamma_2 (x_1 - vt)}{R^2}. \tag{8.46}$$

It can easily be verified that these strain components are derived from the displacement $\boldsymbol{u}^{(2)} = (0, 0, u_3^{(2)})$ where

$$u_3^{(2)} = \tan^{-1}\frac{\gamma_2 x_2}{(x_1 - vt)}. \tag{8.47}$$

The above expressions for the displacement $\boldsymbol{u}^{(2)}$ and strains $\boldsymbol{e}^{(2)}$ when inserted into the equations of motion show that the corresponding body-force $\boldsymbol{f}^{(2)}$ is zero.

The elastic fields $(\boldsymbol{u}^{(1)}, \boldsymbol{e}^{(1)})$ are determined from $\boldsymbol{u}^{(1)} = \boldsymbol{u} + \boldsymbol{u}^{(2)}$, $\boldsymbol{e}^{(1)} = \boldsymbol{e} + \boldsymbol{e}^{(2)}$. The body-force $\boldsymbol{f}^{(1)}$, obtained from (6.13), is given by $\boldsymbol{f}^{(1)} = (0, 0, f_3^{(1)})$, whose non-zero component after appeal to (8.35)–(8.41) becomes

$$\rho f_3^{(1)} = 2\mu\left[E^{**}_{31,1} + E^{**}_{32,2}\right] \tag{8.48}$$

$$= -\frac{\mu}{\mu^{(2)}}\frac{b}{\pi}\frac{\rho\gamma_2 v^2 x_2 (x_1 - vt)}{R^4} + \mu b \mathrm{H}(x_1 - vt)\delta'(x_2). \tag{8.49}$$

Comparison of respective elements in the constituent pairs obtained by the inverse Filon construct for the moving screw dislocation confirms, as expected, that such pairs are not uniquely determined. Indeed, in principle there is an arbitrary large number.

9 Concluding remarks

The original Filon construct for plane isotropic linear elastostatics has been generalised to three-dimensional elastodynamics. Body-forces and especially an auxiliary plastic distortion tensor, crucial for a complete extension, are absent in the original formulation. The construct, however, does not generate the total plastic distortion tensor $\boldsymbol{U}^{(P)}$ but only its symmetric part $\boldsymbol{E}^{(P)}$ which is related to the dislocation density $\boldsymbol{\alpha}$ through (2.23). Expression (2.23), and consequently Filon's construct, is insufficient to uniquely determine $\boldsymbol{\alpha}$ despite probably it being more appropriate as data. In this respect, it is established in [11] that the dislocation density does not uniquely determine the dislocation solution. The full implications of this last remark await investigation not only for the construct itself, but just as importantly for the inverse procedure.

Several other aspects of the generalised Filon construct and its inverse provide scope for future detailed examination. These include the following topics.

The approach adopted in this study has utilised the linearity of the theories involved. Inspection reveals, however, that a fundamental concept is that of internal stress which in turn may be explained by heuristic cut-and-weld operations. Neither notion is intrinsically linear. In consequence, the technique might be extendable to certain non-linear theories.

Other developments that perhaps warrant examination include the application of a generalised Filon construct to elastic inclusions contained in a vibrating body, or to wave motion in laminates. Interrelationships with other dynamic coupled theories, including thermoelasticity, present further open problems.

Acknowledgements The authors are grateful for support from the Research in Groups scheme of the International Centre for Mathematical Sciences, Edinburgh, and also from the Leverhulme Trust for the award of a Visiting Professorship (A.A.) at the University of Bath.

References

1. Filon LNG (1922) On stresses in multiply-connected plates. In: Report of the eighty-ninth meeting of the British association for the advancement of science (1921). London (1922) pp 305–316
2. Muskhelishvili N (1963) Some basic problems of the mathematical theory of elasticity (trans: Radok JRM). Groningen, Noordhoff
3. Boley BA, Weiner JH (1960) Theory of thermal stresses. Wiley, New York
4. Acharya A, Knops RJ (2012) On the three-dimensional Filon construct for dislocations. Int J Eng Sci 59:6–26
5. Knops RJ (2003) Filon's construct for dislocations and related topics. J Eng Math 46:331–354
6. Kosevich AM (1962) The deformation field in an isotropic elastic medium containing moving dislocations. Soviet Phys JETP 15:108–115
7. Kosevich AM (1965) Dynamical theory of dislocations. Soviet Phys Uspekhi 7:837–854
8. Acharya A (2001) A model of crystal plasticity based on the theory of continuously distributed dislocations. J Mech Phys Sols 49:761–784
9. Kröner E (1981) Continuum theory of defects. Course 3. Session 35. In: Balian P, Kleman M, Poirier JP (eds) Physics of defects. Les Houches Summer School, 1980. North-Holland Publ. Co, Amsterdam, pp 217–315
10. Eshelby JD (1956) The continuum theory of lattice defects. In: Seitz F, Turnball D (eds) Progress in solid state physics, vol 3. Academic Press, New York, pp 79–144
11. Acharya A, Knops RJ, Sivalogathan J (2017) Basic structure of linear dislocation field theory (to appear)
12. Michell JH (1899) On the direct determination of stress in an elastic solid, with application to the theory of plates. Proc Lond Maths Soc 31(1):100–124
13. Michell JH (1899) The transmission of stress across a plane of discontinuity in an isotropic elastic solid, and the potential solutions for a plane boundary. Proc Lond Maths Soc 31:183–192
14. Volterra VE (1960) Sur les distorsions des corps élastiques. (théorie et applications.) Mémorial des sciences mathématiques, vol 147. Gauthier-Villars, Paris
15. Gurtin ME (1972) The linear theory of elasticity. In: Flügge S (ed) Handbuch der physik, vol VIa/2. Springer, Berlin, pp 1–295

16. Love AEH (1927) A treatise on the mathematical theory of elasticity, 4th edn. Cambridge University Press, Cambridge
17. Maxwell JC (1869) Scientific Papers, II. Dover Publications, New York
18. Lévy M (1898) Sur la légimaté de la régle dite du trapèze dans l'étude de la résistance des barrages en maconnerie. Comptes Rend Acad Sci Paris 126:1235
19. Timpe A (1905) Probleme der Spannungsverteilung in ebenen systemen, einfach delöst und Hilfe der Airyschen Funktion. Doctoral Dissertation. Göttingen, Leipzig, Teubner
20. Carlson DE (1971) Dependence of linear elastic solutions on the elastic constants. I. Dependence on Poisson's ratio in elastostatics. J Elast 1(2):145–151
21. Carlson DE (1972) Dependence of linear elastic solutions on the elastic constants. II. Dependence on shear modulus in elastostatics. J Elast 2(2):129–134
22. Carlson DE (1973) Dependence of linear elastic solutions on the elastic constants. III. Elastodynamics. J Elast 3(3):169–178
23. Dundurs J (1972) Load and self-stresses in an elastic body. J Elast 2:211–213
24. Markenscoff X (2006) Stress independence of Poisson's ratio and divergence free body-forces. J Elast 45:65–74
25. Bramble JH, Payne LE (1963) Effect of error in measurement of elastic constants on the solutions of problems in classical elasticity. J Res Nat Bureau Stand 67B:157–167
26. Knops RJ, Payne LE (1971) Uniqueness theorems in linear elasticity, vol 19. Springer tracts in natural philosophy. Springer, Berlin
27. Hirth JP, Lothe J (1982) Theory of dislocations, 2nd edn. Kreiger Publishing Company, Malabar, FL
28. Achenbach JD (1975) Wave propagation in elastic solids. North-Holland, Amsterdam
29. Eason G, Fulton J, Sneddon IN (1956) The generation of waves in an infinite elastic solid by variable body forces. Phil Trans R Soc Lond A248:575–607
30. Freund LB (1972) Wave motion in an elastic solid due to a non-uniformly moving line load. Q Appl Math 30:271–281
31. Wu K-C (2002) Moving line force in an anisotropic elastic solid. Proc R Soc Lond A458:1761–1772
32. Eringen AC, Suhubi ES (1975) Elastodynamics, vol II. Linear theory. Academic Press, New York