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CGA-Based Approach to Direct Kinematics of Parallel Mechanisms with the 3-RS structure

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ABSTRACT

This paper presents a unified geometric modeling and solution procedure for direct kinematic analysis of a class of parallel mechanisms based on conformal geometric algebra (CGA). After locking the actuated joints, such parallel mechanisms will be turned into a 3-RS structure, which is composed of two triangular platforms connected by three RS serial chains in parallel. Using the proposed approach, the univariate polynomial equation for the direct kinematic analysis of these parallel mechanisms can be derived in three steps. Firstly, the positions of two of the three spherical joints on the moving platform are formulated by the intersection, dissection and dual of the basic geometric entities under the frame of CGA. Secondly, a coordinate-invariant equation expressed in terms of geometric entities is derived via CGA operation. Thirdly, a univariate polynomial equation is obtained directly from the aforementioned coordinate-invariant equation by using tangent-half-angle substitution. Several case studies are then presented to verify the solution procedure. The novelties of this approach lie in that: (1) The formulation is concise and coordinate-invariant and has intrinsic geometric intuition due to the use of CGA; (2) No algebraic elimination procedure is required to derive the univariate polynomial equation; and (3) The proposed approach is applicable to the direct kinematics of this family of parallel mechanisms with any link parameters.

Key words: parallel mechanisms; direct kinematics; conformal geometric algebra; coordinate-invariant; elimination-free.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>d or D</td>
<td>scalar</td>
</tr>
<tr>
<td>x</td>
<td>a vector in Euclidean space</td>
</tr>
<tr>
<td>X</td>
<td>entity in the conformal space</td>
</tr>
<tr>
<td>A</td>
<td>blades or multi-vectors</td>
</tr>
<tr>
<td>AB</td>
<td>geometric product of A and B</td>
</tr>
<tr>
<td>A ∧ B</td>
<td>outer product of A and B</td>
</tr>
<tr>
<td>A ⋅ B</td>
<td>inner product of A and B</td>
</tr>
<tr>
<td>A∗</td>
<td>dual of A</td>
</tr>
<tr>
<td>e₀</td>
<td>conformal origin</td>
</tr>
<tr>
<td>e∞</td>
<td>conformal infinity</td>
</tr>
<tr>
<td>Ic</td>
<td>the unit pseudo-scalars in CGA</td>
</tr>
</tbody>
</table>

1. Introduction

A parallel mechanisms (PM) consists of two rigid bodies, including the base and the moving platform, connected to each other by at least two kinematic chains or limbs. The most commonly studied and extensively used PMS are the six degree-of-freedom (DoF) mechanisms, which were first introduced in tire testing by Gough [1], later were used by Stewart [2] as flight simulators and now commonly known as the Stewart platform. They have many good characteristics, such as high stiffness and large load capacity, and many researchers have investigated them extensively [3, 4]. Since many industrial applications do not require six-DoF, PMS with less than 6-DoF have attracted the attention of many researchers (see [3, 5–9] for example). Particularly, many 3-DoF PMs have been designed and investigated for applications, including the famous DELTA mechanism with three translational DoF [5],

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the 3-BRR spherical PMs with three rotational DoF [6], the two rotational and one translational DoF PMs, such as 3-RPS PMs exploited as a micromanipulator [7], and 3-PRS PMs used as a machine tool known through the patented Z3 Head by DS Technology [8] and telescope focusing [9], just to name a few.

In this paper, we will revisit the direct kinematics of PMs that generate 3-RS structures when the actuated joints are locked. PMs with a 3-RS structure cover a broad family of PMs, such as the 6-3 Stewart platforms, the 3-RPS PMs, the 3-PRS PMs, the 3-RRS PMs and the 3-RRRS PMs etc., whose moving platform is attached to three/six legs via three spherical joints. For a more comprehensive list of PMs with a 3-RS structure, please refer to [3, 10]. Here, P, R, and S stand for prismatic, revolute, and spherical joint respectively. When a joint is actuated, its symbol is underlined. The direct kinematics of PMs consists of finding the position and orientation of the moving platform with respect to the base platform for a given set of inputs. To this end, it is equivalent to the computation of the coordinates of the centers of the three spherical joints attached on the moving platform. While this problem can be solved by an iterative numerical method, a closed-form solution is clearly preferred. Such a solution is not only more accurate but provides a valuable insight for the design stage.

The direct kinematics problem of the family of PMs has been studied in the literature by many researchers [7-9, 11-26]. The 6-3 Stewart platforms have been studied in [11-19]. The 3-RPS PM has been analyzed in [7, 20-25]. Two different designs of 3-PRS mechanisms have been studied respectively in [8, 9, 21] and 3-RRS mechanisms have been investigated in [21, 26]. It is known that the 6-3 Stewart platform is isomorphic to the above-mentioned 3-DoF PMs and their equivalent architecture is the 3-RS structure. The existing approaches to the direct kinematic solution for a 3-RS structure is usually to establish first a set of three algebraic equations involving trigonometric functions of three angles, and then to convert these equations into a new set of three polynomial equations in three variables by the tangent-half-angle substitution. Then among these equations, two of these variables are eliminated by using resultant method to yield a polynomial equation in one variable of degree 16. Rojas et al. [16] presented the distance-based formulation for the direct kinematics of a family of octahedral Stewart platform and solved this problem using algebraic elimination method. Buruncuk et al. [19] gave a thorough geometric and analytic discussion on the kinematic problem of the symmetric 3-RPS PMs. Bonev et al. [23] used a special orientation representation (called the Tilt-and-Torsion angles) to obtain analytical solutions from a polynomial of degree four for the direct kinematics of the symmetric 3-DoF PMs, i.e., so-called zero-torsion mechanisms and identify the two operation modes of PMs. Schadlbauer et al. [24] presented a complete algebraic description of a symmetric configuration of 3-RPS PMs using Study's kinematic mapping and solved their direct kinematics by Groebner basis. They also identify the two operation modes of PMs. Mamidi et al. [25] pointed out that the problem is equivalent to finding the intersection of a pair of quad-circular octic curves with a circle and also reveal the same geometric properties as [23, 24]. It is concluded from the above-mentioned literature that the modeling and the solution procedure are either formulated algebraically or requires algebraic elimination using resultant method or Groebner basis method etc.

This paper is to present a unified geometric modeling and solution procedure for direct kinematics of a class of PMs, i.e., a CGA-based approach. Using this approach, the positions of two of the three spherical joints on the moving platform are formulated concisely by the intersection, dissection and dual of the geometric entities such as points, spheres and planes and then a coordinate-invariant polynomial equation in the position of another spherical joint on the moving platform is derived and finally it leads directly to a univariate polynomial equation by tangent-half-angle substitution. The whole solution procedure is under the frame of CGA and no algebraic elimination is required. The proposed approach is applicable to the family of PMs with any link parameters. For the symmetric PMs, we will also obtain closed-form solutions from a polynomial of degree four/eight as the result in the literature [23]. However, the method proposed in [23] is not suitable to the direct kinematic problem of general PMs. The modeling procedure has intrinsic geometric intuition since we take advantage of the special geometry of PMs and formulate this problem in the conformal space. Moreover, the solution procedure is compact and simpler. Although the direct kinematic problems of 3-RPS PMs and 6-3 Stewart platforms have been previously solved by Zhang et al. [17, 22] and Wei et al. [18] using CGA, the proposed solution procedure in the paper is more efficient as no algebraic elimination method is required to derive a high-degree univariate polynomial equation.

Conformal geometric algebra (CGA) [27-30] is a relatively new mathematical framework for geometric representation and computation. Essentially, CGA represents visually various geometric entities of points, spheres, planes, lines, circles, and point pair in a systematic hierarchy of multiple grades. More importantly, CGA provides direct algebraic operations on these geometric entities which typically lead to simple, compact and coordinate-invariant formulations. The above-mentioned properties are two superior characteristics of CGA. Hence it is very efficient for geometric modeling and
computation for kinematic problem of mechanisms and robotics. In recent decades, CGA has been mostly applied to solve the inverse kinematics problem of the serial mechanisms [31-35] via CGA operation of the geometric entities. In addition, Taney [36, 37], Jin et al. [38] and Huo et al. [39] employed CGA to study the singularity analysis of PMs. Huo et al. [39] and Li et al. [40] proposed a mobility analysis approach for PMs based on geometric algebra. Shen et al. [41] described the position and orientation characteristics using CGA and gave the direct symbolic algorithm for motion output. Song et al. [42] and Qi et al. [43] applied CGA in topology synthesis of parallel mechanisms. Zhang et al. [17, 22] and Wei et al. [18] applied CGA to solve the direct kinematics of PMs.

The remaining of the paper is organized as follows. In section 2, the fundamentals of CGA will be introduced. In section 3, the CGA-based formulation and solution procedure for the direct kinematic analysis of PMs with the 3-RS structure will be proposed. In addition, we will also discuss the solution to the direct kinematics of PMs with special architecture. Section 4 will provide several case studies to verify the approach. Finally, conclusions and future work will be given in Section 5.

2. Conformal Geometric Algebra

2.1 Fundamental theory of CGA

In geometric algebra, the fundamental algebraic operators are the inner product \( A \cdot B \), the outer product \( A \wedge B \), and the geometric product \( AB = A \cdot B + A \wedge B \).

The 5-dimension (5D) CGA \( G^{5,1} \) is derived from a 3D Euclidean space \( G^3 \) and a 2D Minkowski vector space \( G^{1,1} \). CGA has five orthonormal basis vector given by \( \{e_0, e_1, e_2, e_3, e_\infty\} \) with the following properties:

\[
e_i^2 = e_i \cdot e_i = e_i = 1,
\]
\[
e_i \cdot e_j = 0 (i \neq j; i, j = 1, 2, 3, +, -),
\]
\[
e_i \wedge e_j = -e_i \wedge e_j (i, j = 1, 2, 3, +, -),
\]

where \( \{e_0, e_1, e_2, e_3\} \) are three orthonormal basis vectors in the Euclidean space and \( \{e_\infty, e_\infty\} \) are two orthogonal basis vector in the Minkowski space.

In addition, a null basis can now be introduced by the vectors

\[
e_0 = \frac{1}{2}(e_+ - e_-), e_\infty = e_+ + e_-,
\]

with the properties of:

\[
e_0^2 = e_\infty^2 = 0, \quad e_\infty \cdot e_0 = -1,
\]

where \( e_0 \) is the conformal origin and \( e_\infty \) is the conformal infinity.

**Blades** are the basic computational elements and the basic geometric entities of the geometric algebra. The grade of a blade is simply the number of linearly independent vectors that are “wedged” together. The 5D CGA consists of blades with grades 0, 1, 2, 3, 4 and 5. A linear combination of the \( k \)-blades is called a \( k \)-vector, and a linear combination of blades with different grades is called a multi-vector. The blades with the maximum grade in CGA, i.e., 5-blades, are called pseudo-scalars and denoted by \( I_c (e_{00123}, I_c^2 = -1) \).

The inner and outer products of two vectors \( u, v \) are defined as

\[
\quad u \cdot v = \frac{1}{2}(uv + vu),
\]
\[
\quad u \wedge v = \frac{1}{2}(uv - vu).
\]

As an extension, the inner product of an \( r \)-blade \( u_1 \wedge L \wedge u_r \) with an \( s \)-blade \( v_1 \wedge L \wedge v_s \) can be defined recursively by

\[
(u_1 \wedge L \wedge u_r) \cdot (v_1 \wedge L \wedge v_s) = \begin{cases} (u_1 \wedge L \wedge u_r) \cdot (v_2 \wedge L \wedge v_s) & \text{if } r \geq s \\
(u_1 \wedge L \wedge u_{r\ldots}) \cdot (u_1 \wedge (v_1 \wedge L \wedge v_s)) & \text{if } r < s 
\end{cases}
\]

with
\begin{equation}
(u_i \land L \land u_j) \cdot v_i = \sum_{i=1}^{\infty} (-1)^{i-1} u_i \land L \land u_{i+1} \land (u_j \cdot v_i) \land u_{i+1} \land K \land u_r,
\end{equation}
\begin{equation}
u_j \cdot (v_i \land L \land v_j) = \sum_{i=1}^{\infty} (-1)^{i-1} v_i \land L \land v_{i+1} \land (u_j \cdot v_i) \land v_{i+1} \land K \land v_r.
\end{equation}

We define the dual \( X^* \) of a multi-vector \( X \) by
\begin{equation}
X^* = I_c I_c^{-1} = -X I_c,
\end{equation}
where \( I_c^{-1} \) is the inverse of \( I_c \) and \( I_c I_c = -1 \).

### 2.2 Conformal geometric entities

CGA provides the representation of primitive geometric entities for intuitive expression. The representation of the geometric entities with respect to the inner product null space (IPNS) and with respect to the outer product null space (OPNS) are respectively listed in Table 1. IPNS refers to the geometric entities generated by the intersection of geometric entities, and OPNS refers to the geometric entities represented by the points that belong to the geometric entity. These two representations are dual to each other and therefore can be converted by dual operator. In Table 1, the small italic character represents the point or vector in the Euclidean space while the italic underlined character represents the basic geometric entity in the conformal space. For more information, please refer to the literature [29, 30].

**Table 1 List of conformal geometric entities**

<table>
<thead>
<tr>
<th>Entity</th>
<th>IPNS</th>
<th>Grade</th>
<th>OPNS</th>
<th>Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point</td>
<td>( \mathcal{P} = \rho + \frac{1}{2} \rho^2 e_\rho + e_0 )</td>
<td>1</td>
<td>( \mathcal{P}^* = \mathcal{S_S} \land \mathcal{S_S} \land \mathcal{S_S} )</td>
<td>4</td>
</tr>
<tr>
<td>Sphere</td>
<td>( \mathcal{S} = \rho + \frac{1}{2} (\rho^2 - \rho^3) e_\rho + e_0 )</td>
<td>1</td>
<td>( \mathcal{S}^* = \mathcal{P} \land \mathcal{P} \land \mathcal{P} )</td>
<td>4</td>
</tr>
<tr>
<td>Plane</td>
<td>( \mathcal{P} = n + d e_n )</td>
<td>1</td>
<td>( \mathcal{P}^* = e_\rho \land \mathcal{P} \land \mathcal{P} )</td>
<td>4</td>
</tr>
<tr>
<td>Line</td>
<td>( \mathcal{L} = \mathcal{S} \land \mathcal{S} )</td>
<td>2</td>
<td>( \mathcal{L} = e_\rho \land \mathcal{P} \land \mathcal{P} )</td>
<td>3</td>
</tr>
<tr>
<td>Circle</td>
<td>( \mathcal{C} = \mathcal{S_S} \land \mathcal{S_S} )</td>
<td>2</td>
<td>( \mathcal{C}^* = \mathcal{P} \land \mathcal{P} \land \mathcal{P} )</td>
<td>3</td>
</tr>
<tr>
<td>Point pair</td>
<td>( \mathcal{P} = \mathcal{S_S} \land \mathcal{S_S} \land \mathcal{S_S} )</td>
<td>3</td>
<td>( \mathcal{P}^* = \mathcal{P} \land \mathcal{P} )</td>
<td>2</td>
</tr>
</tbody>
</table>

According to Eqs. (1)-(5), the inner product between two conformal points \( \mathcal{P} \cdot \mathcal{P} \) is calculated as
\begin{equation}
\mathcal{P} \cdot \mathcal{P} = \left( p_1 + \frac{1}{2} p_1^2 e_\rho + e_0 \right) \left( p_2 + \frac{1}{2} p_2^2 e_\rho + e_0 \right) = -\frac{1}{2} (p_1 - p_2)^2 = -\frac{1}{2} d^2.
\end{equation}

where \( d \) denotes the Euclidean distance between the two points.

From Eq. (10), we have \( \mathcal{P} \cdot \mathcal{P} = 0 \).

In the next section, we will formulate the direct kinematics of PMs with a 3-RS structure via CGA operation and derive the univariate polynomial equation. In addition, we will also discuss the solution to the direct kinematics of PMs with some special configurations.

### 3. CGA-Based Solution Procedure for Direct Kinematics of PMs with Triangular Moving Platforms

In this paper, we will focus on two categories of PMs with the 3-RS structure. One category is 3-DOF PMs with three limbs, such as the 3-RPS PMs and the 3-PRS PMs. Another category is 6-DOF Stewart platforms with six limbs, whose three spherical joints on the moving platform are pair-wise, such as 6-3, 5-3, 4-3 and 3-3 (deformable octahedron) Stewart platform, where the former and latter numbers represent the number of the spherical joints on the base and the moving platform respectively [44]. In this section, we will firstly present the CGA-based formulation for direct kinematics of these two categories of PMs respectively although these two categories PMs are isomorphic and then derive the univariate polynomial equation using the same solution procedure.

#### 3.1 CGA-based formulation for direct kinematics of the first category of PMs

In this subsection, we will take a 3-RPS parallel mechanism as the first case. In a 3-RPS PM (Fig. 1), the moving platform is attached to the base platform by three limbs with P joints as actuated joints. The lengths of the P joints are denoted by \( l_i (i = 1, 2, 3) \). Let \( u_i (i = 1, 2, 3) \) denotes the unit vectors along the
axes of the R joints \( A_i \) located on the base; \( a_i \), \((i = 1, 2, 3)\) and \( b_i \), \((i = 1, 2, 3)\) denote the coordinates of the R joints \( A_i \) and the S joints \( B_i \) in the Euclidean space respectively and \( r_i \), \((i = 1, 2, 3)\) denotes the distance between the three S joints \( B_i \) on the moving platform. For a general 3-RPS PM, the axes \( u_i \), \((i = 1, 2, 3)\) in the base are not restricted to lie in a plane. Only the coordinates \( b_i \), \((i = 1, 2, 3)\) are unknown. Next, we will formulate the two spherical joints \( B_i \) and \( A_i \) on the moving platform by the intersection, dissection and dual of the basic geometric entities under the frame of CGA.

As can be seen from Fig. 1, point \( B_3 \) must be located on a sphere \( S_{a_i b_i} \) of radius \( r_i \) with its center at point \( B_1 \), a sphere \( S_{a_i b_i} \) of radius \( l_2 \) with its center at point \( A_2 \), and a plane \( \pi_2 \) passing through point \( A_2 \) and with its normal vector \( n_2 \) align with the unit vector \( u_2 \). Thus, the actual locus of point \( B_3 \) must be located on the intersection of the two spheres \( S_{a_i b_i} \) and \( S_{a_i b_i} \), and one plane \( \pi_2 \). From the knowledge of geometry, it is known that the locus of this intersection will be a point pair \( B_{22} \). Therefore, according to Table 1, the centers of two spheres, \( B_i \) and \( A_i \), and the point pair \( B_{22} \) can be formulated in CGA as

\[
B_1 = b_i + \frac{1}{2} b_i^2 e_n + e_0 \quad (11)
\]

\[
A_3 = a_3 + \frac{1}{2} a_3^2 e_n + e_0 \quad (12)
\]

\[
B_{22} = S_{a_i b_i} \cap S_{a_i b_i} \cap \pi_2 \quad (13)
\]

where the two spheres \( S_{a_i b_i} \) and \( S_{a_i b_i} \) and the plane \( \pi_2 \) can be represented in CGA as

\[
S_{a_i b_i} = B_1 - \frac{1}{2} r_i^2 e_n, \quad S_{a_i b_i} = A_3 - \frac{1}{2} l_2^2 e_n, \quad \pi_2 = u_2 + (a_3 \cdot a_2) e_n.
\]

Let \( B_3 = B_2 \cdot I_c^{-1} = -B_2 \cdot I_c \) be the dual of the point pair \( B_{22} \) (see Eq. 9)). Point \( B_2 \) is dissected from the point pair \( B_{22} \) in the conformal space as [17, 22]

\[
B_2 = T_{a_2} \pm \frac{B_{22}}{A_{a_2}} T_{a_1}
\]

where \( T_{a_2} = e_n \cdot B_{a_2} \), \( T_{a_1} = T_{a_2} \cdot B_{a_2} = T_{a_3} \cdot T_{a_1} \), \( A_{a_2} = T_{a_2} \cdot T_{a_3} \), \( B_{a_2} = B_2 \cdot B_{a_2} \cdot T_{a_3} \) and \( T_{a_3} \) and \( T_{a_2} \) are both 1-blade; \( A_{a_2} \) and \( B_{a_2} \) are both scalars. The geometric meanings of the first term \( T_{a_2} / A_{a_2} \) and the second term \( T_{a_3} / A_{a_2} \) in Eq. (14) are half of the sum and the difference of two points from the point pair respectively.

Please note that the expression of point \( B_2 \) in Eq. (14) is in its standard and normalized form, i.e., the magnitude \((-e_n \cdot B_2)\) is equal to 1.

As seen from Fig. 1, point \( B_3 \) is the intersection point of three spheres \( S_{a_i b_i} \), \( S_{a_i b_i} \), \( S_{a_i b_i} \) and one plane \( \pi_3 \), i.e., the sphere \( S_{a_i b_i} \) of radius \( r_i \) with its center at point \( B_1 \), the sphere \( S_{a_i b_i} \) of radius \( l_i \) with its center at point \( A_i \), the sphere \( S_{a_i b_i} \) of radius \( r_i \) with the center at point \( B_3 \), and the plane \( \pi_3 \) passing through point
and with its normal vector \( \mathbf{n}_1 \) align with the unit vector \( \mathbf{u}_1 \). According to Table 1, the dual of point \( \mathbf{B}_i \) in CGA can be expressed as

\[ \mathbf{B}_i^* = \mathbf{S}_{a_{\mathbf{B}_i}} \wedge \mathbf{S}_{a_{\mathbf{B}_i}} \wedge \mathbf{n}_1 \wedge \mathbf{S}_{a_{\mathbf{B}_i}}, \tag{15} \]

where the three spheres \( \mathbf{S}_{a_{\mathbf{B}_i}} \), \( \mathbf{S}_{a_{\mathbf{B}_i}} \), \( \mathbf{S}_{a_{\mathbf{B}_i}} \) and the plane \( \mathbf{n}_1 \), can be represented in CGA as,

\[
\begin{align*}
\mathbf{S}_{a_{\mathbf{B}_i}} &= \mathbf{B}_i - \frac{1}{2} r_i^2 \mathbf{e}_x, \quad & \mathbf{S}_{a_{\mathbf{B}_i}} &= \mathbf{A}_i - \frac{1}{2} l_i^2 \mathbf{e}_x, \\
\mathbf{S}_{a_{\mathbf{B}_i}} &= \mathbf{B}_i - \frac{1}{2} r_i^2 \mathbf{e}_x, \quad & \mathbf{n}_1 &= \mathbf{u}_1 + (\mathbf{a}_i \cdot \mathbf{u}_1) \mathbf{e}_x.
\end{align*}
\]

According to Table 1, the center of the R joint \( \mathbf{A}_i \), can be written in CGA as \( \mathbf{A}_i = \mathbf{a}_i + \frac{1}{2} \mathbf{r}_i^2 \mathbf{e}_x + \mathbf{e}_y \).

Point \( \mathbf{B}_i \) is reduced from Eqs. (9) and (15) as

\[ \mathbf{B}_i = \mathbf{B}_i^* \mathbf{I}_C^{-1} = -\mathbf{B}_i^* \mathbf{I}_C = -\mathbf{S}_{a_{\mathbf{B}_i}} \cdot \mathbf{B}_i^*, \tag{16} \]

where \( \mathbf{B}_i^* = -\left( \mathbf{S}_{a_{\mathbf{B}_i}} \wedge \mathbf{S}_{a_{\mathbf{B}_i}} \wedge \mathbf{n}_1 \right) \mathbf{I}_C \) and is dual to the point pair \( \mathbf{B}_{a_{\mathbf{B}_i}} \), which is generated by the intersection of two spheres \( \mathbf{S}_{a_{\mathbf{B}_i}} \), \( \mathbf{S}_{a_{\mathbf{B}_i}} \) and a plane \( \mathbf{n}_1 \).

Please note the expression of point \( \mathbf{B}_i \) is not in its standard and normalized form, i.e., the magnitude is not equal to 1 and we can obtain its standard and normalized form by dividing Eq. (16) using its magnitude \((-\mathbf{e}_x \cdot \mathbf{B}_i^*)\).

For other 3-DOF PMs, such as 3-PRS PMs and 3-RRS PMs, the CGA-based formulation for their direct kinematics is the same. Eqs. (14) and (16) are the representation of the coordinates of two spherical joints \( \mathbf{B}_i \) and \( \mathbf{B}_i \) in the coordinates of point \( \mathbf{B}_i \). Therefore, once the coordinates of point \( \mathbf{B}_i \) are known, the coordinates of the other two points are also determined, i.e., the position and the orientation of the moving platform w.r.t the base are known.

### 3.2 CGA-based formulation for direct kinematics of the second category of PMs

In this subsection, we will take a 6-3 Stewart platform as the second case. A 6-3 Stewart platform \( \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4 \mathbf{A}_5 \mathbf{B}_1 \mathbf{B}_2 \mathbf{B}_3 \), shown in Fig. 2, has six SPS limbs meeting in a pair-wise fashion at three-points on the triangular moving platform. The six limb lengths \( l_i \) \((i = 1, 2, 3)\) provided by P-joints are six inputs to control the position and orientation of the moving platform. For the general 6-3 Stewart mechanism, its six spherical joints on the base are not restricted to lie in a plane. Let \( \mathbf{a}_i \) \((i = 1, 2, 3)\) and \( \mathbf{b}_i \) \((i = 1, 2, 3)\) denote the coordinates of the center of the spherical joints \( \mathbf{A}_i \) and \( \mathbf{B}_i \) in the Euclidean space respectively; let \( r_i \) \((i = 1, 2, 3)\) denotes the distance of the three S-joints \( \mathbf{B}_i \) on the moving platform, where the coordinates \( \mathbf{b}_i \) \((i = 1, 2, 3)\) are unknown. In this subsection, we will apply the same denotation for the CGA-based formulation to the direct kinematics in the absence of confusion.

As seen from Fig. 2, in the tetrahedron \( \mathbf{A}_1 \mathbf{A}_2 \mathbf{A}_3 \mathbf{A}_4 \mathbf{A}_5 \mathbf{B}_1 \mathbf{B}_2 \mathbf{B}_3 \), point \( \mathbf{B}_i \) must be located on a sphere \( \mathbf{S}_{a_{\mathbf{B}_i}} \) of radius \( r_i \) with its center at point \( \mathbf{B}_i \), a sphere \( \mathbf{S}_{a_{\mathbf{B}_i}} \) of radius \( l_i \) with its center at point \( \mathbf{A}_1 \), and a sphere \( \mathbf{S}_{a_{\mathbf{B}_i}} \) of radius \( l_i \) with its center at point \( \mathbf{A}_4 \). Thus, the actual locus of point \( \mathbf{B}_i \) must be located on the intersection
of the three spheres \( S_{a_1b_1} \), \( S_{a_2b_2} \), and \( S_{a_3b_3} \). From the knowledge of geometry, it is known that the locus of this intersection will be a point pair \( B_{32} \). And therefore, according to Table 1, the point pair \( B_{32} \) can be formulated in CGA as

\[
B_{32} = S_{a_1b_1} \land S_{a_2b_2} \land S_{a_3b_3},
\]

(17)

where the three spheres \( S_{a_1b_1} \), \( S_{a_2b_2} \) and \( S_{a_3b_3} \) can be represented in CGA as

\[
S_{a_1b_1} = B_1 - \frac{1}{2} l_1^2 e_n \, , \quad S_{a_2b_2} = A_1 - \frac{1}{2} l_2^2 e_n \, , \quad S_{a_3b_3} = A_3 - \frac{1}{2} l_3^2 e_n.
\]

According to Table 1, the three centers of joints \( B_1, A_1 \), and \( A_3 \) can be represented in CGA as

\[
B_1 = b_1 + \frac{1}{2} h_1^2 e_n + e_o, \quad A_1 = a_1 + \frac{1}{2} a_2^2 e_n + e_o, \quad A_3 = a_3 + \frac{1}{2} a_4^2 e_n + e_o.
\]

Let \( B_{32}' = B_{32} I_C^{-1} = -B_{32} I_C \) be the dual of the point pair \( B_{32} \) (see Eq. (9)). Point \( B_3 \) is dissected from the point pair \( B_{32} \) in the conformal space as [17, 22]

\[
B_3 = \frac{T_{32} + \sqrt{B_{33} T_{31}}}{A_{33}},
\]

(18)

where \( T_{31} = e_o \cdot B_{32}' \), \( T_{32} = T_{33} \cdot B_{32}' \), \( A_{33} = T_{31} + T_{32} \), and \( B_{33} = B_{32}' \). The four spheres \( S_{a_1b_1} \), \( S_{a_2b_2} \), and \( S_{a_3b_3} \) can be represented in CGA as

\[
S_{a_1b_1} = B_1 - \frac{1}{2} l_1^2 e_n \, , \quad S_{a_2b_2} = A_1 - \frac{1}{2} l_2^2 e_n \, , \quad S_{a_3b_3} = A_3 - \frac{1}{2} l_3^2 e_n.
\]

According to Table 1, the two centers of joints \( A_2 \), and \( A_3 \) can be represented in CGA as

\[
A_2 = a_2 + \frac{1}{2} a_2^2 e_n + e_o, \quad A_3 = a_3 + \frac{1}{2} a_4^2 e_n + e_o.
\]

Point \( B_3 \) is reduced from Eqs. (9) and (19) as

\[
B_3 = B' I_C^{-1} = -B'I_C = -S_{a_2b_2} \cdot B'_{33}',
\]

(20)

where \( B_{33}' = -\left(S_{a_1b_1} \land S_{a_2b_2} \land S_{a_3b_3}\right) I_C \) and is dual to the point pair \( B_{33} \), which is generated by the intersection of three spheres \( S_{a_2b_2} \), \( S_{a_2b_2} \), and \( S_{a_3b_3} \).

As seen from the above-mentioned CGA-based formulation for two categories of PMS, the representation of the coordinates of two spherical joints \( B_3 \) and \( B_3 \) is the same and the only difference lies in the expression of the bi-vectors \( B'_{32} \) and \( B'_{33} \).

### 3.3 The derivation of the coordinate-invariant polynomial equation

In this subsection, we will derive the coordinate-invariant polynomial equation for the direct kinematics problem of both categories of PMS.

According to Eq. (10), we can readily obtain

\[
\left( B_2 \cdot B_3 \right) = 0 \Leftrightarrow \left( -S_{a_2b_2} \cdot B'_{33} \right) \left( -S_{a_3b_3} \cdot B_{32}' \right) = 0
\]

(21)

Substituting Eqs. (14) or (18) into Eq. (21) and then expanding it, we have

\[
\left( \frac{2}{A_{33}} \left( U \cdot V \right) \pm \frac{1}{A_{33}} \left( U \cdot W \right) \right) \sqrt{B_{33} T_{31}} = \frac{B_{33}}{A_{33}} \left( \frac{U \cdot U}{4} + \frac{1}{A_{33}} \left( V \cdot V \right) - \frac{1}{A_{33}} \left( V \cdot W \right) + \frac{1}{4} \left( W \cdot W \right) \right)
\]

(22)

where \( U = T_{31}, B_{32}' \), \( V = T_{32}, B_{33}' \), and \( W = e_o \cdot B_{32}' \cdot U \), \( V \) and \( W \) are all 1-blade.
Taking the square of both sides of Eq. (22) and combining terms, we obtain

\[
\sum_{i=1}^{2} C_{i} + \frac{C_{1}^{2}}{A_{i}^{2}} + \frac{C_{2}}{A_{i}^{3}} + \frac{C_{3}}{A_{i}^{4}} + C_{0} = 0
\]

where the coefficients \( C_{i} (i = -1, -2, -3, -4) \) are expressed as

\[
C_{-1} = -4B_{vA} (U \cdot V) + B_{vA} (U \cdot U) + 2B_{vA} (U \cdot U)(V \cdot V) + (V \cdot V) \]

\[
C_{-2} = 2B_{vA} \left( U \cdot U \right) (V \cdot W) - 2B_{vA} \left( U \cdot U \right) (V \cdot V) - 2r_{i}^{2} (V \cdot V) (W \cdot W),
\]

\[
C_{-3} = -B_{vA} r_{i}^{4} (U \cdot W) + \frac{B_{vA} r_{i}^{4}}{2} (U \cdot U)(W \cdot W) + \frac{r_{i}^{8}}{2} (V \cdot V)(W \cdot W) + r_{i}^{8} (V \cdot V) \]

\[
C_{-4} = -\frac{r_{i}^{8}}{2} (V \cdot W)(W \cdot W), \quad \text{and} \quad C_{0} = \frac{r_{i}^{8}}{16} (W \cdot W)^{2}.
\]

Simplifying the coefficients \( C_{-1}, C_{-2}, C_{-3} \) and \( C_{-4} \) by using Eqs. (1)-(8), we have

\[
C_{-1} = A_{i}^{2} \left( B_{vA} D_{vA} - G \cdot G + C_{i}^{2} \right) - 4B_{vA} C_{i}^{2} D_{vA},
\]

\[
C_{-2} = A_{i}^{2} \left( -2r_{i}^{2} (B_{vA} D_{vA} - G \cdot G + C_{i}^{2}) (V \cdot W) + 4B_{vA} C_{i}^{2} D_{vA} E_{var} r_{i}^{2} \right),
\]

\[
C_{-3} = -B_{vA} D_{vA} E_{var} r_{i}^{2} + \frac{A_{i}^{2} r_{i}^{4}}{2} \left( B_{vA} D_{vA} - G \cdot G + C_{i}^{2} \right) (W \cdot W) + r_{i}^{4} (V \cdot V) \]

where \( C_{i} = B_{i}^{T} \cdot B_{i} \cdot D_{i}^{T} \cdot D_{i} , \quad D_{i} = B_{i}^{T} \cdot B_{i} , \quad E_{var} = (T_{i}^{T} \wedge e_{i}) \cdot B_{i}^{T} , \) and \( G = B_{i}^{T} \wedge B_{i} , \quad C_{var} , \quad D_{var} \) and \( E_{var} \) are all scalars and \( G \) is a 4-vector. The detailed derivation of Eqs. (24)-(26) is given in the appendix.

Substituting Eqs. (24)-(26) into Eq. (23), after rearranging, simplifying and taking only the numerator, we have

\[
\left( B_{A} D_{A} - G \cdot G + C_{i}^{2} \right) + \frac{r_{i}^{8}}{4} \left( A_{i} r_{i}^{2} (W \cdot W) - 4(V \cdot W) \right) - B_{vA} D_{vA} \left( 2C_{i}^{2} - E_{var} r_{i}^{2} \right) = 0
\]

For the above-mentioned two categories of PMs, Eq. (27) is the same and the only difference lies in the expression of the bi-vectors \( B_{i}^{T} \) and \( B_{i} \). The derivation of Eq. (27) is coordinate-invariant and Eq. (27) depends on only the design parameters, the inputs and the coordinates of point \( B_{i} \).

### 3.4 The representation of point \( B_{i} \) in the variable \( \theta \)

In Fig. 1, it can be clearly seen that point \( B_{1} \) can be only located on a circle with its center \( A_{1} \) and a revolute axis \( u_{1} \). In Fig. 2, in the triangle \( A_{2}B_{1}A_{1} \), point \( B_{1} \) must be located on the intersection of the two spheres, one sphere of radius \( l_{1} \) with its center at \( A_{1} \) and the other sphere of radius \( l_{2} \) with its center at \( A_{2} \).

Thus, the locus of point \( B_{1} \) will be a circle with its center \( A_{1} \), where point \( A_{1} \) denotes the foot point on the line \( A_{1}A_{2} \) with respect to point \( B_{1} \). Therefore, point \( B_{1} \) can be only located on a circle with an axis \( A_{1}A_{2} \). Thus, for these two categories of PMs, point \( B_{1} \) can be only located on a circle.

For convenience, for the first category of PMs, we attach a fixed coordinate frame \( O-XYZ \) to the base platform with its origin \( O \) located on the geometric center of the base and the \( X \)-axis along with the first \( R \) joint. We assume the second \( R \) joint is located in the \( XY \) plane of the fixed frame. For the second category of PMs, we attach a fixed coordinate frame \( O-XYZ \) to the base platform with its origin \( O \) located on the geometric center of the base and the \( Y \)-axis along with the rotational axis \( A_{1}A_{2} \). We assume point \( A_{3} \) is located in the \( XY \) plane of the fixed frame. The coordinates of \( A_{i} (i = 1, 2, L, 6) \) and the coordinates of \( B_{i} (i = 1, 2, 3) \) in the fixed reference frame are denoted by \( a_{i} = (a_{i_{x}}, a_{i_{y}}, a_{i_{z}})^{T} \) and \( b_{i} = (a_{i_{x}}, a_{i_{y}}, a_{i_{z}})^{T} \) respectively, where the coordinates of joints \( A_{i} \) are known.

The variable \( \theta \) is introduced to denote angles between the two lines \( A_{i}B_{j} \) and the \( X \)-axis or two lines \( A_{i}A_{j} \) and the \( X \)-axis. Now the coordinates of point \( B_{i} \) in the fixed frame can be written as

\[
b_{i} = Y(\theta)(l_{i}, 0, 0)^{T} + a_{i_{x}}(l_{i}, \cos \theta, 0, l_{i} \sin \theta)^{T} + a_{i_{y}},
\]

where for the 3-DOF PM, \( l_{i} = [A_{i}, B_{i}] \). For the Stewart platform, \( l_{i} = [A_{i}, B_{i}] \), which can be determined by using the sine law for a planar triangle \( A_{i}B_{i}A_{j} \). \( Y(\theta) \) denotes the rotation matrix representing rotation about the \( Y \)-axis.

According to Table 1, point \( B_{1} \) can be represented in CGA as

\[
b_{i} = b_{i} + \frac{1}{2} b_{i}^{*} e_{n} + e_{0}
\]
3.5 The derivation of a univariate polynomial equation

Introducing the tangent-half-angle substitution, we have \( \cos \theta = \frac{1-x^2}{1+x^2} \) and \( \sin \theta = 2x/(1+x^2) \), where \( x = \tan(\theta/2) \). After substituting Eq. (29) and the above identities into Eq. (27), multiplying by \( (1+x^2)^4 \), and observing Eq. (27), we can see that the total degree in \( x \) do not exceed 16 and the degree of the term \( A_{xx} B_{xx} C_{xx} D_{xx} E_{xx} (G\cdot G), (V\cdot W), (W\cdot W) \) in \( x \) are listed in Table 2. Therefore we can conclude that Eq. (27) yields a 16th-degree polynomial equation in \( x \) as

\[
\sum_{i=0}^{16} m_i x^i = 0
\]

(30)

where the coefficient \( m_i (i = 0, 1, \ldots, 16) \) depends on only the design parameters and inputs of PMs. Eq. (30) agrees with the well-known results in [3].

| Table 2 The degree of the terms for Eq. (27) in \( x \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| term            | \( A_{xx} \)    | \( B_{xx} \)    | \( C_{xx} \)    | \( D_{xx} \)    | \( E_{xx} \)    | \( G\cdot G \)  | \( V\cdot W \)  |
| degree          | 4               | 4               | 4               | 4               | 4               | 8               | 8               | 4               |

However, Eq. (27) will be degenerate to a 12th-degree polynomial equation when the axes of the three revolute joints are parallel each other and will yield the following polynomial equation in \( x \) as

\[
\sum_{i=0}^{12} n_i x^i = 0
\]

(31)

where the coefficient \( n_i (i = 0, 1, \ldots, 12) \) depends on only the design parameters and inputs of the PMs.

3.6 Back substitution

Solving Eq. (30), all the 16 solutions for \( x \) can be obtained. For each solution for \( x \), we obtain the corresponding value for \( \theta \) using \( \theta = 2\tan^{-1} x \). After the values of \( \theta \) are obtained, the coordinates of the joint \( B_1 \) can be directly calculated from Eq. (28). For the coordinates of the joint \( B_2 \), it cannot be determined only in terms of Eq. (14) or (18) after substituting the coordinates of point \( B_1 \). The choice of the positive or negative sign can be obtained from the ratio of two terms

\[
\frac{B_{xx}}{A_{xx}} (U\cdot U) + \frac{1}{A_{xx}} (V\cdot V) - \frac{n^2}{A_{xx}} (V\cdot W) + \frac{\theta^2}{4} (W\cdot W) \quad \text{and} \quad \left(-\frac{2}{A_{xx}} (U\cdot V) + \frac{n^2}{A_{xx}} (U\cdot W) \right) \sqrt{B_{xx}}.
\]

If the ratio is equal to 1, we choose the positive sign and vice versa. If both the two terms are equal to 0, we choose the sign depending on whether the term \( B_{xx} \) equals to 0. If the term \( B_{xx} \) equals to 0, we can choose any sign and it will reduce to the same results. If the term \( B_{xx} \) is not equal to 0, we determine the sign by making sure that the value of \( (-e \cdot B_x) \) is not equal to 0. After we choose the right sign in Eq. (14) or (18), we can obtain the coordinates of point \( B_2 \). The coordinates of the joint \( B_2 \) can be obtained from Eq. (16) or (20) by dividing its magnitude \( (-e_x \cdot B_x) \) after substituting the coordinates of the joints \( B_1 \) and \( B_2 \).

3.7 CGA-based solution procedure for direct kinematics of PMs with the 3-RS structure

Based on the aforementioned theory, we can summarize the procedure of the CGA-based method for direct kinematics of PMs with the 3-RS structure. The detailed steps of the solution procedure are presented as follows:

**Step 1.** Represent the position of the spherical joint \( B_2 \) on the moving platform in CGA by dissecting it from the point pair which is determined by the intersection of three spheres or two spheres and one plane.

**Step 2.** Represent the position of the spherical joint \( B_1 \) on the moving platform in CGA by the intersection of four spheres or three spheres and one plane.

**Step 3.** Derive the coordinate-invariant polynomial equation in the position of the spherical joint \( B_1 \) according to \( B_1 \cdot B_2 = 0 \).

**Step 4.** Represent the position of the spherical joint \( B_1 \) in the variable \( \theta \).

**Step 5.** Derive a univariate polynomial equation from the coordinate-invariant polynomial equation by the tangent-half-angle substitution.
Step 6. Solve the coordinates of the three spherical joints analytically or numerically by back substitution.

3.8 Discussion on the direct kinematics of 3-RPS PMs with symmetric architecture

A symmetric 3-RPS architecture (Fig. 3), consists of an equilateral triangle base and moving platform. The circumradius of the triangle base \( A_1A_2A_3 \) is denoted by \( R_1 \) and the center coincides with the origin of the chosen coordinate frame \( O-XYZ \) in subsection 3.4 for the base platform. The circumradius of the triangle moving platform \( B_1B_2B_3 \) is denoted by \( R_2 \). The axis of the R joint \( A_i \) are tangent to the circumsphere of the triangle \( A_iA_jA_k \). The coordinates of the point \( A_i (i=1, 2, 3) \) in the frame \( O-XYZ \) are expressed as

\[
a_i = (R_1, 0, 0)^T, \quad a_2 = (-R_1 / 2, \sqrt{3}R_1 / 2, 0)^T, \quad a_3 = (-R_1 / 2, -\sqrt{3}R_1 / 2, 0)^T
\]

(32)

The unit vectors along the joint axes of these R joints are \( u_i = (0,1,0)^T, \quad u_2 = (\sqrt{3}/2,1/2,0)^T \) and \( u_3 = (-\sqrt{3}/2,1/2,0)^T \) respectively. The distance \( r_i (i=1, 2, 3) \) between the two spherical joints \( B_i \) is represented by \( r_1 = r_2 = r_3 = \sqrt{3}R_2 \). The three limb lengths are denoted by \( l_i (i=1, 2, 3) \). Therefore, for symmetric architecture of 3-RPS PM, once five parameters are given, we can determine its direct kinematics.

![Diagram of a symmetric 3-RPS parallel mechanism](image)

Substituting Eq. (32) and the unit vectors \( u_i (i=1, 2, 3) \) into Eq. (30), we obtain

\[
(p_{10} + p_{11}X + p_{12}X^2 + p_{13}X^3 + p_{14}X^4)(p_{20} + p_{22}X + p_{23}X^2 + p_{24}X^3 + p_{24}X^4) = 0
\]

(33)

where \( X = x^2 \), \( p_j (i=1, 2, j=0, 1, 2, 3, 4) \) depends on only the five parameters. The expansion of the coefficients \( p_j \) is omitted here due to space limitation.

When the three limb length \( l_i (i=1, 2, 3) \) is identical to each other, i.e., \( l_1 = l_2 = l_3 \), Eq. (33) can be simplified as

\[
(-R_1 + R_2)^2 \left( R_1 + R_2 + l + R_1X + R_2X - lX \right) (-R_1 + R_2 - l - R_1X + R_2X + lX)\]
\[
\left( (9R_1R_2^2 + 9R_2^3 + 4R_1l_1^2 - 8R_1R_2l_1 - 3R_2l_1^2 + 4l_1^3) + (9R_1R_2^2 + 9R_2^3 + 4R_1l_2^2 - 8R_1R_2l_2 - 3R_2l_2^2 + 4l_2^3) \right)\]
\[
\left( (9R_1R_2^2 + 9R_2^3 - 4R_1l_1^2 - 8R_1R_2l_1 + 3R_2l_1^2 - 4l_1^3) + (9R_1R_2^2 + 9R_2^3 - 4R_1l_2^2 - 8R_1R_2l_2 + 3R_2l_2^2 - 4l_2^3) \right) = 0
\]

(34)

Thus, for symmetric 3-RPS PMs, the solutions for \( x \) can be found analytically by solving two sets of polynomial equation of degree four in Eq. (33), which correspond to the two possible operation modes. The results agree well with those in Ref. [23-25]. When the three limb lengths are identical, we can obtain simpler result for \( x \) by solving Eq. (34) analytically.

3.9 Discussion on the direct kinematics of 3-RPS PMs with three parallel revolute joints
In subsection 3.5, we mentioned that for a 3-RPS PM, when the axes of the three R joints are parallel, a 12th-degree univariate polynomial equation was derived for the general case. Next, we will discuss the 3-RPS PM with the following special configuration, shown in Fig. 4. The axes of the three R joints are all perpendicular to the base and the base and the moving platform is similar or identical, i.e., 

\[ \mathbf{u}_i = (0, 0, 1)^T \quad (i = 1, 2, 3). \]

It is found that the number of solutions will be ten. Among the ten solutions, four solutions are the ones similar to the planar 3-RPR PM [45, 46] and the other six solutions are the same as those to the general 3-RPR planar PM where the moving platform is the mirrored version of the one in the 3-RPR PM. For the case, Eq. (30) becomes

\[
(q_{i0} + q_{i1}x + q_{i2}x^2 + q_{i3}x^3 + q_{i4}x^4)(q_{20} + q_{21}x + q_{22}x^2 + q_{23}x^3 + q_{24}x^4 + q_{25}x^5 + q_{26}x^6) = 0
\]

(35)

where \( q_{ij} (i = 1, 2; j = 0, 1, 2, 3, 4, 5, 6) \) depends only on the design parameters and the inputs. The expansion of the coefficients \( q_{ij} \) is not given due to space limitation.

![Fig. 4. A 3-RPS parallel mechanism with special configuration](image)

### 3.10 Discussion on the direct kinematics of 3-PRS PMs with symmetric architecture

A symmetric 3-PRS architecture (Fig. 5), consists of an equilateral triangle base and moving platform, which are connected each other by three limbs with P joints as actuated joints. The three P joints are all perpendicular to the base platform. The circumradius of the triangle base \( \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \) is denoted by \( R_1 \) and the center coincides with the origin of the chosen coordinate frame \( O-XYZ \) in subsection 3.4 for the base platform. The circumradius of the triangle moving platform \( \mathbf{B}_1\mathbf{B}_2\mathbf{B}_3 \) is denoted by \( R_2 \). The axis \( \mathbf{u}_i \) of the R joint \( \mathbf{A}_i \) are tangent to the circumcircle of the triangle \( \mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \). The three limb lengths and the three inputs are denoted by \( l_i \) \( (i = 1, 2, 3) \) and \( \rho_i \) \( (i = 1, 2, 3) \) respectively. The three limb lengths are identical, i.e., \( l_1 = l_2 = l_3 \). The coordinates of the point \( \mathbf{A}_i \) \( (i = 1, 2, 3) \) in the frame \( O-XYZ \) are expressed as

\[
\mathbf{a}_i = (R_i, 0, \rho_i)^T, \quad \mathbf{a}_2 = (-R_i / 2, \sqrt{3}R_i / 2, \rho_i)^T, \quad \mathbf{a}_3 = (-R_i / 2, -\sqrt{3}R_i / 2, \rho_i)^T
\]

(36)

The unit vectors along the axes of three R joints are \( \mathbf{u}_1 = (0, 1, 0)^T, \quad \mathbf{u}_2 = (\sqrt{3}/2, 1/2, 0)^T \) and \( \mathbf{u}_3 = (-\sqrt{3}/2, 1/2, 0)^T \) respectively. The distance \( r_i \) \( (i = 1, 2, 3) \) between the two spherical joints \( \mathbf{B}_i \) is represented by \( r_1 = r_2 = r_3 = \sqrt{3}R_i \). Therefore, for a symmetric architecture of 3-PRS PM, once design parameters and inputs are given, we can determine its direct kinematics.
For the direct kinematics of the symmetric 3-PRS PM, according to the values of the inputs \( \rho_i \) (\( i=1,2,3 \)), we clarify them into four cases.

For the first case, all the inputs are not identical and not equal to 0, i.e., \( \rho_1 \neq \rho_2 \neq \rho_3 \neq 0 \), and \( 2\rho_1 - \rho_2 - \rho_3 \neq 0 \), Eq. (30) becomes:

\[
\begin{align*}
\left( g_{10} + g_{11}x + g_{12}x^2 + g_{13}x^3 + g_{14}x^4 + g_{15}x^5 + g_{16}x^6 + g_{17}x^7 + g_{18}x^8 \right) \\
\left( g_{20} + g_{21}x + g_{22}x^2 + g_{23}x^3 + g_{24}x^4 + g_{25}x^5 + g_{26}x^6 + g_{27}x^7 + g_{28}x^8 \right) &= 0
\end{align*}
\]  

(37)

where \( g_{ij} (i=1,2; j=0,1,2,3,4,8) \) only depends on the link parameters and inputs. The expansion of the coefficient \( g_{ij} \) is not given here due to space limitation.

For the second case, all the inputs are not identical, i.e., \( \rho_1 \neq \rho_2 \neq \rho_3 \), but \( 2\rho_1 - \rho_2 - \rho_3 = 0 \), Eq. (30) becomes:

\[
\begin{align*}
\left( f_{10} + f_{11}X + f_{12}X^2 + f_{13}X^3 + f_{14}X^4 \right) \left( f_{20} + f_{21}X + f_{22}X^2 + f_{23}X^3 + f_{24}X^4 \right) &= 0
\end{align*}
\]  

(38)

where \( X = x^2 \). \( f_{ij} (i=1,2; j=0,1,2,3,4) \) only depends on the link parameters and inputs. The expansion of the coefficient \( f_{ij} \) is omitted here due to space limitation.

For the third case, two of three inputs are identical, when \( \rho_1 = \rho_2 \neq \rho_3 \), or \( \rho_1 = \rho_3 \neq \rho_2 \), Eq. (30) becomes

\[
\begin{align*}
\left( R_1 + R_2 + l_1 + R_3x^2 + R_4x^2 - l_4x^2 \right)^2 \left( -R_1 + R_2 - l_1 - R_3x^2 + R_4x^2 + l_4x^2 \right)^2 \\
\left( c_{10} + c_{11}x + c_{12}x^2 + c_{13}x^3 + c_{14}x^4 \right) \left( c_{20} + c_{21}x + c_{22}x^2 + c_{23}x^3 + c_{24}x^4 \right) &= 0
\end{align*}
\]  

(39)

When \( \rho_2 = \rho_3 \neq \rho_1 \), Eq. (30) becomes

\[
\begin{align*}
\left( d_{10} + d_{11}x + d_{12}x^2 \right)^2 \left( d_{20} + d_{21}x + d_{22}x^2 \right)^2 \\
\left( d_{30} + d_{31}x + d_{32}x^2 + d_{41}x^3 + d_{42}x^4 \right) \left( d_{40} + d_{41}x + d_{42}x^2 + d_{44}x^4 \right) &= 0
\end{align*}
\]  

(40)

where \( c_{ij} (i=1,2; j=0,1,2,3,4) \) and \( d_{ij} (i=1,2,3,4; j=0,1,2,3,4) \) depend on only the design parameters and the inputs. Their expansion is not given due to space limitation.

For the fourth case, all the inputs are identical to each other, i.e., \( \rho_1 = \rho_2 = \rho_3 \), Eq. (30) is the same as Eq. (34).

Thus, for symmetric 3-PRS PMs, the solutions for \( x \) can be found by solving two sets of polynomial equation of degree eight or four, which correspond to the two possible operation modes. The results agree well with those in Ref. [24].
3.11 Comparison between CGA-based approach and the existing methods for direct kinematics of PMs with the 3-RS structure

The proposed CGA-based approach is relatively universal and unified for direct kinematics of PMs with the 3-RS structure. As compared with the existing methods in the literature, the characteristics of the proposed CGA-based approach are:

1. The direct kinematics of PMs with the same structure can be solved in a unified and simplified approach. For a family of Stewart platforms with the 3-RS structure, the architecture is not required to change the equivalent 3-RS structure for the direct kinematics and the formulation is derived directly in terms of geometric relationship.

2. The kinematic formulation has geometric meaning due to the intuitiveness of CGA.

3. The univariate polynomial equation is derived via CGA operation and it does not require algebraic elimination.

4. The solution procedure is applicable to the direct kinematics of this family of PMs with any link parameters. For the PMs with the planar moving platform, we can obtain an even 16th-degree univariate polynomial equation and it is equivalent to solve an 8th-degree univariate polynomial equation. For general PMs, a 16th-degree univariate polynomial equation is derived.

4. Case Studies

In order to validate the solution procedure, five case studies are given. The first case study is the planar 3-RPS PM with symmetric architecture from Ref. [23]; the second case study is the planar 3-RPS PM with the axes of three R joints perpendicular to the base platform, and the link parameters and inputs are the same as the first case; the third case study is the 3-PRS PM with symmetric architecture from Ref. [8], whose three prismatic actuators are parallel and all perpendicular to the base platform; the fourth case study is the planar 6-3 Stewart platform from Ref. [13] and the last case is the general Stewart platform from Ref. [14]. For the former three case studies, we can obtain their analytical solution, and for the latter two case studies, we can obtain 16 solutions by solving 8/16-degree equation in x by using Newton-Raphson numerical technique.

4.1 Case study 1

The link parameters and inputs of the symmetric 3-RPS parallel mechanisms with equilateral triangular base and moving platforms (Fig. 3), are \( R_1 = 2.50, R_2 = 1.00, l_1 = 3.20, l_2 = 2.80, l_3 = 3.60 \). Substituting the link parameters and inputs into Eq. (33), we can obtain two groups of polynomials of degree four as

\[
6.2512\times 10^6 - 7.0504\times 10^6 X + 2.7292\times 10^6 X^2 - 3.9711\times 10^5 X^3 + 1.3357\times 10^4 X^4 = 0 \tag{41}
\]

\[
7.6698\times 10^7 + 3.3975\times 10^6 X + 2.7722\times 10^5 X^2 + 6.9152\times 10^4 X^3 + 5.4535\times 10^3 X^4 = 0 \tag{42}
\]

The two groups of polynomials correspond to the two operation modes of the 3-RPS PM. Solving Eq. (41) analytically, we can obtain the 8 real solutions for \( x \). Table 3 shows the four real solutions to the direct kinematics of the 3-RPS PM with the moving platform above the base. The other four real solutions correspond to 4 assembly modes with the moving platform below the base and are omitted here. The results agree with the one in Ref. [23]. Solving Eq. (42), we can obtain 8 complex solutions. Due to space limitation, they are not given here. The computation speed is faster due to the closed-form solution.

<table>
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<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( B_3 )</th>
</tr>
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<td>-0.2515</td>
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<td>0.7424</td>
<td>-0.6135</td>
<td>0.4543</td>
</tr>
<tr>
<td>3</td>
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<td>1.0888</td>
<td>-0.3626</td>
<td>-0.4344</td>
</tr>
<tr>
<td>4</td>
<td>1.5775</td>
<td>1.1347</td>
<td>-0.1877</td>
<td>-0.4142</td>
</tr>
</tbody>
</table>

Table 3 Four assembly modes with the moving platform above the base for case 1.
4.2 Case study 2

The link parameters and inputs of the 3-RPS PM with special structure mentioned in the subsection 3.8 (Fig. 4), are the same as the case study 1. Substituting the link parameters and inputs into Eq. (35), we can obtain two groups of polynomials of degree four and six respectively as

$$1.6024 \times 10^7 - 1.5317 \times 10^7 x + 1.6333 \times 10^7 x^2 - 2.0147 \times 10^7 x^3 + 1.7131 \times 10^7 x^4 = 0$$ (43)

$$205122 - 892.486x - 80809.9 x^2 - 1784.97 x^3 + 9015.33 x^4 - 892.49 x^5 + 35.4847 x^6 = 0$$ (44)

The two groups of polynomials correspond to the direct kinematics of the two planar 3-RPR PMs. Solving Eq. (43) analytically, we can obtain 2 real solutions and 2 complex solutions. The four solutions correspond to the ones for the planar 3-RPR PM with similar platform. Solving Eq. (44), we can obtain another 2 real solutions and 4 complex solutions. The six solutions correspond to the ones for the general 3-RPR planar PM where the moving platform is the mirrored version of the one. Table 4 only presents the four real solutions to the direct kinematics of the 3-RPS PM due to space limitation.

<table>
<thead>
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<th>x</th>
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<th>B₂</th>
<th>B₃</th>
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<tr>
<td></td>
<td>Y</td>
<td>0.3643</td>
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<td></td>
<td>Z</td>
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<td>0</td>
<td>0</td>
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<td>-0.4407</td>
<td>-0.2843</td>
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<td></td>
<td>Y</td>
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<tr>
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<td>Z</td>
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<td>0</td>
<td>0</td>
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<tr>
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<tr>
<td></td>
<td>Y</td>
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<td>0</td>
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<tr>
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<td>Z</td>
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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

4.3 Case study 3

The link parameters and inputs of the symmetric 3-PRS PM (shown in Fig. 5) with equilateral triangular base and moving platform, are: \( R_1 = \sqrt{3}/4, R_2 = \sqrt{3}/10, l_1 = l_2 = l_3 = 1.00, \rho_1 = \rho_2 = \rho_3 = 0 \). For the given inputs, substituting the link parameters into Eq. (34), we obtain

$$\begin{pmatrix} 3.5201 - 2.4999X \end{pmatrix} \begin{pmatrix} -7.9978 + 1.6222X \end{pmatrix} \begin{pmatrix} 1.6062 - 0.3938X \end{pmatrix} ^T \begin{pmatrix} -1.2598 + 0.7402X \end{pmatrix} = 0$$ (45)

Solving Eq. (45), we can obtain 4 groups of different real solutions. The latter two groups consist of three identical solutions respectively. Among the 16 solutions to the direct kinematics of a 3-PRS PM, there are four real ones as shown in Table 5. The proposed method is simpler than the one in Ref. [8].

<table>
<thead>
<tr>
<th>i</th>
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<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
</tr>
</thead>
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<td></td>
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<td>-0.0866</td>
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<tr>
<td></td>
<td>Y</td>
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<td>0.1500</td>
<td>-0.1500</td>
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<td>Z</td>
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<td>0.9657</td>
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<td>-0.1318</td>
<td>0.0866</td>
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<td></td>
<td>Y</td>
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<td>0.2282</td>
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</tr>
<tr>
<td></td>
<td>Z</td>
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<td>Z</td>
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<td>0.7488</td>
<td>0.9657</td>
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</tbody>
</table>

4.4 Case study 4

The link parameters and inputs of the planar Stewart platform (Fig. 2) are given in Table 6. The reference frame is not consistent with the one in the subsection 3.4. Thus the coordinates of the six spherical joints in the base are expressed in the reference system \( O-XYZ \) as shown in Table 7. With the given link parameters and inputs, Eq. (30) is an even 16th-degree equation as:

$$4.0229 \times 10^3 x^{16} + 1.3689 \times 10^6 x^4 - 4.5162 \times 10^6 x^{12} + 5.3161 \times 10^6 x^{10} - 2.5990 \times 10^6 x^8 + 4.4940 \times 10^5 x^6 - 2.1902 \times 10^4 x^4 - 7.8035 \times 10^2 x^2 + 6.2936 \times 10^1 = 0$$ (46)
Solving Eq. (46), we can obtain 16 solutions. For the given link parameters and inputs, we can only obtain 4 real solutions and Table 8 lists all the 4 real solutions. The results agree with those in Ref. [17].

Table 6 Link parameters and inputs for case 4

<table>
<thead>
<tr>
<th>O-X-Y-Z</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
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<tr>
<td>$a_\alpha$</td>
<td>-2.9</td>
<td>-1.2</td>
<td>1.3</td>
<td>-1.2</td>
<td>2.5</td>
<td>3.2</td>
</tr>
<tr>
<td>$a_\beta$</td>
<td>-2.9</td>
<td>3.0</td>
<td>-2.3</td>
<td>-3.7</td>
<td>4.1</td>
<td>1.0</td>
</tr>
<tr>
<td>$a_\gamma$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The distances $r_i$ between the points $B_i$

$r_i = 2.0, r_2 = 2.0, r_3 = 3.0$

The lengths of six legs $l_i$

$l_1 = 5.0, l_2 = 4.5, l_3 = 5.0, l_4 = 5.5, l_5 = 5.5, l_6 = 5.7$

Table 7 Inputs for case 4 in the reference system O-XYZ

<table>
<thead>
<tr>
<th>O-X-Y-Z</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_\alpha$</td>
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<td>0</td>
<td>4.0803</td>
<td>1.4802</td>
<td>5.7676</td>
<td>6.2278</td>
</tr>
<tr>
<td>$a_\beta$</td>
<td>-2.9</td>
<td>-1.2</td>
<td>1.3</td>
<td>-1.2</td>
<td>2.5</td>
<td>3.2</td>
</tr>
<tr>
<td>$a_\gamma$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8 Four real solutions for case 4

<table>
<thead>
<tr>
<th>$i$</th>
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<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
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<td>1.5287</td>
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</tr>
<tr>
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<td>0.4002</td>
<td>3.1079</td>
<td>1.5287</td>
<td>1.1961</td>
</tr>
<tr>
<td>4</td>
<td>0.9641</td>
<td>0.1570</td>
<td>1.6474</td>
<td>2.0248</td>
</tr>
</tbody>
</table>

4.5 Case study 5

The link parameters and inputs of the general Stewart platform are given in Table 9. Using the above solution procedure, we have obtained 4 real solutions as shown in Table 10. The results agree well with those in Ref. [13]. For the general case, the computation speed is slower than the planar Stewart mechanism, due to the maximum degree is 16 not 8.

Table 9 Link parameters and inputs for case 5

<table>
<thead>
<tr>
<th>O-X-Y-Z</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
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<td>$a_\beta$</td>
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<td>-20</td>
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<td>31</td>
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<td>$a_\gamma$</td>
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<td>40</td>
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<td>68</td>
<td>-93</td>
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</table>

The distances $r_i$ between the points $B_i$

$r_i = 135, r_2 = 190, r_3 = 141$

The lengths of six legs $l_i$

$l_1 = 76, l_2 = 160, l_3 = 139, l_4 = 55, l_5 = 128, l_6 = 217$

Table 10 Four real solutions for case 5

<table>
<thead>
<tr>
<th>$i$</th>
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5. Conclusion

The paper has proposed a CGA-based approach for the direct kinematics of a class of PM with the 3-RS structure. This approach is general enough to cover several PMs in the literature. Thanks to the intuitiveness of CGA, the representations of the positions of two spherical joints have explicit geometric meaning. A coordinate-invariant polynomial equation was derived via CGA operation. The univariate polynomial equation has been derived directly without any algebraic elimination from the aforementioned coordinate-invariant equation by using tangent-half-angle substitution. For the symmetric architectures of PMs with the 3-RS structure, we can obtain their analytical solutions from two groups of equations of degree 4 corresponding to two possible operational modes.

In future, we will extend this approach to both the inverse and direct kinematics of more complicated PMs.

Acknowledgements

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References


Appendix: Derivation of Equations (24)-(26)

Eq. (24) is derived as
\[ C_{4} = -4B_{12} (U \cdot V)^{2} + B_{12}^{2} (U \cdot U)^{2} + 2B_{12} (U \cdot U)(V \cdot V) + (V \cdot V)^{2} \]
\[ = \left( -4B_{12} (U \cdot V)^{2} + 4B_{12} (U \cdot U)(V \cdot V) \right) + \left( B_{12}^{2} (U \cdot U)^{2} - 2B_{12} (U \cdot U)(V \cdot V) + (V \cdot V)^{2} \right) \]
\[ = -4B_{12} \left( (U \cdot V) - (U \cdot U)(V \cdot V) \right) + (-B_{12} (U \cdot U) + (V \cdot V))^{2} \]
\[ = -4B_{12} \left( A_{12} B_{12} + B_{12}^{2} \right) \left( A_{12} B_{12} + (B_{12} D_{12} - G \cdot G + C_{12}^{2}) \right) \]
\[ = A_{12}^{2} \left( -B_{12} C_{12} D_{12} + B_{12} D_{12} - G \cdot G + C_{12}^{2} \right). \]

where \( U \wedge V = A_{12} B_{12} \), \( -B_{12} (U \cdot U) + (V \cdot V) = A_{12} \left( B_{12} D_{12} - G \cdot G + C_{12}^{2} \right) \), \( C_{12} = B_{12}^{2} \cdot B_{12} \).

The expansion of \( U \wedge V \) is expressed as
\[ U \wedge V = (T_{12} \cdot B_{12}^{3}) \wedge (T_{12} \cdot B_{12}^{3}) = (T_{12} \cdot B_{12}^{3}) \wedge (T_{12} \cdot B_{12}^{3}) \]
\[ = ((T_{12} \cdot B_{12}^{3}) B_{12}^{3} - (T_{12} \cdot B_{12}^{3}) B_{12}^{3}) \cdot ((T_{12} \cdot B_{12}^{3}) B_{12}^{3} - (T_{12} \cdot B_{12}^{3}) B_{12}^{3}) \]
\[ = (T_{12} \cdot B_{12}^{3}) (T_{12} \cdot B_{12}^{3}) (B_{12}^{3} \wedge B_{12}^{3}) - (T_{12} \cdot B_{12}^{3}) (T_{12} \cdot B_{12}^{3}) (B_{12}^{3} \wedge B_{12}^{3}) \]
\[ = (T_{12} \cdot B_{12}^{3}) (T_{12} \cdot B_{12}^{3}) (B_{12}^{3} \wedge B_{12}^{3}) \]
\[ = (T_{12} T_{12} B_{12}^{3} B_{12}^{3}) B_{12}^{3} = A_{12} \left( B_{12}^{3} \cdot B_{12}^{3} \right) B_{12}^{3} = A_{12} \left( C_{12} \right) B_{12}^{3}, \]

where according to Table 1, we have \( B_{12}^{3} = B_{12} \wedge B_{12} \).

The expansion of \( -B_{12} (U \cdot U) + (V \cdot V) \) is simplified as
\[ -B_{12} (U \cdot U) + (V \cdot V) = -B_{12} \left( T_{12} \cdot B_{12}^{3} \right) \cdot (T_{12} \cdot B_{12}^{3}) + (T_{12} \cdot B_{12}^{3}) \cdot (T_{12} \cdot B_{12}^{3}) \]
\[ = -B_{12} \left( T_{12} \cdot B_{12}^{3} \right) \left( T_{12} \cdot B_{12}^{3} \right) + (T_{12} \wedge B_{12}^{3}) \cdot (T_{12} \wedge B_{12}^{3}) - (T_{12} \cdot B_{12}^{3}) \left( B_{12}^{3} \wedge B_{12}^{3} \right) \]
\[ = A_{12} \left( B_{12}^{3} \wedge B_{12}^{3} \right) \left( B_{12}^{3} \wedge B_{12}^{3} \right) + A_{12} \left( B_{12}^{3} \wedge B_{12}^{3} \right)^{2} + A_{12} \left( B_{12}^{3} \wedge B_{12}^{3} \right) \]
\[ = A_{12} \left( B_{12}^{3} \cdot B_{12}^{3} - G \cdot G + C_{12}^{2} \right), \]

where we use the following expressions.
\[
T_{s2} \cdot T_{s2} = \left( T_{s2} \cdot B_{s2}^+ \right) \left( T_{s2} \cdot B_{s2}^- \right) = -A_{\text{var}} \left( B_{s2}^- \cdot B_{s2}^+ \right) = -A_{\text{var}} R_{\text{var}},
\]
\[
T_{s2} \wedge B_{s2}^+ \cdot T_{s2} \wedge B_{s2}^- = \left( T_{s2} \wedge \left( B_{s2}^+ \wedge B_{s2}^- \right) \right) = \left( \left( T_{s2} \cdot B_{s2}^+ \right) \wedge B_{s2}^- \right) = \left( \left( T_{s2} \cdot B_{s2}^+ \right) \wedge B_{s2}^- \right) - \left( \left( T_{s2} \cdot B_{s2}^- \right) \wedge B_{s2}^+ \right)
\]
\[
= \left( \left( T_{s2} \cdot B_{s2}^+ \right) \wedge B_{s2}^- \right) - 2 \left( \left( T_{s2} \cdot \left( B_{s2}^+ \wedge B_{s2}^- \right) \right) \cdot \left( \left( T_{s2} \cdot B_{s2}^- \right) \wedge B_{s2}^+ \right) \right)
\]
Eq. (25) is obtained as
\[
C_{s2} = 4B_{\text{var}} \frac{r_{2}^{2}}{2} \left( (U \cdot V)(U \cdot W) - 2B_{\text{var}} \frac{r_{2}^{2}}{2} \left( (U \cdot U)(V \cdot V) \right) - 2r_{2}^{4} \left( V \cdot V \right) \left( V \cdot W \right) \right)
\]
\[
= 2r_{2}^{4} \left( B_{\text{var}} \left( U \cdot V \right)(U \cdot W) - (V \cdot V)(V \cdot W) \right) + 2r_{2}^{4} B_{\text{var}} \left( (U \cdot V)(U \cdot W) - (U \cdot U)(V \cdot W) \right)
\]
\[
= 2r_{2}^{4} \left( B_{\text{var}} \left( U \cdot V \right)(U \cdot W) - (U \cdot V)(U \cdot W) \right) - 2r_{2}^{4} \left( U \cdot V \right)(U \cdot W)
\]
where \( U \wedge W = E_{\text{var}} B_{s2}^+ \), and \( E_{\text{var}} = \left( T_{s2} \cdot e_{\text{var}} \right) \cdot B_{s2}^+ \).

The expansion of \( U \wedge W \) is expressed as
\[
U \wedge W = \left( T_{s2} \cdot B_{s2}^+ \right) \wedge \left( e_{\text{var}} \cdot B_{s2}^+ \right) = \left( T_{s2} \cdot \left( B_{s2}^+ \wedge B_{s2}^- \right) \right) \wedge \left( e_{\text{var}} \cdot \left( B_{s2}^+ \wedge B_{s2}^- \right) \right)
\]
\[
= \left( \left( T_{s2} \cdot B_{s2}^+ \right) \wedge B_{s2}^- \right) \left( \left( e_{\text{var}} \cdot B_{s2}^+ \right) \wedge B_{s2}^- \right) \wedge \left( \left( T_{s2} \cdot B_{s2}^- \right) \wedge B_{s2}^+ \right) \left( \left( e_{\text{var}} \cdot B_{s2}^- \right) \wedge B_{s2}^+ \right)
\]
\[
= \left( \left( T_{s2} \cdot e_{\text{var}} \right) \left( B_{s2}^+ \wedge B_{s2}^- \right) \right) B_{s2}^+ = \left( T_{s2} \cdot e_{\text{var}} \right) \cdot B_{s2}^+ = E_{\text{var}} B_{s2}^+
\]

Eq. (26) is obtained as
\[
C_{s2} = -B_{\text{var}} r_{2}^{4} \left( U \cdot W \right)^{2} + \frac{B_{\text{var}} r_{2}^{4}}{2} \left( U \cdot U \right)(W \cdot W) + \frac{r_{2}^{4}}{2} (V \cdot V)(W \cdot W) + r_{2}^{4} (V \cdot W)^{2}
\]
\[
= -B_{\text{var}} r_{2}^{4} \left( U \cdot W \right)^{2} - 2B_{\text{var}} \frac{r_{2}^{4}}{2} \left( U \cdot U \right)(V \cdot V)(W \cdot W) + 2r_{2}^{4} (V \cdot V)(V \cdot W)(W \cdot W) + r_{2}^{4} (V \cdot W)^{2}
\]
\[
= -B_{\text{var}} r_{2}^{4} \left( U \cdot W \right)^{2} - 2B_{\text{var}} \frac{r_{2}^{4}}{2} \left( U \cdot U \right)(V \cdot V)(W \cdot W) + 2r_{2}^{4} (V \cdot V)(V \cdot W)(W \cdot W) + r_{2}^{4} (V \cdot W)^{2}
\]

**Figure captions**

Fig. 1. A 3-RPS parallel mechanism.
Fig. 2. A 6-3 Stewart platform.
Fig. 3. A symmetric 3-RPS parallel mechanism.
Fig. 4. A 3-RPS parallel mechanism with special configuration.
Fig. 5. A symmetric 3-RPS parallel mechanism.
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Table 2 The degree of the terms for Eq. (25) in $x$.
Table 3 Four assembly modes with the moving platform above the base for case 1.
Table 4 Four assembly modes with the moving platform above the base for case 2.
Table 5 Four assembly modes with the moving platform above the base for case 3.
Table 6 Link parameters and inputs for case 4.
Table 7 Inputs for case 4 in the reference system $O$-XYZ.
Table 8 Four real solutions for case 4.
Table 9 Link parameters and inputs for case 5.
Table 10 Four real solutions for case 5.
Fig. 1. A 3-RPS parallel mechanism
Fig. 2. A 6-3 Stewart platform

Figure 3
Fig. 3. A symmetric 3-RPS parallel mechanism
Fig. 4. A 3-RPS parallel mechanism with special configuration

Figure 5
Fig. 5. A symmetric 3-PRS parallel mechanism

Table 1
<table>
<thead>
<tr>
<th>Entity</th>
<th>IPNS</th>
<th>Grade</th>
<th>OPNS</th>
<th>Grade</th>
</tr>
</thead>
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<tr>
<td>Point</td>
<td>$p = p + \frac{1}{2} p^* e_s + e_0$</td>
<td>1</td>
<td>$p^* = S_0 \land S_1 \land S_1 \land S_1$</td>
<td>4</td>
</tr>
<tr>
<td>Sphere</td>
<td>$S = p + \frac{1}{2} (p^* - \rho^2) e_s + e_0$</td>
<td>1</td>
<td>$S^* = P_0 \land P_1 \land P_1 \land P_1$</td>
<td>4</td>
</tr>
<tr>
<td>Plane</td>
<td>$\pi = n + d e_u$</td>
<td>1</td>
<td>$\pi^* = e_u \land P_1 \land P_1 \land P_1$</td>
<td>4</td>
</tr>
<tr>
<td>Line</td>
<td>$L = \pi_1 \land \pi_2$</td>
<td>2</td>
<td>$L = e_u \land P_1 \land P_1$</td>
<td>3</td>
</tr>
<tr>
<td>Circle</td>
<td>$C = S_1 \land S_1$</td>
<td>2</td>
<td>$C^* = P_0 \land P_1 \land P_1$</td>
<td>3</td>
</tr>
<tr>
<td>Point pair</td>
<td>$P = S_0 \land S_1 \land S_1$</td>
<td>3</td>
<td>$P^* = L \land P_1$</td>
<td>2</td>
</tr>
</tbody>
</table>
Table 2 The degree of the terms for Eq. (25) in $x$.

<table>
<thead>
<tr>
<th>Term</th>
<th>$A_{xx}$</th>
<th>$B_{xx}$</th>
<th>$C_{xx}$</th>
<th>$D_{xx}$</th>
<th>$E_{xx}$</th>
<th>$G \cdot G$</th>
<th>$V \cdot W$</th>
<th>$W \cdot W$</th>
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</thead>
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<td>Degree</td>
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<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>4</td>
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</tbody>
</table>
Table 3 Four assembly modes with the moving platform above the base for case 1

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<th>B₂</th>
<th>B₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-0.4121</td>
<td>-0.6838</td>
<td>-0.2515</td>
</tr>
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<td>Y</td>
<td>0</td>
<td>1.1844</td>
<td>-0.4356</td>
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<tr>
<td></td>
<td>Z</td>
<td>1.3266</td>
<td>2.5608</td>
<td>2.9953</td>
</tr>
<tr>
<td>2</td>
<td>1.8539</td>
<td>0.7424</td>
<td>-0.6135</td>
<td>0.4543</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>0</td>
<td>1.0626</td>
<td>0.7869</td>
</tr>
<tr>
<td></td>
<td>Z</td>
<td>2.6741</td>
<td>2.4939</td>
<td>1.1583</td>
</tr>
<tr>
<td>3</td>
<td>1.6056</td>
<td>1.0888</td>
<td>-0.3626</td>
<td>-0.4344</td>
</tr>
<tr>
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<td>Y</td>
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<td>0.6281</td>
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<td>Z</td>
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<td>2.1657</td>
<td>3.2093</td>
</tr>
<tr>
<td>4</td>
<td>1.5775</td>
<td>1.1347</td>
<td>-0.1877</td>
<td>-0.4142</td>
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<tr>
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<td>Y</td>
<td>0</td>
<td>0.3251</td>
<td>-0.7173</td>
</tr>
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<td>Z</td>
<td>2.8941</td>
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<td>3.1883</td>
</tr>
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</table>

Table 4
Table 4 Four assembly modes with the moving platform above the base for case 2

<table>
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<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>-17.5133</td>
<td>-0.6792</td>
<td>0.8929</td>
<td>-0.5228</td>
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<tr>
<td></td>
<td></td>
<td>Y 0.3643</td>
<td>0.3628</td>
<td>1.3607</td>
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<td></td>
<td></td>
<td>Z 0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4.8665</td>
<td>X -0.4407</td>
<td>-0.2843</td>
<td>1.1314</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Y 1.2618</td>
<td>-0.4631</td>
<td>0.5348</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Z 0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>3</td>
<td>1.8740</td>
<td>X 0.7184</td>
<td>1.3097</td>
<td>-0.3958</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Y 2.6582</td>
<td>1.0302</td>
<td>1.3321</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Z 0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3.1989</td>
<td>X -0.1302</td>
<td>1.5488</td>
<td>1.0776</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Y 1.8226</td>
<td>2.2480</td>
<td>0.5812</td>
</tr>
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<td></td>
<td></td>
<td>Z 0</td>
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<td>0</td>
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</table>

Table 5
Table 5 Four assembly modes with the moving platform above the base for case 3

<table>
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<tr>
<th>i</th>
<th>$x$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1866</td>
<td>0.2635</td>
<td>0.0866</td>
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<tr>
<td></td>
<td>Y</td>
<td>0</td>
<td>-0.1500</td>
<td>0.1500</td>
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<tr>
<td></td>
<td>Z</td>
<td>0.9855</td>
<td>0.7953</td>
<td>0.7953</td>
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<tr>
<td>2</td>
<td>2.2204</td>
<td>-0.2297</td>
<td>-0.0866</td>
<td>-0.0866</td>
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<td></td>
<td>Y</td>
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<td>0.1500</td>
<td>-0.1500</td>
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<td></td>
<td>Z</td>
<td>0.7488</td>
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<td>0.9657</td>
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<td>3</td>
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<td>-0.1318</td>
<td>0.0866</td>
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<tr>
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<td>Y</td>
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<td>0.2282</td>
<td>0.1500</td>
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<td>Z</td>
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<td>0.9855</td>
<td>0.7953</td>
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<td>4</td>
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<td>Y</td>
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<td>-0.1990</td>
<td>-0.1500</td>
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<td>Z</td>
<td>0.9657</td>
<td>0.7488</td>
<td>0.9657</td>
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</table>

Table 6
Table 6 Link parameters and inputs for case 4

<table>
<thead>
<tr>
<th>O-X-Y-Z</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{ix}$</td>
<td>-2.9</td>
<td>-1.2</td>
<td>1.3</td>
<td>-1.2</td>
<td>2.5</td>
<td>3.2</td>
</tr>
<tr>
<td>$a_{iy}$</td>
<td>-0.9</td>
<td>3.0</td>
<td>-2.3</td>
<td>-3.7</td>
<td>4.1</td>
<td>1.0</td>
</tr>
<tr>
<td>$a_{iz}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The distances $r_i$ between the points $B_i$:

$r_1 = 2.0, r_2 = 2.0, r_3 = 3.0$

The lengths of six legs $l_i$:

$l_1 = 5.0, l_2 = 4.5, l_3 = 5.0, l_4 = 5.5, l_5 = 5.5, l_6 = 5.7$

Table 7
Table 7 Inputs for case 4 in the reference system $O$-XYZ

<table>
<thead>
<tr>
<th>$O$-XYZ</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_x$</td>
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<td>0</td>
<td>4.0803</td>
<td>1.4802</td>
<td>5.7676</td>
<td>6.2278</td>
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<td>$a_y$</td>
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<td>3.2366</td>
<td>-2.3929</td>
<td>-3.5971</td>
<td>3.8962</td>
<td>0.7516</td>
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<tr>
<td>$a_z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>
Table 8 Four real solutions for case 4

<table>
<thead>
<tr>
<th>i</th>
<th>x</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9641</td>
<td>0.1570</td>
<td>1.6474</td>
<td>2.0248</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>1.8880</td>
<td>1.6311</td>
<td>2.0987</td>
</tr>
<tr>
<td></td>
<td>Z</td>
<td>4.2903</td>
<td>1.6994</td>
<td>3.6070</td>
</tr>
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<td>1.5287</td>
<td>1.1961</td>
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<tr>
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<td>Z</td>
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<td>1.1961</td>
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<td>Z</td>
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<td>1.6474</td>
<td>2.0248</td>
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<td>Z</td>
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<td>-3.6070</td>
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Table 9
### Table 9 Link parameters and inputs for case 5

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<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
<th>$A_5$</th>
<th>$A_6$</th>
</tr>
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<td>−50</td>
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<td>−20</td>
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<td>−93</td>
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</table>

The distances $r_i$ between the points $B_i$:

- $r_1 = 135$, $r_2 = 190$, $r_5 = 141$

The lengths of six legs $l_i$:

- $l_1 = 76$, $l_2 = 160$, $l_3 = 139$, $l_4 = 55$, $l_5 = 128$, $l_6 = 217$

---

Table 10
Table 10 Four real solutions for case 5

<table>
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<th>( x )</th>
<th>( B_1 )</th>
<th>( B_2 )</th>
<th>( B_3 )</th>
</tr>
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