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1 **FRACTIONAL PATLAK-KELLER-SEGEL EQUATIONS FOR**
2 **CHEMOTACTIC SUPERDIFFUSION***

3 GISSELL ESTRADA-RODRIGUEZ[†], HEIKO GIMPERLEIN^{†‡}, AND KEVIN J. PAINTER^{†§}

4 **Abstract.** The long range movement of certain organisms in the presence of a chemoattractant
5 can be governed by long distance runs, according to an approximate Lévy distribution. This article
6 clarifies the form of biologically relevant model equations: We derive Patlak-Keller-Segel-like equa-
7 tions involving nonlocal, fractional Laplacians from a microscopic model for cell movement. Starting
8 from a power-law distribution of run times, we derive a kinetic equation in which the collision term
9 takes into account the long range behaviour of the individuals. A fractional chemotactic equation
10 is obtained in a biologically relevant regime. Apart from chemotaxis, our work has implications for
11 biological diffusion in numerous processes.

12 **Key words.** Chemotaxis, Patlak-Keller-Segel equation, velocity-jump model, nonlocal diffusion,
13 Lévy walk, cell motility.

14 **AMS subject classifications.** 92C17, 35R11, 35Q92

15 **1. Introduction.** Chemotaxis is the directed movement response of a cell or
16 organism to some chemical concentration gradient, and has been identified in areas
17 as diverse as microbiology [4, 9, 32], developmental biology [16], immunosurveillance
18 [28, 46], cancer development [50] and animal movement [25, 47].

19 Motivated by these applications, chemotaxis and related phenomena have received
20 significant attention in the theoretical community, e.g. see the reviews [20, 21]. Mod-
21 elling approaches range from microscopic to macroscopic, with the early and seminal
22 contributions of Patlak [38] and Keller and Segel [23, 24], respectively, providing ex-
23 amples. Chemotactic models derived from microscopic perspectives have tended to
24 follow standard assumptions on the behaviour of individuals, usually assuming that
25 the search strategy follows a biased random walk. In particular, the distribution
26 of times between reorientations by the cells/organisms is taken to follow a Poisson
27 distribution, as backed up by observations of *E. coli* such as [9], and the result is a
28 Fickian-type diffusive flux when a continuous approximation is derived.

29 In this work, motivated by real world examples (discussed in Subsection 1.1), we
30 assume the motion follows a long-tailed distribution of run times. From a microscopic
31 model for chemosensitive movement we derive fractional Patlak-Keller-Segel equations
32 for the density (\bar{u}) of some chemotactic population in the presence of a chemoattrac-
33 tant (of concentration ρ). The fractional Patlak-Keller-Segel system obtained here

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34 is

$$35 \quad (1) \quad \begin{aligned} \partial_t \bar{u} &= c_0 \nabla \cdot (D_\alpha \nabla^{\alpha-1} \bar{u} - \chi \bar{u} \nabla \rho), \\ \partial_t \rho &= D_\rho \Delta \rho + f(\bar{u}, \rho). \end{aligned}$$

36 $\nabla^{\alpha-1}$ denotes a fractional gradient which interpolates between ballistic motion ($\alpha =$
 37 1) and ordinary diffusion ($\alpha = 2$); note that the case $\alpha = 2$ corresponds to the
 38 classic formulation of Patlak-Keller-Segel equations proposed phenomenologically in
 39 [23]. The second equation is of standard reaction-diffusion form, assuming that the
 40 chemical diffusion with coefficient D_ρ is not affected by the nonlocal behaviour of the
 41 organisms. The chemotactic population is governed by a diffusion term with coefficient
 42 D_α (defined at the end of Section 6) that represents a random component to motility,
 43 and a chemotactic flux of advective type, where the advection is proportional to
 44 the chemical gradient. The function χ is commonly referred to as the chemotactic
 45 sensitivity. In the case of constant D_α we obtain an honest fractional Laplacian,
 46 namely, $\nabla \cdot \nabla^{\alpha-1} = c(-\Delta)^{\alpha/2}$ for $1 < \alpha < 2$. Unlike recent analyses which obtain
 47 fractional behaviour in different contexts (e.g. [43] and [6]), we consider the derivation
 48 from a fully microscopic description, according to how the concentration and gradient
 49 of chemoattractant influences the movement of individual organisms.

50 Starting from a velocity jump model in which an individual performs occasional
 51 long jumps according to an approximate Lévy distribution, we derive the appropriate
 52 kinetic-transport equation where the collision term describes the nonlocal motion. We
 53 then use a perturbation argument and an appropriate hyperbolic scaling in space and
 54 time, obtaining system (1) in the limit.

55 **1.1. Lévy walks and motivating examples.** Our work is motivated by exper-
 56 imental results which indicate the presence of behaviour with characteristics similar
 57 to a Lévy walk as an alternative search strategy, particularly when chemoattractants,
 58 food or other targets are sparse or rare; examples include [19, 29, 30, 40]. In contrast
 59 to Brownian motion, a Lévy walk includes a non-negligible probability for long po-
 60 sitional jumps. In a biological context these “long jumps” correspond to persisting
 61 in a single direction of motion for a substantially longer time than in typical random
 62 walks. The distribution of runs asymptotically behaves like a power-law distribution
 63 with finite mean, but unbounded variance. While for Brownian motion the mean
 64 squared displacement $\langle x^2 \rangle$ of a particle is a linear function of time, for a power-law
 65 distribution with power $1 < \alpha < 2$, $\langle x^2 \rangle \sim t^{2/\alpha}$ grows faster for large times. The
 66 exponent $\alpha = 1$ corresponds to ballistic transport, while $\alpha = 2$ is the case of normal
 67 diffusion. For a review of Lévy walk models and their ubiquitous appearance we refer
 68 to [51].

69 To motivate the present modelling, we describe two systems in which organisms
 70 with well documented chemotactic responses have been suggested to display Lévy
 71 walk characteristics. Moreover, we note that Lévy walk behaviour has been suggested
 72 in numerous biological contexts, e.g. immune cells [19], ecology [11] and human
 73 populations [41].

74 **1.1.1. *E. coli*.** The chemotactic behaviour of the bacterium *E. coli* has been
 75 extensively studied, such that more is known for its signalling pathways and mech-
 76 anistic control of chemotaxis than for any other system [26]. Motile *E. coli* carry
 77 long flagella that allow them to move in “run and tumble” fashion: counterclock-
 78 wise rotation of flagella results in their bundling, and smooth swimming occurs with
 79 an approximately fixed heading; rotation clockwise results in outward flaying, and

80 the bacterium tumbles randomly while maintaining an almost fixed position. In the
 81 presence of a chemoattractant, rotation is controlled by a signalling pathway, where
 82 attachment of the chemical to a membrane bound receptor induces signalling to the
 83 flagellum’s rotatory machinery. Chemotaxis is achieved by increasing the run time
 84 when the cell experiences an increasing attractant gradient, so that on average an
 85 individual spends more time moving up gradients than down them.

86 While classic experiments [9] indicate that the distribution of tumbling events
 87 for *E. coli* follows a Poisson distribution, more recent experiments [27] have shown
 88 that (for mediums where the attractant is absent) the distribution of runs can have a
 89 heavy tail, suggesting that the bacteria may follow a Lévy walk in particular environ-
 90 ments. A theoretical study carried out in [44] suggested that temporal fluctuations
 91 of a key protein in the signalling pathway of *E. coli* can induce power-law distribu-
 92 tions of the run times, in agreement with the previous experimental results [27]. A
 93 modelling study in [33] suggested that the switch from local (Brownian) to nonlocal
 94 (Lévy) search in particular depends on CheR activity (a cytoplasmic signalling protein
 95 regulating receptor activity). In the case of fluctuating CheR, the running behaviour
 96 of *E. coli* followed a power-law distribution, while for constant CheR it followed a
 97 Brownian motion. Simulations showed that for the case of fluctuating CheR, bacteria
 98 subsequently found food faster as switching between long and short runs allowed them
 99 to leave nutrient depleted patches and reinitiate searching.

100 **1.1.2. *Dictyostelium discoideum* (*Dd*).** Similar findings have been suggested
 101 in the searching strategy of certain amoeboid cells, and in particular the cellular slime
 102 mold *Dd*. *Dd* is the classic model system for studying chemotaxis behaviour in eu-
 103 karyotic cells, displaying a complex life cycle in which it switches between unicellular
 104 (“vegetative swarming”, in which single cells migrate, consume and divide) and mul-
 105 ticellular (cells self-organise into a collective of $\sim 10^5$ cells and behave as a single
 106 organism) phases. The chemotactic response of *Dd* to the chemoattractant cAMP
 107 is well known to control multicellular phases [10], but chemotaxis is also a crucial
 108 mechanism for finding food during vegetative phases; for example, chemotaxis to folic
 109 acid allows *Dd* to seek out folate-secreting bacteria prey [37].

110 A study of Levandowski et al. [29], where a group of 17 soil amoeba of 8 different
 111 types were isolated and tracked in a medium free of nutrients, revealed that the mean
 112 squared displacement could follow a power-law distribution, suggesting the Lévy walk
 113 model as an approach to describe the movement of these organisms. Nevertheless,
 114 the study also remarked that the duration of the experiment may not be sufficient to
 115 see if cells switched to a normal distribution at longer times.

116 A more recent research explored the motility of *Dd* and *Polysphondylium palla-*
 117 *dium* cells in a food free medium [30], with authors concluding that cells bias their
 118 motion by remembering the last turn and subsequently performing long runs without
 119 changing direction for ~ 9 min. Experiments tracked 12 cells, obtaining the trajec-
 120 tories for each cell at different run times (results reproduced in Figure 1). For short
 121 run times ($0.4 \text{ min} < \tau < 5 \text{ min}$) the trajectories appear to be almost ballistic, while
 122 for larger run times ($\tau > 30 \text{ min}$) the trajectories lie between normal diffusion and
 123 ballistic transport, suggesting a superdiffusion-like behaviour. These results are also
 124 reflected in their measurements of the cell velocities at different run times. Character-
 125 istics of the type of motion observed in [30] suggest that these cells do not specifically
 126 perform a Lévy walk, but a form of long directionally persistent random walk.

127 Finally, a study on the search strategy of wild type AX3 *Dictyostelium* cells in
 128 the absence of attractant [45] indicated that starved cells search for food in larger

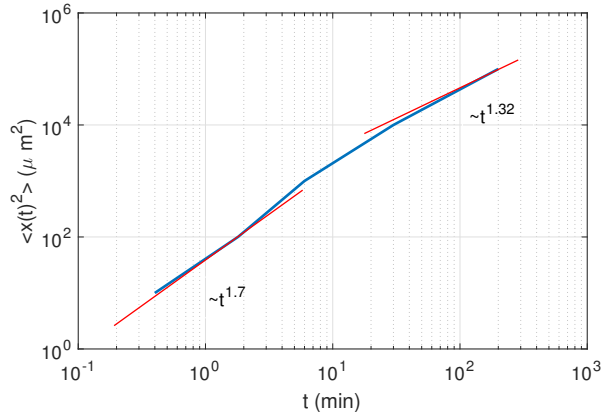


FIG. 1. Reproduction of the data reported in [30]. Average of the mean squared displacement $\langle x(t)^2 \rangle$ of 12 cells, where $x(t)^2$ was averaged over all pairs of time points for each trajectory. As described in [30], each cell was followed for 8 – 10 hrs with a sampling interval of 10 s.

129 areas not by increasing their speed but by biasing towards very long runs. In other
 130 words, cells changed their strategy from making a very localized search to expanding
 131 the search area through persisting in their motion in a single direction.

132 **1.2. Outline.** This paper is organized in the following way. In Section 2 we
 133 discuss the assumptions for the type of motion of the organism that we are modelling.
 134 In Section 3 we derive the resulting kinetic equation, and in Section 4 we introduce the
 135 relevant scaling regime. Section 5 deals with the derivation of the “collision operator”
 136 that describes the actual dynamics of the organism or cell, and finally in Section 6 we
 137 obtain the fractional Patlak-Keller-Segel equation.

138 **2. Model assumptions.** Motivated by the experimental results in [27] and [30]
 139 we model a population of organisms moving in a medium in \mathbb{R}^n , containing some
 140 chemical (with concentration $\rho = \rho(\mathbf{x}, t)$) that acts as an attractant. We assume that
 141 each individual performs a biased random walk according to the distribution of ρ with
 142 the following properties:

- 143 1. The interactions between individuals are taken to be negligible. This assump-
 144 tion is reasonable for the descriptions of experiments on *Dd* above (tracking
 145 spatially distributed cells) and for swimming *E. coli*, where the intracellu-
 146 lar separation is often at least one order of magnitude greater than the cell
 147 diameter (e.g. [9]).
- 148 2. Starting at position \mathbf{x} and time t , we assume an individual runs in direction
 149 θ for some time τ , called the “run time”. Typical trajectories are shown in
 150 Figure 2 for different run time distributions.
- 151 3. The individuals are assumed to move with constant forward speed c , following
 152 a straight line motion between reorientations.
- 153 4. Each time the individual stops it selects a new direction η according to a
 154 distribution $k(\mathbf{x}, t, \theta; \eta)$ which only depends on $|\theta - \eta|$. The choice of new
 155 direction is taken here to be independent of the chemical concentration or
 156 gradient.
- 157 5. The reorientation is assumed to be (effectively) instantaneous.

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6. The running¹ probability ψ , which is defined as the probability that an individual moving in some fixed direction does not stop until time τ , is taken to depend on the environment surrounding the individual (specifically, the concentration ρ and its gradient $D^\theta \rho$). Consequently, the stopping rate β will also depend on ρ and $D^\theta \rho$.



FIG. 2. Illustration of trajectories of a Brownian motion (2a) and a Lévy walk (2b) in two dimensions. The Brownian motion trajectory was obtained using a running probability function $\psi = e^{-\tau}$ and for the Lévy walk $\psi = \left(\frac{1}{1+\tau}\right)^\alpha$ for $\alpha = 1.5$. The angle θ for the new direction is chosen from a uniform distribution in both cases.

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The above assumptions are particularly relevant for the run and tumble motion of *E. coli* and similar bacteria which bias their run length according to the chemical concentration. For example, the speed c of such bacteria is typically between 10 – 30 $\mu\text{m/s}$ and the average length of a run is approximately ten times the cell body length [8, 9, 31]. Further, tumbling durations are known to be approximately 0.11 s, an order of magnitude shorter than typical run times (~ 1.3 s), so reorientations can be assumed to be instantaneous [8, 9]. Nevertheless, the above assumptions can be modified for cells like *Dictyostelium* and *leukocytes* [1], under appropriate re-examination. Even the motion of larger organisms, such as butterflies, can be described by a persistent or correlated random walk with underlying characteristics similar to those described above [22, 35].

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2.1. Turn angle distribution. Recall that when an individual reorients in an isotropic medium, the new direction chosen, η , is symmetrically distributed with respect to the previous direction, θ [1]. In this case

177 (2)

$$k(\mathbf{x}, t, \theta; \eta) = \ell(\mathbf{x}, t, |\eta - \theta|),$$

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where ℓ represents a distribution and $|\eta - \theta|$ denotes the distance between two directions on the unit sphere S .

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For *E. coli* and certain other cells, an inhomogeneous medium (i.e. heterogeneous chemoattractant concentration) generates a variable run length, such that the run length is increased when the cell experiences an increasing concentration of ligand. The turn angle, however, is not affected by the concentration [7] since the bacteria

¹In probability this is also known as survival probability, where the “event” in this case is to stop. Hence “survival” in that context refers to the probability of continuing to move in the same direction for some time τ .

184 are believed to be too small to directly sense a chemoattractant gradient [17]. Hence,
 185 during the reorientation we assume that cells choose a new direction from the distri-
 186 bution ℓ , while the stopping frequency β during the subsequent run is taken to depend
 187 on the ligand concentration.

188 We should note that in the case of larger cells, such as *leukocytes* or *Dd*, a cell can
 189 sense a chemoattractant gradient without moving (i.e. the cell is large enough to assess
 190 it directly), and hence their next direction at a turn can also be directly influenced
 191 by the gradient. In this case the turn angle distribution k would not be symmetric
 192 but biased according to the concentration and total gradient of the attractant. We
 193 do not consider this extension here.

194 **2.2. Running probability.** As described earlier, the motion of *E. coli* depends
 195 on the concentration of chemoattractant via intracellular signalling molecules that
 196 control the tumbling phase. As shown in [27], under certain conditions *E. coli* can
 197 perform occasional long jumps with a corresponding power-law distribution of run
 198 lengths. To describe motion in such environmental conditions, we assume the following
 199 power distribution with exponent α for the running probability:

$$200 \quad (3) \quad \psi(\cdot, \theta, \tau) = \left(\frac{\mathcal{S}(\rho, D^\theta \rho)}{\mathcal{S}(\rho, D^\theta \rho) + \tau} \right)^\alpha.$$

201 ψ describes the probability that an individual running in direction θ stops after time
 202 τ . Here $\mathcal{S}(\rho, D^\theta \rho) = \tau_0(\rho) + \tau_1(\rho)D^\theta \rho$. The dot denotes dependence on space and
 203 time $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

204 As a remark, we note that the above choice of ψ is possibly more relevant when
 205 the concentration of ρ is small, i.e. when individuals need to do more searching. In
 206 regions of large ρ it may be relevant to revert to a more classic (exponential/Poisson)
 207 choice, e.g. as in [1]: see the discussion at the end of the paper.

208 The running probability ψ is related to the stopping frequency via

$$209 \quad (4) \quad \psi(\cdot, \theta, \tau) = \exp \left(- \int_0^\tau \beta(\mathbf{x} + cs\theta, t + s, \theta, s) ds \right).$$

210 This means that the probability of running for time τ without stopping is equal
 211 to the exponential of the cumulative stopping frequency. Therefore, the stopping
 212 frequency during a run phase is given by

$$213 \quad (5) \quad \beta(\cdot, \theta, \tau) = \frac{\alpha}{\tau_0 + \tau_1 D^\theta \rho + \tau},$$

214 in a quasi-static approximation.

215 *Remark 2.1.* As discussed in [36], the manner by which the chemoattractant
 216 concentration affects motion (and consequently the equations) heavily depends on
 217 the magnitude of the perturbation that is considered in the stopping frequency:
 218 $\beta(\cdot, \theta, \tau) = \beta_0 + \beta_1(\cdot, \theta, \tau, D^\theta \rho)$ where β_1 is of lower order in ε , in the sense of [Section 6](#).
 219 Note that here we consider that only the second term β_1 depends on ρ .
 220 Additive perturbations of ψ , e.g.

$$221 \quad (6) \quad \psi(\cdot, \theta, \tau) = (1 - \varepsilon^z) \left(\frac{\varepsilon^\mu \tau_0(\rho) + \varepsilon^\mu \tau_1(\rho) D_\varepsilon^\theta \rho}{\varepsilon^\mu \tau_0(\rho) + \varepsilon^\mu \tau_1(\rho) D_\varepsilon^\theta \rho + \tau} \right)^\alpha + \varepsilon^z e^{-\tau \mathcal{C}(\rho, D_\varepsilon^\theta \rho)},$$

222 give rise to perturbations in β for large τ as well. For the choice (6) with $z \leq 0$ we
 223 would also obtain a fractional Patlak-Keller-Segel type of equation as in [Section 6](#).

224 **3. Model equations.** We consider the assumptions from [Section 2](#) and are
 225 guided by Alt's approach in [\[1, 2\]](#). For a population of total density $\sigma(\mathbf{x}, t, \theta, \tau)$, where
 226 individuals at (\mathbf{x}, t) move in direction θ for some time τ , the governing equations of
 227 motion are given by

$$228 \quad (7) \quad (\partial_\tau + \partial_t + c\theta \cdot \nabla) \sigma(\cdot, \theta, \tau) = -(\beta\sigma)(\cdot, \theta, \tau),$$

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$$230 \quad (8) \quad \sigma(\cdot, \eta, 0) = \int_0^t \int_S (\beta\sigma)(\cdot, \theta, \tau) k(\cdot, \theta; \eta) d\theta d\tau.$$

The kinetic-transport equation [\(7\)](#) is analogous to the Boltzmann equation, where the collision term in this case describes the behaviour of the individuals for classical velocity jump models of bacteria. It is well known that in a suitable asymptotic limit one obtains diffusion-like equations for the macroscopic (or observable) density of bacteria,

$$\bar{u}(\mathbf{x}, t) := \frac{1}{|S|} \int_S \int_0^t \sigma(\cdot, \theta, \tau) d\tau d\theta.$$

231 The left hand side of [\(7\)](#) describes the temporal variation and transport of the
 232 density σ , while the right hand side gives the density of individuals "left behind" due to
 233 tumbling, occurring with frequency $\beta(\cdot, \theta, \tau)$. The individuals that tumble undertake
 234 a reorientation process and choose a new direction η with probability $k(\cdot, \theta; \eta)$, i.e. the
 235 turn angle distribution. This process is explicitly described by the initial conditions
 236 in the run time τ in [Equation \(8\)](#), where the left hand side is the total density of
 237 individuals starting a new run ($\tau = 0$). This density is equal to the total population
 238 at (\mathbf{x}, t) oriented across all directions on the surface S and with different run times τ .

239 Note that we consider [Equations \(7\) and \(8\)](#) in the whole space \mathbb{R}^n , thereby
 240 avoiding any specification of boundary conditions and allowing our approach to be
 241 applicable to a wide variety of systems. Further discussion of the boundary conditions
 242 is provided in the conclusions.

243

244 Using the method of characteristics, we can find the solution of [equation \(7\)](#),

$$245 \quad (9) \quad \sigma(\cdot, \theta, \tau) = \sigma(\mathbf{x} - c\theta\tau, t - \tau, \theta, 0) \exp\left(\int_0^\tau -\beta(\mathbf{x} + cs\theta, t + s, \theta, s) ds\right).$$

246 Experimentally measuring the density σ at each τ is infeasible, and therefore we
 247 write system [\(7\)-\(8\)](#) in terms of a new density

$$248 \quad (10) \quad \bar{\sigma}(\cdot, \theta) = \int_0^t \sigma(\cdot, \theta, \tau) d\tau,$$

249 which describes the density of the population in \mathbf{x} at time t and moving in direction
 250 θ . Integrating over τ in [\(7\)-\(8\)](#) we obtain

$$251 \quad (11) \quad \partial_t \bar{\sigma} + c\theta \cdot \nabla \bar{\sigma} = \sigma(\cdot, \theta, 0) - \int_0^t (\beta\sigma)(\cdot, \theta, \tau) d\tau,$$

252

253 where $\sigma(\mathbf{x}, t, \theta, 0)$ is analogous to [\(8\)](#) and is given by

$$254 \quad (12) \quad \sigma(\cdot, \theta, 0) = \int_0^t \int_S (\beta\sigma)(\cdot, \eta, \tau) k(\cdot, \eta; \theta) d\eta d\tau.$$

Biologically, it is crucial that the stopping frequency β will depend not only on the concentration at a given point (\mathbf{x}, t) , but also on the gradient of the concentration along a run [42], such that

$$\beta(\cdot, \theta, \tau) = \beta_0(\rho(\mathbf{x}, t), D^\theta \rho(\mathbf{x}, t)), \text{ where } D^\theta \rho = \partial_t \rho + c\theta \cdot \nabla \rho.$$

This dependence of β on ρ reflects a memory process in the intracellular signalling pathway that allows the individual to assess the variation in the chemoattractant concentration along the run.

The turn angle operator T describes the effect of changing from direction θ to a new direction η . It is given by

$$(13) \quad T\phi(\eta) = \int_S k(\cdot, \theta; \eta)\phi(\theta)d\theta.$$

Some of its basic properties are discussed in [Appendix A](#). Using T , the differential-integral equation (11) can be re-written, with (12), as

$$\begin{aligned} \partial_t \bar{\sigma} + c\theta \cdot \nabla \bar{\sigma} &= \int_S k(\cdot, \eta; \theta) \int_0^t (\beta\sigma)(\cdot, \eta, \tau) d\tau d\eta - \int_0^t (\beta\sigma)(\cdot, \theta, \tau) d\tau \\ &= T \left(\int_0^t (\beta\sigma)(\cdot, \theta, \tau) d\tau \right) - \int_0^t (\beta\sigma)(\cdot, \theta, \tau) d\tau \\ &= -(\mathbf{1} - T) \int_0^t (\beta\sigma)(\cdot, \theta, \tau) d\tau. \end{aligned}$$

4. Scaling. Assume that \mathcal{X} and \mathcal{T} are the macroscopic space and time scales respectively. Let us also consider that the mean run time $\bar{\tau}$ is small compared with the macroscopic time \mathcal{T} , i.e., $\varepsilon = \bar{\tau}/\mathcal{T} \ll 1$ where ε is a small parameter. Suppose further that the concentration ρ is already dimensionless in the sense that it stands for ρ/ρ_0 where ρ_0 is an averaged value of ρ over \mathbb{R}^n .

The new dimensionless variables are

$$t_n = \frac{t}{\mathcal{T}}, \quad \mathbf{x}_n = \frac{\mathbf{x}}{\mathcal{X}}, \quad \tau_n = \frac{\tau}{\bar{\tau}} \text{ and } c_n = \frac{c}{s}.$$

We consider the scaling

$$t_n = \varepsilon t, \quad \mathbf{x}_n = \frac{\varepsilon \mathbf{x}}{s}, \quad c_n = \varepsilon^{-\gamma} c_0 \text{ and } \tau_n = \tau \varepsilon^\mu,$$

for $\mu > 0$ and $0 < \gamma < 1$. Equations (3) and (5) become, after substituting the new variables,

$$(15) \quad \psi_\varepsilon(\cdot, \theta, \tau) = \left(\frac{\tau_0 \varepsilon^\mu + \varepsilon^\mu \tau_1 D_\varepsilon^\theta \rho}{\tau_0 \varepsilon^\mu + \varepsilon^\mu \tau_1 D_\varepsilon^\theta \rho + \tau} \right)^\alpha$$

and

$$(16) \quad \beta_\varepsilon(\cdot, \theta, \tau) = \frac{\alpha \varepsilon^\mu}{\tau_0 \varepsilon^\mu + \tau_1 \varepsilon^\mu D_\varepsilon^\theta \rho + \tau}.$$

Here $D_\varepsilon^\theta \rho = \varepsilon \partial_t \rho + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \rho$. The parameters μ and γ will be chosen appropriately in [Section 6](#). Note that the scaling chosen here suggests that the macroscopic equation is valid in the scale of the experiments shown in [Figure 1](#).

280 The scaling of (14) gives

$$281 \quad (17) \quad \varepsilon \partial_t \bar{\sigma} + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \bar{\sigma} = -(\mathbf{1} - T) \int_0^t \beta_\varepsilon \sigma d\tau.$$

282 Under the appropriate scaling we will pass to the limit when $\varepsilon \rightarrow 0$ and obtain a
283 fractional Patlak-Keller-Segel equation describing the singular limit.

284 To do so, we first obtain a conservation equation by integrating (17) over θ in the
285 whole sphere S and use the conservation of particles, (50). This gives

$$286 \quad \varepsilon \partial_t \frac{1}{|S|} \int_S \bar{\sigma} d\theta + \varepsilon^{1-\gamma} \frac{c_0}{|S|} \nabla \cdot \int_S \theta \bar{\sigma} d\theta = 0.$$

288 The mean direction $\bar{w} = \frac{1}{|S|} \int_S \theta \bar{\sigma} d\theta$ (Appendix A) is calculated in Section 6 in terms
289 of a new density $\bar{u} = \frac{1}{|S|} \int_S \bar{\sigma} d\theta$. After substituting the mean direction into the
290 conservation equation we will obtain a nonlocal diffusion equation for \bar{u} .

291 **5. Derivation of the turning operator.** In this section we derive the turning
292 operator, given by a kernel \mathcal{B} , that describes the behaviour of the individuals.

293 We define the density of cells leaving the point \mathbf{x} for all times τ from 0 to t , also
294 called the escape rate, as

$$295 \quad (18) \quad i(\mathbf{x}, t, \theta) = \int_0^t \beta(\mathbf{x}, t, \theta, \tau) \sigma(\mathbf{x}, t, \theta, \tau) d\tau.$$

296 Recalling the expression for the running probability (4) and its relationship to
297 the stopping density function φ ,

$$298 \quad (19) \quad \varphi(\mathbf{x}, t, \theta, \tau) = -\partial_\tau \psi(\mathbf{x}, t, \theta, \tau),$$

299 we can write β as

$$300 \quad \beta(\mathbf{x}, t, \theta, \tau) = \frac{\varphi(\mathbf{x}, t, \theta, \tau)}{\psi(\mathbf{x}, t, \theta, \tau)}.$$

301 Substituting this expression into (18) and using the solution (9) obtained from
302 the method of characteristics which is given by

$$303 \quad (20) \quad \sigma(\mathbf{x}, t, \theta, \tau) = \sigma(\mathbf{x} - c\theta\tau, t - \tau, \theta, 0) \psi(\mathbf{x}, t, \theta, \tau),$$

304 we get

$$305 \quad i(\mathbf{x}, t, \theta) = \int_0^t \varphi(\mathbf{x}, t, \theta, \tau) \sigma(\mathbf{x} - c\theta\tau, t - \tau, \theta, 0) d\tau$$

$$306 \quad (21) \quad = \int_0^t \varphi(\mathbf{x}, t, \theta, t - s) e^{-(t-s)c\theta \cdot \nabla} \sigma(\mathbf{x}, s, \theta, 0) ds,$$

308 by letting $\tau = t - s$. In order to find the Laplace transform of (21) we expand the
309 term φ in a quasi-static approximation by freezing coefficients at $t = t_0$,

$$310 \quad i(\mathbf{x}, t, \theta) = \int_0^t \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \partial_t^{(k)} \varphi(\mathbf{x}, t_0, \theta, t-s) e^{-(t-s)c\theta \cdot \nabla} \sigma(\mathbf{x}, s, \theta, 0) ds$$

$$311 \quad (22) \quad = \int_0^t \varphi(\mathbf{x}, t_0, \theta, t-s) e^{-(t-s)c\theta \cdot \nabla} \sigma(\mathbf{x}, s, \theta, 0) ds + \mathcal{O}((t-t_0)\varphi'(\mathbf{x}, t_0, \theta, t-s)).$$

312

313 We later let $t_0 \rightarrow t$ and keep the leading approximation in the quasi-static regime.
 314 The Laplace transform of (21) is

$$315 \quad \hat{i}(\mathbf{x}, \lambda, \theta) = \hat{\varphi}(\mathbf{x}, t_0, \theta, \lambda + c\theta \cdot \nabla) \Big|_{t_0=t} \hat{\sigma}(\mathbf{x}, \lambda, \theta, 0) \\
 317 \quad (23) \quad + \mathcal{O}((t - t_0)(\lambda + c\theta \cdot \nabla)\hat{\varphi}(\mathbf{x}, t_0, \theta, \lambda + c\theta \cdot \nabla)).$$

318 On the other hand, using the definition of $\bar{\sigma}$ given in (10) we also have

$$319 \quad \bar{\sigma}(\mathbf{x}, t, \theta) = \int_0^t \sigma(\mathbf{x} - c\theta\tau, t - \tau, \theta, 0)\psi(\mathbf{x}, t, \theta, \tau)d\tau \\
 320 \quad (24) \quad = \int_0^t e^{-(t-s)c\theta \cdot \nabla} \sigma(\mathbf{x}, s, \theta, 0)\psi(\mathbf{x}, t, \theta, t - s)ds. \\
 321$$

322 Following the same approximation as in (22), we obtain the Laplace transform of $\bar{\sigma}$
 323 as follows

$$324 \quad \hat{\bar{\sigma}}(\mathbf{x}, \lambda, \theta) = \hat{\sigma}(\mathbf{x}, \lambda, \theta, 0)\hat{\psi}(\mathbf{x}, t_0, \theta, \lambda + c\theta \cdot \nabla) \Big|_{t_0=t} \\
 325 \quad (25) \quad + \mathcal{O}\left((t - t_0)(\lambda + c\theta \cdot \nabla)\hat{\psi}(\mathbf{x}, t_0, \theta, \lambda + c\theta \cdot \nabla)\right). \\
 326$$

327 Finally, from (23) and (25) we get

$$328 \quad (26) \quad \hat{i}(\mathbf{x}, \lambda, \theta) = \hat{\mathcal{B}}(\mathbf{x}, t, \theta, \lambda + c\theta \cdot \nabla)\hat{\sigma}(\mathbf{x}, \lambda, \theta) + \text{l.o.t.},$$

329 where we neglect the lower order terms and \mathcal{B} denotes the turning operator defined
 330 as

$$331 \quad (27) \quad \hat{\mathcal{B}}(\mathbf{x}, t, \theta, \lambda + c\theta \cdot \nabla) = \frac{\hat{\varphi}(\mathbf{x}, t, \theta, \lambda + c\theta \cdot \nabla)}{\hat{\psi}(\mathbf{x}, t, \theta, \lambda + c\theta \cdot \nabla)} + \text{l.o.t.}$$

332 Applying the inverse Laplace transform to (26) we have

$$333 \quad (28) \quad i(\mathbf{x}, t, \theta) = \int_0^t \mathcal{B}(\mathbf{x}, t, \theta, t - s)\bar{\sigma}(\mathbf{x} - c\theta(t - s), s, \theta)ds.$$

334 Next we find $\hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda)$ and $\hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \lambda)$ in order to obtain an explicit form for
 335 $\hat{\mathcal{B}}_\varepsilon$. The subscript ε denotes that these quantities are scaled as indicated in Section 4.
 336 For $a = \tau_0(\rho)\varepsilon^\mu + \tau_1(\rho)\varepsilon^\mu D_\varepsilon^\theta \rho$, the Laplace transform of ψ_ε given in (15) is

$$337 \quad (29) \quad \hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda) = a^\alpha \lambda^{\alpha-1} e^{a\lambda} \Gamma(-\alpha + 1, a\lambda),$$

338 in the quasi-static approximation that $D_\varepsilon^\theta \rho$ varies slowly along a run. Using the
 339 following asymptotic expansion for the incomplete Gamma function

$$340 \quad (30) \quad \Gamma(b, z) = \Gamma(b) \left(1 - z^b e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b + k + 1)} \right), \\
 341$$

342 where b is positive non-integer [15], and recalling that $b\Gamma(b) = \Gamma(b+1)$, we can rewrite
 343 the expression (29) as

$$344 \quad \hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda) = -\frac{a}{1 - \alpha} - \frac{a^2 \lambda}{(1 - \alpha)(2 - \alpha)} + a^\alpha \lambda^{\alpha-1} \Gamma(-\alpha + 1) + \mathcal{O}(a^3 \lambda^2).$$

345 Note that in the above we have considered that $e^{a\lambda} = 1 + \mathcal{O}(a\lambda)$.

346 To simplify notation, let us define the quantities

$$347 \quad \zeta = -\frac{a}{1-\alpha}, \quad \vartheta = \frac{a^2}{(1-\alpha)(2-\alpha)}, \quad \eta = a^\alpha \Gamma(-\alpha + 1),$$

348

349 which are respectively of order a , a^2 , and a^α . Then $\hat{\psi}_\varepsilon$ is

$$350 \quad (31) \quad \hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda) = \zeta - \vartheta\lambda + \eta\lambda^{\alpha-1} + \mathcal{O}(a^3\lambda^2).$$

351 From a geometric expansion in $a\lambda \neq 0$ and the binomial theorem we have

$$352 \quad \left(\hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda)\right)^{-1} = \frac{1}{\zeta} \sum_{k=0}^{\infty} \left(\frac{\vartheta}{\zeta}\lambda - \frac{\eta}{\zeta}\lambda^{\alpha-1}\right)^k$$

$$353 \quad (32) \quad = \frac{1}{\zeta} + \frac{1}{\zeta} \sum_{k=2}^{\frac{1}{\alpha-1}} \left(\frac{\eta}{\zeta}\right)^k \lambda^{k(\alpha-1)} - \frac{\eta}{\zeta^2}\lambda^{\alpha-1} + \frac{\vartheta}{\zeta^2}\lambda + \mathcal{O}(a^{-1}(a\lambda)^\alpha)$$

354

355 for $k \in \mathbb{N}$ and $|\vartheta\lambda - \eta\lambda^{\alpha-1}| < \zeta$, since the left hand side is of higher order in $a\lambda$. The
 356 terms in the sum over k will eventually be of lower order in the scaling parameter ε
 357 in [Section 6](#). We neglect them in the following.

358 Solving [\(19\)](#) we obtain

$$359 \quad (33) \quad \varphi_\varepsilon(\mathbf{x}, t, \theta, \tau) = \frac{\alpha a^\alpha}{(a + \tau)^{\alpha+1}},$$

360 and the Laplace transform of [\(33\)](#) is

$$361 \quad \hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \lambda) = \alpha(a\lambda)^\alpha \Gamma(-\alpha, a\lambda) e^{a\lambda}.$$

362 Again using the expansion for the incomplete Gamma function [\(30\)](#) we see that

$$363 \quad (34) \quad \hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \lambda) = 1 + \frac{a\lambda}{1-\alpha} + \mathcal{O}(a^\alpha \lambda^\alpha).$$

364 As a consequence, from [\(32\)](#) and [\(34\)](#) we conclude

$$365 \quad (35) \quad \frac{\hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \lambda)}{\hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \lambda)} = \frac{\alpha-1}{a} - \frac{\lambda}{2-\alpha} - a^{\alpha-2}\lambda^{\alpha-1}(\alpha-1)^2\Gamma(-\alpha+1) + \mathcal{O}(a^{\alpha-1}\lambda^\alpha),$$

366 up to lower order terms in $a\lambda$.

367 **5.1. Fractional diffusion equation.** Using the form of β obtained in the pre-
 368 vious part, the scaled model equation [\(17\)](#) takes the form

$$369 \quad \varepsilon \partial_t \bar{\sigma} + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \bar{\sigma} = -(\mathbf{1} - T) \int_0^t \mathcal{B}_\varepsilon(\mathbf{x}, t, \theta, t-s) \bar{\sigma}(\mathbf{x} - c\theta(t-s), s, \theta) ds.$$

370

371 Computing the Laplace transform of the above expression, we obtain

$$372 \quad (36) \quad [\varepsilon\lambda + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla] \hat{\bar{\sigma}}(\mathbf{x}, \lambda, \theta) - \varepsilon \bar{\sigma}_0(\mathbf{x}, \theta) = -(\mathbf{1} - T) \hat{\mathcal{B}}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon\lambda + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla) \hat{\bar{\sigma}}(\mathbf{x}, \lambda, \theta).$$

373 We substitute $\mathcal{B}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)$ with (27) in the above expression and use
 374 (32) and (34) to obtain

$$375 \quad [\varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla]\hat{\sigma}(\mathbf{x}, \lambda, \theta) - \varepsilon\bar{\sigma}_0(\mathbf{x}, \theta) = -(\mathbb{1} - T)\left[\frac{1}{\zeta} + \frac{\vartheta}{\zeta^2}(\varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)\right. \\
 (37) \quad \left. - \frac{\eta}{\zeta^2}(\varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)^{\alpha-1} + \mathcal{O}(a^{\alpha-1}\lambda^\alpha)\right]\hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)\hat{\sigma}(\mathbf{x}, \lambda, \theta).$$

378 Recalling that $1 - \gamma < 1$ we find that to leading order in ε

$$379 \quad (\varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)^{\alpha-1} = (\varepsilon^{1-\gamma}c_0\theta \cdot \nabla)^{\alpha-1} + \mathcal{O}\left(\varepsilon^{1+(\alpha-1)(1-\gamma)}\right). \\
 380$$

381 Hence,

$$382 \quad [\varepsilon\lambda + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla]\hat{\sigma}(\mathbf{x}, \lambda, \theta) - \varepsilon\bar{\sigma}_0(\mathbf{x}, \theta) = -(\mathbb{1} - T)\left[\frac{1}{\zeta} + \frac{\vartheta}{\zeta^2}\varepsilon^{1-\gamma}c_0\theta \cdot \nabla\right. \\
 (38) \quad \left. - \frac{\eta}{\zeta^2}(\varepsilon^{1-\gamma}c_0\theta \cdot \nabla)^{\alpha-1} + \mathcal{O}\left(\varepsilon^{1+(\alpha-1)(-\mu-\gamma+1)}\right)\right]\hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)\hat{\sigma}(\mathbf{x}, \lambda, \theta).$$

385 Transforming (38) back to the time domain, we conclude

$$386 \quad (39) \quad \varepsilon\partial_t\bar{\sigma} + \varepsilon^{1-\gamma}c_0\theta \cdot \nabla\bar{\sigma} = -(\mathbb{1} - T)\mathcal{T}_\varepsilon\bar{\sigma}, \\
 387$$

388 where to leading order

$$389 \quad \mathcal{T}_\varepsilon = \frac{\hat{\varphi}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)}{\hat{\psi}_\varepsilon(\mathbf{x}, t, \theta, \varepsilon^{1-\gamma}c_0\theta \cdot \nabla)}.$$

390 **6. Scaling analysis.** We expand $\bar{\sigma}_\varepsilon$ using the eigenfunction representation from
 391 [Appendix A](#):

$$392 \quad \bar{\sigma}_\varepsilon = \frac{1}{|S|}(\bar{u} + \varepsilon^\kappa n\theta \cdot \bar{w}) + \text{l.o.t.},$$

393 where $\kappa > 0$. Note that the lower order terms are orthogonal to all linear polynomials
 394 in θ . Substituting the expansion into [Equation \(39\)](#),

$$(40) \quad \frac{\varepsilon}{|S|}\partial_t(\bar{u} + \varepsilon^\kappa n\theta \cdot \bar{w}) + \frac{\varepsilon^{1-\gamma}c_0}{|S|}\theta \cdot \nabla(\bar{u} + \varepsilon^\kappa n\theta \cdot \bar{w}) = -\frac{1}{|S|}(\mathbb{1} - T)\mathcal{T}_\varepsilon(\bar{u} + \varepsilon^\kappa n\theta \cdot \bar{w}),$$

396 up to lower order terms. By integrating over θ and recalling (50) ([Appendix A](#)),
 397 as well as the above-mentioned orthogonality, we find the macroscopic conservation
 398 equation

$$399 \quad (41) \quad \varepsilon\partial_t\bar{u} + \varepsilon^{\kappa+1-\gamma}c_0n\nabla \cdot \bar{w} = 0.$$

400 This equation is nontrivial only for $\kappa = \gamma$, so that

$$401 \quad \bar{\sigma}_\varepsilon = \frac{1}{|S|}(\bar{u} + \varepsilon^\gamma n\theta \cdot \bar{w}) + \text{l.o.t.}.$$

402 To obtain an equation for the mean direction \bar{w} , we multiply (40) by θ and integrate
 403 over the whole sphere S :

$$404 \quad (42) \quad n\varepsilon^{\gamma+1}\partial_t\bar{w} + \varepsilon^{1-\gamma}c_0\nabla\bar{u} = -\frac{1}{|S|}\int_S \theta(\mathbb{1} - T)\mathcal{T}_\varepsilon(\bar{u} + \varepsilon^\gamma n\theta \cdot \bar{w})d\theta. \\
 405$$

406 Using Equation (42) and the appropriate values for μ and γ , we get an expression
 407 for \bar{w} which, on substitution into the conservation equation (41), leads to the fractional
 408 Patlak-Keller-Segel equation.

409 To see this, we first determine \mathcal{T}_ε . Considering (35) we have,

$$410 \quad \mathcal{T}_\varepsilon = \frac{\alpha - 1}{a} - \frac{\varepsilon^{1-\gamma} c_0}{2 - \alpha} (\theta \cdot \nabla) - a^{\alpha-2} (\alpha - 1)^2 \Gamma(-\alpha + 1) (\varepsilon^{1-\gamma} c_0 \theta \cdot \nabla)^{\alpha-1}$$

$$411 \quad (43) \quad \quad \quad + \mathcal{O}(\varepsilon^{\mu-\gamma+1}).$$

We notice that $D_\varepsilon^\theta \rho = \varepsilon \partial_t \rho + \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \rho \simeq \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \rho$ since $1 - \gamma < 1$. Then, expanding the term

$$a^{\alpha-2} = (\tau_0 \varepsilon^\mu)^{\alpha-2} \left(1 + \frac{\tau_1}{\tau_0} \varepsilon^{1-\gamma} c_0 \theta \cdot \nabla \rho \right)^{\alpha-2}$$

413 using a binomial expansion, we find

$$414 \quad a^{\alpha-2} = (\tau_0 \varepsilon^\mu)^{\alpha-2} \left(1 + \varepsilon^{1-\gamma} (\alpha - 2) \frac{\tau_1}{\tau_0} c_0 \theta \cdot \nabla \rho + \mathcal{O}(\varepsilon^{2(1-\gamma)}) \right).$$

415 Similarly we can write

$$416 \quad a^{-1} = \frac{\varepsilon^{-\mu}}{\tau_0} \left(1 - \varepsilon^{1-\gamma} \frac{\tau_1}{\tau_0} c_0 \theta \cdot \nabla \rho + \mathcal{O}(\varepsilon^{2(1-\gamma)}) \right).$$

417 Therefore, the operator in (43) becomes

$$418 \quad (44) \quad \mathcal{T}_\varepsilon = \left[\frac{\varepsilon^{-\mu}}{\tau_0} (\alpha - 1) - \frac{\tau_1}{\tau_0^2} (\alpha - 1) \varepsilon^{-\mu-\gamma+1} c_0 \theta \cdot \nabla \rho - \frac{\varepsilon^{1-\gamma} c_0}{2 - \alpha} \theta \cdot \nabla \right.$$

$$\quad \left. + \left(-\tau_0^{\alpha-2} \varepsilon^{\mu(\alpha-2)+(1-\gamma)(\alpha-1)} + \tau_0^{\alpha-3} \tau_1 (2 - \alpha) \varepsilon^{\mu(\alpha-2)+\alpha(1-\gamma)} c_0 \theta \cdot \nabla \rho \right) \right.$$

$$\quad \left. (1 - \alpha)^2 \Gamma(-\alpha + 1) c_0^{\alpha-1} (\theta \cdot \nabla)^{\alpha-1} \right] + \mathcal{O}(\varepsilon^{\mu-\gamma+1}).$$

419 The physically relevant scaling regime involves transport in the equation for \bar{w} . For
 420 $-\mu - \gamma + 1 = -\mu + \gamma$ we obtain $\gamma = 1/2$, and therefore $\mu = \frac{2-\alpha}{2(\alpha-1)}$. This scaling leads
 421 to

$$422 \quad \mathcal{T}_\varepsilon (\bar{u} + \varepsilon^{-\gamma} n \theta \cdot \bar{w}) = \frac{\varepsilon^{\frac{\alpha-2}{2(\alpha-1)}}}{\tau_0} (\alpha - 1) \bar{u} - \frac{\tau_1}{\tau_0^2} (\alpha - 1) \varepsilon^{\frac{\alpha-2}{2(\alpha-1)} + \frac{1}{2}} c_0 (\theta \cdot \nabla \rho) \bar{u}$$

$$423 \quad - \tau_0^{\alpha-2} (1 - \alpha)^2 \Gamma(-\alpha + 1) \varepsilon^{\frac{\alpha-2}{2(\alpha-1)} + \frac{1}{2}} c_0^{\alpha-1} (\theta \cdot \nabla)^{\alpha-1} \bar{u} + \frac{\varepsilon^{\frac{\alpha-2}{2(\alpha-1)} + \frac{1}{2}}}{\tau_0} (\alpha - 1) n \theta \cdot \bar{w}$$

$$424 \quad + \mathcal{O}(\varepsilon^{\frac{\alpha-2}{2(\alpha-1)} + 1}).$$

426 We now compare the leading powers of ε in Equation (42). For the coefficient of the
 427 leading term $\varepsilon^{\frac{\alpha-2}{2(\alpha-1)}}$ we find

$$428 \quad (45) \quad 0 = -\frac{1}{|S|} \int_S \theta (\mathbf{1} - T) \frac{\alpha - 1}{\tau_0} \bar{u} d\theta,$$

430 while the subleading term is of order $\varepsilon^{\frac{2\alpha-3}{2(\alpha-1)}}$ with coefficients

$$431 \quad 0 = -\frac{1}{|S|} \int_S \theta(\mathbb{1} - T) \left[-\frac{\tau_1}{\tau_0^2} (\alpha - 1) c_0 (\theta \cdot \nabla \rho) \bar{u} \right. \\ 432 \quad (46) \quad \left. - \tau_0^{\alpha-2} (1 - \alpha)^2 \Gamma(-\alpha + 1) c_0^{\alpha-1} (\theta \cdot \nabla)^{\alpha-1} \bar{u} + \frac{\alpha-1}{\tau_0} n \theta \cdot \bar{w} \right] d\theta. \\ 433$$

434 We see from (45):

$$435 \quad (47) \quad -\frac{1}{|S|} \int_S \theta(\mathbb{1} - T) \frac{\alpha-1}{\tau_0} \bar{u} d\theta = -\frac{1}{|S|} \frac{\alpha-1}{\tau_0} \bar{u} \int_S \theta(\mathbb{1} - T) d\theta = 0,$$

436 due to the conservation condition (50). Similarly, (46) becomes, using the represen-
437 tation of T in terms of its eigenfunctions as in (55) in Appendix B,

$$438 \quad 0 = -\frac{1}{|S|} \int_S \theta \left(\frac{\tau_1}{\tau_0^2} (\alpha - 1) c_0 (\theta \cdot \nabla \rho) \bar{u} (\nu_1 - 1) - \tau_0^{\alpha-2} (1 - \alpha)^2 \Gamma(-\alpha + 1) (c_0 \theta \cdot \nabla)^{\alpha-1} \bar{u} \right. \\ 439 \quad \left. + \tau_0^{\alpha-2} (1 - \alpha)^2 \Gamma(-\alpha + 1) c_0^{\alpha-1} \left(\frac{\mathbb{D}^{\alpha-1}}{|S|} + \frac{n^2 \nu_1}{|S|} \theta \cdot \nabla^{\alpha-1} \right) \bar{u} - \frac{\alpha-1}{\tau_0} n \theta \cdot \bar{w} (\nu_1 - 1) \right) d\theta \\ 440 \quad = -\frac{\tau_1}{\tau_0^2} (\alpha - 1) c_0 \bar{u} (\nu_1 - 1) \nabla \rho - \tau_0^{\alpha-2} (1 - \alpha)^2 \Gamma(-\alpha + 1) c_0^{\alpha-1} \nabla^{\alpha-1} \bar{u} \left(\frac{n^2 \nu_1}{|S|} - 1 \right) \\ 441 \quad + \frac{\alpha-1}{\tau_0} n \bar{w} (\nu_1 - 1). \\ 442$$

443 We can solve this for the mean flux $c_0 \bar{w}$, which is given by

$$444 \quad (48) \quad c_0 \bar{w} = \frac{\tau_1}{n \tau_0} c_0^2 \bar{u} \nabla \rho + \frac{\pi \tau_0^{\alpha-1} (\alpha - 1) (n^2 \nu_1 - |S|)}{\sin(\pi \alpha) \Gamma(\alpha) n |S| (\nu_1 - 1)} c_0^\alpha \nabla^{\alpha-1} \bar{u},$$

445 where we have used $\Gamma(-\alpha + 1) = \frac{\pi}{\sin(\pi \alpha) \Gamma(\alpha)}$. The conservation equation (41) can
446 therefore be written as

$$447 \quad (49) \quad \partial_t \bar{u} = c_0 \nabla \cdot (D_\alpha \nabla^{\alpha-1} \bar{u} - \chi \bar{u} \nabla \rho)$$

448 for

$$449 \quad D_\alpha = -\frac{\pi (\tau_0 c_0)^{\alpha-1} (\alpha - 1) (n^2 \nu_1 - |S|)}{\sin(\pi \alpha) \Gamma(\alpha) |S| (\nu_1 - 1)} \quad \text{and} \quad \chi = \frac{\tau_1 c_0}{\tau_0}.$$

450 Note that $D_\alpha > 0$ since $\sin(\pi \alpha) < 0$ for $1 < \alpha < 2$. Figure 3 shows the behaviour of
451 the diffusion coefficient for different values of $c_0 \tau_0$, depending on α .

452 Note further that using a Cattaneo approximation to approximate the effective
453 contribution of higher order terms leads to an additional diffusive term in (49). How-
454 ever, the coefficient of this term turns out to be of lower order in the scaling variable
455 ε and hence can be neglected.

456 **7. Conclusions & Outlook.** In this paper we have derived effective macro-
457 scopic diffusion equations for organisms with long range behaviour, in the presence of
458 some chemoattractant or nutrient. Beginning with a microscopic model in which run
459 time distributions follow a power-law as observed, for example, for E. coli and Dd at
460 low nutrient concentrations, we obtain the form of the scattering operator and the
461 resulting kinetic equation. The fractional Patlak-Keller-Segel system (49) emerges in
462 a realistic hyperbolic limit.

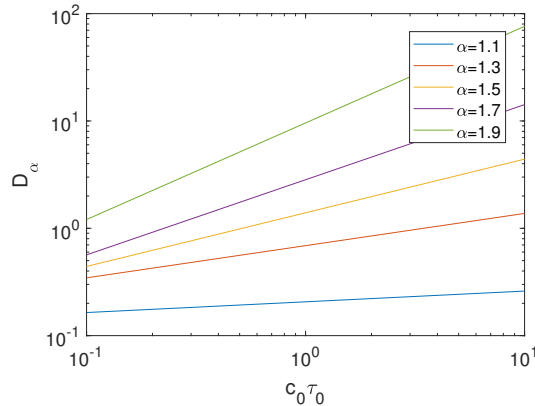


FIG. 3. Diffusion coefficient D_α for different values of $c_0\tau_0$, where the remaining parameters have being left fixed.

463 Unlike in [6], where the authors derived a similar fractional diffusion equation
 464 starting from a kinetic equation, our approach starts from a model for the individual
 465 organisms, reflecting the experimentally observed movement patterns. This model
 466 can subsequently be made concrete in a wide range of different biological contexts.

467 Our discussion in this article focused on organisms in an unbounded domain or
 468 sufficiently far away from physical boundaries. This would seem reasonable for cells
 469 tracked in vitro under the microscope, where the containing disk is multiple orders of
 470 magnitude larger than a cell. However, the nonlocality of (49) will lead to a signif-
 471 icantly increased influence of the boundary as well as the surrounding environment,
 472 compared to standard diffusion [2]. The actual interaction between an organism and
 473 the boundary is expected to vary considerably according to the organism and the na-
 474 ture of the system (for example between an experiment and a natural environment).
 475 Any meaningful discussion of boundary conditions would therefore have to focus on
 476 the context of modelling a particular biological system. For an example of an anom-
 477 alous effective diffusion equation with Dirichlet boundary conditions, see [14]. A more
 478 detailed discussion of relevant boundary conditions for fractional Patlak-Keller-Segel
 479 equations will be considered in future work.

480 More generally, it will certainly be fruitful to tailor the model to particular bi-
 481 ological systems. For example in the case of *E. coli*, the stopping probability could
 482 be specifically linked to molecular components (e.g. CheR) which enter as internal
 483 variables. We refer to work in this direction by Perthame et. al. [39] for the run-and-
 484 tumble of bacteria including a biochemical pathway. For a more detailed discussion
 485 of modelling bacterial chemotaxis including internal variables we refer to [49] and
 486 references therein.

487 In the current paper our assumptions have been largely motivated by the motion
 488 of *E. coli*, which offers an opportunistic case study due to its well characterised be-
 489 haviour. While the results give some insight into the expected equations for other
 490 cells or organisms, extending to such systems in a more meaningful way would require
 491 re-evaluation of the core assumptions. For example, eukaryotic cells such as *Dd* or
 492 immune cells can be large enough to directly sense a spatial gradient, so that the turn-
 493 ing distribution is potentially biased with respect to the chemoattractant gradient.
 494 Nevertheless, moving to such cell types provides an exciting focus for applications,

495 with T cell movement in the central nervous system (CNS) being one such example.
 496 CNS resistance to the encephalitis causing pathogen *Toxoplasma gondii* demands that
 497 patrolling T cells locate potentially sparsely distributed infection sites. Data in [19]
 498 suggest that the immune cells optimise searching via a generalised Lévy walk involving
 499 fixed velocity straight runs with distances randomly chosen from a Lévy distribution,
 500 as in our above assumptions, but also interspersed by pauses that are also drawn from
 501 a Lévy distribution. Adapting the model to this system, however, would allow us to
 502 quantitatively investigate how this behaviour increases searching efficiency.

503 Evidence for Lévy walk type behaviour often seem to arise under very specific
 504 conditions: for example, the presence or absence of food or chemoattractant in organ-
 505 ism movement. Modelling-wise, this suggests that generalized running probabilities
 506 could include switches from a power-law type distribution to exponential law, where
 507 the control is specifically mediated by the chemical concentration and/or gradient.
 508 Such “switching” behaviour between local and nonlocal search has been suggested in
 509 [30]. Its accurate mathematical modelling remains an open challenge.

510 Furthermore, the impact of interactions among individuals in swarming bacteria
 511 appears to be related to the emergence of superdiffusion: See [18] for a first work
 512 in this direction. Based on experimental results in [3], the authors show that Lévy
 513 walks can emerge as a cooperative effect without assuming a power-law distribution
 514 of run distances. Nevertheless, the appearance of Lévy walks in the case of systems
 515 of interacting self-propelled particles remains unknown.

516 Also, mathematically, the analysis of equations of the form (49) is of high current
 517 interest and has been extensively studied. In [12] the authors proved existence of
 518 global in time solutions for certain initial data, for the case of a fractional parabolic-
 519 elliptic Keller-Segel equation. Travelling wave solutions in the case of equations like
 520 (49) are expected to lead to new phenomena and in particular could be expected
 521 to speed up with time, see for instance [13]. In the absence of processes such as
 522 proliferation, travelling bands of bacteria dissipate over time in classical Keller-Segel
 523 equations, unless bacteria are given “extreme” sensitivity responses [48]: this dissipa-
 524 tion occurs as individuals drop away from the main band and lose contact with
 525 the chemoattractant. It is tempting to speculate that giving such “lost” bacteria an
 526 improved searching through fractional diffusion may allow them to reconnect with the
 527 main band.

528 Other aspects of chemotaxis equations in general, such as pattern formation, are
 529 intensely studied, see for example [5], and relevant for biological and ecological appli-
 530 cations [11]. Numerical investigations should allow us to address some of the previous
 531 questions about the dynamics, pattern formation and travelling wave solutions in
 532 realistic systems. This is the topic of ongoing work.

533 **Appendix A. Turn angle operator.**

534 This section recalls some basic spectral properties of the turn angle operator T
 535 defined in (13). Crucially, its kernel $k(\mathbf{x}, t, \theta; \eta) = \ell(\mathbf{x}, t, |\eta - \theta|)$ only depends on the
 536 distance $|\eta - \theta|$:

$$537 \quad T\phi(\eta) = \int_S k(\mathbf{x}, t, \theta; \eta)\phi(\theta)d\theta = \int_S \ell(\mathbf{x}, t, |\eta - \theta|)\phi(\theta)d\theta.$$

539 Because ℓ is a probability distribution, it is normalized to $\int_S \ell(\mathbf{x}, t, |\theta - e_1|)d\theta = 1$,
 540 where $e_1 = (1, 0, \dots, 0)$. We immediately observe

$$541 \quad (50) \quad \int_S (\mathbf{1} - T)\phi d\theta = 0$$

543 for all $\phi \in L^2(S)$. Biologically, (50) corresponds to the conservation of the number of
 544 organisms in the tumbling phase.

545 We also require some more detailed information about the spectrum of T . Recall
 546 that in n -dimensions, the surface area of the unit sphere S is given by

$$547 \quad |S| = \begin{cases} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, & \text{for } n \text{ even,} \\ \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, & \text{for } n \text{ odd.} \end{cases}$$

548

549 **LEMMA A.1.** *Assume that ℓ is continuous. Then T is a symmetric compact op-*
 550 *erator. In particular, there exists an orthonormal basis of $L^2(S)$ consisting of eigen-*
 551 *functions of T .*

552 *With $\theta = (\theta_0, \theta_1, \dots, \theta_{n-1}) \in S$, we have*
 (51)

$$553 \quad \begin{aligned} \phi_0(\theta) &= \frac{1}{|S|} \quad \text{is an eigenfunction to the eigenvalue } \nu_0 = 1, \\ \phi_1^j(\theta) &= \frac{n\theta_j}{|S|} \quad \text{are eigenfunctions to the eigenvalue } \nu_1 = \int_S \ell(\cdot, |\eta - 1|)\eta_1 d\eta < 1. \end{aligned}$$

554 *Any function $\bar{\sigma} \in L^2(\mathbb{R}^n \times \mathbb{R}^+ \times S)$ admits a unique decomposition*

$$555 \quad (52) \quad \bar{\sigma} = \frac{1}{|S|} (\bar{u} + n\theta \cdot \bar{w}) + \hat{z},$$

556 *where \hat{z} is orthogonal to all linear polynomials in θ . Explicitly,*

$$557 \quad \bar{u}(\mathbf{x}, t) = \int_S \bar{\sigma}(\mathbf{x}, t, \theta) \phi_0(\theta) d\theta, \quad \bar{w}^j(\mathbf{x}, t) = \int_S \bar{\sigma}(\mathbf{x}, t, \theta) \phi_1^j(\theta) d\theta,$$

558 *and $\bar{w} = (\bar{w}^1, \dots, \bar{w}^n)$.*

559 We interpret \bar{u} as the density of organisms independent of the direction and \bar{w} as
 560 their mean direction.

561 **Appendix B. Fractional operators.** We recall some basic definitions concern-
 562 ing fractional differential operators, as well as their relation to the turning operator
 563 T .

564 **DEFINITION B.1.** *For $s \in (0, 2)$ and $f \in C^2(\mathbb{R}^n)$ define the fractional gradient of*
 565 *f as*

$$566 \quad (53) \quad \nabla^s f(\mathbf{x}) = \frac{1}{|S|} \int_S \theta \mathbf{D}_\theta^s f(\mathbf{x}) d\theta = \frac{1}{|S|} \int_S \theta (\theta \cdot \nabla)^s f(\mathbf{x}) d\theta,$$

567 *where $\mathbf{D}_\theta^s = (\theta \cdot \nabla)^s$ is the fractional directional derivative of order s . The fractional*
 568 *Laplacian of f is given by*

$$569 \quad (54) \quad \mathbb{D}^s f(\mathbf{x}) = \frac{1}{|S|} \int_S \mathbf{D}_\theta^s f(\mathbf{x}) d\theta.$$

570 \mathbb{D}^s is associated to $(-\Delta)^{\alpha/2}$ as follows,

$$571 \quad \mathbb{D}^s f(\mathbf{x}) = \Xi_\alpha (-\Delta)^{\alpha/2}$$

572 where, in two dimensions, for $1 < \alpha < 2$,

$$573 \quad \Xi_\alpha = -2\sqrt{\pi} \cos\left(\frac{\pi\alpha}{2}\right) \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)}.$$

574 See [34] and [43] for further information.

575 Using Lemma A.1 and the definitions (53) and (54), we obtain for sufficiently
576 smooth functions f and ρ :

$$577 \quad \begin{aligned} T(\theta \cdot \nabla)f &= \nu_1(\theta \cdot \nabla)f, \quad T(\theta \cdot \nabla\rho)f = \nu_1(\theta \cdot \nabla\rho)f, \\ T(\theta \cdot \nabla)^s f &\simeq \frac{1}{|S|} \int_S \frac{1}{|S|} (\eta \cdot \nabla)^s f \, d\eta + \nu_1 \frac{n\theta}{|S|} \int_S \frac{n\eta}{|S|} (\eta \cdot \nabla)^s f \, d\eta \\ &= \frac{\mathbb{D}^s f}{|S|} + \nu_1 \frac{n^2\theta}{|S|} \cdot \nabla^s f. \end{aligned}$$

578

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