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Compressible Fluids Interacting with a Linear-Elastic Shell

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Abstract

We study the Navier–Stokes equations governing the motion of an isentropic compressible fluid in three dimensions interacting with a flexible shell of Koiter type. The latter one constitutes a moving part of the boundary of the physical domain. Its deformation is modeled by a linearized version of Koiter's elastic energy. We show the existence of weak solutions to the corresponding system of PDEs provided the adiabatic exponent satisfies $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). The solution exists until the moving boundary approaches a self-intersection. This provides a compressible counterpart of the results in Lengeler and Růžička (Arch Ration Mech Anal 211(1):205–255, 2014) on incompressible Navier–Stokes equations.

1. Introduction

Fluid structure interactions have been studied intensively by engineers, physicists and also mathematicians. This is motivated by a plethora of applications anytime a fluid force is balanced by some flexible material; for instance in hydro- and aero-elasticity [7,16] or biomechanics [4]. In this work we consider the motion of an isentropic compressible fluid (in particular a gas) in a three-dimensional body. A part of the boundary is assumed to be changing in time. The displacement of the boundary is prescribed via a two dimensional surface representing a Kirchhof–Love shell. Its material properties are deduced by assuming small strains and plane stresses parallel to the middle surface. We prove the existence of a weak solution to the coupled compressible Navier–Stokes system interacting with the Kirchhof–Love shell on a part of the boundary. The time interval of existence is only restricted once a self-intersection of the moving boundary (namely the shell) is approached.

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1.1. Motivation and State of Art

Over the last century mathematicians have been fascinated by the dynamics of fluid flows. The theory of (long-time) weak solutions started with the pioneering work of Leray concerning incompressible Navier–Stokes equations [34]. A compressible counterpart has been provided by Lions [37]. Lions' results have later been extended by Feireisl et al. [19,21] to physically important situations (including, in particular, monoatomic gases). Today, there exists an abundant amount of literature for both incompressible as well as compressible fluids. In the last decades fluid structure interactions have been the subject of active research. The interactions of fluids and elastic solids are of particular interest. A major mathematical difficulty is the parabolic–hyperbolic nature of the system resulting in regularity incompatibilities between the fluid- and the solid-phase. First results concerning weak solutions in the incompressible case consider regularized or damped elasticity laws, see [3,5,8,33]. The fluid interacts with an elastic shell which constitutes a moving part of the boundary of the physical domain in Lagrangian coordinates. The existence of strong solutions in short time was shown in [9]. We also refer to related studies in [13,14], where the motion of an elastic body in an incompressible fluid is considered. Long-time weak solutions in a similar setting have finally been obtained in [32] assuming a linearized elastic behavior of the shell. The authors of [32] consider a general three dimensional body in Eulerian coordinates. The elastic shell is a possibly large part of the boundary and may deform in the direction of the outer normal. Its material behavior depends on membrane and bending forces. A solution exists provided the magnitude of the displacement stays below some bound (depending only on the geometry of the reference domain) which excludes self-intersections. The results from [32] have been extended to some incompressible non-Newtonian cases in [31]; see also [24]. Results for incompressible fluids in cylindrical domains have been shown in [38,39] and [6]. The paper [38] deals with a cylindrical linear elastic/viscoelastic Koiter shell in two dimensions (the shell is prescribed by a one-dimensional curve). The papers [6,39] extend this to cylindrical three-dimensional fluid flows. Note that in [39] even nonlinear elastic behavior of the shell is allowed.

In contrast to the growing literature on incompressible fluids the knowledge about compressible fluids interacting with elastic solids is quite limited. To the best of our knowledge, the only related result is [28]. Here, a compressible fluid interacts with a structure modeled by a linear wave equation in Lagrangian coordinates. The result of [28] concerns the existence of short-time strong solutions. It is related to earlier results about the incompressible setting, see [13]. Results on long-time weak solutions from problems coupling compressible fluids with a priori unknown elastic structures seem to be missing. The aim of the present paper is to open this field by developing a compressible counterpart of the theory from [32]. More, precisely we are going to prove the existence of a weak solution to the compressible Navier–Stokes system coupled with a linear elastic Koiter-type shell. The two dimensional shell is connected to the velocity field via boundary values on the free part of the boundary. Moreover, momentum forces acting on the boundary are in equilibrium with the membrane forces and bending forces (flexural forces) of the shell.

1.2. The Model

We consider the Navier–Stokes system of an isentropic compressible viscous fluid interacting with a shell of Koiter-type of thickness $\varepsilon_0 > 0$. The Koiter shell model is a version of the Kirchhoff-Love shell. More precisely, it is a model reduction assuming small strains and plane stresses parallel to the middle surface of the shell. Physically this means that the shell consists of a homogeneous, isotropic material. Its mathematical formulation is as follows. Let $\Omega \subset \mathbb{R}^3$ be the initial physical domain and let $T > 0$. We devide $\partial\Omega$ into the fixed in time part Γ and its compact complement M, the part where the shell is located. The shell is assumed to be driven solely in the direction of the outer normal ν of Ω, cf. [8,23]. This allows one to write the energy for elastic shells via a scalar function $\eta : M \to (a, b)$. Here, the numbers a, b are fixed and depend only on the geometry of Ω, such that self intersections of the boundary are not possible. For example, in case of a ball $\Omega = B_r$ the interval is $(-r, \infty)$. The elastic energy of the deformation is then modeled via Koiter's energy [26, eqs. (4.2), (8.1), (8.3)]

$$K(\eta) = \frac{1}{2}\varepsilon_0 \int_M \mathbf{C} : \boldsymbol{\sigma}(\eta\nu) \otimes \boldsymbol{\sigma}(\eta\nu)\, \mathrm{d}\mathcal{H}^2 + \frac{1}{6}\varepsilon_0^3 \int_M \mathbf{C} : \boldsymbol{\theta}(\eta\nu) \otimes \boldsymbol{\theta}(\eta\nu)\, \mathrm{d}\mathcal{H}^2,$$

which is the sum of two terms reflecting different material properties. The first term is the membrane part of the energy which would remain even in the case of thin films. Indeed, the term $\boldsymbol{\sigma}(\eta\nu)$ depends linearly on the pullback of the first fundamental form of the two dimensional surface $\eta(M)$. The second term reflects the flexural part of the energy. Respectively, the argument $\boldsymbol{\theta}$ depends linearly on the pullback of the second fundamental form (the change of curvature). The coefficient tensor $\mathbf{C}$ is a non-linear function of the first fundamental form. For more details on the derivation of this model we refer to [25,26], where Koiter's energy for nonlinear elastic shells has been introduced; see also [11,12] for a more recent exposition. Following [12, Thm. 4.2-1 and Thm. 4.2-2] one can linearize $\boldsymbol{\sigma}$ and $\boldsymbol{\theta}$ with respect to η and obtain

$$K'(\eta) = m\Delta_\Gamma^2 \eta + B\eta \tag{1.1}$$

for the L^2-gradient K' of K. Here, $m > 0$ depends on the shell material (to be precise on ε_0 and the Lamé constants), B is a second order differential operator and Δ_Γ is the Laplace operator associated to the covariant derivative of the surface. In particular, it is shown in [12, Thm. 4.4-2], that

$$K(\eta) = \frac{1}{2}\int_M K'(\eta)\, \eta\ \mathrm{d}\mathcal{H}^2 \geqq c_0 \int_M |\nabla^2\eta|^2\ \mathrm{d}\mathcal{H}^2$$

for all $\eta \in H_0^2(M)$ with some $c_0 > 0$. Equation (1.1) is a Kirchhoff–Love shell equation for transverse displacements, cf. [10]. The technical restriction, that we only allow forces to act on the shell in (a fixed) normal direction, is the most severe restriction in our paper. Under this assumption we will, however, show long time weak solutions; they exist as long as the shell does not approach a self intersection. We also observe that long time existence results seem to be unavailable for less restrictive geometric assumptions. Even for one dimensional boundaries and for incompressible fluids no long time existence result seems to be available for less severe restrictions. Finally, observe that since η is assumed to have zero boundary values on ∂M, there is a canonical extension by zero to $\partial\Omega$, which we will use in the following without further remark.

We denote by $\Omega_{\eta(t)}$ the variable in time domain. With a slight abuse of notation we denote by $I \times \Omega_\eta = \bigcup_{t\in I} \{t\} \times \Omega_{\eta(t)}$ the deformed time-space cylinder, defined via its boundary

$$\partial\Omega_{\eta(t)} = \{x + \eta(t,x)\nu : x \in \partial\Omega\}.$$

Recall that Ω is a given (smooth) reference domain with outer normal ν.

Along this cylinder we observe the flow of an isentropic compressible fluid subject to the volume force $\mathbf{f} : I \times \Omega_\eta \to \mathbb{R}^3$. We seek the density $\varrho : I \times \Omega_\eta \to \mathbb{R}$ and velocity field $\mathbf{u} : I \times \Omega_\eta \to \mathbb{R}^3$ solving the following system:

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \quad \text{in } I \times \Omega_\eta, \tag{1.2}$$

$$\begin{aligned} \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) &= \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} \\ &\quad - \nabla p(\varrho) + \varrho\mathbf{f} \quad \text{in } I \times \Omega_\eta, \end{aligned} \tag{1.3}$$

$$\mathbf{u}(t, x + \eta(x)\nu(x)) = \partial_t\eta(t,x)\nu(x) \quad \text{on } I \times M, \tag{1.4}$$

$$\mathbf{u} = 0 \quad \text{on } I \times \Gamma, \tag{1.5}$$

$$\varrho(0) = \varrho_0, \quad (\varrho\mathbf{u})(0) = \mathbf{q}_0 \quad \text{in } \Omega_{\eta_0}. \tag{1.6}$$

Here, $p(\varrho)$ is the pressure which is assumed to follow the γ-law, for simplicity $p(\varrho) = a\varrho^\gamma$, where $a > 0$ and $\gamma > 1$. Note that in (1.3) we suppose Newton's rheological law

$$\mathbf{S} = \mathbf{S}(\nabla\mathbf{u}) = 2\mu\Big(\frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2} - \frac{1}{3}\operatorname{div}\mathbf{u}\,\mathcal{I}\Big) + \Big(\lambda + \frac{2}{3}\mu\Big)\operatorname{div}\mathbf{u}\,\mathcal{I},$$

with viscosity coefficients $\mu, \ \lambda$ satisfying

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu > 0,$$

see Remark 1.3 for the case $\lambda + \frac{2}{3}\mu = 0$. The shell should respond optimally with respect to the forces, which act on the boundary. Therefore we have

$$\varepsilon_0\varrho_S\partial_t^2\eta + K'(\eta) = g + \nu\cdot\mathbf{F} \quad \text{on} \quad I \times M, \tag{1.7}$$

where $\varrho_S > 0$ is the density of the shell. In (1.7) $g : [0,T] \times M \to \mathbb{R}$ is a given force density and we have

$$\begin{aligned} \mathbf{F} &:= \big(-\boldsymbol{\tau}\nu_\eta\big) \circ \boldsymbol{\Psi}_{\eta(t)}|\det D\boldsymbol{\Psi}_{\eta(t)}| \\ \boldsymbol{\tau} &:= \mathbf{S}(\nabla\mathbf{u}) - p(\varrho)\mathcal{I}. \end{aligned}$$

Here, $\boldsymbol{\Psi}_{\eta(t)} : \partial\Omega \to \partial\Omega_{\eta(t)}$ is a change of coordinates and $\boldsymbol{\tau}$ is the Cauchy stress. To simplify the presentation in (1.7) we will assume that

$$\varepsilon_0\varrho_S = 1$$

throughout the paper. We assume the following boundary and initial values for η:

$$\eta(0,\cdot) = \eta_0, \quad \partial_t\eta(0,\cdot) = \eta_1 \quad \text{on} \quad M, \tag{1.8}$$
$$\eta = 0, \quad \nabla\eta = 0 \quad \text{on} \quad \partial M, \tag{1.9}$$

where $\eta_0, \eta_1 : M \to \mathbb{R}$ are given functions such that

$$\mathrm{Im}(\eta_0) \subset (a, b).$$

In view of (1.4) we have to suppose the compatibility condition

$$\eta_1(x)\nu(x) = \frac{\mathbf{q}_0}{\varrho_0}(x + \eta(x)\nu(x)) \quad \text{on} \quad M. \tag{1.10}$$

By the canonical extension of η and $\partial_t\eta$ by 0 to $\partial\Omega$ we can unify (1.4) and (1.5) to

$$\mathbf{u}(t, x + \eta(t,x)\nu(x)) = \partial_t\eta(t,x)\nu \quad \text{on} \quad I \times \partial\Omega. \tag{1.11}$$

Our main result is the following existence theory for the system (1.2)–(1.9) which can be written in a natural way as a weak solution. The precise formulation can be found in Section 7, cf. (7.1) and (7.2):

Theorem 1.1. *For regular data and* $\gamma \in \left(\frac{12}{7}, \infty\right)$ *there exists a weak solution* $(\eta, \mathbf{u}, \varrho)$ *to* (1.2)–(1.9) *satisfying the energy estimate*

$$\begin{aligned}
&\sup_I \int_{\Omega_\eta} \varrho|\mathbf{u}|^2 \,\mathrm{d}x + \sup_I \int_{\Omega_\eta} \varrho^\gamma \,\mathrm{d}x + \int_I \int_{\Omega_\eta} |\nabla\mathbf{u}|^2 \,\mathrm{d}x \,\mathrm{d}\sigma \\
&\quad + \sup_I \int_M \frac{|\partial_t\eta|^2}{2} \,\mathrm{d}\mathcal{H}^2 + \sup_I \int_M |\nabla^2\eta|^2 \,\mathrm{d}\mathcal{H}^2 \\
&\quad \leqq c(\mathbf{f}, g, \mathbf{q}_0, \varrho_0, \eta_0, \eta_1).
\end{aligned} \tag{1.12}$$

The interval of existence is of the form $I = [0, t)$*, where* $t < T$ *only in case* $\Omega_{\eta(s)}$ *approaches a self-intersection when* $s \to t$.

The function space for weak solutions to (1.2)–(1.9) is determined by the left-hand side of (1.12), taking into account the variable domain. For the precise assumptions on the given data, as well as the precise definition of a weak solution see Theorem 7.1 in the last section of the paper. We remark that in the three dimensional case the bound $\gamma > \frac{12}{7}$ is less restrictive then the bound $\gamma > \frac{9}{5}$ appearing in the pioneering work of Lions [37], but more restrictive than the bound $\gamma > \frac{3}{2}$ arising in the theory by Feireisl et al. [21]. A detailed explanation can be found in the next subsection. For more information on the restrictions for the growth condition of the pressure see Remark 1.2 at the end of this section.

1.3. Mathematical Significance and Novelties

A primary task is to understand how to pass to the limit in a sequence of solutions $(\eta_n, \mathbf{u}_n, \varrho_n)$ to (1.2)–(1.9) which enjoys suitable regularity properties and satisfies the uniform estimate (1.12). The passage to the limit in the convective terms $\varrho_n\mathbf{u}_n$ and $\varrho_n\mathbf{u}_n \otimes \mathbf{u}_n$ follows by local arguments combined with global integrability, see (6.13) and (6.14). Here, problems with the moving boundary can be avoided. Note that this is totally different to the incompressible system studied in [32], where huge difficulties arise due to the divergence-free constraint. As it is common for the compressible Navier–Stokes system, the major difficulty is to pass to the limit in the nonlinear pressure. A key step is to improve the (space)-integrability of the density to ensure that $p(\varrho_n)$ actually converges to a measurable function (and not just to a measure). Locally, where the effect of the moving boundary disappears, this can be done by the standard method, see Proposition 6.3. However, our test-functions in the weak formulation are not compactly supported. This is crucial for the coupling of fluid and shell. Note, in particular, that this is different from [18], where the interaction of compressible fluids and rigid bodies is studied. In [18], at least the symmetric gradients of test-functions are supported away from the area of interaction. In our case, however, it is essential to exclude the concentration of $p(\varrho_n)$ at the boundary. On account of the limited regularity of the moving boundary (it is not even Lipschitz continuous in three dimensions, see (1.12)) the common approach based on the Bogovskiĭ operator fails. We solve this problem inspired by a method introduced in [29] for compressible Navier–Stokes equations in irregular domains; see Propositions 6.4 and 7.4. Consequently, we can exclude the concentration of the pressure at the boundary. This, in turn, allows us to prove that the weak continuity of the effective viscous flux

$$p(\varrho) - (\lambda + 2\mu)\,\mathrm{div}\,\mathbf{u}$$

holds globally, see (6.35) and (7.29). In order to combine this with the renormalized continuity equation we are confronted with another problem: we do not have zero boundary conditions for the velocity at the shell. In general, it seems to be extremely difficult if not impossible to combine the properties of the effective viscous flux with the renormalized continuity equation in this case (see the remarks in [42, Section 7.12.5]). This is due to the additional boundary term which appears when extending the continuity equation to the whole space. However, due to the natural interplay between fluid flow and elastic shell, our situation can be understood as no-slip boundary conditions with respect to the moving shell. Hence, the just mentioned boundary term disappears due to the Lagrangian background of the material derivative. To make this observation accessible, a careful study of the damped continuity equation in time dependent domains is necessary. We refer to Section 3.2 and in particular Theorem 3.1, which collects the necessary regularity results for the density function on time changing domains; it implies, for instance, the respective renormalized formulation. This is the second essential tool which allows one to show strong convergence of an approximate sequence ϱ_n and hence to establish the correct form of the pressure in the limit equation.

The third difficulty is to construct a sequence of solutions. In the present case, this is rather difficult to do, since the geometry and the solution are highly coupled via the partial differential equations. Hence, in order to use the ideas explained above rigorously, we need a four layer approximation of the system as follows:

- Artificial pressure (δ-layer): replace $p(\varrho) = a\varrho^\gamma$ by $p_\delta(\varrho) = a\varrho^\gamma + \delta\varrho^\beta$ where β is chosen large enough.
- Artificial viscosity (ε-layer): add $\varepsilon\Delta\varrho$ to the right-hand side of (1.2).
- Regularization of the boundary (κ-layer): replace the underlying domain Ω_η by Ω_{η_κ} where η_κ is a suitable regularization of η. Accordingly, the convective terms and the pressure have to be regularized as well.
- Finite-dimensional approximation (N-layer): the momentum equation has to be solved by means of a Galerkin-approximation.

The first two layers are common in the theory of compressible Navier–Stokes equations, see [21]. The third layer is needed additionally due to the low regularity of the shell described by η. By (1.12) we have $\eta \in W^{2,2}(M)$ such that Sobolev's embedding implies $\eta \in W^{1,q}(M)$ for all $q < \infty$ but not necessarily $\eta \in W^{1,\infty}(M)$. So, we do not have a Lipschitz boundary. In addition, it is necessary to regularize the convective terms in (1.2) and (1.3) (see the comments on the N-layer below for a detailed explanation).

On the last layer we are confronted with the problem that the function space depends on the solution itself. As a consequence a finite-dimensional Galerkin approximation is not possible, as the Ansatz functions depend on the solution itself. Motivated by [32] we apply a fixpoint argument in η and $\mathbf{u}$ for a linearized problem. (Roughly speaking we replace $\varrho\mathbf{u} \otimes \mathbf{u}$ in (1.3) by $\varrho\mathbf{u} \otimes \mathbf{v}$ and $\varrho\mathbf{u}$ in (1.2) by $\varrho\mathbf{v}$ for $\mathbf{v}$ given, see Section 4.3 since it is crucial for our fixed point argument that the momentum equation is linear in $\mathbf{u}$. For $(\zeta, \mathbf{v})$ given we solve the system on the domain Ω_ζ. The domain still varies in time but is independent of the solution. Note here that ϱ is computed by solving the continuity equation with convective term independent of $\mathbf{u}$. The existence of a weak solution $(\eta, \mathbf{u})$ to the decoupled system can than be shown by the Galerkin approximation without further problems. This is due to the good a-priori information for ϱ from Theorem 3.1, see Theorem 4.4. The next difficulty is to find a fixed point of the mapping $(\zeta, \mathbf{v}) \mapsto (\eta, \mathbf{u})$ in an appropriate function space. The compactness of the mapping situated on the shell is rather easy as we apply a proper regularization with arbitrary smoothness. The main issue is the compactness of the velocity. Inspired by ideas from [32] we can prove compactness of $\mathbf{u}_n$ in $L^2(I \times \mathbb{R}^3)$ (where $\mathbf{u}_n$ is extended by zero). This is based on Lemma 2.8, where we prove a variant of the Aubin–Lions compactness theorem for variable domains. It is noteworthy that we are unable to exclude a vacuum even in the situation of a damped continuity equation. To prevent the problem with the vacuum we replace on the κ-level the momentum $\partial_t(\varrho\mathbf{u})$ by $\partial_t((\varrho + \kappa)\mathbf{u})$ in the momentum equation, which allows us to show that $\mathbf{u}_n$ is strongly compact in $L^2(I \times \mathbb{R}^3)$.

1.4. Outline of the Paper

In Section 2 we present basics concerning variable domains as well as the functional analytic set-up. In Section 3 we study the continuity equation (with artificial viscosity) on variable domains. The renormalized formulation is of particular importance. Section 4 is concerned with the decoupled system, its finite dimensional approximation and the fixed point argument. The main result of this section is the existence of a weak solution to the regularized system with artificial viscosity and pressure. In Section 5 we pass to the limit in the regularization (of domain and convective terms) and gain a weak solution to the system with artificial viscosity. Compactness of the density can be shown as in the fixed point argument. Hence we can pass to the limit in all nonlinearities without further difficulties. The proceeding Sections 6 and 7 deal with the vanishing artificial viscosity and vanishing artificial pressure limit respectively. Both follow a similar scheme, where the major difficulty is the strong convergence of the density. The argumentation is based on the weak continuity of the effective viscous flux, oscillation defect measures and the renormalized continuity equation. The main result of this paper (the existence of weak solutions to (1.2)–(1.9)) follows after passing to the limit with $\delta \to 0$ in Section 7. The full statement is given in Theorem 7.1.

Remark 1.2. The restriction $\gamma > \frac{12}{7}$ in three space dimensions is needed to exclude concentrations of the pressure near the moving boundary. Indeed, such concentrations are excluded by constructing a test-function $\boldsymbol{\varphi}_n$ whose divergence explodes at the boundary while $\int p(\varrho_n)\,\mathrm{div}\,\boldsymbol{\varphi}_n$ is still bounded. This requires, in particular, estimation of the integral $\int_I \int_{\Omega_{\eta_n}} \varrho_n \mathbf{u}_n\, \partial_t \boldsymbol{\varphi}_n \,\mathrm{d}x\,\mathrm{d}t$. Naturally, the function $\boldsymbol{\varphi}_n$ depends on the distance to the boundary and as such on the shape of the moving boundary which only has low regularity. Indeed, the given a priori estimates imply that $\partial_t \boldsymbol{\varphi}_n$ can only be bounded in $L^2(I; L^q)$ for all $q < 4$ (using (1.4)). Hence, we need to know that $\varrho_n \mathbf{u}_n$ is bounded in $L^2(I; L^p)$ uniformly in n for some $p > \frac{4}{3}$. This follows from the a priori estimates provided we have $\gamma > \frac{12}{7}$ (using that $\varrho_n \mathbf{u}_n \in L^2(I; L^{6\gamma/(\gamma+6)})$ in three dimensions). In the two dimensional case we have instead $\partial_t \boldsymbol{\varphi}_n \in L^\infty(I; L^q)$ for all $q < \infty$. Consequently, no additional restrictions on γ are needed and the result holds for all $\gamma > 1$.

Remark 1.3. Our proof requires the bulk viscosity $\lambda + \frac{2}{3}\mu$ to be strictly positive. In case $\lambda + \frac{2}{3}\mu = 0$ it is necessary to control the full gradient by the deviatoric part of the symmetric gradient. Such a Korn-type inequality is well-known for Lipschitz domains, see [43]. In our context of domains with less regularity, a Korn-type inequality for symmetric gradients is shown in [31, Prop. 2.9] following ideas of [1]. The integrability of the full gradient is, however, less than the one of the symmetric gradient. We believe that a corresponding trace-free version can be shown following similar ideas. Thus, the case $\lambda + \frac{2}{3}\mu = 0$ could be included for the price that the velocity only belongs to $W^{1,p}$ for all $p < 2$.

2. Preliminaries

The variable domain Ω_η can be parametrized in terms of the reference domain Ω via a mapping $\boldsymbol{\Psi}_\eta$ such that

$$\begin{aligned} \boldsymbol{\Psi}_\eta : \Omega \to \Omega_\eta \text{ is invertible} \\ \text{and } \boldsymbol{\Psi}_\eta|_{\partial\Omega} : \partial\Omega \to \partial\Omega_\eta \text{ is invertible.} \end{aligned} \tag{2.1}$$

The explicit construction can be found below in (2.8).

Throughout the paper we will make heavily use of Reynolds transport theorem, which we will use without any further reference. This theorem says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\eta(t)}} g \,\mathrm{d}x = \int_{\Omega_{\eta(t)}} \partial_t g \,\mathrm{d}x + \int_{\partial\Omega_{\eta(t)}} \partial_t \eta \nu \circ \boldsymbol{\Psi}_\eta^{-1} \cdot \nu_\eta g \,\mathrm{d}\mathcal{H}^2, \tag{2.2}$$

provided all terms are well-defined. The above can easily be justified by transposition and a chain rule. The heuristic beyond is that ν is the direction in which the domain changes (which in our model is a fixed prescribed direction) and $\partial_t\eta$ describes the velocity of change. Therefore, the scalar $\partial_t \eta \nu \circ \boldsymbol{\Psi}_\eta^{-1} \cdot \nu_\eta$ is the derivative of the change of the domain; i.e. the forces acting in direction of the outer normal of $\partial\Omega_{\eta(t)}$. For a couple of functions (φ, b) which satisfy $\mathrm{tr}_\eta(\varphi) = b$ in the sense of Lemma 2.4 we have

$$\int_{\partial\Omega_\eta} \varphi \,\mathrm{d}\mathcal{H}^2 = \int_{\partial\Omega} \varphi \circ \boldsymbol{\Psi}_\eta \,|\mathrm{det}(D\boldsymbol{\Psi}_\eta)| \,\mathrm{d}\mathcal{H}^2 = \int_{\partial\Omega} b \,|\mathrm{det}(D\boldsymbol{\Psi}_\eta)| \,\mathrm{d}\mathcal{H}^2.$$

2.1. Formal a Priori Estimates and Weak Solutions

We introduce the weak formulation of the momentum equation, which will be coupled to the material law of the shell. This is motivated by the a priori estimates. We will now derive these estimates formally. First, we multiply the momentum equation by $\mathbf{u}$ and integrate with respect to space (at a fixed time). We multiply the continuity equation by $\frac{|\mathbf{u}|^2}{2}$ and integrate with respect to space (at the same time). Subtracting both and applying a chain rule yields

$$\begin{aligned} &\int_{\Omega_\eta} \partial_t\Big(\varrho\frac{|\mathbf{u}|^2}{2}\Big)\,\mathrm{d}x \\ &\quad= \int_{\Omega_\eta} \partial_t(\varrho\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x - \int_{\Omega_\eta}\partial_t\varrho\frac{|\mathbf{u}|^2}{2}\,\mathrm{d}x \\ &\quad= \int_{\Omega_\eta}\big(-\mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u} - a\nabla\varrho^\gamma + \varrho\mathbf{f}\big)\cdot\mathbf{u}\,\mathrm{d}x \\ &\qquad+ \int_{\Omega_\eta}\mathrm{div}(\varrho\mathbf{u})\,\tfrac{|\mathbf{u}|^2}{2}\,\mathrm{d}x \\ &\quad= -\int_{\partial\Omega_\eta}\varrho\tfrac{|\mathbf{u}|^2}{2}\mathbf{u}\cdot\nu_\eta\,\mathrm{d}\mathcal{H}^2 + \int_{\Omega_\eta}\mathrm{div}\,\boldsymbol{\tau}\cdot\mathbf{u}\,\mathrm{d}x + \int_{\Omega_\eta}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\partial\Omega_\eta} \varrho \tfrac{|\mathbf{u}|^2}{2} \mathbf{u} \cdot \nu_\eta \, \mathrm{d}\mathcal{H}^2 + \int_{\partial\Omega_\eta} \boldsymbol{\tau}\mathbf{u} \cdot \nu_\eta \, \mathrm{d}\mathcal{H}^2 - \int_{\Omega_\eta} \boldsymbol{\tau} : \nabla\mathbf{u} \, \mathrm{d}x \\
&\quad + \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x.
\end{aligned}
$$

By Reynolds' transport theorem we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \frac{|\mathbf{u}|^2}{2} \, \mathrm{d}x &= \int_M \boldsymbol{\tau}\, (\mathbf{u} \cdot \nu_\eta) \circ \boldsymbol{\Psi}_\eta |\det D\boldsymbol{\Psi}_\eta| \, \mathrm{d}\mathcal{H}^2 - \mu \int_{\Omega_\eta} |\nabla\mathbf{u}|^2 \, \mathrm{d}x \\
&\quad - (\lambda + \mu) \int_{\Omega_\eta} |\operatorname{div} \mathbf{u}|^2 \, \mathrm{d}x + \int_{\Omega_\eta} a\varrho^\gamma \operatorname{div} \mathbf{u} \, \mathrm{d}x + \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x.
\end{aligned}
$$

To control the pressure term, we multiply the continuity equation by $\gamma\varrho^{\gamma-1}$ and gain

$$
\begin{aligned}
0 &= \int_{\Omega_\eta} \partial_t \varrho^\gamma \, \mathrm{d}x + \int_{\Omega_\eta} \left(\operatorname{div} \mathbf{u}\, \gamma\varrho^\gamma + \mathbf{u} \cdot \nabla\varrho^\gamma \right) \mathrm{d}x \\
&= \int_{\Omega_\eta} \partial_t \varrho^\gamma \, \mathrm{d}x + (\gamma - 1) \int_{\Omega_\eta} \varrho^\gamma \operatorname{div} \mathbf{u} \, \mathrm{d}x + \int_{\Omega_\eta} \operatorname{div} \left(\mathbf{u}\varrho^\gamma \right) \mathrm{d}x.
\end{aligned}
$$

We obtain by Reynold's transport theorem and the assumed boundary values that

$$
0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho^\gamma \, \mathrm{d}x + (\gamma - 1) \int_{\Omega_\eta} \varrho^\gamma \operatorname{div} \mathbf{u} \, \mathrm{d}x.
$$

Later we will make this step rigorous via the use of so-called renormalized formulation of the continuity equation, see Section 3.1 below. Hence, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \left(\varrho \frac{|\mathbf{u}|^2}{2} + \frac{a}{\gamma - 1} \varrho^\gamma \right) \mathrm{d}x \\
&\quad = \int_M \boldsymbol{\tau}\, (\mathbf{u} \cdot \nu^\eta) \circ \boldsymbol{\Psi}_\eta |\det D\boldsymbol{\Psi}_\eta| \, \mathrm{d}\mathcal{H}^2 - \mu \int_{\Omega_\eta} |\nabla\mathbf{u}|^2 \, \mathrm{d}x \\
&\qquad - (\lambda + \mu) \int_{\Omega_\eta} |\operatorname{div} \mathbf{u}|^2 \, \mathrm{d}x + \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x.
\end{aligned}
$$

The boundary term represents the forces which are acting on the shell. Naturally these have to be in equilibrium with the bending and membrane potential of the shell. Formally, this is achieved by multiplying the shell equation (1.7) with $\partial_t\eta$.[1] Using once more that $\mathbf{u} \circ \boldsymbol{\Psi}_\eta = \partial_t\eta\nu$ on M we find that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_M \frac{|\partial_t\eta|^2}{2} \, \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} K(\eta) = \int_M \mathbf{F} \cdot \nu\partial_t\eta \, \mathrm{d}x = \int_M \mathbf{F} \cdot \mathbf{u} \circ \boldsymbol{\Psi}_\eta \, \mathrm{d}x.
$$

[1] Recall that we assume $\varepsilon_0\rho_S = 1$.

Thus, the right-hand sides of both equations cancel. Finally, we gain

$$
\begin{aligned}
&\int_{\Omega_\eta} \frac{\varrho(t)|\mathbf{u}(t)|^2}{2}\,\mathrm{d}x + \int_{\Omega_\eta} \frac{a}{\gamma-1}\varrho^\gamma(t)\,\mathrm{d}x + \mu\int_0^t\int_{\Omega_\eta} |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\quad + (\lambda+\mu)\int_0^t\int_{\Omega_\eta} |\operatorname{div}\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma + \int_M \frac{|\partial_t\eta|^2}{2}\,\mathrm{d}\mathcal{H}^2 + \frac{K(\eta)}{2}\\
&= \int_0^t\int_{\Omega_\eta} \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma + \int_0^t\int_M g\,\partial_t\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma\\
&\quad + \int_{\Omega_{\eta_0}} \frac{|\mathbf{q}_0|^2}{2\varrho_0}\,\mathrm{d}x + \frac{K(\eta_0)}{2} + \int_M \frac{|\eta_1|^2}{2}\,\mathrm{d}\mathcal{H}^2.
\end{aligned}
$$

This implies, by Hölder's inequality and absorption, (1.12). In coherence with the a-priori estimates we introduce the following weak formulation of the coupled momentum equation:

$$
\begin{aligned}
&\int_I \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\Omega_\eta} \varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\bigg)\,\mathrm{d}t\\
&\quad + \int_I\int_{\Omega_\eta} \Big(\mu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t - a\varrho^\gamma\,\operatorname{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_M \partial_t\eta b\,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t\eta\,\partial_t b\,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\,b\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\
&\quad = \int_I\int_{\Omega_\eta} \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_M g\,b\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.3}
$$

for all test-functions $(b,\boldsymbol{\varphi}) \in C_0^\infty(M)\times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\varphi} = b\nu$.

2.2. Geometry

In this section we present the background for variable domains, see [32] for further details. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class C^4 with outer unit normal ν. In the following Ω will be called the reference domain. We define for $\alpha > 0$ the set

$$S_\alpha := \{x\in\mathbb{R}^3 : \ \mathrm{dist}(x,\partial\Omega) < \alpha\}.$$

There exists a positive number $L > 0$ such that the mapping

$$\Lambda : \partial\Omega\times(-L,L)\to S_L,\quad \Lambda(q,s) = q + s\nu(q) \tag{2.4}$$

is a C^3-diffeomorphism. It is the so called Hanzawa transform. The details of this construction may be found in [30]. This is due to the fact, that for C^2-domains the closest point projection is well defined in a strip around the boundary. Indeed, its inverse Λ^{-1} will be written as $\Lambda^{-1}(x) = (q(x), s(x))$. Here $q(x) = \arg\min\{|q-x||q\in\partial\Omega\}$ is the closest boundary point to x (which is an orthogonal projection) and $s(x) = (x - q(x)) \cdot \nu(q(x))$. For the sake of simpler notation we assume with no loss of generality that

$$\Lambda(q, s) \in \Omega \text{ for all } s \in [-L, 0).$$

Hence, $s(x)$ is the negative distance to the boundary if $x \in \Omega$ and the positive distance, if $x \notin \Omega$. The orthogonality of the mapping is best characterized via the equation $\nabla s(x) = \nu(q(x))$. For a continuous function $\eta : \partial\Omega \to [-L, L]$ we define the variable domain

$$\Omega_\eta := \Omega \setminus S_L \cup \{x \in S_L : \; s(x) < \eta(q(x))\}. \tag{2.5}$$

Now

$$\nu_\eta(x) \text{ is defined as the outer normal at the point } x \in \partial\Omega_\eta. \tag{2.6}$$

Definition 2.1. (*Function spaces*) For $I = (0, T)$, $T > 0$, and $\eta \in C(\overline{I} \times \partial\Omega)$ with $\|\eta\| < L$ we set $I \times \Omega_\eta := \bigcup_{t \in I} \{t\} \times \Omega_{\eta(t)} \subset \mathbb{R}^4$. We define for $1 \leqq p, r \leqq \infty$

$$\begin{aligned} L^p(I; L^r(\Omega_\eta)) &:= \Big\{v \in L^1(I \times \Omega_\eta) : \; v(t, \cdot) \in L^r(\Omega_{\eta(t)}) \\ &\qquad \text{for a.e. } t, \; \|v(t, \cdot)\|_{L^r(\Omega_{\eta(t)})} \in L^p(I)\Big\}, \\ L^p(I; W^{1,r}(\Omega_\eta)) &:= \Big\{v \in L^p(I; L^r(\Omega_\eta)) : \; \nabla v \in L^p(I; L^r(\Omega_\eta))\Big\}. \end{aligned}$$

Lemma 2.2. ([32], p. 210, 211 and references given there.) *Let $\eta : \partial\Omega \to (-L, L)$ be a continuous function such that:*

(a) *There is a homomorphism $\mathbf{\Psi}_\eta : \overline{\Omega} \to \overline{\Omega}_\eta$ such that $\mathbf{\Psi}_\eta|_{\Omega \setminus S_L}$ is the identity;*
(b) *If $\eta \in C^k(\partial\Omega)$ for $k \in \{1, 2, 3\}$ then $\mathbf{\Psi}_\eta$ is a C^k-diffeomorphism.*

As the impact of the geometry on the PDE is quite severe we will include an explicit construction of $\mathbf{\Psi}_\eta$. Since we will use the parametrisation below locally we will assume $\mathrm{Im}(\eta) \subset [-\frac{L}{2}, \frac{L}{2}]$, where L is a fixed size, such that Λ given in (2.4) is well defined on the set $\partial\Omega \times [-L, L]$. We relate to any $\eta : \partial\Omega \to (-L, L)$ the mapping $\mathbf{\Psi}_\eta : \Omega \to \Omega_\eta$, such that

$$\begin{aligned} \mathbf{\Psi}_\eta &: \Omega \to \Omega_\eta \text{ is invertible}, \\ \mathbf{\Psi}_\eta &: \partial\Omega \to \partial\Omega_\eta \text{ is invertible}. \end{aligned} \tag{2.7}$$

This can be constructed as follows. Let $\varphi \in C^\infty\big((-\frac{3L}{4}, \infty), [0, 1]\big)$ such that $\varphi \equiv 0$ in $[-\frac{3L}{4}, -\frac{L}{2}]$ and $\varphi \equiv 1$ in $[-\frac{L}{4}, \infty)$. Moreover, we assume that φ is a C^k diffeomorphism on $[-\frac{L}{2}, -\frac{L}{4}]$ with $\varphi^{(l)}(-\frac{L}{2}) = 0 = \varphi^{(l)}(-\frac{L}{4})$ for all $l \in \{1, \dots, k\}$. We relate to any $\eta : \partial\Omega \to (-L, L)$ the mapping $\mathbf{\Psi}_\eta : \Omega \to \Omega_\eta$ given by

$$\mathbf{\Psi}_\eta(x) = \begin{cases} q(x) + \Big(s(x) + \eta(q(x))\varphi(s(x))\Big)\nu(q(x)), & \text{if } \operatorname{dist}(x, \partial\Omega) < L, \\ x, & \text{elsewhere} \end{cases}. \tag{2.8}$$

Hence, the two one-to-one relations in (2.7) are satisfied.

If $\|\eta\|_\infty < \frac{L}{2}$, the mapping $\boldsymbol{\Psi}_\eta$ can be extended such that

$$\boldsymbol{\Psi}_\eta : \Omega_{\frac{L}{2}-\eta} \to \Omega \cup S_{\frac{L}{2}} \text{ is invertible.} \tag{2.9}$$

Due to the assumption $\|\eta\|_\infty < \frac{L}{2}$ we have that $\overline{\Omega} \subset \Omega_{\frac{L}{2}-\eta} \subset \Omega \cup S_L$. The extension is obtained by setting

$$\boldsymbol{\Psi}_\eta(x) = \begin{cases} q(x) + (s(x) + \eta(q(x)))\nu(q(x)), & \text{if } x \notin \Omega \text{ and } s(x) + \eta(q(x)) \leqq \frac{L}{2}, \\ q(x) + \Big(s(x) + \eta(q(x))\varphi(s(x))\Big)\nu(q(x)), & \text{if } x \in \Omega \text{ and } s(x) < L, \\ x, & \text{elsewhere.} \end{cases} \tag{2.10}$$

We collect a few properties of the above mapping $\boldsymbol{\Psi}_\eta$.

Lemma 2.3. *Let* $1 < p \leqq \infty$ *and* $\sigma \in (0, 1]$. *Then:*

(a) *If* $\eta \in W^{2,2}(\partial\Omega)$ *with* $\|\eta\|_\infty < L$, *then the linear mapping* $\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta$ *(*$\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta^{-1}$*) is continuous from* $L^p(\Omega_\eta)$ *to* $L^r(\Omega)$ *(from* $L^p(\Omega)$ *to* $L^r(\Omega_\eta)$*) for all* $1 \leqq r < p$*;*
(b) *If* $\eta \in W^{2,2}(\partial\Omega)$ *with* $\|\eta\|_\infty < L$, *then the linear mapping* $\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta$ *(*$\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta^{-1}$*) is continuous from* $W^{1,p}(\Omega_\eta)$ *to* $W^{1,r}(\Omega)$ *(from* $W^{1,p}(\Omega)$ *to* $W^{1,r}(\Omega_\eta)$*) for all* $1 \leqq r < p$*;*
(c) *If* $\eta \in C^{0,1}(\partial\Omega)$ *with* $\|\eta\|_\infty < L$, *then the linear mapping* $\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta$ *(*$\mathbf{v} \mapsto \mathbf{v} \circ \boldsymbol{\Psi}_\eta^{-1}$*) is continuous from* $W^{\sigma,p}(\Omega_\eta)$ *to* $W^{\sigma,p}(\Omega)$ *(from* $W^{\sigma,p}(\Omega)$ *to* $W^{\sigma,p}(\Omega_\eta)$*).*

The continuity constants depend only on Ω, p, r, σ *and the respective norms of* η.

Proof. The first two properties can be found in [32, Lemma 2.6]. The third assertion follows by transposition rule and the fact, that $\nabla\boldsymbol{\Psi}_\eta$, $\nabla\boldsymbol{\Psi}_\eta^{-1}$ are uniformly bounded. Indeed, let us assume that $f \in W^{\sigma,p}(\Omega)$ for some $\sigma \in (0, 1)$. Recall that this means that

$$|f|^p_{W^{\sigma,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{3+p\sigma}} \,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{2.11}$$

Then

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|f \circ \boldsymbol{\Psi}_\eta(x) - f \circ \boldsymbol{\Psi}_\eta(y)|^p}{|x - y|^{3+p\sigma}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_{\Omega_\eta} \int_{\Omega_\eta} \frac{|f(a) - f(b)|^p}{|\boldsymbol{\Psi}_\eta^{-1}(a) - \boldsymbol{\Psi}_\eta^{-1}(b)|^{3+r\sigma}} |\det(D\boldsymbol{\Psi}_\eta^{-1})|\,\mathrm{d}a\,|\det(D\boldsymbol{\Psi}_\eta^{-1})|\,\mathrm{d}b \\
&\quad = \int_{\Omega_\eta} \int_{\Omega_\eta} \frac{|f(a) - f(b)|^p}{|a - b|^{3+p\sigma}} \frac{|\boldsymbol{\Psi}_\eta(\boldsymbol{\Psi}^{-1}(a)) - \boldsymbol{\Psi}_\eta(\boldsymbol{\Psi}^{-1}(b))|^{3+r\sigma}}{|\boldsymbol{\Psi}_\eta^{-1}(a) - \boldsymbol{\Psi}_\eta^{-1}(b)|^{3+r\theta}} \\
&\qquad |\det(D\boldsymbol{\Psi}_\eta^{-1})|\,\mathrm{d}a\,|\det(D\boldsymbol{\Psi}_\eta^{-1})|\,\mathrm{d}b
\end{aligned}$$

$$\leqq c\|\det(D\boldsymbol{\Psi}_\eta^{-1})\|_\infty^2\|\nabla\boldsymbol{\Psi}_\eta\|_\infty\int_\Omega\int_\Omega\frac{|f(a)-f(b)|^p}{|a-b|^{3+p\sigma}}\,\mathrm{d}a\,\mathrm{d}b.$$

In case $\sigma = 1$, the result follows directly by the transposition rule. Since the argument can be applied analogously to $f\circ\boldsymbol{\Psi}_\eta^{-1}$, the proof is completed. □

The following lemma is a modification of [32, Cor. 2.9]:

Lemma 2.4. *Let $1 < p < 3$ and $\eta \in W^{2,2}(\partial\Omega)$ with $\|\eta\|_{L^\infty(\partial\Omega)} < L$. Then the linear mapping $\mathrm{tr}_\eta : v \mapsto v\circ\boldsymbol{\Psi}_\eta|_{\partial\Omega}$ is well defined and continuous from $W^{1,p}(\Omega_\eta)$ to $W^{1-\frac{1}{r},r}(\partial\Omega)$ for all $r \in (1,p)$ and well defined and continuous from $W^{1,p}(\Omega_\eta)$ to $L^q(\partial\Omega)$ for all $1 < q < \frac{2p}{3-p}$. The continuity constants depend only on Ω, p, and $\|\eta\|_{W^{2,2}}$.*

Proof. The claim is a consequence of Lemma 2.3 and the continuity of the trace operator on the reference domain Ω, which is assumed to be smooth. □

The following lemma allows us to extend functions defined on the variable domain to the whole space $\mathbb{R}^3$ this is not trivial for $\eta \in W^{2,2}(\partial\Omega)$ because the boundary is not Lipschitz continuous, however, it requires the additional assumption $\|\eta\|_{L^\infty(\partial\Omega)} < \frac{L}{2}$:

Lemma 2.5. *Let $1 \leqq r < p < \infty$ and $\eta \in W^{2,2}(\partial\Omega)$ with $\|\eta\|_{L^\infty(\partial\Omega)} < \frac{L}{2}$. There is a continuous linear operator $\mathscr{E}_\eta : W^{1,p}(\Omega_\eta) \to W^{1,r}(\mathbb{R}^3)$ such that $\mathscr{E}_\eta\big|_{\Omega_\eta} = \mathrm{Id}$.*

Proof. If $\eta \in W^{1,\infty}(\partial\Omega)$ the result is standard. There is a continuous linear operator

$$E_{p,A} : W^{1,p}(A) \to W^{1,p}(\mathbb{R}^3)$$

for any bounded Lipschtz domain A and $1 \leqq p < \infty$, see, for instance [2, Thm. 5.28], where even slightly less regularity of the boundary is required.

For the general case we use the extension above to transfer from a functions space over the variable domain to a function from a functions space over the reference domain. Indeed Lemma 2.3 implies that the mapping

$$W^{1,p}(\Omega_\eta) \ni u \mapsto u^\eta \in W^{1,\tilde{r}}(\Omega), \quad u^\eta(x) = u(\boldsymbol{\Psi}_\eta(x))$$

is continuous for any $1 \leqq \tilde{r} < p$. Since Ω is assumed to have a uniform Lipschitz boundary it is possible to extend the function u^η to the whole space. Hence, using the Extension operator on Ω, we find

$$E_{\tilde{r},\Omega}(u^\eta) \in W^{1,\tilde{r}}(\mathbb{R}^3) \text{ and } E_{\tilde{r},\Omega}(u^\eta)|_\Omega = u^\eta.$$

In order to transform back we use the fact, that $\boldsymbol{\Psi}_\eta$ is invertible on Ω_η^* where $\overline{\Omega} \subset \Omega_\eta^*$ due to the assumption $\|\eta\|_{L^\infty(\partial\Omega)} < \frac{L}{2}$, cf. (2.9). By Lemma 2.3 we find that the mapping

$$W^{1,\tilde{r}}(\Omega_\eta^*) \ni v \mapsto v_\eta \in W^{1,r}(\Omega \cup S_{\frac{L}{2}}), \quad v_\eta(x) = v(\boldsymbol{\Psi}_\eta^{-1}(x))$$

is continuous for any $1 \leqq r < \tilde{r} < p$. Finally, we set

$$\mathscr{E}_\eta u = E_{r,\Omega\cup S_{\frac{L}{2}}}\left((E_{\tilde{r},\Omega}u^\eta)\big|_{\Omega^*_\eta}\right)_\eta \quad u \in W^{1,p}(\Omega_\eta).$$

It is now easy to check that $\mathscr{E}_\eta$ has the required properties. $\square$

Remark 2.6. If $\eta \in L^\infty(I; W^{2,2}(\partial\Omega))$ we obtain non-stationary variants of the results stated above.

2.3. Convergence on Variable Domains

Due to the variable domain the framework of Bochner spaces is not available. Hence, we cannot use the Aubin–Lions compactness theorem. In this subsection we are concerned with the question how to get compactness anyway. We start with the following definition of convergence in variable domains. Convergence in Lebesgue spaces follows from an extension by zero.

Definition 2.7. Let $(\eta_i) \subset C(\overline{I} \times \partial\Omega; [-\theta L, \theta L])$, $\theta \in (0, 1)$, be a sequence with $\eta_i \to \eta$ uniformly in $\overline{I} \times \partial\Omega$. Let $p \in [1, \infty]$ and $k \in \mathbb{N}_0$. Then:

(a) We say that a sequence $(g_i) \subset L^p(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^p(I, L^q(\Omega_{\eta_i}))$ strongly with respect to (η_i), in symbols $g_i \to^\eta g$ in $L^p(I, L^q(\Omega_{\eta_i}))$, if

$$\chi_{\Omega_{\eta_i}} g_i \to \chi_{\Omega_\eta} g \quad \text{in} \quad L^p(I, L^q(\mathbb{R}^3));$$

(b) Let $p, q < \infty$. We say that a sequence $(g_i) \subset L^p(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^p(I, L^q(\Omega_{\eta_i}))$ weakly with respect to (η_i), in symbols $g_i \rightharpoonup^\eta g$ in $L^p(I, L^q(\Omega_{\eta_i}))$, if

$$\chi_{\Omega_{\eta_i}} g_i \rightharpoonup \chi_{\Omega_\eta} g \quad \text{in} \quad L^p(I, L^q(\mathbb{R}^3));$$

(c) Let $p = \infty$ and $q < \infty$. We say that a sequence $(g_i) \subset L^\infty(I, L^q(\Omega_{\eta_i}))$ converges to g in $L^\infty(I, L^q(\Omega_{\eta_i}))$ weakly* with respect to (η_i), in symbols $g_i \rightharpoonup^{*,\eta} g$ in $L^\infty(I, L^q(\Omega_{\eta_i}))$, if

$$\chi_{\Omega_{\eta_i}} g_i \rightharpoonup^* \chi_{\Omega_\eta} g \quad \text{in} \quad L^\infty(I, L^q(\mathbb{R}^3)).$$

Note that in case of one single η (i.e. not a sequence) the space $L^p(I, L^q(\Omega_\eta))$ (with $1 \leqq p < \infty$ and $1 < q < \infty$) is reflexive and we have the usual duality pairing

$$L^p(I, L^q(\Omega_\eta)) \cong L^{p'}(I, L^{q'}(\Omega_\eta)), \tag{2.12}$$

provided that η is smooth enough, see [41]. Definition 2.7 can be extended in a canonical way to Sobolev spaces. We say that a sequence $(g_i) \subset L^p(I, W^{1,q}(\Omega_{\eta_i}))$ converges to $g \in L^p(I, W^{1,q}(\Omega_{\eta_i}))$ strongly with respect to (η_i), in symbols

$$g_i \to^\eta g \quad \text{in} \quad L^p(I, W^{1,p}(\Omega_{\eta_i})),$$

if both g_i and ∇g_i converges (to g and ∇g respectively) in $L^p(I, L^q(\Omega_{\eta_i}))$ strongly with respect to (η_i) (in the sense of Definition 2.7 a)). We also define weak and weak* convergence in Sobolev spaces with respect to (η_i) with an obvious meaning. Note that an extension to higher order Sobolev spaces is possible but not needed for our purposes.

2.4. A Lemma of Aubin–Lions Type for Time Dependent Domains

For the next compactness lemma we require the following assumptions on the functions describing the boundary:

(A1) The sequence $(\eta_i) \subset C(\overline{I} \times M; [-\theta L, \theta L])$, $\theta \in (0,1)$, satisfies

$$\begin{aligned} \eta_i &\rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I, W_0^{2,2}(M)),\\ \partial_t \eta_i &\rightharpoonup^* \partial_t \eta \quad \text{in} \quad L^\infty(I, L^2(M)); \end{aligned}$$

(A2) Let (v_i) be a sequence such that for some $p, s \in [1,\infty)$ we have

$$v_i \rightharpoonup^\eta v \quad \text{in} \quad L^p(I; W^{1,s}(\Omega_{\eta_i}));$$

(A3) Let (r_i) be a sequence such that for some $m, b \in [1,\infty)$ we have

$$r_i \rightharpoonup^\eta r \quad \text{in} \quad L^m(I; L^b(\Omega_{\eta_i})).$$

Assume further that there are sequences $(\mathbf{H}_i^1)$, $(\mathbf{H}_i^2)$ and (h_i), bounded in $L^m(I; L^b(\Omega_{\eta_i}))$, such that $\partial_t r_i = \operatorname{div}\operatorname{div}\mathbf{H}_i^1 + \operatorname{div}\mathbf{H}_i^2 + h_i$ in the sense of distributions, i.e.,

$$\begin{aligned} \int_I \int_{\Omega_{\eta_i}} r_i\, \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t &= \int_I \int_{\Omega_{\eta_i}} \mathbf{H}_i^1 \cdot \nabla^2 \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\Omega_{\eta_i}} \mathbf{H}_i^2 \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_I \int_{\Omega_{\eta_i}} h_i\, \varphi \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $\varphi \in C_0^\infty(I \times \Omega_{\eta_i})$.

Lemma 2.8. *Let (η_i), (v_i) and (r_i) be sequences satisfying (A1)–(A3) where $\frac{1}{s^*} + \frac{1}{b} = \frac{1}{a} < 1$ and $\frac{1}{m} + \frac{1}{p} = \frac{1}{q} < 1$.*[2] *Then there is a subsequence with*

$$v_i r_i \rightharpoonup vr \textit{ weakly in } L^q(I, L^a(\Omega_{\eta_i})). \tag{2.13}$$

Remark 2.9. Assumption (A3) in Lemma 2.8 can be extended in an obvious way to the case of higher order distributional derivatives. We have chosen the version above as it is most suitable for our applications.

Proof. First we show local compactness. Consider a cube $Q = J \times B$ such that $Q \Subset \Omega_{\eta_i}^I$ for all i large enough. By (A3) we know that r_i is bounded in $L^m(I; L^b(B))$ and that $\partial_t r_i$ is bounded in $L^m(I; W^{-2,b}(B))$. We can apply the classical Aubin–Lions compactness Theorem [35] for the triple

$$L^b(B) \hookrightarrow\hookrightarrow W^{-1,b}(B) \hookrightarrow W^{-2,b}(B),$$

and gain

$$r_i \to r \quad \text{in} \quad L^m(J; W^{-1,b}(B)). \tag{2.14}$$

[2] Here, we set $s^* = \frac{3s}{3-s}$, if $s \in (1,3)$ and otherwise s^* can be fixed in $(1,\infty)$ conveniently.

Note that we do not have to take a subsequence since the original sequence already converges by (A3). Now note that (A1) implies

$$\eta_i \to \eta \quad \text{in} \quad C^\alpha(\overline{I} \times M)$$

for some $\alpha \in (0,1)$ by interpolation. Consequently, for a given $\kappa > 0$ there is $i_0 = i_0(\kappa)$ such that

$$\mathcal{L}^4\Big(I \times (\Omega_\eta \setminus \Omega_l)\Big) \leqq \kappa \quad \forall l \geqq i_0, \tag{2.15}$$

where we have set

$$\Omega_l = \bigcap_{i \geqq l} \Omega_{\eta_i}.$$

Now we fix a measurable set $A_\kappa \Subset I \times \Omega_l$ with $\mathcal{L}^4\Big(\big(I \times \Omega_l\big) \setminus A_\kappa\Big) \leqq \kappa$ and cover it by at most countable many cubes $Q_k = J_k \times B_k$ such that

$$A_\kappa \subset \bigcup_k Q_k \Subset \big(I \times \Omega_l\big).$$

They can be chosen in such a way that we find a partition of unity (ψ_k) associated to the family Q_k such that $\psi_k \in C_0^\infty(Q_k)$ and

$$\sum \psi_k = 1 \text{ in } A_\kappa.$$

In particular, by taking a diagonal sequence, we can assume that (2.14) holds with $Q = Q_k$. For $w \in C_0^\infty(I \times \mathbb{R}^3)$, we have

$$\begin{aligned}
&\int_I \int_{\mathbb{R}^3} \big(\chi_{\Omega_{\eta_i}} r_i v_i - \chi_{\Omega_\eta} r v\big)\, w \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \sum_k \int_I \int_{\mathbb{R}^3} \big(r_i v_i - r v\big)\, \psi_k w \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_I \int_{\mathbb{R}^3} \big(\chi_{\Omega_{\eta_i}} r_i v_i - \chi_{\Omega_\eta} r v\big) \sum_k (1 - \psi_k) w \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

On account of (2.14) (with $Q = Q_k$) and (A2) the first integral on the right-hand side converges to zero. Due to (2.15) the second integral can be bounded in terms of κ. Here, we took into account the boundedness of $\chi_{\Omega_{\eta_i}} r_i v_i$ in $L^r(I \times \mathbb{R}^3)$ for some $r > 1$ which is a consequence of the assumptions $\frac{1}{s} + \frac{1}{b} < 1$ and $\frac{1}{m} + \frac{1}{p} < 1$. As κ is arbitrary we have shown

$$\lim_{i \to \infty} \int_I \int_{\mathbb{R}^3} \big(\chi_{\Omega_{\eta_i}} r_i v_i - \chi_{\Omega_\eta} r v\big)\, w \,\mathrm{d}x\,\mathrm{d}t = 0,$$

which means we have

$$\chi_{\Omega_{\eta_i}} r_i v_i \to \chi_{\Omega_\eta} r v \quad \text{in} \quad \mathcal{D}'(I \times \mathbb{R}^3). \tag{2.16}$$

However, our assumptions imply that $\chi_{\Omega_{\eta_i}} r_i v_i$ converges weakly in $L^q(I, L^a(\mathbb{R}^3))$ at least after taking a subsequence. As a consequence of (2.16) we can identify the limit and the claim follows. $\square$

Remark 2.10. In the case $r_i = v_i$, we find that

$$\int_0^T \int_{\Omega_{\eta_i}} |v_i|^2 \, dx \, dt \to \int_0^T \int_{\Omega_\eta} |v|^2 \, dx \, dt.$$

Since weak convergence and norm convergence implies strong convergence, we find (by interpolation) that

$$v_i \to^\eta v \text{ strongly in } L^2(I, L^2(\Omega_{\eta_i})).$$

Showing such a result for the velocity field in the context of incompressible fluid mechanics is the main achievement of the paper [32]. As opposed to (A3), the time derivative is only a distribution acting on divergence-free test-functions in the incompressible case. In contrast to the compactness arguments in [32], the proof of Lemma 2.8 does not face this difficulty.

3. The Continuity Equation in Variable Domains

3.1. Renormalized Solutions in Time Dependent Domains

This subsection is concerned with the study of the continuity equation

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \tag{3.1}$$

in a variable domain Ω_η with $\eta \in L^\infty(I; W^{2,2}(\partial\Omega))$ and $\mathbf{u} \in L^2(I; W^{1,2}(\Omega_\eta))$. Observe the following interplay of the two terms of the material derivative, that shall be used many times within this work. A (strong) solution to (3.5) satisfies for any $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$

$$\begin{aligned}
&\int_{\Omega_\eta} \Big(\partial_t \varrho \psi + \operatorname{div}(\varrho \mathbf{u}) \psi \Big) \,\mathrm{d}x \\
&\quad = \int_{\Omega_\zeta} \Big(\partial_t (\varrho \psi) + \operatorname{div}(\varrho \mathbf{u} \psi) \Big) \,\mathrm{d}x - \int_{\Omega_\eta} \Big(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \Big) \,\mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \psi \,\mathrm{d}x + \int_{\partial\Omega_\eta} \varrho \psi (\mathbf{u} - (\partial_t \eta \nu) \circ \mathbf{\Psi}_\eta) \cdot \nu_\eta \mathrm{d}\mathcal{H}^2 \,\mathrm{d}t \\
&\qquad - \int_{\Omega_\eta} \Big(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \Big) \,\mathrm{d}x.
\end{aligned}$$

In the case of our consideration we find, due to $\operatorname{tr}_\zeta(\mathbf{u}) = \partial_t \eta \nu$, that

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \psi \,\mathrm{d}x \,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \Big) \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{3.2}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$. Equation (3.2) will serve as a weak formulation of (3.1). It is worth mentioning, that by taking $\psi \equiv \chi_{[0,t]}$ in (3.2) we find that the total mass is conserved in the sense that

$$\int_{\Omega_{\eta(t)}} \varrho(t) \,\mathrm{d}x = \int_{\Omega_{\eta(0)}} \varrho(0) \,\mathrm{d}x$$

for all $t \in I$. Following DiPerna and Lions [15] we will introduce a renormalized formulation which will be of crucial importance for the reminder of the paper. An important observation is that the formulation in (3.2) can be extended to the whole space despite the fact that $\mathbf{u}$ does not have zero boundary values (this will be essential to prove strong convergence of the density, see Section 6.4). In fact, we have

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varrho\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{3.3}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ provided we extend ϱ by zero. It is essential to introduce the principle of normalized solutions on variable domains. To be explicit we wish to study the family of solutions $\theta(\varrho)$, where $\theta \in C^2(\mathbb{R}^+;\mathbb{R}^+)$, such that $\theta'' = 0$ for large values and $\theta(0) = 0$. In what follows we only argue formally. For a rigorous derivation of the renormalized continuity equation we refer to the next subsection and the derivations in Sections 6.3 and 7.2. We may use the test function $\theta'(\varrho)\psi$ with $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ in (3.3) and hence find that

$$\begin{aligned}
0 &= \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varrho\theta'(\varrho)\,\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \Big(\partial_t(\varrho\theta'(\varrho)\,\psi) - \partial_t\varrho\theta'(\varrho)\psi \\
&\quad + \operatorname{div}(\varrho\theta'(\varrho)\mathbf{u}\,\psi)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\mathbb{R}^3} \Big(\nabla\varrho\theta'(\varrho)\cdot\mathbf{u}\,\psi + \varrho\theta'(\varrho)\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varrho\theta'(\varrho)\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \Big(\partial_t(\varrho\theta'(\varrho)\,\psi) + \operatorname{div}(\varrho\theta'(\varrho)\mathbf{u}\,\psi)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_I \int_{\mathbb{R}^3} \Big(\partial_t\theta(\varrho)\psi + \operatorname{div}(\theta(\varrho)\mathbf{u})\,\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^3} (\varrho\theta'(\varrho) - \theta(\varrho))\operatorname{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Now, integration by parts and Reynolds' transport theorem imply that the first line vanishes. Again, integration by parts and Reynolds' transport theorem transfer the second line in the appropriate weak formulation. Hence, we find the renormalized formulation is

$$\begin{aligned}
0 &= \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \theta(\varrho)\,\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\theta(\varrho)\partial_t\psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_I \int_{\Omega_\eta} (\varrho\theta'(\varrho) - \theta(\varrho))\operatorname{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.4}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$.

3.2. The Damped Continuity Equation in Time Dependent Domains

We will need very explicit a-priori information about our approximation of the density ϱ. The necessary result is collected in Theorem 3.1 below. For the analogous results for fixed in time domains see [21, section 2.1]. We will assume that the moving boundary is prescribed by a function ζ of class $C^3(\overline{I}\times M)$. For a given function $\mathbf{w}\in L^2(I;W^{1,2}(\Omega_\zeta))$ and $\varepsilon>0$ we consider the equation

$$\begin{aligned}&\partial_t\varrho+\operatorname{div}(\varrho\mathbf{w})=\varepsilon\Delta\varrho \quad \text{in} \quad I\times\Omega_\zeta, \quad \varrho(0)=\varrho_0 \text{ in } \Omega_{\zeta(0)},\\ &\partial_{\nu_\zeta}\varrho\big|_{\partial\Omega_\zeta}=\tfrac{1}{\varepsilon}\varrho(\mathbf{w}-(\partial_t\zeta\nu)\circ\mathbf{\Psi}_\zeta^{-1})\cdot\nu_\zeta \quad \text{on} \quad I\times\partial\Omega_\zeta.\end{aligned}\tag{3.5}$$

A solution to (3.5) satisfies (formally) for any $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3)$

$$\begin{aligned}&\int_{\Omega_\zeta}\Big(\partial_t\varrho\psi+\operatorname{div}(\varrho\mathbf{w})\psi\Big)\,\mathrm{d}x\\ &\quad=\int_{\Omega_\zeta}\Big(\partial_t(\varrho\psi)+\operatorname{div}(\varrho\mathbf{w}\psi)\Big)\,\mathrm{d}x-\int_{\Omega_\zeta}\Big(\varrho\partial_t\psi+\varrho\mathbf{w}\cdot\nabla\psi\Big)\,\mathrm{d}x\\ &\quad=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\zeta}\varrho\psi\,\mathrm{d}x+\int_{\partial\Omega_\zeta}\varrho\psi(\mathbf{w}-(\partial_t\zeta\nu)\circ\mathbf{\Psi}_\zeta)\cdot\nu_\zeta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\ &\qquad-\int_{\Omega_\zeta}\Big(\varrho\partial_t\psi+\varrho\mathbf{w}\cdot\nabla\psi\Big)\,\mathrm{d}x.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{\Omega_\zeta}\varepsilon\Delta\varrho\,\psi\,\mathrm{d}x&=\int_{\Omega_\zeta}\varepsilon\partial_\nu\varrho\,\psi\,\mathrm{d}\mathcal{H}^2-\int_I\int_{\Omega_\zeta}\varepsilon\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\ &=\int_{\partial\Omega_\zeta}\varrho\psi(\mathbf{w}-(\partial_t\zeta\nu)\circ\mathbf{\Psi}_\zeta)\cdot\nu_\zeta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\ &\quad-\int_I\int_{\Omega_\zeta}\varepsilon\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This motivates the choice of the Neumann boundary values in (3.5) which implies the following neat weak formulation:

$$\begin{aligned}&\int_I\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\zeta}\varrho\psi\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\zeta}\big(\varrho\partial_t\psi+\varrho\mathbf{w}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=-\int_I\int_{\Omega_\zeta}\varepsilon\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{3.6}$$

for all $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3)$. We wish to emphasize that this weak formulation is canonical with respect to the moving boundary, as it is the only formulation which preserves mass. This turns out to be the essential property to gain the necessary estimates and correlations.

Theorem 3.1. *Let $\zeta\in C^3(\overline{I}\times M,[\frac{L}{2},\frac{L}{2}])$ be the function describing the boundary. Assume that $\mathbf{w}\in L^2(I;W^{1,2}(\Omega_\zeta))\cap L^\infty(I\times\Omega_\zeta)$ and $\varrho_0\in L^2(\Omega_{\zeta(0)})$. Then:*

(a) *There is a unique weak solution ϱ to (3.6) such that*

$$\varrho\in L^\infty(0,T;L^2(\Omega_\zeta))\cap L^2(0,T;W^{1,2}(\Omega_\zeta));$$

(b) *Let* $\theta \in C^2(\mathbb{R}_+;\mathbb{R}_+)$ *be such that* $\theta'(s) = 0$ *for large values of* s *and* $\theta(0) = 0$. *Then the following holds, for the canonical zero extension of* $\varrho \equiv \varrho\chi_{\Omega_\zeta}$*:*

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\theta(\varrho)\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{I\times\mathbb{R}^3}\theta(\varrho)\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_{I\times\mathbb{R}^3}\big(\varrho\theta'(\varrho)-\theta(\varrho)\big)\,\mathrm{div}\,\mathbf{w}\,\psi\,\mathrm{d}x + \int_{I\times\mathbb{R}^3}\theta(\varrho)\mathbf{w}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_{I\times\mathbb{R}^3}\varepsilon\chi_{\Omega_\zeta}\nabla\theta(\varrho)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_{I\times\mathbb{R}^3}\varepsilon\chi_{\Omega_\zeta}\theta''(\varrho)|\nabla\varrho|^2\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{3.7}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$*;*

(c) *Assume that* $\varrho_0 \geqq 0$ *a.e. in* $\Omega_{\zeta(0)}$. *Then we have* $\varrho \geqq 0$ *a.e. in* $I\times\Omega_\zeta$.

Proof. In order to find a solution to (3.6) we discretise the system. It is standard to find a smooth orthonormal basis $(\tilde{\omega}_k)_{k\in\mathbb{N}}$ of $W^{1,2}(\Omega)$. Now define pointwise in t

$$\omega_k := \tilde{\omega}_k \circ \mathbf{\Psi}_\zeta^{-1}.$$

By Lemma 2.2 we still know that ω_k belongs to the class $C^3(\overline{\Omega}_\zeta(t))$. Obviously, $(\omega_k)_{k\in\mathbb{N}}$ forms a basis of $W^{1,2}(\Omega_\zeta(t))$. We fix the initial values as the $L^2(\Omega_\zeta(0))$-projection onto $W_N = \mathrm{Span}\{\omega_1,\dots\omega_N\}$ such that

$$\varrho_0^N \to \varrho_0 \quad\text{in}\quad L^2(\Omega_{\zeta(0)}).$$

We are looking for a function $\varrho_N = \sum \beta_k\omega_k$ satisfying for all $l = 1,\dots,N$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\zeta}\varrho_N\,\omega_l\,\mathrm{d}x &- \int_{\Omega_\zeta}\big(\varrho_N\,\partial_t\omega_l + \varrho_N\mathbf{w}\cdot\nabla\omega_l\big)\,\mathrm{d}x\\
&= -\int_{\Omega_\zeta}\varepsilon\nabla\varrho_N\cdot\nabla\omega_l\,\mathrm{d}x,
\end{aligned}\tag{3.8}$$

with initial data ϱ_0^N. This is equivalent to

$$\begin{aligned}
\frac{\mathrm{d}\beta_k}{\mathrm{d}t}\int_{\Omega_\zeta}\omega_k\omega_l\,\mathrm{d}x &= -\beta_k\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\zeta}\omega_k\omega_l\,\mathrm{d}x + \beta_k\int_{\Omega_\zeta}\big(\omega_k\partial_t\omega_l + \omega_k\mathbf{w}\cdot\nabla\omega_l\big)\,\mathrm{d}x\\
&\quad-\beta_k\int_{\Omega_\zeta}\varepsilon\nabla\omega_k\cdot\nabla\omega_l\,\mathrm{d}x,
\end{aligned}\tag{3.9}$$

and $\beta^l(0) = \beta_0^l$. Here, the β_k's are the unknowns (as functions only on time). Now, we define the matrices $\mathcal{A},\mathcal{B}\in\mathbb{R}^{N\times N}$ by

$$\begin{aligned}
\mathcal{A}_{k,l} &= \int_{\Omega_\zeta}\omega_k\omega_l\,\mathrm{d}x,\\
\mathcal{B}_{k,l} &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\zeta}\omega_k\omega_l\,\mathrm{d}x + \int_{\Omega_\zeta}\big(\omega_k\partial_t\omega_l + \omega_k\mathbf{w}\cdot\nabla\omega_l\big)\,\mathrm{d}x
\end{aligned}$$

$$- \int_{\Omega_\zeta} \varepsilon \nabla \omega_k \cdot \nabla \omega_l \, \mathrm{d}x.$$

Because (ω_k) is a basis of $W^{1,2}(\Omega_{\zeta(t)})$ the matrix $\mathcal{A}$ is positive definite. Hence (3.9) can be written as $\boldsymbol{\beta}' = \mathcal{A}^{-1}\mathcal{B}\boldsymbol{\beta}$ where $\boldsymbol{\beta}$ is the vector containing the β_k's. This is a linear system of ODEs which has a unique solution. In order to pass to the limit $N \to \infty$ we need uniform a priori estimates. So, we multiply (3.9) by β_l and sum over l such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\zeta} \frac{|\varrho_N|^2}{2} \, \mathrm{d}x + \int_{\Omega_\zeta} \varepsilon |\nabla \varrho_N|^2 \, \mathrm{d}x \\
&\quad = \int_{\partial\Omega_\zeta} \frac{|\varrho_N|^2}{2} (\partial_t \zeta \nu) \circ \boldsymbol{\Psi}_\zeta^{-1} \cdot \nu_\zeta \mathrm{d}\mathcal{H}^2 + \int_{\Omega_\zeta} \varrho_N \mathbf{w} \cdot \nabla \varrho_N \, \mathrm{d}x \\
&\quad \leqq c \int_{\partial\Omega_\zeta} |\varrho_N|^2 \mathrm{d}\mathcal{H}^2 + c \int_{\Omega_\zeta} |\varrho_N||\nabla \varrho_N| \, \mathrm{d}x =: (I)_N + (II)_N.
\end{aligned}$$

We use Lemma 2.3 and the trace theorem $W^{\frac{1}{2},2}(\Omega_\zeta) \to L^2(\partial\Omega_\zeta)$ (note that ζ is Lipschitz continuous uniformly in time) to conclude that (see [2, Chapter 7.7])

$$(I)_N = \|\varrho_N\|^2_{L^2(\partial\Omega_\zeta)} \leqq c(\zeta) \|\varrho_N\|^2_{W^{\frac{1}{2},2}(\Omega_\zeta)} = c(\zeta)\big(\|\varrho_N\|^2_{L^2(\Omega_\zeta)} + |\varrho_N|^2_{W^{\frac{1}{2},2}(\Omega_\zeta)}\big),$$

where $|\cdot|_{W^{\frac{1}{2},2}(\Omega_\zeta)}$ is given by (2.11). Interpolating $W^{\frac{1}{2},2}$ between L^2 and $W^{1,2}$ (see see [2, Chapter 7.3]) we obtain for $\kappa > 0$ arbitrary

$$(I)_N = \int_{\partial\Omega_\zeta} |\varrho_N|^2 \mathrm{d}\mathcal{H}^2 \leqq \kappa \int_{\Omega_\zeta} |\nabla \varrho_N|^2 \, \mathrm{d}x + c(\kappa) \int_{\Omega_\zeta} |\varrho|^2 \, \mathrm{d}x. \tag{3.10}$$

The same estimate holds for $(II)_N$ by a simple application of Young's inequality. Combining both and applying Gronwall's lemma we have shown

$$\sup_I \int_{\Omega_\zeta} |\varrho_N|^2 \, \mathrm{d}x + \int_I \int_{\Omega_\zeta} \varepsilon |\nabla \varrho_N|^2 \, \mathrm{d}x \, \mathrm{d}t \leqq C \int_{\Omega_{\zeta(0)}} |\varrho_0|^2 \, \mathrm{d}x$$

uniformly in N, where C depends on ξ, $\|\mathbf{w}\|_\infty$ and $|I|$ only. Hence, we obtain the existence of a limit function

$$\varrho \in L^\infty(I; L^2(\Omega_\zeta)) \cap L^2(I; W^{1,2}(\Omega_\zeta))$$

using (2.12). Moreover, ϱ^N converges weakly (weakly*) to ϱ. The passage to the limit in (3.8) is obvious as it is a linear equation. The uniqueness is shown in the following way: assume that we have two solutions ρ_1, ρ_2. The differences of the two solutions $\varrho_1 - \varrho_2$ and $\rho_2 - \rho_1$ are both solutions with zero initial datum. Now we may take $\varphi \equiv 1$ as a test-function for both equations and find that

$$0 \leqq \int_{\Omega_{\zeta(t)}} \big(\varrho_1(t) - \varrho_2(t)\big) \, \mathrm{d}x \leqq 0 \quad \text{for all} \quad t \in I.$$

Hence, a) is shown.

Next we show b). We extend ϱ by zero to $I \times \mathbb{R}^3$ and obtain

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varrho\, \psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \big(\varrho\, \partial_t \psi + \varrho \mathbf{w} \cdot \nabla \psi\big) \,\mathrm{d}x\,\mathrm{d}t$$
$$= - \int_I \int_{\mathbb{R}^3} \varepsilon \chi_{\Omega_\zeta} \nabla \varrho \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}t$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$. Now, we mollify the equation in space using a standard convolution with parameter $\kappa > 0$. Then we find that the following PDE is satisfied:

$$\partial_t \varrho_\kappa + \operatorname{div} \big(\varrho \mathbf{w} \chi_{\Omega_\zeta}\big)_\kappa = \operatorname{div} \big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \quad \text{in} \quad I \times \mathbb{R}^3. \tag{3.11}$$

We observe that this equation implies in particular, that $\partial_t \varrho_\kappa$ is a smooth function in space. To proceed we need to use an extension operator on $\mathbf{w}$. Since Ω_ζ is uniformly in C^2 there exists a continuous linear extension operator

$$\mathscr{E}_\zeta : W^{1,2}(\Omega_\zeta) \to W^{1,2}(\mathbb{R}^3),$$

see, for instance, [2, Thm. 5.28]. Using this operator, we can reformulate (3.11) by:

$$\partial_t \varrho_\kappa + \operatorname{div} \big(\varrho_\kappa \mathscr{E}_\zeta \mathbf{w}\big) = \mathbf{r}_\kappa + \varepsilon \operatorname{div} \big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \quad \text{in} \quad I \times \mathbb{R}^3, \tag{3.12}$$

where $\mathbf{r}_\kappa = \operatorname{div}(\varrho_\kappa \mathscr{E}_\zeta \mathbf{w}) - \operatorname{div}(\varrho \mathscr{E}_\zeta \mathbf{w})_\kappa$. We infer from the commutator lemma (see e.g. [36, Lemma 2.3]) that for a.e. t

$$\|\mathbf{r}_\kappa\|_{L^q(\mathbb{R}^3)} \leqq \|\mathbf{w}\|_{W^{1,2}(\mathbb{R}^3)} \|\varrho\|_{L^{10/3}(\mathbb{R}^3)}, \quad \tfrac{1}{q} = \tfrac{1}{2} + \tfrac{3}{10},$$

as well as

$$\mathbf{r}_\kappa \to 0 \quad \text{in} \quad L^q(\mathbb{R}^3) \tag{3.13}$$

a.e. in I. Note that a) implies that $\varrho \in L^{10/3}(I \times \Omega_\zeta)$. Now we multiply (3.12) by $\theta'(\varrho_\kappa)$ and obtain

$$\begin{aligned}\partial_t \theta(\varrho_\kappa) &+ \operatorname{div} \big(\theta(\varrho_\kappa) \mathscr{E}_\zeta \mathbf{w}\big) + \big(\varrho_\kappa \theta'(\varrho_\kappa) - \theta(\varrho_\kappa)\big) \operatorname{div} \mathscr{E}_\zeta \mathbf{w} \\ &= \mathbf{r}_\kappa \theta'(\varrho_\kappa) + \operatorname{div} \big(\varepsilon \big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \theta'(\varrho_\kappa)\big) \\ &\quad - \big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \cdot \theta''(\varrho_\kappa) \nabla \varrho_\kappa. \end{aligned} \tag{3.14}$$

Due to the properties of the mollification and $\theta \in C^2$ we have (at least after taking a subsequence)

$$\theta(\varrho_\kappa) \to \theta(\varrho) \quad \text{in} \quad L^q(I \times \mathbb{R}^3),$$
$$\theta(\varrho_\kappa) \rightharpoonup^* \theta(\varrho) \quad \text{in} \quad L^\infty(I \times \mathbb{R}^3)$$

for all $q < \infty$. The same is true for $\theta'(\varrho_\kappa)$ and $\theta''(\varrho_\kappa)$. Consequently, we have

$$\big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \cdot \theta''(\varrho_\kappa) \nabla \varrho_\kappa \rightharpoonup \chi_{\Omega_\zeta} \theta''(\varrho) |\nabla \varrho|^2 \quad \text{in} \quad L^1(I \times \mathbb{R}^3)$$
$$\text{and } \big(\chi_{\Omega_\zeta} \nabla \varrho\big)_\kappa \theta'(\varrho_\kappa) \rightharpoonup \chi_{\Omega_\zeta} \theta'(\varrho) \nabla \varrho \quad \text{in} \quad L^2(I \times \mathbb{R}^3).$$

Hence, multiplying (3.14) by $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and integrating over $I \times \mathbb{R}^3$ implies

$$\begin{aligned}
&\int_I \partial_t \int_{\mathbb{R}^3} \theta(\varrho)\, \psi \,\mathrm{d}x\,\mathrm{d}t - \int_{I\times\mathbb{R}^3} \theta(\varrho)\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{I\times\mathbb{R}^3} \big(\varrho\theta'(\varrho) - \theta(\varrho)\big) \operatorname{div} \mathscr{E}_\zeta \mathbf{w}\, \psi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{I\times\mathbb{R}^3} \theta(\varrho) \mathscr{E}_\zeta \mathbf{w} \cdot \nabla\psi \\
&\quad - \int_{I\times\mathbb{R}^3} \varepsilon \chi_{\Omega_\zeta} \nabla\theta(\varrho) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t - \int_{I\times\mathbb{R}^3} \chi_{\Omega_\zeta} \theta''(\varrho) |\nabla\varrho|^2 \psi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

This proves b) since $\mathscr{E}_\zeta \mathbf{w} \equiv \mathbf{w}$ in Ω_ζ. In order to prove c) we use (3.7) for $\psi = \chi_{[0,t]}$ and $\theta = \theta_n$ where θ_n is a smooth approximation to $\theta(z) = z^- = -\min\{z, 0\}$. It is possible to define θ_n as a convex function such that

$$\theta_n \to \theta, \quad \theta_n' \to \theta' \tag{3.15}$$

pointwise as $n \to \infty$ as well as

$$|\theta_n(z)| \leqq c\,(1 + |z|), \quad |\theta_n'(z)| \leqq c, \tag{3.16}$$

uniformly in n and z. This yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} \theta_n(\varrho(t)) \,\mathrm{d}x - \int_{\mathbb{R}^3} \theta_n(\varrho_0) \,\mathrm{d}x \\
&= -\int_0^t \int_{\mathbb{R}^3} \big(\varrho\theta_n'(\varrho) - \theta_n(\varrho)\big) \operatorname{div} \mathbf{w} \,\mathrm{d}x - \int_0^t \int_{\mathbb{R}^3} \varepsilon \chi_{\Omega_\zeta} \theta_n''(\varrho) |\nabla\varrho|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leqq -\int_0^t \int_{\mathbb{R}^3} \big(\varrho\theta_n'(\varrho) - \theta_n(\varrho)\big) \operatorname{div} \mathbf{w} \,\mathrm{d}x.
\end{aligned}$$

On account of (3.15) and (3.16) we can pass to the limit by dominated convergence, so we have

$$\int_{\mathbb{R}^3} \theta(\varrho(t)) \,\mathrm{d}x - \int_{\mathbb{R}^3} \theta(\varrho_0) \,\mathrm{d}x \leqq -\int_0^t \int_{\mathbb{R}^3} \big(\varrho\theta'(\varrho) - \theta(\varrho)\big) \operatorname{div} \mathbf{w} \,\mathrm{d}x = 0,$$

which implies $\theta(\varrho) = 0$ a.e. by the definition of θ and the non-negativity assumption on ϱ_0. This implies c). $\square$

4. The Regularized System

The aim of this section is to prepare the existence of a weak solution to the regularized system with artificial viscosity and pressure. In order to do so we have to regularize the convective terms and the variable domain. We start by introducing a suitable regularization. Here and in the following we will use, whenever necessary, zero-extensions to the whole space for quantities which we wish to regularize via convolution without further reference.

4.1. Definition of the Regularized System

We will construct a mollification of both ζ and $\mathbf{v}$. At first, for any

$$\zeta \in C\Big(\overline{I} \times \partial\Omega; \Big[-\frac{L}{2}, \frac{L}{2}\Big]\Big),$$

we introduce a standard regularizer. Since we cannot extend ζ to $\mathbb{R}$ in time, we use convolution with half intervals. Firstly, we take $\tau_\kappa^- \in C_0^\infty((-\kappa, 0], \mathbb{R}_+)$ and $\tau_\kappa^+ \in C_0^\infty([0, \kappa), \mathbb{R}_+)$ with $\int \tau_\kappa^\pm = 1$. Secondly, we take $\psi \in C^\infty([0, T], [0, 1])$ such that $\psi = 0$ on $[0, T/4]$, $\psi = 1$ on $[3/4T, T]$ Then we define $\tau_\kappa = \psi\tau_\kappa^+ + (1-\psi)\tau_\kappa^-$. Now, we convolute ζ with the product of τ_κ and a standard mollification kernel φ_κ on $\partial\Omega$ (i.e. a smooth function with $\varphi_\kappa \rightharpoonup^* \delta_0$ and $\int \varphi_\kappa = 1$) and define $\mathscr{R}_\kappa\zeta(t, q) = (\tau_\kappa\varphi_\kappa * \zeta)(t, q)$. By classical theory we have the following properties:

Lemma 4.1. (a) *We have* $\mathscr{R}_\kappa\zeta \in C^4(\overline{I} \times M)$.
(b) *If* $\kappa \to 0$ *we have* $\mathscr{R}_\kappa\zeta \to \zeta$ *uniformly.*
(c) *If* $\zeta \in L^2(I; W_0^{2,2}(M))$ *then we have* $\mathscr{R}_\kappa\zeta \to \zeta$ *in* $L^2(I; W_0^{2,2}(M))$ *for* $\kappa \to 0$.
(d) *If* $\partial_t\zeta \in L^p(I \times M)$ *we have* $\partial_t\mathscr{R}_\kappa\zeta = \mathscr{R}_\kappa(\partial_t\zeta) \to \partial_t\zeta$ *in* $L^p(I \times M)$ *for* $\kappa \to 0$.
(e) *If* $\zeta \in C^\gamma(\overline{I} \times M)$ *for some* $\gamma \in (0, 1)$ *we have* $\mathscr{R}_\kappa\zeta \to \zeta$ *in* $C^\gamma(\overline{I} \times M)$ *as* $\kappa \to 0$.
(f) *We have* $\max |\mathscr{R}_\kappa\zeta| \leqq \max |\zeta|$.

On the other hand, for functions belonging to $L^2(I; L^2(\mathbb{R}^3))$ we define ψ_κ to be the standard space-time mollification kernel with parameter κ. Note that functions defined on the variable domain can be extended to the whole space by zero (i.e. a smooth function with $\psi_k \rightharpoonup^* \delta_0$ and $\int \psi_\kappa = 1$). To be precise, we will use the definition for the regularization

$$(\mathscr{R}_\kappa v)(x) := \int_{\mathbb{R}^{n+1}} \psi_\kappa(t-s, y-x)\chi_{I\times\Omega_{\mathscr{R}_\kappa\zeta}}(s, y)v(s, y)\mathrm{d}y\,\mathrm{d}\sigma.$$

Since we may assume that ψ_κ is an even function, we find that, for $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$,

$$\int_{I\times\Omega_{\mathscr{R}_\kappa\zeta}} \mathscr{R}_\kappa\, vu\,\mathrm{d}x\,\mathrm{d}t = \int_{I\times\Omega_{\mathscr{R}_\kappa\zeta}} v\,\mathscr{R}_\kappa\, u\,\mathrm{d}x\,\mathrm{d}t.$$

With no loss of generality, we assume that ϱ_0, $\mathbf{q}_0$ are defined in the whole space $\mathbb{R}^3$. We also set $\mathbf{u}_0 = \frac{\mathbf{q}_0}{\varrho_0}$ and assume that $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$. Finally, in accordance with (1.10), we assume that

$$\mathrm{tr}_{\mathscr{R}_\kappa\eta_0}\mathbf{u}_0 = \eta_1\nu.$$

This can be achieved as done in [32][p. 234, 235] (in fact, our situation is easier as we do not have to take into account the divergence-free constraint).

The aim is therefore to get a solution to the following system: we are looking for a triple $(\eta, \varrho, \mathbf{u})$ such that

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathscr{R}_\kappa \mathbf{u}) &= \varepsilon \Delta \varrho \quad \text{in } I \times \Omega_{\mathscr{R}_\kappa \eta},\\
\partial_t((\varrho+\kappa)\mathbf{u}) + \operatorname{div}(\varrho \mathscr{R}_\kappa \mathbf{u} \otimes \mathbf{u}) &= \varepsilon \Delta(\rho \mathbf{u}) + \mu \Delta \mathbf{u} + (\lambda+\mu)\nabla \operatorname{div} \mathbf{u}\\
&\quad - \mathscr{R}_\kappa \nabla(a\varrho^\gamma + \delta \varrho^\beta) + \varrho \mathbf{f} \quad \text{in } I \times \Omega_{\mathscr{R}_\kappa \eta},\\
\partial_{\nu_{\mathscr{R}_\kappa \eta}} \varrho(\cdot, \cdot + \mathscr{R}_\kappa \eta \nu) = 0, &\qquad \mathbf{u}(\cdot, \cdot + \mathscr{R}_\kappa \eta \nu) = \partial_t \eta \nu \quad \text{in } I \times \partial\Omega,\\
\varrho(0) = \varrho_0, &\qquad (\varrho \mathbf{u})(0) = \mathbf{q}_0 \quad \text{in } \Omega_{\mathscr{R}_\kappa \eta(0)},\\
\partial_t^2 \eta + K'(\eta) &= -\nu \cdot \big(- \boldsymbol{\tau} \nu \big) \circ \boldsymbol{\Psi}_{\mathscr{R}_\kappa \eta}^{-1} |\det D\boldsymbol{\Psi}_{\mathscr{R}_\kappa \eta}| \quad \text{in } I \times M,\\
\boldsymbol{\tau} &= \varepsilon \nabla(\rho \mathbf{u}) - 2\mu \boldsymbol{\varepsilon}^D(\mathbf{u}) - \lambda \operatorname{div} \mathbf{u}\, \mathbb{I}\\
&\quad + \mathscr{R}_\kappa (a\varrho^\gamma + \delta \varrho^\beta)\mathbb{I}\\
\eta = 0, &\qquad \nabla \eta = 0 \quad \text{in } I \times \partial M,\\
\eta(0) = \eta_0, &\qquad \partial_t \eta(0) = \eta_1 \quad \text{in } M.
\end{aligned} \tag{4.1}
$$

The choice of the regularization of the above system will be clear by defining the weak formulation. In fact, the weak form of the above system can be written in two equations. Every other piece of information will be imposed upon by the choice of convenient function spaces. For this reason we define the following function spaces: we set

$$
Y^I := W^{1,\infty}(I; L^2(M)) \cap L^\infty(I; W_0^{2,2}(M))
$$

and for $\zeta \in Y^I$ with $\|\zeta\|_\infty < L$ we define

$$
X_\zeta^I := L^2(I; W^{1,2}(\Omega_{\zeta(t)})).
$$

A weak solution to (4.1) is a triplet $(\eta, \mathbf{u}, \varrho) \in Y^I \times X^I_{\mathscr{R}_\kappa \eta} \times X^I_{\mathscr{R}_\kappa \eta}$ that satisfies the following:

(K1) The regularized weak momentum equation

$$
\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\mathscr{R}_\kappa \eta}} (\varrho+\kappa)\mathbf{u} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t - \kappa \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} \varrho \mathscr{R}_\kappa \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} \mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} (\lambda+\mu) \operatorname{div} \mathbf{u} \, \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} (a\varrho^\gamma + \delta\varrho^\beta) \, \operatorname{div} \mathscr{R}_\kappa \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I \bigg(\frac{\mathrm{d}}{\mathrm{d}t} \int_M \partial_t \eta \, b \,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t \eta \, \partial_t b \,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\, b \,\mathrm{d}\mathcal{H}^2 \bigg) \mathrm{d}t\\
&= - \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} \varepsilon \nabla(\varrho \mathbf{u}) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_{\Omega_{\mathscr{R}_\kappa \eta}} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$+\int_I\int_M g\,b\,\mathrm{d}x\,\mathrm{d}t \tag{4.2}$$

for all test-functions $(b,\boldsymbol{\varphi})\in C_0^\infty(M)\times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_{\mathscr{R}_\kappa\eta}\boldsymbol{\varphi}=b\nu$. Moreover, we have $(\varrho\mathbf{u})(0)=\mathbf{q}_0$, $\eta(0)=\eta_0$ and $\partial_t\eta(0)=\eta_1$.

(K2) The regularized continuity equation

$$\begin{aligned}&\int_I\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{R}_\kappa\eta}\varrho\psi\,\mathrm{d}x-\int_{\mathscr{R}_\kappa\eta}\Big(\varrho\,\partial_t\psi+\varrho\mathscr{R}_\kappa\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\right)\mathrm{d}t\\&\quad+\varepsilon\int_I\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t=0\end{aligned}\tag{4.3}$$

for all $\psi\in C^\infty\big(\overline{I}\times\mathbb{R}^3\big)$ and we have $\varrho(0)=\varrho_0$.

(K3) The boundary condition $\mathrm{tr}_{\mathscr{R}_\kappa\,\eta}\mathbf{u}=\partial_t\eta\nu$ holds in the sense of Lemma 2.4

For more details on the interplay of the convective term and the time derivative on the boundary we refer to the next subsection.

4.2. *Formal a Priori Estimates for the Regularized System*

To understand the particular regularization we briefly discuss how to obtain formal a priori estimates for (4.1). By taking $\frac{|\mathbf{u}|^2}{2}$ in the continuity equation and subtracting it from the momentum equation tested by the couple $(\mathbf{u},\partial_t\eta)$ we find

$$\begin{aligned}&\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\Big(\frac{\varrho(t)}{2}+\kappa\Big)|\mathbf{u}(t)|^2\,\mathrm{d}x+\int_0^t\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\big(\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)|\mathrm{div}\,\mathbf{u}|^2\big)\,\mathrm{d}x\,\mathrm{d}\sigma\\&\quad+\int_0^t\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\varepsilon\varrho|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t+\int_M\frac{|\partial_t\eta(t)|^2}{2}\,\mathrm{d}\mathcal{H}^2+\frac{K(\eta(t))}{2}\\&\quad-\int_0^t\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}(\varrho^\gamma+\delta\varrho^\beta)\ \mathrm{div}\,\mathscr{R}_\kappa\,\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma\\&\quad=\int_{\Omega_{\mathscr{R}\,\kappa\,\eta(0)}}\frac{|\mathbf{q}_0|^2}{2}\,\mathrm{d}x+\int_M\frac{|\eta_0|^2}{2}\,\mathrm{d}\mathcal{H}^2+\int_M\frac{|\eta_1|^2}{2}\,\mathrm{d}\mathcal{H}^2+\frac{K(\eta_0)}{2}\\&\qquad+\int_0^t\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^t\int_M g\,\partial_t\eta\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma.\end{aligned}\tag{4.4}$$

The right-hand side of the inequality is as wanted, since all dependencies on η, $\mathbf{u}$ can be absorbed to the left hand side. Therefore, the only term that needs an extra treatment is the pressure term. We multiply the continuity equation by $\varrho^{\gamma-1}$ to obtain

$$\begin{aligned}0=&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\varrho^\gamma\,\mathrm{d}x+\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}(\gamma-1)\varrho^\gamma\,\mathrm{div}\,\mathscr{R}_\kappa\,\mathbf{u}\,\mathrm{d}x\\&+\varepsilon\int_{\Omega_{\mathscr{R}\,\kappa\,\eta}}\gamma(\gamma-1)\varrho^{\gamma-2}|\nabla\varrho|^2\,\mathrm{d}x.\end{aligned}$$

Repeating the above for $\theta(\varrho) = \varrho^\beta$, we can estimate the pressure term in (4.4) accordingly and deduce the following a priori estimate:

$$
\begin{aligned}
&\sup_{t\in I}\int_{\Omega_{\mathscr{R}_\kappa\eta}}(\varrho+\kappa)|\mathbf{u}|^2\,\mathrm{d}x+\sup_{t\in I}\int_{\Omega_{\mathscr{R}_\kappa\eta}}\big(a\varrho^\gamma+\delta\varrho^\beta\big)\,\mathrm{d}x+\int_I\int_{\Omega_{\mathscr{R}_\kappa\eta}}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\varepsilon\int_I\int_{\Omega_{\mathscr{R}_\kappa\eta}}\varrho^{\gamma-2}|\nabla\varrho|^2+\varrho|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in I}\int_M|\partial_t\eta|^2\,\mathrm{d}\mathcal{H}^2+\sup_{t\in I}K(\eta)\\
&\qquad\leqq c\bigg(\int_{\Omega_{\mathscr{R}_\kappa\eta(0)}}\frac{|\mathbf{q}_0|^2}{\varrho_0}\,\mathrm{d}x+\int_{\Omega_{\mathscr{R}_\kappa\eta(0)}}\big(\varrho_0^\gamma+\delta\varrho_0^\beta\big)\,\mathrm{d}x+\int_I\|\mathbf{f}\|^2_{L^2(\Omega_{\mathscr{R}_\kappa\eta})}\,\mathrm{d}t\bigg)\\
&\qquad+c\bigg(\int_M|\eta_0|^2\,\mathrm{d}\mathcal{H}^2+\int_M|\eta_1|^2\,\mathrm{d}\mathcal{H}^2+K(\eta_0)+\int_I\|g\|^2_{L^2(M)}\,\mathrm{d}t\bigg),
\end{aligned}
\tag{4.5}
$$

with a constant c that is independent of $\kappa,\delta,\varepsilon$. The rest of this section is now dedicated to the proof of the following existence theorem:

Theorem 4.2. *Suppose that* $\eta_0,\eta_1,\varrho_0,\mathbf{q}_0,\mathbf{f}$ *and* g *are regular enough to give sense to the right-hand side of* (4.5)*, that* $\varrho_0\geqq 0$ *a.e. and* (1.10) *is satisfied. Then there exists a solution* $(\eta,\mathbf{u},\varrho)\in Y^I\times X^I_{\mathscr{R}_\kappa\eta}\times X^I_{\mathscr{R}_\kappa\eta}$ *to (K1)–(K3). Here, we have* $I=(0,T_*)$*, where* $T_*<T$ *only if* $\lim_{t\to T^*}\|\eta(t,\cdot)\|_{L^\infty(\partial\Omega)}=\frac{L}{2}$*. The solution satisfies the energy estimate* (4.5)*.*

Remark 4.3. The restriction $\|\eta\|_\infty<\frac{L}{2}$ is needed for the construction of our extension operator, see Lemma 2.5. The latter one is used for the renormalized continuity equation, see Theorem 3.1 b) and, in particular, Section 6.3. This is why we keep the assumption $\|\eta\|_\infty<\frac{L}{2}$ during the whole construction and only relax it at the very end in Section 7.4.

4.3. Definition of the Decoupled System

The strategy for proving Theorem 4.2 is to first construct a weak solution to a decoupled system and eventually apply a fixed point theorem. Let us consider a given deformation $\zeta\in C(\overline{I}\times M)$ and a given function $\mathbf{v}\in L^2(I;\mathbb{R}^3)$. We will decouple (4.1), by replacing there $\mathscr{R}_\kappa\eta$ with $\mathscr{R}_\kappa\zeta$ and $\mathscr{R}_\kappa\mathbf{u}$ by $\mathscr{R}_\kappa\mathbf{v}$. Firstly, we find from Theorem 3.1 that there exists a unique $\varrho\in X_{\mathscr{R}_\kappa\zeta}$ that satisfies

$$
\begin{aligned}
\int_I\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}_\kappa\zeta}}\varrho\,\psi\,\mathrm{d}x-\int_{\Omega_{\mathscr{R}_\kappa\zeta}}\Big(\varrho\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t+\varrho\,\mathscr{R}_\kappa\mathbf{v}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t&\\
+\varepsilon\int_I\int_{\Omega_{\mathscr{R}_\kappa\zeta}}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t=0&
\end{aligned}
\tag{4.6}
$$

for all $\psi\in C^\infty\big(\overline{I}\times\mathbb{R}^3\big)$. Observe that ϱ exists independently of $\mathbf{u},\eta$.

Secondly, we repeat the interplay of the boundary deformation with the convective term for the momentum equation and find that smooth functions satisfy

$$
\begin{aligned}
&\int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\partial_t((\varrho+\kappa)\mathbf{u})\cdot\boldsymbol{\varphi} + \operatorname{div}(\varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\mathbf{u})\cdot\boldsymbol{\varphi}\Big)\,\mathrm{d}x \\
&= \int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\partial_t((\varrho+\kappa)\mathbf{u}\cdot\boldsymbol{\varphi}) + \operatorname{div}(\varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\mathbf{u}\,\boldsymbol{\varphi})\Big)\,\mathrm{d}x \\
&\quad - \int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big((\varrho+\kappa)\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}_\kappa \zeta}} (\varrho+\kappa)\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega_{\mathscr{R}_\kappa \zeta}} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\Big(\mathscr{R}_\kappa\,\mathbf{v}\cdot\nu_{\mathscr{R}_\kappa \zeta} - (\partial_t\,\mathscr{R}_\kappa\,\zeta\nu)\circ\boldsymbol{\Psi}^{-1}_{\mathscr{R}_\kappa \zeta}\Big)\,\mathrm{d}\mathcal{H}^2 \\
&\quad + \kappa\int_{\partial\Omega_{\mathscr{R}_\kappa \zeta}} \mathbf{u}\cdot\boldsymbol{\varphi}\,\partial_t\,\mathscr{R}_\kappa\,\zeta\nu\circ\boldsymbol{\Psi}^{-1}_{\mathscr{R}_\kappa \zeta}\,\mathrm{d}\mathcal{H}^2 - \int_{\Omega_{\mathscr{R}_\kappa \zeta}} \kappa\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.7}
$$

Observe that in the case of a fixed point $\mathbf{u}\equiv\mathbf{v},\ \eta\equiv\zeta$ we find that

$$
\Big(\mathscr{R}_\kappa\,\mathbf{u} - (\partial_t\,\mathscr{R}_\kappa\,\eta\nu)\circ\boldsymbol{\Psi}^{-1}_{\mathscr{R}_\kappa \eta}\Big)\cdot\nu_{\mathscr{R}_\kappa \eta} \equiv 0 \text{ on } \partial\Omega_{\mathscr{R}_\kappa \eta},
$$

which implies that the boundary integrals will vanish. For this reason, we will solve the decoupled momentum equation with boundary values of $\mathbf{u}$, which are implicitly defined by removing the first boundary term (this is analogous to the Neumann boundary data of the decoupled continuity equation, see Section 3). For the same reason we neglect the second boundary integral as well as the very last integral. Concerning the other terms of the momentum equation, when adapting partial integration we get force terms acting on the boundary in normal direction (pressure, diffusion, exterior forces). These are then assumed to be equalized by the elastic forces of the shell. Observe here that $\boldsymbol{\tau}$ is identical for the decoupled system and the coupled system.

All together we require from $(\eta,\mathbf{u},\varrho)\in X^I_{\mathscr{R}_\kappa \zeta}\times X^I_{\mathscr{R}_\kappa \zeta}\times Y^I$ that it satisfies the following:

(N1) The regularized decoupled momentum equation

$$
\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}_\kappa \zeta}} (\varrho+\kappa)\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I\int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\mu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\,\operatorname{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_I\int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big((\varrho^\gamma + \delta\varrho^\beta)\,\operatorname{div}\mathscr{R}_\kappa\,\boldsymbol{\varphi} - \varepsilon\nabla(\varrho\mathbf{u}) : \nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_I\Big(\frac{\mathrm{d}}{\mathrm{d}t}\int_M \partial_t\eta b\,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t\eta\,\partial_t b\,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\,b\,\mathrm{d}\mathcal{H}^2\Big)\,\mathrm{d}t
\end{aligned}
$$

$$= \int_I \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta}} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t + \int_I \int_M g\, b \,\mathrm{d}x\,\mathrm{d}t \tag{4.8}$$

holds for all test-functions $(b, \boldsymbol{\varphi}) \in C_0^\infty(M) \times C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_{\mathscr{R}_\kappa\,\zeta} \boldsymbol{\varphi} = b\nu$. Moreover, we have $(\varrho \mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t \eta(0) = \eta_1$.

(N2) The decoupled regularized continuity equation (4.6) is satisfied with initial datum $\varrho(0) = \varrho_0$.

(N3) The boundary condition $\mathrm{tr}_{\mathscr{R}_\kappa \zeta} \mathbf{u} = \partial_t \eta \nu$ holds in the sense of Lemma 2.4

The a priori estimates are formally available as before for the regularized system in Section 4.2. First, one uses $(\mathbf{u}, \partial_t \eta)$ as test-function in the momentum equation and subtract the continuity equation tested with $\frac{|\mathbf{u}|^2}{2}$. Second, one uses the renormalized formulation (3.4) to estimate the pressure term.

Theorem 4.4. *For any $\zeta \in C(\overline{I} \times M; [-\frac{L}{2}, \frac{L}{2}]))$ and $\mathbf{v} \in L^2(I; L^2(\mathbb{R}^3))$ there exists a solution $(\eta, \mathbf{u}, \varrho) \in Y^I \times X^I_{\mathscr{R}_\kappa\,\zeta} \times X^I_{\mathscr{R}_\kappa\,\zeta}$ to (N1)–(N3). Here, we have $I = (0, T_*)$, where $T_* < T$ only if $\lim_{t\to T^*} \|\eta(t,\cdot)\|_{L^\infty(M)} = \frac{L}{2}$. The solution satisfies the energy estimate*

$$\begin{aligned}
&\int_{\Omega_{\mathscr{R}\,\kappa\,\zeta}} \Big(\frac{\varrho(t)}{2} + \kappa\Big) |\mathbf{u}(t)|^2 \,\mathrm{d}x \\
&\quad + \int_0^t \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta}} \Big((\mu + \varepsilon \varrho) |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\mathrm{div}\, \mathbf{u}|^2 \Big) \,\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad + \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta}} \Big(\frac{a}{\gamma - 1} \varrho^\gamma(t) + \frac{\delta}{\beta - 1} \varrho^\beta(t) \Big) \,\mathrm{d}x + \int_M \frac{|\partial_t \eta(t)|^2}{2} \,\mathrm{d}\mathcal{H}^2 + \frac{K(\eta(t))}{2} \\
&\leqq \int_I \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta}} \rho \mathbf{f} \cdot \mathbf{u} \,\mathrm{d}x\,\mathrm{d}\sigma + \int_I \int_M g \partial_t \eta \;\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma + \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta(0)}} \frac{|\mathbf{q}_0|^2}{\varrho_0} \,\mathrm{d}x \\
&\quad + \int_{\Omega_{\mathscr{R}\,\kappa\,\zeta(0)}} \Big(\frac{a}{\gamma - 1} \varrho_0^\gamma + \frac{\delta}{\beta - 1} \varrho_0^\beta \Big) \,\mathrm{d}x + \frac{K(\eta_0)}{2} + \int_M \frac{|\eta_0|^2}{2} \,\mathrm{d}\mathcal{H}^2 \\
&\quad + \int_M \frac{|\eta_1|^2}{2} \,\mathrm{d}\mathcal{H}^2
\end{aligned}$$

for all $t \in [0, T^*]$, provided that $\eta_0, \eta_1, \varrho_0, \mathbf{q}_0, \mathbf{f}$ and g are regular enough to give sense to the right-hand side, that $\varrho_0 \geqq 0$ a.e and (1.10) is satisfied. Here, the constant c is independent of all involved quantities; in particular, it is independent of $\mathbf{v}$ and ζ.

Proof. In order to prove Theorem 4.4 we discretise the system. It is standard to find a smooth orthonormal basis $(\tilde{\mathbf{X}}_k)_{k\in\mathbb{N}}$ of $W_0^{1,2}(\Omega)$ and a smooth orthonormal basis $(\tilde{Y}_k)_{k\in\mathbb{N}}$ of $W_0^{2,2}(M)$. We define vector fields $\tilde{\mathbf{Y}}_k$ by solving the homogeneous Laplace equation on Ω with boundary datum $\tilde{Y}_k \nu$ (which is extended by zero to $\partial\Omega$). Note that standard results on the inverse Laplace operator guarantee that $\tilde{\mathbf{Y}}_k$ is smooth. Now we define, pointwise in t,

$$\mathbf{X}_k := \tilde{\mathbf{X}}_k \circ \boldsymbol{\Psi}^{-1}_{\mathscr{R}_\kappa \zeta}, \quad \mathbf{Y}_k := \tilde{\mathbf{Y}}_k \circ \boldsymbol{\Psi}^{-1}_{\mathscr{R}_\kappa \zeta}.$$

By Lemma 2.2 we still know that $\mathbf{X}_k$ and $\mathbf{Y}_k$ belong to the class $C^3(\overline{\Omega}_{\mathscr{R}_\kappa\zeta}(t))$. Obviously, $(\mathbf{X}_k)_{k\in\mathbb{N}}$ forms a basis of $W_0^{1,2}(\Omega_{\mathscr{R}_\kappa\zeta}(t))$. Now we choose an enumeration $(\boldsymbol{\omega}_k)_{k\in\mathbb{N}}$ of $\mathbf{X}_k \oplus \mathbf{Y}_k$. In return we associate $w_k := \boldsymbol{\omega}_k \circ \boldsymbol{\Psi}_{\mathscr{R}_\kappa\zeta}|_{\partial\Omega_{\mathscr{R}_\kappa\zeta}} \cdot \nu$. Analogous to the arguments in [32, p. 237] we find that

$$Z := \operatorname{span}\Big\{(\varphi w_k, \varphi\boldsymbol{\omega}_k)\,|\varphi \in C^1(I),\ k \in \mathbb{N}\Big\}$$

is dense in the solution space

$$Z_{\mathscr{R}_\kappa\zeta} := \Big\{(\xi, \boldsymbol{\varphi}) \in Y^I \times X^I_{\mathscr{R}_\kappa\zeta} :\ \partial_t\xi\,\nu_{\mathscr{R}_\kappa\zeta} = \operatorname{tr}_{\mathscr{R}_\kappa\zeta}\boldsymbol{\varphi}\Big\},$$

and in the space of test-functions

$$\begin{aligned}Z^*_{\mathscr{R}_\kappa\zeta} :=\Big\{&(\xi, \boldsymbol{\varphi}) \in C(\overline{I}; W_0^{2,2}(M)) \times L^2(I, W^{1,2}(\Omega_{\mathscr{R}_\kappa\zeta})) \cap C(\overline{I}; L^2(\Omega_{\mathscr{R}_\kappa\zeta})) :\\ &\xi\nu = \operatorname{tr}_{\mathscr{R}_\kappa\zeta}\boldsymbol{\varphi}\Big\}.\end{aligned}$$

Now, we can begin with the construction of the solution. First, we fix $\varrho = \varrho(\mathscr{R}_\kappa\zeta, \mathscr{R}_\kappa\mathbf{v})$ as the unique solution to the continuity equations subject to the initial datum ϱ_0 existence of which is guaranteed by Theorem 3.1, where $\zeta \equiv \mathscr{R}_\kappa\zeta$ and $\mathbf{w} \equiv \mathscr{R}_\kappa\mathbf{v}$. Next we seek for a couple of discrete solutions $(\eta^N, \mathbf{u}^N) \in \mathbf{Z}_{\mathscr{R}_\kappa\zeta}$ of the form

$$\eta_N = \eta_0 + \sum\nolimits_{k=1}^N \int_0^t \alpha_{kN} w_k \,\mathrm{d}\sigma, \quad \mathbf{u}_N = \sum\nolimits_{k=1}^N \alpha_{kN}\boldsymbol{\omega}_k,$$

which solves the following discrete version of (4.8):

$$\begin{aligned}&\int_{\Omega_{\mathscr{R}_\kappa\zeta}} (\varrho(t) + \kappa)\mathbf{u}_N(t)\cdot\boldsymbol{\omega}_k(t)\,\mathrm{d}x\\ &\quad - \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big(\varrho\mathbf{u}_N\cdot\partial_t\boldsymbol{\omega}_k + \varrho\,\mathscr{R}_\kappa\mathbf{v}\otimes\mathbf{u}^N : \nabla\boldsymbol{\omega}_k\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big(\mu\nabla\mathbf{u}_N : \nabla\boldsymbol{\omega}_k + (\lambda+\mu)\operatorname{div}\mathbf{u}_N\,\operatorname{div}\boldsymbol{\omega}_k\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\\ &\quad - \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big((\varrho^\gamma + \delta\varrho^\beta)\,\operatorname{div}\mathscr{R}_\kappa\boldsymbol{\omega}_k - \varepsilon\nabla(\varrho\mathbf{u}_N) : \nabla\boldsymbol{\omega}_k\Big)\,\mathrm{d}x\,\mathrm{d}\sigma \qquad (4.9)\\ &\quad + \int_0^t\int_M \Big(K'(\eta_N)w_k - \partial_t\eta_N\,\partial_t w_k\Big)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma + \int_M \partial_t\eta_N(t)w_k(t)\,\mathrm{d}\mathcal{H}^2\\ &\quad = \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \varrho\mathbf{f}\cdot\boldsymbol{\omega}_k\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_M g\,w_k\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad + \int_{\Omega_{\mathscr{R}_\kappa\zeta(0)}} \mathbf{q}_0\cdot\boldsymbol{\omega}_k(0,\cdot)\,\mathrm{d}x + \int_M \eta_1\,w_k\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

We can choose $\alpha_{kN}(0)$ in such a way that $\mathbf{u}_N(0)$ converges to $\mathbf{q}_0/\varrho_0$.
The system (4.9) is equivalent to a system of integro-differential equations for the vector $\boldsymbol{\alpha}_N = (\alpha_{kN})_{k=1}^N$; it reads as

$$\begin{aligned}\mathcal{A}(t)\boldsymbol{\alpha}_N(t) &= \int_0^t \mathcal{B}(\sigma)\boldsymbol{\alpha}_N(\sigma)\,\mathrm{d}\sigma + \int_0^t\int_0^\sigma \tilde{\mathcal{B}}(s,\sigma)\boldsymbol{\alpha}_N(s)\mathrm{d}s\,\mathrm{d}\sigma \\ &\quad + \int_0^t \mathbf{c}(\sigma)\,\mathrm{d}\sigma + \tilde{\mathbf{c}},\end{aligned} \tag{4.10}$$

with

$$\begin{aligned}\mathcal{A}_{ij} &= \int_{\Omega_{\mathscr{R}_\kappa\zeta}} (\varrho(t)+\kappa)\boldsymbol{\omega}_i(t)\cdot\boldsymbol{\omega}_j(t)\,\mathrm{d}x + \int_M w_i(t)w_j(t)\,\mathrm{d}\mathcal{H}^2\\
\mathcal{B}_{ij} &= \int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big(\varrho\boldsymbol{\omega}_i\cdot\partial_t\boldsymbol{\omega}_j + \varrho\,\mathscr{R}_\kappa\,\mathbf{v}\otimes\boldsymbol{\omega}_i : \nabla\boldsymbol{\omega}_j\Big)\,\mathrm{d}x\\
&\quad+\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big(\mu\nabla\boldsymbol{\omega}_i:\nabla\boldsymbol{\omega}_j + (\lambda+\mu)\,\mathrm{div}\,\boldsymbol{\omega}_i\ \mathrm{div}\,\boldsymbol{\omega}_j\Big)\,\mathrm{d}x\\
&\quad-\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \varepsilon(\nabla\varrho\otimes\boldsymbol{\omega}_i + \varrho\nabla\boldsymbol{\omega}_i):\nabla\boldsymbol{\omega}_j\,\mathrm{d}x\,\mathrm{d}\sigma - \int_{\partial\Omega_{\mathscr{R}_\kappa\zeta}} w_i\,\partial_t w_j\,\mathrm{d}\mathcal{H}^2\\
\tilde{\mathcal{B}}_{ij} &= \int_M K'(w_i(s))w_j(\sigma))\,\mathrm{d}\mathcal{H}^2,\\
c_i &= \int_{\Omega_{\mathscr{R}_\kappa\zeta}} (\varrho^\gamma+\delta\varrho^\beta)\ \mathrm{div}\,\mathscr{R}_\kappa\,\boldsymbol{\omega}_i\,\mathrm{d}x + \int_{\Omega_{\mathscr{R}_\kappa\zeta}} \varrho\mathbf{f}\cdot\boldsymbol{\omega}_i\,\mathrm{d}x\,\mathrm{d}t + \int_M g\,w_i\,\mathrm{d}\mathcal{H}^2\\
\tilde{c}_i &= \int_{\Omega_{\mathscr{R}_\kappa\zeta(0)}} \mathbf{q}_0\cdot\boldsymbol{\omega}_i(0,\cdot)\,\mathrm{d}x + \int_M \eta_1\,w_i\,\mathrm{d}\mathcal{H}^2.\end{aligned}$$

As $(\varrho+\kappa)$ is strictly positive (recall Theorem 3.1) and the $\boldsymbol{\omega}_k$ and w_k from a basis the matrix $\mathcal{A}$ is bounded (by the integrability of ϱ) and positive definite (due to $\kappa > 0$). Hence the inverse $\mathcal{A}^{-1}$ exists and is bounded as well. We find a continuous solution $\boldsymbol{\alpha}_N$ to (4.10) by standard arguments for ordinary integro-differential equations. Since we wish to use it as a testfunction in the momentum equation we have to show, that $\partial_t\boldsymbol{\alpha}_N \in L^2(I)$, for some $s > 1$. The difficulty here is that ϱ is not weakly differentiable in time. This has to be circumvented. First observe that by the Leibnitz rule, we find that

$$\partial_t\boldsymbol{\alpha} = \mathcal{A}^{-1}\Big(\partial_t(\mathcal{A}\boldsymbol{\alpha}) - \partial_t\mathcal{A}\boldsymbol{\alpha}\Big).$$

Due to (4.10) and the integrability of ϱ from Theorem 3.1 we have $\partial_t(\mathcal{A}\boldsymbol{\alpha}_N) \in L^\infty(I)$. Moreover, $\mathcal{A}^{-1}$ is uniformly bounded (due to $\kappa > 0$). Consequently, it suffices to prove that $\partial_t\mathcal{A}_{i,j} \in L^2(I)$ to conclude the differentiability of α_N. By taking the test function $\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j$ in (4.6) we find that

$$\partial_t\Big(\mathcal{A}_{i,j} - \int_M w_iw_j\,\mathrm{d}\mathcal{H}^2\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \varrho\,\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \kappa\,\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{\Omega_{\mathscr{R}_\kappa\zeta}} \varrho \Big(\partial_t(\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j) + \mathscr{R}_\kappa \mathbf{v}\cdot\nabla(\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j) + \varepsilon\nabla\varrho\cdot\nabla(\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j)\,\mathrm{d}x \\
&\quad + \kappa\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \big(\partial_t\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j + \boldsymbol{\omega}_i\cdot\partial_t\boldsymbol{\omega}_j\big)\,\mathrm{d}x + \kappa\int_{\partial\Omega_{\mathscr{R}_\kappa\zeta}} \partial_t\,\mathscr{R}_\kappa\,\zeta\,\boldsymbol{\omega}_i\cdot\boldsymbol{\omega}_j\,\mathrm{d}x.
\end{aligned}
$$

Since the right hand side is in $L^2(I)$ (note that the $\boldsymbol{\omega}_i$ are smooth also in time) and the w_i are smooths in time we find that $\partial_t\alpha_N \in L^s(I)$ and hence $\partial_t\mathbf{u}_N \in L^s(I\times\Omega_{\mathscr{R}_\kappa\zeta})$. The a priori estimates are now achieved by differentiating (4.9) in time, testing with $(\partial_t\eta_N,\mathbf{u}_N)$ and subtracting (4.6) tested by $\frac{1}{2}|\mathbf{u}_N|^2$. The terms with the time derivative and the convective terms cancel and we obtain

$$
\begin{aligned}
&\int_{\Omega_{\mathscr{R}_\kappa\zeta}} (\varrho(t)+\kappa)\frac{|\mathbf{u}_N(t)|^2}{2}\,\mathrm{d}x + \int_M \frac{|\partial_t\eta_N(t)|^2}{2}\,\mathrm{d}\mathcal{H}^2 + \frac{K(\eta_N(t))}{2} \\
&\quad + \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big((\mu+\varepsilon\varrho)|\nabla\mathbf{u}_N|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}_N|^2\Big)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&= \int_I\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \rho\mathbf{f}\cdot\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}\sigma + \int_I\int_M g\,\partial_t\eta_N\;\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma + \int_{\Omega_{\mathscr{R}_\kappa\zeta(0)}} |\mathbf{q}_0|^2\,\mathrm{d}x \\
&\quad + \int_M \frac{|\eta_0|^2}{2}\,\mathrm{d}\mathcal{H}^2 + \int_M \frac{|\eta_1|^2}{2}\,\mathrm{d}\mathcal{H}^2 + \frac{K(\eta_0)}{2} \\
&\quad + \int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} (a\varrho^\gamma+\delta\varrho^\beta)\,\operatorname{div}\mathscr{R}_\kappa\,\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned}
\tag{4.11}
$$

Finally, we use Theorem 3.1 b) in order to rewrite the last integral. Choosing $\varphi = \chi_{[0,t]}$ yields

$$
\begin{aligned}
&\int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} \big(\varrho\theta'(\varrho)-\theta(\varrho)\big)\operatorname{div}\mathscr{R}_\kappa\,\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\leqq \int_{\Omega_{\mathscr{R}_\kappa\zeta(0)}} \theta(\varrho(0))\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_{\mathscr{R}_\kappa\zeta}} \theta(\varrho(t))\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for any convex $\theta \in C^2(\mathbb{R}_+;\mathbb{R}_+)$ such that $\theta'(s) = 0$ for large values of s and $\theta(0) = 0$. We approximate the function $s \mapsto \frac{as^\gamma}{\gamma-1} + \frac{\delta s^\beta}{\beta-1}$ by a sequence of such functions and obtain

$$
\begin{aligned}
&\int_0^t\int_{\Omega_{\mathscr{R}_\kappa\zeta}} (a\varrho^\gamma+\delta\varrho^\beta)\,\operatorname{div}\mathscr{R}_\kappa\,\mathbf{u}_N\,\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad\leqq \int_{\Omega_{\mathscr{R}_\kappa\zeta(0)}} \Big(\frac{a}{\gamma-1}\varrho_0^\gamma + \frac{\delta}{\beta-1}\varrho_0^\beta\Big)\,\mathrm{d}x \\
&\qquad - \int_{\Omega_{\mathscr{R}_\kappa\zeta}} \Big(\frac{a}{\gamma-1}\varrho_0^\gamma(t) + \frac{\delta}{\beta-1}\varrho^\beta(t)\Big)\,\mathrm{d}x.
\end{aligned}
$$

By Young's inequality we can absorb the terms that depend on $\mathbf{u}_N$ or $\partial_t \eta_N$ in the left hand side of (4.11) such that

$$\begin{aligned}
&\sup_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} (\varrho+\kappa)\frac{|\mathbf{u}_N|^2}{2}\,\mathrm{d}x + \sup_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} \Big(\frac{a}{\gamma-1}\varrho^\gamma + \frac{\delta}{\beta-1}\varrho^\beta\Big)\,\mathrm{d}x \\
&\quad + \int_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} |\nabla \mathbf{u}_N|^2\,\mathrm{d}x\,\mathrm{d}\sigma + \int_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} (\mu+\varepsilon\varrho)|\nabla \mathbf{u}_N|^2)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&\quad + \int_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} (\lambda+\mu)|\operatorname{div}\mathbf{u}_N|^2\,\mathrm{d}x + \sup_I \int_M |\partial_t \eta_N|^2\,\mathrm{d}\mathcal{H}^2 + \sup_I \frac{K(\eta_N)}{2} \\
&\quad \leqq c\Big(\int_I \int_{\Omega_{\mathscr{R}_\kappa \zeta}} |\mathbf{f}|^2\,\mathrm{d}x\,\mathrm{d}\sigma + \int_I \int_M |g|^2\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}\sigma\Big) \\
&\qquad + c\Big(\int_{\Omega_{\mathscr{R}_\kappa \zeta(0)}} \frac{|\mathbf{q}_0|^2}{\varrho_0}\,\mathrm{d}x + \int_M |\eta_0|^2\,\mathrm{d}\mathcal{H}^2\Big) + c\Big(\int_M |\eta_1|^2\,\mathrm{d}\mathcal{H}^2 + K(\eta_0)\Big) \\
&\qquad + c\int_{\Omega_{\mathscr{R}_\kappa \zeta(0)}} \Big(\frac{a}{\gamma-1}\varrho_0^\gamma + \frac{\delta}{\beta-1}\varrho_0^\beta\Big)\,\mathrm{d}x.
\end{aligned} \tag{4.12}$$

This implies that there is a subsequence such that

$$\eta_N \rightharpoonup^* \eta \quad \text{in} \quad Y^I, \quad \mathbf{u}_N \rightharpoonup \mathbf{u} \quad \text{in} \quad X^I_{\mathscr{R}_\kappa \zeta}$$

for some limit function $(\eta, \mathbf{u})$. As (4.9) is linear in $(\eta_N, \mathbf{u}_N)$ we can pass to the limit and see that $(\eta, \mathbf{u})$ solves (4.8). □

4.4. A Fixed Point Argument

Now we are seeking for a fixed point of the solutions map $(\mathbf{v}, \zeta) \mapsto (\mathbf{u}, \eta)$ on $L^2(I, L^2(\mathbb{R}^3)) \times C(\overline{I} \times \partial\Omega)$ from Theorem 4.4. As we do not know about uniqueness of the solutions constructed in Theorem 4.4 we will use the following foxed point theorem for set-valued mappings:

Theorem 4.5. ([22]) *Let C be a convex subset of a normed vector space Z and let $F : C \to \mathfrak{P}(C)$ be an upper-semicontinuous set-valued mapping, that is, for every open set $W \subset C$ the set $\{c \in C : F(c) \in W\} \subset C$ is open. Moreover, let $F(C)$ be contained in a compact subset of C, and let $F(c)$ be non-empty, convex and compact for all $c \in C$. Then F possesses a fixed point, that is, there exists some $c_0 \in C$ with $c_0 \in F(c_0)$.*

4.5. Proof of Theorem 4.2

We will prove Theorem 4.2 by finding a fixed point of a suitable mapping defined below. We denote $I_* = [0, T_*]$ with T_* sufficiently small. We do not know about the uniqueness of solutions. Hence, we apply Theorem 4.5 to get a fixed point. We consider the sets

$$D := \Big\{(\zeta, \mathbf{v}) \in C(\overline{I}_* \times \partial\Omega) \times L^2(I_*, L^2(\mathbb{R}^3)) :$$

$$\zeta(0)=\eta_0,\ \ \|\zeta\|_{L^\infty}\leqq M,\ \ \|\mathbf{v}\|_{L^2(I_*;L^2(\mathbb{R}^3))}\leqq K\Big\}$$

for $M=(\|\eta_0\|_\infty+L)/2$ and $K>0$ to be chosen later. Note that the coupling at the boundary between velocity and shell is not contained in the definition of D. This is a feature which one only gains via the fixed point and not before. Let

$$F:D\to\mathfrak{P}(D)$$

with

$$\begin{aligned}F:(\mathbf{v},\zeta)\mapsto\Big\{(\mathbf{u},\eta):\ (\mathbf{u},\eta)\text{ solves (4.8)}\\ \text{with }(\mathbf{v},\zeta)\text{ and satisfies the energy estimate}\Big\}.\end{aligned}$$

Note that we extend $\mathbf{u}$ and η by zero to $\mathbb{R}^3$ and $\partial\Omega$ respectively. First, we have to check that $F(D)\subset D$. We will use the a priori estimate from Theorem 4.4 to conclude

$$\begin{aligned}&\sup_{I_*}\int_{\Omega_{\mathscr{R}_\kappa\zeta}}(\varrho+\kappa)\frac{|\mathbf{u}|^2}{2}\,\mathrm{d}x+\sup_{I_*}\int_{\Omega_{\mathscr{R}_\kappa\zeta}}\Big(a\varrho^\gamma+\delta\varrho^\beta\Big)\,\mathrm{d}x\\&\quad+\int_{I_*}\int_{\Omega_{\mathscr{R}_\kappa\zeta}}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\sigma+\sup_{I_*}\int_M\Big(|\partial_t\eta|^2+|\nabla^2\eta|^2\Big)\,\mathrm{d}\mathcal{H}^2\\&\quad\leqq c(\mathbf{f},g,\mathbf{q}_0,\eta_0,\eta_1,\varrho_0)\end{aligned}$$

independently of L, K and the size of I_*. This implies that $\eta\in C^\alpha(\overline{I}\times M)$, by Sobolev embedding for some $\alpha>0$, with Hölder norm independent of L and K. We obtain

$$|\eta(t,x)|\leqq|\eta(t,x)-\eta_0(0,x)|+|\eta_0(0,x)|\leqq c(T^*)^\alpha+\|\eta_0\|_\infty. \tag{4.13}$$

Therefore, we find for T^* small enough (but independent of $\mathbf{v}$ and ζ) such that

$$\|\eta\|_{L^\infty(I_*\times\partial\Omega)}\leqq M.$$

Hence we gain $F(D)\subset D$ for an appropriate choice of $K\in\mathbb{R}_+$.

Next, since the problem is linear and the left-hand side of the energy inequality is convex, we find that $F(\zeta,\mathbf{v})$ is a convex and closed subset of $\mathbf{Z}$. It remains to show that $F(D)$ is relatively compact. Consider $(\eta_n,\mathbf{u}_n)_n\subset F(D)$. Then there exists a corresponding sequence $(\zeta_n,\mathbf{v}_n)_n\subset D$, such that $(\eta_n,\mathbf{u}_n)$ solve (4.8), with respect to $(\mathbf{v}_n,\zeta_n)$. Due to the energy estimate we may choose subsequences such that

$$\eta_n\rightharpoonup^*\eta\quad\text{in}\quad L^\infty(I_*,W^{1,2}_0(M)), \tag{4.14}$$

$$\partial_t\eta_n\rightharpoonup^*\partial_t\eta\quad\text{in}\quad L^\infty(I_*,L^2(M)), \tag{4.15}$$

$$\mathbf{u}_n\rightharpoonup^{*,\eta}\mathbf{u}\quad\text{in}\quad L^\infty(I_*;L^2(\Omega_{\mathscr{R}_\kappa\zeta})), \tag{4.16}$$

$$\nabla\mathbf{u}_n\rightharpoonup^{\eta}\nabla\mathbf{u}\quad\text{in}\quad L^2(I_*;L^2(\Omega_{\mathscr{R}_\kappa\zeta})). \tag{4.17}$$

Note also that we can extend $\mathbf{u}_n$ and $\mathbf{u}$ by zero to the whole space and gain

$$\mathbf{u}_n \rightharpoonup^* \mathbf{u} \quad \text{in} \quad L^\infty(I_*; L^2(\mathbb{R}^3)). \tag{4.18}$$

The compactness of η_n in $C(\overline{I}_* \times \partial\Omega)$ follows immediately by Arcela–Ascoli's theorem, since we know that η_n is uniformly Hölder continuous. The proof of the compactness of $\mathbf{u}_n$ is much more sophisticated. We first need to show compactness of ϱ_n, where ϱ_n is the unique solution to (4.6) with $\mathbf{v} = \mathbf{v}_n$. A direct application of Theorem 3.1 a) shows

$$\begin{aligned} \varrho_n &\rightharpoonup^\eta \varrho \quad \text{in} \quad L^2(I_*; W^{1,2}(\Omega_{\mathscr{R}_\kappa\, \zeta_n})), \\ \varrho_n &\rightharpoonup^{*,\eta} \varrho \quad \text{in} \quad L^\infty(I_*; L^2(\Omega_{\mathscr{R}_\kappa\, \zeta_n})), \end{aligned} \tag{4.19}$$

at least after taking a subsequence. Firstly, we find for all $k, l \in \mathbb{N}$ that $\|\partial_t^l \nabla^k \mathscr{R}_\kappa\, \zeta_n\|_{L^\infty(I\times\partial\Omega)} \leqq c(\kappa, k, l)$. Hence, there is a (not relabeled) subsequence such that

$$\mathscr{R}_\kappa\, \zeta_n \to \mathscr{R}_\kappa\, \zeta \quad C^2(\overline{I}_* \times \partial\Omega). \tag{4.20}$$

Next, we claim that

$$\varrho_n \to^\eta \varrho \quad \text{in} \quad L^q(I_*; L^q \Omega_{\mathscr{R}_\kappa \zeta_n}) \tag{4.21}$$

for any $q < \frac{10}{3}$. In fact, the assumptions of Lemma 2.8 are satisfied due to (4.6). In particular, (A3) holds with $\mathbf{H}_n^1 = 0$, $\mathbf{H}_n^2 = \mathscr{R}_\kappa \mathbf{v}_n + \varepsilon\nabla\varrho_n$ and $h_n = 0$. Due to the uniform bounds on ϱ_n in (4.19) and the bounds on $\mathbf{v}_n$ encoded in the definition of D we gain strong convergence of ϱ_n in L^2 by Remark 2.10 at least for a subsequence. Combining this with (4.19) proves (4.21). Now, again by Lemma 2.8, we find for the couple $(\kappa + \varrho_n)\mathbf{u}_n$ and $\mathbf{u}_n$, that

$$(\kappa + \varrho_n)|\mathbf{u}_n|^2 \rightharpoonup (\kappa + \varrho)|\mathbf{u}|^2 \text{ in } L^s(I_* \times \Omega_{\mathscr{R}_\kappa\, \zeta_n}) \tag{4.22}$$

for some $s > 1$. To be precise, we infer from (4.8) that

$$\begin{aligned} \partial_t((\varrho_n + \kappa)\mathbf{u}_n) = &- \operatorname{div}(\varrho_n \mathscr{R}_\kappa \mathbf{v}_n \otimes \mathbf{u}_n) + \varepsilon\Delta(\varrho_n \mathbf{u}_n) + \mu\Delta\mathbf{u}_n + (\lambda + \mu)\nabla \operatorname{div} \mathbf{u}_n \\ &- \mathscr{R}_\kappa \nabla(\varrho_n^\gamma + \delta\varrho_n^\beta) + \varrho_n \mathbf{f} \end{aligned}$$

holds locally in the sense of distributions. In particular, (A3) is satisfied with

$$\mathbf{H}_n^1 = \varepsilon\varrho_n\mathbf{u}_n + \mathcal{R}\mathbf{u}_n, \quad \mathbf{H}_n^2 = -\varrho_n\mathscr{R}_\kappa\mathbf{v}_n \otimes \mathbf{u}_n - \mathscr{R}_\kappa(\varrho_n^\gamma + \delta\varrho_n^\beta)I, \quad \mathbf{h}_n = \varrho_n\mathbf{f},$$

choosing $p = s = 2$, m arbitrary and $b \in \left(\frac{6}{5}, \frac{10}{3}\right)$. Here $\mathcal{R} \in \mathbb{R}^{3\times3}$ is chosen appropriately. We obtain (4.22). On account of (4.21) and (4.22) we conclude (extending ϱ with 0 outside of $\Omega_{\mathscr{R}_\kappa\, \zeta}$)

$$\begin{aligned} \lim_{n\to\infty} \int_{I^*} \int_{\mathbb{R}^3} |\mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t = &\lim_{n\to\infty} \int_{I^*} \int_{\mathbb{R}^3} \frac{\kappa + \varrho_n}{\kappa + \varrho} |\mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \lim_{n\to\infty} \int_{I^*} \int_{\mathbb{R}^3} \frac{\varrho - \varrho_n}{\kappa + \varrho} |\mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$= \int_{I^*} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Since strong norm convergence and weak convergence imply strong convergence the compactness is shown and the existence of a fixpoint follows by Theorem 4.5.

This gives the claim of Theorem 4.2.

5. The Viscous Approximation

In this section we want to get rid of the regularization operator $\mathscr{R}_\kappa$ in order to find a solution $(\eta, \mathbf{u}, \varrho) \in Y^I \times X_\eta^I \times X_\eta^I$ to the viscous approximation satisfying the following:

(E1) The regularized momentum equation holds in the sense that

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_I \int_{\Omega_\eta} \Big(\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\lambda + \mu) \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_I \int_{\Omega_\eta} (\varrho^\gamma + \delta \varrho^\beta) \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_{\Omega_\eta} \varepsilon \nabla (\varrho \mathbf{u}) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_M \partial_t \eta b \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}t - \int_I \int_M \partial_t \eta \, \partial_t b \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}t \\
&\quad + \int_I \int_M K'(\eta) \, b \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}t \\
&= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_M g \, b \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{5.1}$$

for all test-functions $(b, \boldsymbol{\varphi}) \in C_0^\infty(M) \times C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\operatorname{tr}_\eta \boldsymbol{\varphi} = b\nu$. Moreover, we have $(\varrho \mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t \eta(0) = \eta_1$.

(E2) The regularized continuity equation in the sense that

$$\begin{aligned}
&\int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \psi \, \mathrm{d}x - \int_I \int_{\Omega_\eta} \Big(\varrho \, \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi \Big) \, \mathrm{d}x \right) \mathrm{d}t \\
&\quad + \varepsilon \int_I \int_{\Omega_\eta} \nabla \varrho \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = 0
\end{aligned} \tag{5.2}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and we have $\varrho(0) = \varrho_0$.

(E3) The boundary condition $\operatorname{tr}_\eta \mathbf{u} = \partial_t \eta \nu$ in the sense of Lemma 2.4.

Theorem 5.1. *There is a solution $(\eta, \mathbf{u}, \varrho) \in Y^I \times X_\eta^I \times X_\eta^I$ to (E1)–(E3). Here, we have $I = (0, T_*)$, where $T_* < T$ only if $\lim_{t \to T^*} \|\eta(t, \cdot)\|_{L^\infty(\partial\Omega)} = \frac{L}{2}$. The* *solution satisfies the energy estimate*

$$\begin{aligned}
&\sup_{t\in I}\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\sup_{t\in I}\int_{\Omega_\eta}\big(a\varrho^\gamma+\delta\varrho^\beta\big)\,\mathrm{d}x+\int_I\int_{\Omega_\eta}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\varepsilon\int_I\int_{\Omega_\eta}\big(\varrho^{\gamma-2}|\nabla\varrho|^2+\varrho|\nabla\mathbf{u}|^2\big)\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in I}\int_M|\partial_t\eta|^2\,\mathrm{d}\mathcal{H}^2+\sup_{t\in I}K(\eta)\\
&\quad\leqq c\bigg(\int_{\Omega_\eta}|\mathbf{q}_0|^2\,\mathrm{d}x+\int_{\Omega_\eta}\big(\varrho_0^\gamma+\delta\varrho_0^\beta\big)\,\mathrm{d}x+\int_I\|\mathbf{f}\|^2_{L^2(\Omega_\eta)}\,\mathrm{d}t\bigg)\\
&\qquad+c\bigg(\int_M|\eta_0|^2\,\mathrm{d}\mathcal{H}^2+\int_M|\eta_1|^2\,\mathrm{d}\mathcal{H}^2+K(\eta_0)+\int_I\|g\|^2_{L^2(M)}\,\mathrm{d}t\bigg),
\end{aligned}\tag{5.3}$$

provided that $\eta_0,\eta_1,\varrho_0,\mathbf{q}_0,\mathbf{f}$ *and* g *are regular enough to give sense to the right-hand side, that* $\varrho_0\geqq 0$ *a.e. and* (1.10) *is satisfied. The constant* c *is independent of* δ,ε*.*

Lemma 5.2. *Under the assumptions of Theorem 5.1, the continuity equation holds in the renormalized sense that is*

$$\begin{aligned}
&\int_I\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta}\theta(\varrho)\psi\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\Big(\theta(\varrho)\partial_t\psi+\theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqq-\int_I\int_{\Omega_\eta}(\varrho\theta'(\varrho)-\theta(\varrho_n))\,\mathrm{div}\,\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\int_I\int_{\Omega_\eta}\nabla\varrho\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{5.4}$$

for all $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3,[0,\infty))$ *and all convex* $\theta\in C^1(\mathbb{R})$*, with* $\theta(0)=0$ *and* $\theta'(z)=0$ *for* $z\geqq M_\theta$*.*

Proof (Proof of Theorem 5.1). In Theorem 4.2 we take $\kappa:=1/n$ where $1/n$ is the regularizing parameter. We call the corresponding solution $(\eta_n,\mathbf{u}_n,\varrho_n)$. If $n\to\infty$ then $\mathscr{R}_{1/n}\to\mathrm{id}$. Now we analyze the convergence of $(\eta_n,\mathbf{u}_n,\varrho_n)$. The estimate from Theorem 4.2 holds uniformly with respect to n. Additionally, by testing the continuity equation with ϱ_ε and using $\beta\geqq 4$ we find that

$$\varrho_n\in L^\infty(I;L^2(\Omega_{\mathscr{R}_{1/n}\eta_n})),\ \nabla\varrho_n\in L^2(I;L^2(\Omega_{\mathscr{R}_{1/n}\eta_n}))\text{ uniformly.}\tag{5.5}$$

Hence, we find that there is a subsequence such that for some $\alpha\in(0,1)$ fixed we have

$$\eta_n\rightharpoonup^*\eta\quad\text{in}\quad L^\infty(I;W^{2,2}(M))\tag{5.6}$$

$$\eta_n\rightharpoonup^*\eta\quad\text{in}\quad W^{1,\infty}(I;L^2(M)),\tag{5.7}$$

$$\eta_\varepsilon\to\eta\quad\text{in}\quad C^\alpha(\overline{I}\times M),\tag{5.8}$$

$$\mathbf{u}_n\rightharpoonup^\eta\mathbf{u}\quad\text{in}\quad L^2(I;W^{1,2}(\Omega_{\mathscr{R}_{1/n}\eta_n})),\tag{5.9}$$

$$\frac{1}{\sqrt{n}}\mathbf{u}_n\to^\eta 0\quad\text{in}\quad L^\infty(I;L^2(\Omega_{\mathscr{R}_{1/n}\eta_n})),\tag{5.10}$$

$$\varrho_n\rightharpoonup^{*,\eta}\varrho\quad\text{in}\quad L^\infty(I;L^\beta(\Omega_{\mathscr{R}_{1/n}\eta_n})).\tag{5.11}$$

Moreover, Remark 2.10 and (5.5) imply

$$\varrho_n \rightharpoonup^{*,\eta} \varrho \quad \text{in} \quad L^\infty(I; L^2(\Omega_{\mathscr{R}_{1/n}\eta_n})), \tag{5.12}$$

$$\varrho_n \rightharpoonup^{\eta} \varrho \quad \text{in} \quad L^2(I; W^{1,2}(\Omega_{\eta_n})), \tag{5.13}$$

$$\varrho_n \to^{\eta} \varrho \quad \text{in} \quad L^2(I; L^2(\Omega_{\mathscr{R}_{1/n}\eta_n})). \tag{5.14}$$

The last convergence, (5.9) and Lemma 2.8 imply

$$\varrho_n \mathbf{u}_n \rightharpoonup^{\eta} \varrho \mathbf{u} \quad \text{in} \quad L^2(I; L^2(\Omega_{\mathscr{R}_{1/n}\eta_n})), \tag{5.15}$$

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup^{\eta} \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{in} \quad L^1(I; L^1(\mathscr{R}_{1/n}\eta_n)). \tag{5.16}$$

Therefore, we can pass to the limit in the equation and obtain a weak solution to the viscous approximation. The energy inequality is a consequence of lower semi-continuity. $\square$

Proof of Lemma 5.2. First, observe that since ϱ_n is a renormalized solution to the continuity equation by Theorem 3.1 b), i.e. we have

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \theta(\varrho_n)\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \Big(\theta(\varrho_n)\partial_t \psi + \theta(\varrho_n)\mathbf{u}_n \cdot \nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_I \int_{\mathbb{R}^3} \Big((\varrho_n \theta'(\varrho_n) - \theta(\varrho_n))\,\mathrm{div}\,\mathbf{u}_n\,\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_I \int_{\mathbb{R}^3} \Big(\varepsilon\theta''(\varrho_n)|\nabla\varrho_n|^2\psi + \varepsilon\nabla\varrho_n \cdot \nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

As θ is convex this yields

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \theta(\varrho_n)\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\mathbb{R}^3} \Big(\theta(\varrho_n)\partial_t \psi + \theta(\varrho_n)\mathbf{u}_n \cdot \nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqq -\int_I \int_{\mathbb{R}^3} (\varrho_n \theta'(\varrho_n) - \theta(\varrho_n))\,\mathrm{div}\,\mathbf{u}_n\,\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \varepsilon \int_I \int_{\mathbb{R}^3} \nabla\varrho_n \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{5.17}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and all $\theta \in C^2(\mathbb{R})$ with $\theta(0) = 0$ and $\theta'(z) = 0$ for $z \geqq M_\theta$. By approximation its is easy to see that the assumption $\theta \in C^1(\mathbb{R})$ suffices for (5.17). Due to the convergences (5.8), (5.9), (5.13) and (5.14) we can pass to the limit in (5.17). This implies (5.4). $\square$

6. The Vanishing Viscosity Limit

The aim of this Section is to study the limit $\varepsilon \to 0$ in the approximate system (5.1)–(5.2) and establish the existence of a weak solution $(\eta, \varrho, \mathbf{u})$ to the system with artificial viscosity in the following sense. We define

$$\widetilde{W}^I_\eta = C_w(\overline{I}; L^\beta(\Omega_\eta)).$$

A weak solution is a triple $(\eta, \mathbf{u}, \varrho) \in Y^I \times X^I_\eta \times \widetilde{W}^I_\eta$ that satisfies the following:

(D1) The momentum equation in the sense that

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\Omega_\eta}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I\int_{\Omega_\eta}\Big(\mu\nabla\mathbf{u}:\nabla\boldsymbol{\varphi} + (\lambda+\mu)\operatorname{div}\mathbf{u}\,\operatorname{div}\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_I\int_{\Omega_\eta}(\varrho^\gamma + \delta\varrho^\beta)\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_I\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\int_M \partial_t\eta b\,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t\eta\,\partial_t b\,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\,b\,\mathrm{d}\mathcal{H}^2\bigg)\,\mathrm{d}t\\
&\quad = \int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_I\int_M g\,b\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{6.1}$$

for all $(b,\boldsymbol{\varphi}) \in C_0^\infty(M)\times C^\infty(\overline{I}\times\mathbb{R}^3)$ with $\mathrm{tr}_\eta\boldsymbol{\varphi} = b\nu$. Moreover, we have $(\varrho\mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t\eta(0) = \eta_1$.

(D2) The continuity equation holds in the sense that

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta}\varrho\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{6.2}$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$ and we have $\varrho(0) = \varrho_0$.

(D3) The boundary condition $\mathrm{tr}_\eta\mathbf{u} = \partial_t\eta\nu$ holds in the sense of Lemma 2.4.

Theorem 6.1. *There is a solution $(\eta,\mathbf{u},\varrho) \in Y^I \times X_\eta^I \times \widetilde{W}_\eta^I$ to (D1)–(D3). Here, we have $I = (0,T_*)$, where $T_* < T$ only if $\lim_{t\to T^*}\|\eta(t,\cdot)\|_{L^\infty(\partial\Omega)} = \frac{L}{2}$. The solution satisfies the energy estimate*

$$\begin{aligned}
&\sup_{t\in I}\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\,\mathrm{d}x + \sup_{t\in I}\int_{\Omega_\eta}\big(a\varrho^\gamma + \delta\varrho^\beta\big)\,\mathrm{d}x\\
&\quad + \int_I\int_{\Omega_\eta}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t + \sup_{t\in I}\int_M|\partial_t\eta|^2\,\mathrm{d}\mathcal{H}^2 + \sup_{t\in I}K(\eta)\\
&\leqq c\bigg(\int_{\Omega_\eta}\frac{|\mathbf{q}_0|^2}{\varrho_0}\,\mathrm{d}x + \int_{\Omega_\eta}\big(a\varrho_0^\gamma + \delta\varrho_0^\beta\big)\,\mathrm{d}x + \int_I\|\mathbf{f}\|^2_{L^\infty(\Omega_\eta)}\,\mathrm{d}t\bigg)\\
&\quad + c\bigg(\int_M|\eta_0|^2\,\mathrm{d}\mathcal{H}^2 + \int_M|\eta_1|^2\,\mathrm{d}\mathcal{H}^2 + K(\eta_0) + \int_I\|g\|^2_{L^2(M)}\,\mathrm{d}t\bigg),
\end{aligned}$$

provided that $\eta_0, \eta_1, \varrho_0, \mathbf{q}_0, \mathbf{f}$ and g are regular enough to give sense to the right-hand side, that $\varrho_0 \geqq 0$ a.e. and (1.10) is satisfied. The constant c is independent of δ.

Lemma 6.2. *Under the assumptions of Theorem 6.1 the continuity equation holds in the renormalized sense that is*

$$\begin{aligned}
&\int_I\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta}\theta(\varrho)\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta}\Big(\theta(\varrho)\partial_t\psi + \theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = -\int_I\int_{\Omega_\eta}(\varrho\theta'(\varrho) - \theta(\varrho))\operatorname{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{6.3}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ *and all* $\theta \in C^1(\mathbb{R})$ *with* $\theta(0) = 0$ *and* $\theta'(z) = 0$ *for* $z \geqq M_\theta$.

The proof will be split in several parts. For a given ε we gain a weak solutions $(\eta_\varepsilon, \mathbf{u}_\varepsilon, \varrho_\varepsilon)$ to (5.1)–(5.2) by Theorem 5.1. The estimate from Theorem 5.1 holds uniformly with respect to ε. In particular,

$$\begin{aligned}\sup_{t\in I}\int_{\Omega_{\eta_\varepsilon}} \varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x + \sup_{t\in I}\int_{\Omega_{\eta_\varepsilon}} \big(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta\big)\,\mathrm{d}x + \int_I\int_{\Omega_{\eta_\varepsilon}} |\nabla\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\ + \sup_{t\in I}\int_M |\partial_t\eta_\varepsilon|^2\,\mathrm{d}\mathcal{H}^2 + \sup_{t\in I} K(\eta_\varepsilon) \\ \leqq C(\eta_1, g, \mathbf{f}, \varrho_0, \mathbf{q}_0)\end{aligned} \tag{6.4}$$

is satisfied uniformly in ε for the time interval I. Hence, we may take a subsequence such that for some $\alpha \in (0, 1)$ we have

$$\eta_\varepsilon \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I; W^{2,2}(M)) \tag{6.5}$$

$$\eta_\varepsilon \rightharpoonup^* \eta \quad \text{in} \quad W^{1,\infty}(I; L^2(M)), \tag{6.6}$$

$$\eta_\varepsilon \to \eta \quad \text{in} \quad C^\alpha(\overline{I} \times M)), \tag{6.7}$$

$$\mathbf{u}_\varepsilon \rightharpoonup^\eta \mathbf{u} \quad \text{in} \quad L^2(I; W^{1,2}(\Omega_{\eta_\varepsilon})), \tag{6.8}$$

$$\varrho_\varepsilon \rightharpoonup^{*,\eta} \varrho \quad \text{in} \quad L^\infty(I; L^\beta(\Omega_{\eta_\varepsilon})). \tag{6.9}$$

Now, using the a-priori estimates (6.4) and the bounds that one gains (using the renormalized continuity equation from Lemma 5.2 with $\theta(z) = z^2$ and testing with $\psi \equiv 1$) we find, due to $\beta > 4$, that

$$\int_I\int_{\Omega_{\eta_\varepsilon}} \varepsilon|\nabla\varrho_\varepsilon|^2\,dx\,dt \leqq C, \tag{6.10}$$

with C independent of ε. This and (6.8) imply

$$\varepsilon\nabla\varrho_\varepsilon \to^\eta 0 \quad \text{in} \quad L^2(I \times \Omega_{\eta_\varepsilon}), \tag{6.11}$$

$$\varepsilon\nabla(\mathbf{u}_\varepsilon\varrho_\varepsilon) \to^\eta 0 \quad \text{in} \quad L^1(I \times \Omega_{\eta_\varepsilon}). \tag{6.12}$$

We observe that the a-priori estimates (6.4) imply uniform bounds of $\varrho_\varepsilon\mathbf{u}_\varepsilon$ in $L^\infty(I, L^{\frac{2\beta}{\beta+1}})$. Therefore, we may apply Lemma 2.8 with the choice $v_i \equiv \mathbf{u}_\varepsilon$, $r_i = \varrho_\varepsilon$, $p = s = 2$, $b = \beta$ and m sufficiently large to obtain

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \rightharpoonup^\eta \varrho\mathbf{u} \qquad \text{in} \qquad L^q(I, L^a(\Omega_{\eta_\varepsilon})), \tag{6.13}$$

where $a \in (1, \frac{2\beta}{\beta+1})$ and $q \in (1, 2)$. We apply Lemma 2.8 once more with the choice $v_i \equiv \mathbf{u}_\varepsilon$, $r_i = \varrho_\varepsilon\mathbf{u}_\varepsilon$, $p = s = 2$, $b = \frac{2\beta}{\beta+1}$ and m sufficiently large to find that

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup^\eta \varrho\mathbf{u} \otimes \mathbf{u} \qquad \text{in} \qquad L^1(I \times \Omega_{\eta_\varepsilon}). \tag{6.14}$$

6.1. Equi-Integrability of the Pressure

First, we have to handle the problem that the pressure is merrily bounded in L^1 in space. Consequently, it might converge to a measure and not a measurable function. This is usually excluded by showing that the pressure possesses higher integrability properties. From this we deduce a weakly converging subsequence (in some Lebesgue space) and hence get a function as a limit object. In the case of a moving domain standard procedures do not apply and global higher integrability on the moving domain can not be achieved. The solution is two divide the problem in two steps: the first step is to improve the space integrability of the pressure inside the moving domain; the second step is to show that the mass of the pressure can not be concentrated on the boundary. Combining the two results will imply equi-integrability of the pressure which is equivalent to weak compactness L^1. The next two lemmata settle that matter. The first one is happily a localized version of the standard procedure.

Lemma 6.3. *Let $Q = J \times B \Subset I \times \Omega_\eta$ be a parabolic cube. The following holds for any $\varepsilon \leqq \varepsilon_0(Q)$:*

$$\int_Q \left(a\varrho_\varepsilon^{\gamma+1} + \delta\varrho_\varepsilon^{\beta+1}\right) \mathrm{d}x\, \mathrm{d}t \leqq C(Q), \tag{6.15}$$

with constant independent of ε.

Proof. We consider a parabolic cube $\tilde{Q} = \tilde{J} \times \tilde{B}$ with $Q \Subset \tilde{Q} \Subset I \times \Omega_\eta$. Due to (6.7) we can assume that $\tilde{Q} \Subset I \times \Omega^I_{\eta_\varepsilon}$ (by taking ε small enough). Next we wish to test with $\psi \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon$ where $\psi \in C_0^\infty(\tilde{Q})$ with $\psi = 1$ in Q and $\Delta_{\tilde{B}}^{-1} \varrho_\varepsilon$ is defined as the unique $W^{2,\beta}(\tilde{B}) \cap W_0^{1,\beta^*}(\tilde{B})$-solution to the equation

$$-\Delta v = \varrho_\varepsilon \quad \text{in} \quad \tilde{B}. \tag{6.16}$$

The test-function $(\psi \nabla \Delta^{-1} \varrho_\varepsilon, 0)$ is indeed admissible in (5.1) since ψ has compact support. Moreover, regularity follows from local theory for the respective parabolic equation. In order to deal with the term involving the time derivative we use the continuity equation. We find that

$$-\Delta \partial_t v = \partial_t \varrho_\varepsilon = -\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon + \varepsilon \nabla \varrho_\varepsilon) \tag{6.17}$$

in the sense of distributions such that $\partial_t v = -\partial_t \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon = \nabla \Delta_{\tilde{B}}^{-1} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon + \varepsilon \nabla \varrho_\varepsilon)$. Hence, we have

$$
\begin{aligned}
J_0 :=& \int_I \int_{\mathbb{R}^3} \psi \big(a\varrho_\varepsilon^{\gamma+1} + \delta\varrho_\varepsilon^{\beta+1}\big) \,\mathrm{d}x \,\mathrm{d}\sigma \\
=& \,\mu \int_I \int_{\mathbb{R}^3} \psi \nabla \mathbf{u}_\varepsilon : \nabla^2 \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ \mu \int_I \int_{\mathbb{R}^3} \nabla \mathbf{u}_\varepsilon : \nabla\psi \otimes \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \psi \operatorname{div} \mathbf{u}_\varepsilon \, \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \operatorname{div} \mathbf{u}_\varepsilon \, \nabla\psi \cdot \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla^2 \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\psi \otimes \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ \varepsilon \int_I \int_{\mathbb{R}^3} \nabla(\mathbf{u}_\varepsilon \varrho_\varepsilon) : \nabla^2 \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \psi \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ \varepsilon \int_I \int_{\mathbb{R}^3} \nabla(\mathbf{u}_\varepsilon \varrho_\varepsilon) : \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \otimes \nabla\psi \,\mathrm{d}x \,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \big(a\varrho_\varepsilon^{\gamma} + \delta\varrho_\varepsilon^{\beta}\big) \nabla\psi \cdot \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{f} \cdot \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
&+ \int_I \int_{\mathbb{R}^3} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \nabla \Delta_{\tilde{B}}^{-1} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon + \varepsilon \nabla \varrho_\varepsilon) \,\mathrm{d}x \,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \partial_t \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}\sigma \\
=:& \, J_1 + \cdots + J_{12}.
\end{aligned}
\tag{6.18}
$$

Our goal is to find an estimate for the expectation of J_0 which means that we have to find suitable bounds for all the other terms. Using the continuity of the operator $\nabla \Delta_{\tilde{B}}^{-1}$ and Sobolev's embedding theorem, we obtain for some $p > 3$ that

$$
\|\nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon\|_{L^\infty(\tilde{B})} \leqq C \,\|\nabla^2 \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon\|_{L^p(\tilde{B})} \leqq C \,\|\varrho_\varepsilon\|_{L^p(\tilde{B})}, \tag{6.19}
$$

using (6.9) and $\beta > 3$. Note that, in particular, we have shown that $\psi \nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon \in L^\infty(I \times \mathbb{R}^3)$ uniformly in ε. As $\varrho_\varepsilon \in L^2(I \times \tilde{Q})$ uniformly due to $\beta \geqq 2$ we deduce that $|J_1| \leqq C$ as a consequence of uniform bounds on $\mathbf{u}_\varepsilon$ in (6.8) and the continuity of the operator $\nabla^2 \Delta_{\tilde{B}}^{-1}$. Similar arguments lead to the bound for J_2, J_3, J_4. The most critical is the convective term J_5. It can be estimated using the continuity of $\nabla^2 \Delta_{\tilde{B}}^{-1}$, Sobolev's embedding theorem (combined with Poincaré's inequality and the fact that $\mathbf{u}_\varepsilon = 0$ on Γ), Hölder's inequality and (6.4)

$$
\begin{aligned}
|J_5| &\leqq C \int_{\tilde{J}} \|\varrho_\varepsilon\|_{L^3(\tilde{B})} \|\mathbf{u}_\varepsilon\|_{L^6(\tilde{B})}^2 \|\varrho_\varepsilon\|_{L^3(\tilde{B})} \,\mathrm{d}s \\
&\leqq C \sup_{\tilde{J}} \|\varrho_\varepsilon\|_{L^3(\tilde{B})}^2 \int_{\tilde{Q}} |\nabla \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}s \leqq C.
\end{aligned}
$$

The term J_6 is estimated similarly. For J_7 we obtain

$$|J_7| \leqq \sup_{\tilde{J}} \|\nabla^2 \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon\|_{L^3(\tilde{B})} \int_{\tilde{J}} \Big(\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\tilde{B})}^2 + \varepsilon^2 \|\nabla \varrho_\varepsilon\|_{L^2(\tilde{B})}^2 + \varepsilon^2 \|\varrho_\varepsilon\|_{L^6(\tilde{B})}^2 \Big) \, \mathrm{d}t,$$

which is uniformly bounded due to (6.8), (6.9) and (6.10). Similarly, J_8 is bounded by

$$\begin{aligned} |J_8| \leqq \sup_{\tilde{J}} \|\nabla \Delta_{\tilde{B}}^{-1} \varrho_\varepsilon\|_{L^\infty(\tilde{B})}^2 \bigg(\int_{\tilde{J}} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\tilde{B})}^2 + \|\mathbf{u}_\varepsilon\|_{L^6(\tilde{B})}^2 \\ + \varepsilon^2 \|\nabla \varrho_\varepsilon\|_{L^2(\tilde{B})}^2 + \varepsilon^2 \|\varrho_\varepsilon\|_{L^6(\tilde{B})}^2 \, \mathrm{d}t \bigg), \end{aligned}$$

taking into account (6.19). The terms of J_9, J_{10} can be estimated using by the bounds on the operator $\nabla \Delta_{\tilde{B}}^{-1}$ and Hölder's and Young's inequalities. The same is used to estimate

$$J_{11} \leqq C \int_{\tilde{Q}} |\varrho_\varepsilon \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t + \varepsilon C \bigg(\int_{\tilde{Q}} |\varrho_\varepsilon \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \bigg)^{\frac{1}{2}} \bigg(\int_{\tilde{Q}} |\nabla \varrho_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \bigg)^{\frac{1}{2}},$$

which is finite since we have uniformly in ε

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \in L^2 \Big(\tilde{J}; L^{\frac{6\beta}{\beta+6}}(\tilde{Q}) \Big). \tag{6.20}$$

The latter bound is a consequence of the fact that

$$\varrho_\varepsilon \in L^\infty(\tilde{J}; L^\beta(\tilde{B}))), \qquad \mathbf{u}_\varepsilon \in L^2(\tilde{J}; L^6(\tilde{B})))$$

uniformly in ε. Finally, J_{12} can be estimated using (6.19) and (6.20). Plugging all of this together we obtain (6.15) uniformly in ε. $\square$

The standard method as used in the proof of Lemma 6.15 does not apply up to the boundary. Also, the usage of the Bogovskiĭ-operator—common in literature as well—does not help (recall that our boundary depends on time and is not Lipschitz-continuous). In the following Lemma we show equi-integrability at the boundary related to the method from [29]:

Lemma 6.4. *Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that we have for all $\varepsilon \leqq \varepsilon_0$*

$$\int_{I \times \mathbb{R}^3 \setminus A_\kappa} \big(a \varrho_\varepsilon^\gamma + \delta \varrho_\varepsilon^\beta \big) \chi_{\Omega_{\eta_\varepsilon}} \, \mathrm{d}x \, \mathrm{d}t \leqq \kappa. \tag{6.21}$$

Proof. We construct a test-function which has a positive and arbitrarily large divergence. For this let $\varphi \in C_0^\infty(S_L; [0, 1])$, such that $\chi_{S_{\frac{L}{2}}} \leqq \varphi \leqq \chi_{\Omega_0 \cup S_L}$ and $|\nabla \varphi| \leqq \frac{c}{L}$. Since we know that $|\eta_\varepsilon| \leqq \frac{L}{2}$, we find that $\varphi(x) \equiv 1$ in $S_{\frac{L}{2}} \cap \Omega_{\eta_\varepsilon}$. We extend φ by zero to $\mathbb{R}^3$ and define

$$\boldsymbol{\varphi}_\varepsilon(t, x) = \varphi \min \{ K(s(x) - \eta_\varepsilon(t, q(x))), 1\} \nu(q(x)),$$

where $K > 0$ will be chosen later. It is well defined, since $\varphi \neq 0$ only in S_L, where the mapping $x \mapsto (q(x), s(x))$ is well defined, see Section 2.2. Observe that we take coordinates with respect to the reference geometry Ω and with respect to the reference outer normal ν on $\partial\Omega$. On account of $\nabla s(x) = \nu(q(x))$ we have

$$
\begin{aligned}
\partial_j \boldsymbol{\varphi}^l_\varepsilon(t,x) &= \partial_j \varphi(x) \min\{K(s(x)-\eta_\varepsilon(t,q(x))),1\}\nu_l(q(x)) \\
&\quad + K\chi_{\{K(s(x)-\eta_\varepsilon(t,q(x)))\leqq 1\}}\nu_j(q(x))\nu_l(q(x))\varphi(x) \\
&\quad - K\chi_{\{K(s(x)-\eta_\varepsilon(t,q(x)))\leqq 1\}}\nabla\eta_\varepsilon(t,q(x))\cdot\partial_j q(x)\nu_l(q(x))\varphi(x) \\
&\quad + \varphi(x)\min\{K(s(x)-\eta_\varepsilon(t,q(x))),1\}\partial_j\nu_l(q(x)) \\
&= \xi^1_{jl}(t,x)+\xi^2_{jl}(t,x)+\xi^3_{jl}(t,x)+\xi^4_{jl}(t,x).
\end{aligned}
$$

Observe that $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^4$ are uniformly bounded by some constant $c_{\boldsymbol{\xi}}$. Moreover, for every $p\in(1,\infty)$, $q>p$, it holds that

$$
\begin{aligned}
\bigg(\int_I\int_{\Omega_{\eta_\varepsilon}}|\boldsymbol{\xi}^3|^p\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{p}} &\leqq cK\bigg(\int_I\int_{\Omega_{\eta_\varepsilon}}\chi_{\{K(s-\eta_\varepsilon\circ q)\leqq 1\}}|\nabla\eta_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{p}} \\
&\leqq cK\bigg(\int_I\int_{\Omega_{\eta_\varepsilon}}|\nabla\eta_\varepsilon|^q\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{q}}\Big|\{K(s-\eta_\varepsilon\circ q)\leqq 1\}\Big|^{\frac{1}{q'p}} \\
&\leqq c_pK^{1-\frac{1}{pq'}}
\end{aligned}
$$

uniformly in ε, cf. (6.5). Estimating $\boldsymbol{\xi}^2$ in a similar way we gain

$$
\bigg(\int_I\int_{\Omega_{\eta_\varepsilon}}|\nabla\boldsymbol{\varphi}_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{p}}\leqq c_p\big(K^{1-\frac{1}{pp'}}+1\big) \tag{6.22}
$$

for all $p<\infty$ uniformly in ε. Finally, we use the fact that ∇q^i are all living in the tangentplane of $\partial\Omega$ and are therefore orthogonal to $\nu(q(x))$. Hence, we have $\xi^3_{jj}=0$. This implies that for every $K>0$ there is a κ such that we have

$$
\operatorname{div}\boldsymbol{\varphi}_\varepsilon\geqq K-c_{\boldsymbol{\xi}}\quad\text{in}\quad\Omega_{\eta_\varepsilon}\setminus\Big\{x\in\Omega_{\eta_\varepsilon}:\operatorname{dist}(\partial\Omega_{\eta_\varepsilon})\geqq\frac{1}{K}\Big\}. \tag{6.23}
$$

Finally, we calculate

$$
\partial_t\boldsymbol{\varphi}_\varepsilon(t,x)=-K\chi_{\{K(s(x)-\eta_\varepsilon(t,q(x)))\leqq 1\}}\partial_t\eta_\varepsilon(t,q(x))\nu(q(x)).
$$

Due to (6.6) we have

$$
\begin{aligned}
\bigg(\sup_I\int_{\Omega_{\eta_\varepsilon}}|\partial_t\boldsymbol{\varphi}_\varepsilon|^r\,\mathrm{d}x\bigg)^{\frac{1}{r}} &\leqq cK\bigg(\sup_I\int_{\Omega_{\eta_\varepsilon}}|\partial_t\eta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{2}} \\
&\qquad\Big|\{K(s-\eta_\varepsilon\circ q)\leqq 1\}\Big|^{\frac{2-r}{2r}} \\
&\qquad\leqq c\,K^{1-\frac{2-r}{2r}}
\end{aligned} \tag{6.24}
$$

for all $r<2$, is a fashion similar to (6.22). Now, using $\boldsymbol{\varphi}_\varepsilon$ as a test-function (note that $\boldsymbol{\varphi}_\varepsilon=0$ on $\partial\Omega_{\eta_\varepsilon}$), we obtain by smooth approximation that

$$
\begin{aligned}
\int_I\int_{\Omega_{\eta_\varepsilon}}(\varrho^\gamma+\delta\varrho^\beta)\operatorname{div}\boldsymbol{\varphi}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t &\leqq -\int_I\int_{\Omega_{\eta_\varepsilon}}\varrho_\varepsilon\mathbf{u}_\varepsilon\,\partial_t\boldsymbol{\varphi}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\big(K^{1-\lambda}+1\big)
\end{aligned} \tag{6.25}
$$

for some fixed $C > 0$ and $\lambda \in (0, 1)$, where C, λ are independent of ε. Here, we used the uniform integrability bounds of all other terms of the momentum equation (5.3) and (6.22). Taking (6.20) and $\beta > 3$ into account we see that the remaining integral in (6.25) is uniformly p-integrable for some exponent $p > 1$ in terms of (6.24) cf. (6.4). This means we have

$$\int_I \int_{\Omega_{\eta_\varepsilon}} (\varrho^\gamma + \delta\varrho^\beta)\ \mathrm{div}\, \boldsymbol{\varphi}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqq C(K^{1-\lambda} + 1) \tag{6.26}$$

uniformly for some $\lambda \in (0, 1)$. Now, we set

$$A_\kappa = \big\{x \in \Omega_{\eta_\varepsilon} : \mathrm{dist}(\partial\Omega_{\eta_\varepsilon}) \geqq \frac{1}{K}\big\},$$

where $K = K(\kappa)$ is the solution to

$$\frac{C(K^{1-\lambda} + 1)}{K - c_{\boldsymbol{\xi}}} = \kappa$$

with C given in (6.26). Note that such a K always exists if κ is small enough. As a consequence of (6.23) and (6.25) we gain

$$\begin{aligned}\int_{I\times\mathbb{R}^3\setminus A_\kappa} (\varrho^\gamma + \delta\varrho^\beta)\,\mathrm{d}x\,\mathrm{d}t &\leqq \frac{1}{K - c_{\boldsymbol{\xi}}} \int_{I\times\mathbb{R}^3\setminus A_\kappa} (\varrho^\gamma + \delta\varrho^\beta)\ \mathrm{div}\, \boldsymbol{\varphi}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\\ &\leqq \frac{C(K^{1-\lambda} + 1)}{K - c_{\boldsymbol{\xi}}} = \kappa.\end{aligned}$$

The claim follows. □

We connect Lemmas 6.3 and 6.4 to get the following corollary:

Corollary 6.5. *Under the assumptions of Theorem 6.1 there existence of a function $\overline{p}$ such that*

$$\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta \rightharpoonup^\eta \overline{p} \text{ in } L^1(I; L^1(\Omega_{\eta_\varepsilon})),$$

at least for a subsequence. Additionally, for $\kappa > 0$ arbitrary, there is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that $\overline{p}\varrho \in L^1(A_\kappa)$ and

$$\int_{(I\times\Omega_\eta)\setminus A_\kappa} \overline{p}\,\mathrm{d}x\,\mathrm{d}t \leqq \kappa. \tag{6.27}$$

Combining Corollary 6.5 with the convergences (6.5)–(6.14) we can pass to the limit in (5.1)–(5.2) and obtain the following. There is $(\eta, \mathbf{u}, \varrho, \overline{p}) \in Y^I \times X_\eta^I \times \tilde{W}_\eta^I \times L^1(I \times \Omega_\eta)$ that satisfies (in the sense of Lemma 2.4)

$$\mathbf{u}(\cdot, \cdot + \eta\nu) = \partial_t \eta \nu_\eta \text{ in } I \times \partial\Omega,$$

the continuity equation

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho\psi \,\mathrm{d}x\,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{6.28}$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and the coupled weak momentum equation

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_{\Omega_\eta} \Big(\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\lambda + \mu) \operatorname{div} \mathbf{u} \, \operatorname{div} \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_I \int_{\Omega_\eta} \overline{p} \, \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \qquad\qquad (6.29) \\
&+ \int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_M \partial_t \eta b \, \mathrm{d}\mathcal{H}^2 - \int_M \partial_t \eta \, \partial_t b \, \mathrm{d}\mathcal{H}^2 + \int_M K'(\eta) \, b \, \mathrm{d}\mathcal{H}^2 \, \mathrm{d}t \\
&= \int_I \int_{\Omega_\eta} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_I \int_M g \, b \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

for all $(b, \boldsymbol{\varphi}) \in C_0^\infty(M) \times C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta \, \boldsymbol{\varphi} = b\nu$.

It remains to show that $\overline{p} = a\varrho^\gamma + \delta \varrho^\beta$. This will be achieved in the following two subsections.

6.2. The Effective Viscous Flux

We fix $\varepsilon_0 > 0$ and consider in the following just $\varepsilon \in (0, \varepsilon_0)$. Next, we define

$$\Omega_{\varepsilon_0} = \bigcap_{\varepsilon \leqq \varepsilon_0} \Omega_{\eta_\varepsilon}.$$

It is the aim of this subsection to show that for $\psi \in C_0^\infty(I \times \Omega_{\varepsilon_0})$ we have

$$\begin{aligned}
&\int_{I \times \Omega_{\eta_\varepsilon}} \psi^2 \Big(a\varrho_\varepsilon^\gamma + \delta \varrho_\varepsilon^\beta - (\lambda + 2\mu) \operatorname{div} \mathbf{u}_\varepsilon \Big) \varrho_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad \longrightarrow \int_{I \times \Omega_\eta} \psi^2 \big(\overline{p} - (\lambda + 2\mu) \operatorname{div} \mathbf{u} \big) \varrho \, \mathrm{d}x \, \mathrm{d}t
\end{aligned} \qquad (6.30)$$

as $\varepsilon \to 0$. Testing the momentum equation with $\psi \nabla \Delta^{-1}(\psi \varrho_\varepsilon)$ implies

$$\begin{aligned}
J_0 &:= \int_I \int_{\mathbb{R}^3} \psi^2 \big(\varrho_\varepsilon^\gamma + \delta \varrho_\varepsilon^\beta \big) \varrho_\varepsilon \, \mathrm{d}x \, \mathrm{d}\sigma \\
&= \mu \int_I \int_{\mathbb{R}^3} \psi \nabla \mathbf{u}_\varepsilon : \nabla^2 \Delta^{-1}(\psi \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}\sigma \\
&\quad + \mu \int_I \int_{\mathbb{R}^3} \nabla \mathbf{u}_\varepsilon : \nabla \psi \otimes \nabla \Delta^{-1} \psi \varrho_\varepsilon \, \mathrm{d}x \, \mathrm{d}\sigma \\
&\quad + (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \psi^2 \operatorname{div} \mathbf{u}_\varepsilon \, \varrho_\varepsilon \, \mathrm{d}x \, \mathrm{d}\sigma \\
&\quad + (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \operatorname{div} \mathbf{u}_\varepsilon \, \nabla \psi \cdot \nabla \Delta^{-1}(\psi \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}\sigma \\
&\quad - \int_I \int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla^2 \Delta^{-1}(\psi \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}\sigma
\end{aligned}$$

$$
\begin{aligned}
&- \int_I \int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\psi \otimes \nabla\Delta^{-1}(\psi\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \big(\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta\big)\nabla\psi \cdot \nabla\Delta^{-1}(\psi\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{f} \cdot \nabla\Delta^{-1}(\psi\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \partial_t\psi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\Delta^{-1}(\psi\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&- \int_I \int_{\mathbb{R}^3} \nabla\Delta^{-1}(\partial_t\psi\varrho_\varepsilon) \cdot \varrho_\varepsilon \mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma \\
&+ \int_I \int_{\mathbb{R}^3} \psi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\Delta^{-1}(\psi\,\mathrm{div}(\mathbf{u}_\varepsilon\varrho_\varepsilon))\,\mathrm{d}x\,\mathrm{d}\sigma \\
&+ \varepsilon\int_I \int_{\mathbb{R}^3} \psi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\Delta^{-1}\big(\psi\Delta\varrho_\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}\sigma \\
&+ \varepsilon\int_I \int_{\mathbb{R}^3} \Big(\psi\nabla(\mathbf{u}_\varepsilon\varrho_\varepsilon) \cdot \nabla^2\Delta^{-1}(\psi\varrho_\varepsilon) \\
&+ \nabla(\mathbf{u}_\varepsilon\varrho_\varepsilon) : \nabla\Delta^{-1}(\psi\varrho_\varepsilon) \otimes \nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}\sigma \\
= {}& J_1 + \cdots + J_{11} + E_1 + E_2.
\end{aligned}
$$

We rewrite

$$
\begin{aligned}
J_1 &= \mu\int_I \int_{\mathbb{R}^3} \psi^2\,\mathrm{div}\,\mathbf{u}_\varepsilon\,\psi\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}\sigma + J_1', \\
J_1' &= \mu\int_I \int_{\mathbb{R}^3} \Big(\mathbf{u}_\varepsilon \cdot \nabla\psi\,\psi\varrho_\varepsilon - \mathbf{u}_\varepsilon \otimes \nabla\psi : \nabla^2\Delta^{-1}(\psi\varrho_\varepsilon)\Big)\,\mathrm{d}x\,\mathrm{d}\sigma,
\end{aligned}
$$

as well as

$$
\begin{aligned}
J_{11} &= \int_I \int_{\mathbb{R}^3} \psi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\Delta^{-1}\,\mathrm{div}(\psi\mathbf{u}_\varepsilon\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma + J_{11}' \\
J_{11}' &= -\int_I \int_{\mathbb{R}^3} \psi\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\Delta^{-1}(\nabla\psi \cdot \mathbf{u}_\varepsilon\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned}
$$

We define the operator $\mathcal{R}$ by $\mathcal{R}_{ij} := \partial_j\Delta^{-1}\partial_i$ and obtain

$$
\begin{aligned}
&\int_I \int_{\mathbb{R}^3} \psi^2\big(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta - (\lambda + 2\mu)\,\mathrm{div}\,\mathbf{u}_\varepsilon\big)\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = J_1' + J_2 + J_4 + J_6 + \cdots + J_{10} + J_{11}' + E_1 + E_2 \\
&\qquad\quad + \sum_{i,j}\int_I \int_{\mathbb{R}^3} u_\varepsilon^i\big(\psi\varrho_\varepsilon\mathcal{R}_{ij}[\psi\varrho_\varepsilon u_\varepsilon^j] - \psi\varrho_\varepsilon u_\varepsilon^j\mathcal{R}_{ij}[\psi\varrho_\varepsilon]\big)\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned}
\tag{6.31}
$$

Similarly, we obtain by testing the limit equation (6.29) by $\psi\nabla\Delta^{-1}(\psi\varrho)$

$$
K_0 := \int_I \int_{\mathbb{R}^3} \psi^2\overline{p}\varrho\,\mathrm{d}x\,\mathrm{d}\sigma
$$

$$
\begin{aligned}
&= \mu \int_I \int_{\mathbb{R}^3} \psi \nabla \mathbf{u} : \nabla^2 \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad + \mu \int_I \int_{\mathbb{R}^3} \nabla \mathbf{u} : \nabla \psi \otimes \nabla \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad + (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \psi^2 \operatorname{div} \mathbf{u} \varrho\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad + (\lambda + \mu) \int_I \int_{\mathbb{R}^3} \operatorname{div} \mathbf{u}\, \nabla \psi \cdot \nabla(\Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad - \int_I \int_{\mathbb{R}^3} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla^2 \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad - \int_I \int_{\mathbb{R}^3} \varrho \mathbf{u} \otimes \mathbf{u} : \nabla \psi \otimes \nabla \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad - \int_I \int_{\mathbb{R}^3} \overline{p} \nabla \psi \cdot \nabla \Delta^{-1}(\psi \varrho))\, \mathrm{d}x\, \mathrm{d}\sigma - \int_I \int_{\mathbb{R}^3} \varrho \mathbf{f} \cdot \nabla \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad - \int_I \int_{\mathbb{R}^3} \partial_t \psi \varrho \mathbf{u} \cdot \nabla \Delta^{-1}(\psi \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma - \int_I \int_{\mathbb{R}^3} \nabla \Delta^{-1}(\partial_t \psi \varrho) \cdot \varrho \mathbf{u}\, \mathrm{d}x\, \mathrm{d}\sigma \\
&\quad + \int_I \int_{\mathbb{R}^3} \psi \varrho \mathbf{u} \cdot \nabla \Delta^{-1}(\psi \operatorname{div}(\mathbf{u} \varrho))\, \mathrm{d}x\, \mathrm{d}\sigma = K_1 + \cdots + K_{11}.
\end{aligned}
$$

In a manner similar to (6.31) we obtain

$$
\begin{aligned}
&\int_I \int_{\mathbb{R}^3} \psi^2 \big(\overline{p} - (\lambda + 2\mu) \operatorname{div} \mathbf{u}\big) \varrho\, \mathrm{d}x\, \mathrm{d}t \\
&\qquad = K_1' + K_2 + K_4 + K_6 + \cdots + K_{10} + K_{11}' \\
&\qquad\quad + \sum_{i,j} \int_I \int_{\mathbb{R}^3} u^i \big(\psi \varrho \mathcal{R}_{ij}[\psi \varrho u^j] - \psi \varrho u^j \mathcal{R}_{ij}[\psi \varrho]\big)\, \mathrm{d}x\, \mathrm{d}\sigma,
\end{aligned} \tag{6.32}
$$

where

$$
\begin{aligned}
K_1' &= \mu \int_I \int_{\mathbb{R}^3} \Big(\mathbf{u} \cdot \nabla \psi\, \psi \varrho - \mathbf{u} \otimes \nabla \psi : \nabla^2 \Delta^{-1}(\psi \varrho) \Big)\, \mathrm{d}x\, \mathrm{d}\sigma, \\
K_{11}' &= - \int_I \int_{\mathbb{R}^3} \psi \varrho \mathbf{u} \cdot \nabla \Delta^{-1}(\nabla \psi \cdot \mathbf{u} \varrho)\, \mathrm{d}x\, \mathrm{d}\sigma.
\end{aligned}
$$

Hence we now have that

$$
\begin{aligned}
&\int_{I \times \Omega_{\eta \varepsilon}} \psi^2 \big(a \varrho_\varepsilon^\gamma + \delta \varrho_\varepsilon^\beta - (\lambda + 2\mu) \operatorname{div} \mathbf{u}_\varepsilon\big)\, \varrho_\varepsilon\, \mathrm{d}x\, \mathrm{d}t \\
&\quad - \int_{I \times \Omega_\eta} \psi^2 \big(\overline{p} - (\lambda + 2\mu) \operatorname{div} \mathbf{u}\big)\, \varrho\, \mathrm{d}x\, \mathrm{d}t \\
&= J_1' - K_1' + J_2 - K_2 + J_4 - K_4 + J_6 - K_6 + \cdots + J_{10} - K_{10} \\
&\quad + J_{11}' - K_{11}' + E_1 + E_2 \\
&\quad + \sum_{i,j} \int_I \int_{\mathbb{R}^3} u_\varepsilon^i \big(\psi \varrho_\varepsilon \mathcal{R}_{ij}[\psi \varrho_\varepsilon u_\varepsilon^j] - \psi \varrho_\varepsilon u_\varepsilon^j \mathcal{R}_{ij}[\psi \varrho_\varepsilon]\big)\, \mathrm{d}x\, \mathrm{d}\sigma
\end{aligned}
$$

$$-\int_I\int_{\mathbb{R}^3} u^i\big(\psi\varrho\mathcal{R}_{ij}[\psi\varrho u^j]-\psi\varrho u^j\mathcal{R}_{ij}[\psi\varrho]\big)\,\mathrm{d}x\,\mathrm{d}\sigma. \tag{6.33}$$

We will now show that the right hand side converges to 0 as $\varepsilon\to 0$. Observe that after this preparation everything is localized and the known approach can be enforced to our problem. Nevertheless, to keep the result self contained we repeat the main steps of the argument here.

First, by the assumption $\beta>3$ and the continuity of $\nabla\Delta^{-1}$ we find that

$$\begin{aligned}
|E_2| &\leqq C\sqrt{\varepsilon}\Big(\|\nabla^2\Delta^{-1}(\psi\varrho_\varepsilon)\|^3_{L^\infty(\tilde{I},(L^3(\tilde{B}))}+\|\nabla\Delta^{-1}(\psi\varrho_\varepsilon)\|^3_{L^\infty(\tilde{I},(L^3(\tilde{B}))}\\
&\qquad+\|\nabla\mathbf{u}_\varepsilon\|^3_{L^2(\tilde{Q})}+\|\mathbf{u}_\varepsilon\|^3_{L^2(\tilde{I},(L^6(\tilde{B}))}+\|\sqrt{\varepsilon}\nabla\varrho_\varepsilon\|^3_{L^2(\tilde{Q})}+\|\sqrt{\varepsilon}\varrho_\varepsilon\|^3_{L^2(\tilde{I},(L^6(\tilde{B}))}\\
&\leqq\sqrt{\varepsilon}\Big(\|\tilde{\varrho}_\varepsilon\|^3_{L^\infty(\tilde{I};L^3(\tilde{B}))}+\|\nabla\mathbf{u}_\varepsilon\|^3_{L^2(\tilde{Q})}+\|\sqrt{\varepsilon}\nabla\varrho_\varepsilon\|^3_{L^2(\tilde{Q})}+\|\sqrt{\varepsilon}\varrho_\varepsilon\|^3_{L^2(\tilde{I},(L^6(\tilde{B}))}\Big),\\
&\leqq C\sqrt{\varepsilon}
\end{aligned}$$

using, additionally, the fact that $\sqrt{\varepsilon}\nabla\varrho_\varepsilon$, $\nabla\mathbf{u}_\varepsilon$ are uniformly bounded in L^2, cf. (6.8) and (6.10). Similarly, we find $|E_1|\leqq C\sqrt{\varepsilon}$ as well. Hence, we have $E_1,E_2\to 0$. All other couples converge to 0 (by the known weak and strong convergences we have) except for the last couple on the right hand side of (6.33). The crucial point is to estimate the commutator term. We will prove that $\psi\varrho_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon\mathbf{u}_\varepsilon]-\psi\varrho_\varepsilon\mathbf{u}_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon]$ converges strongly in $L^2(W^{-1,2})$. Then the crucial term converges, since $\psi\mathbf{u}$ converges weakly to 0 in $L^2(W^{1,2})$. For the identification of the limit we make use of the div-curl lemma. From (6.9) and (6.13) we obtain that

$$\begin{aligned}
\varrho_\varepsilon &\rightharpoonup \varrho \quad\text{in}\quad L^\beta(\mathbb{R}^3)\quad\text{a.e. in } I,\\
\varrho_\varepsilon\mathbf{u}_\varepsilon &\rightharpoonup \varrho\mathbf{u}\quad\text{in}\quad L^{\frac{2\beta}{\beta+1}}(\mathbb{R}^3)\quad\text{a.e. in } I.
\end{aligned}$$

Hence we can apply [21, Lemma 3.4] (to the sequences $\psi\varrho_\varepsilon$ and $\psi\varrho_\varepsilon u^j_\varepsilon$) to conclude that

$$\begin{aligned}
&\psi\varrho_\varepsilon\mathcal{R}_{ij}[\psi\varrho_\varepsilon u^j_\varepsilon]-\psi\varrho_\varepsilon u^j_\varepsilon\mathcal{R}_{ij}[\psi\varrho_\varepsilon]\\
&\quad\rightharpoonup\psi\varrho\mathcal{R}_{ij}[\psi\varrho u^j]-\psi\varrho u^j\mathcal{R}_{ij}[\psi\varrho]\quad\text{in}\quad L^r(\mathbb{R}^3)
\end{aligned}$$

a.e. in t, where

$$\frac{1}{r}=\frac{1}{\beta}+\frac{\beta+1}{2\beta}<\frac{6}{5}.$$

Therefore $L^r(\mathcal{O})$ is compactly embedded into $W^{-1,2}(\mathcal{O})$ for $\mathcal{O}\Subset\mathbb{R}^3$. As a consequence we have

$$\begin{aligned}
&\psi\varrho_\varepsilon\mathcal{R}_{ij}[\psi\varrho_\varepsilon u^j_\varepsilon]-\psi\varrho_\varepsilon u^j_\varepsilon\mathcal{R}_{ij}[\psi\varrho_\varepsilon)\rightharpoonup\psi\varrho\mathcal{R}_{ij}[\psi\varrho u^j]\\
&\quad-\psi\varrho u^j\mathcal{R}_{ij}[\psi\varrho]\quad\text{in}\quad W^{-1,2}(\mathbb{R}^3)
\end{aligned}$$

a.e. in t using the compact support of the involved functions. Moreover, it is possible to show that for some $p>2$,

$$\begin{aligned}
&\int_I\big\|\psi\varrho_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon\mathbf{u}_\varepsilon]-\psi\varrho_\varepsilon\mathbf{u}_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon]\big\|^p_{W^{-1,2}(\mathbb{R}^3)}\,\mathrm{d}t\\
&\quad\leqq C\int_{\tilde{I}}\|\varrho_\varepsilon\|^{2pr}_{L^{\beta+1}(\tilde{B})}\mathrm{d}t+C\sup_{t\in\tilde{I}}\|\varrho_\varepsilon\mathbf{u}_\varepsilon\|^{2pr}_{L^{\frac{2\beta}{\beta+1}}(\tilde{B})}\,\mathrm{d}t\leqq C,
\end{aligned}$$

using (6.13) and (6.15) together with $\beta \geqq 4$. This gives the desired convergence

$$\psi\varrho_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon\mathbf{u}_\varepsilon] - \psi\varrho_\varepsilon\mathbf{u}_\varepsilon\mathcal{R}[\psi\varrho_\varepsilon] \to \psi\varrho\mathcal{R}[\psi\varrho\mathbf{u}] \\ -\psi\varrho\mathbf{u}\mathcal{R}[\psi\varrho] \quad \text{in} \quad L^2(I; W^{-1,2}(\mathbb{R}^3))$$

as $\varepsilon \to 0$. Thus, we conclude that

$$\begin{aligned} &\int_{I\times\mathbb{R}^3} \psi\varrho_\varepsilon\, u_\varepsilon^i \big(\mathcal{R}_{ij}[\psi\varrho_\varepsilon u_\varepsilon^j] - \psi\varrho_\varepsilon u_\varepsilon^j \mathcal{R}_{ij}[\psi\varrho_\varepsilon]\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \to \int_{I\times\mathbb{R}^3} \psi\varrho\, u^i \big(\mathcal{R}_{ij}[\psi\varrho u^j] - \psi\varrho u^j \mathcal{R}_{ij}[\psi\varrho]\big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{6.34}$$

and, accordingly,

$$\begin{aligned} &\int_{I\times\mathbb{R}^3} \psi^2\big(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\varepsilon\big)\,\varrho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \longrightarrow \int_{I\times\mathbb{R}^3} \psi^2\big(\overline{p} - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}\big)\,\varrho\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{6.35}$$

as $\varepsilon \to 0$.

6.3. Renormalized Solutions

The aim of this subsection is to prove Lemma 6.2. Similarly to Lemma 3.1 b) the proof is based on mollification and Lions' commutator estimate. Due to (6.7), (6.9) and (6.13) it is easy to pass to the limit in (5.2). Hence, we obtain

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta} \varrho\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\Omega_\eta} \big(\varrho\,\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$. We extend ϱ by zero to $I\times\mathbb{R}^3$ and $\mathbf{u}$ by means of the extension operator

$$\mathscr{E}_\eta : W^{1,2}(\Omega_\eta) \to W^{1,p}(\mathbb{R}^3),$$

constructed in Lemma 2.5 where $1 < p < 2$ (but may be chosen close to 2). Hence, we find that

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \varrho\,\psi\,\mathrm{d}x\,\mathrm{d}t - \int_I\int_{\mathbb{R}^3} \big(\varrho\,\partial_t\psi + \varrho\mathscr{E}_\eta\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\psi \in C^\infty(\overline{I}\times\mathbb{R}^3)$. Now, analogous to the proof in Theorem 3.1 we mollify the equation in space using a standard convolution with parameter $\kappa > 0$ in space. The following holds:

$$\partial_t\varrho_\kappa + \mathrm{div}\,\big(\varrho_\kappa\mathscr{E}_\eta\mathbf{u}\big) = \mathbf{r}_\kappa \quad \text{in} \quad I\times\mathbb{R}^3, \tag{6.36}$$

where $\mathbf{r}_\kappa = \mathrm{div}(\varrho_\kappa\mathscr{E}_\eta\mathbf{u}) - \mathrm{div}(\varrho\mathscr{E}_\eta\mathbf{u})_\kappa$. Due to $\beta > 2$ we can infer from the commutator lemma (see e.g. [36, Lemma 2.3]) that for a.e. t

$$\|\mathbf{r}_\kappa\|_{L^q(\mathbb{R}^3)} \leqq \|\mathscr{E}_\eta\mathbf{u}\|_{W^{1,p}(\mathbb{R}^3)}\|\varrho\|_{L^\beta(\mathbb{R}^3)}, \quad \tfrac{1}{q} = \tfrac{1}{p} + \tfrac{1}{\beta},$$

as well as

$$\mathbf{r}_\kappa \to 0 \quad \text{in} \quad L^q(\mathbb{R}^3) \tag{6.37}$$

a.e. in I. Now, we multiply (6.36) by $\theta'(\varrho_\kappa)$ where θ, satisfies $\theta(0) = 0$ and obtain

$$\partial_t \theta(\varrho_\kappa) + \operatorname{div}\big(\theta(\varrho_\kappa)\mathscr{E}_\eta \mathbf{w}\big) + \big(\varrho_\kappa \theta'(\varrho_\kappa) - \theta(\varrho_\kappa)\big) \operatorname{div} \mathscr{E}_\eta \mathbf{u} = \mathbf{r}_\kappa \theta'(\varrho_\kappa). \tag{6.38}$$

Due to the properties of the mollification and $\theta \in C^1$ the terms $\theta(\varrho_\kappa)$ and $\theta'(\varrho_\kappa)$ converge to the correct limits (at least after taking a subsequence). Hence, multiplying (6.38) by $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and integrating over $I \times \mathbb{R}^3$ implies

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \theta(\varrho)\, \psi \,\mathrm{d}x\,\mathrm{d}t - \int_{I\times\mathbb{R}^3} \theta(\varrho)\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{I\times\mathbb{R}^3} \big(\varrho\theta'(\varrho) - \theta(\varrho)\big) \operatorname{div} \mathscr{E}_\eta \mathbf{u}\, \psi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{I\times\mathbb{R}^3} \theta(\varrho) \mathscr{E}_\eta \mathbf{u} \cdot \nabla \psi.
\end{aligned} \tag{6.39}$$

6.4. Strong Convergence of the Density

In order to deal with the local nature of (6.35) we use ideas from [18]. First of all, by the monotonicity of the mapping $z \mapsto az^\gamma + \delta z^\beta$, we find for arbitrary non-negative $\psi \in C_0^\infty(\Omega_{\varepsilon_0})$ that

$$\begin{aligned}
&(\lambda + 2\mu) \liminf_{\varepsilon\to 0} \int_{I\times\mathbb{R}^3} \psi\big(\operatorname{div} \mathbf{u}_\varepsilon\, \varrho_\varepsilon - \operatorname{div} \mathbf{u}\, \varrho\big) \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{\varepsilon\to 0} \int_{I\times\Omega_{\eta_\varepsilon}} \Big(\psi\big(\overline{p} - (\lambda+2\mu) \operatorname{div}\mathbf{u}\big)\varrho \\
&\quad - \psi\big(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta - (\lambda+2\mu)\operatorname{div}\mathbf{u}_\varepsilon\big)\,\varrho_\varepsilon\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \liminf_{\varepsilon\to 0} \int_{I\times\Omega_{\eta_\varepsilon}} \psi\big(a\varrho_\varepsilon^{\gamma+1} + \delta\varrho_\varepsilon^{\beta+1} - \overline{p}\varrho\big) \,\mathrm{d}x\,\mathrm{d}t \\
&= \liminf_{\varepsilon\to 0} \int_{I\times\Omega_{\eta_\varepsilon}} \psi\big(a\varrho_\varepsilon^\gamma + \delta\varrho_\varepsilon^\beta - \overline{p}\big)\big(\varrho_\varepsilon - \varrho\big) \,\mathrm{d}x\,\mathrm{d}t \geqq 0,
\end{aligned}$$

using (6.35). As ψ is arbitrary we conclude

$$\overline{\operatorname{div}\mathbf{u}\,\varrho} \geqq \operatorname{div}\mathbf{u}\,\varrho \quad \text{a.e. in} \quad I \times \Omega_\eta, \tag{6.40}$$

where

$$\operatorname{div}\mathbf{u}_\varepsilon\, \varrho_\varepsilon \rightharpoonup^\eta \overline{\operatorname{div}\mathbf{u}\,\varrho} \quad \text{in} \quad L^1(\Omega; L^1(\Omega_{\eta_\varepsilon}));$$

recall (6.8) and (6.9). Now, we compute both sides of (6.40) by means of the corresponding continuity equations. Due to Lemma 5.2 with $\theta(z) = z\ln z$ and $\psi = \chi_{[0,t]}$ we have

$$\int_0^t \int_{\mathbb{R}^3} \operatorname{div}\mathbf{u}_\varepsilon\, \varrho_\varepsilon \,\mathrm{d}x\,\mathrm{d}\sigma \leqq \int_{\mathbb{R}^3} \varrho_0 \ln(\varrho_0)\,\mathrm{d}x - \int_{\mathbb{R}^3} \varrho_\varepsilon(t) \ln(\varrho_\varepsilon(t))\,\mathrm{d}x. \tag{6.41}$$

Similarly, equation (6.39) yields

$$\int_0^t \int_{\mathbb{R}^3} \operatorname{div} \mathbf{u}\, \varrho \,\mathrm{d}x \,\mathrm{d}\sigma = \int_{\mathbb{R}^3} \varrho_0 \ln(\varrho_0) \,\mathrm{d}x - \int_{\mathbb{R}^3} \varrho(t) \ln(\varrho(t)) \,\mathrm{d}x. \tag{6.42}$$

Combining (6.40)–(6.42) shows

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^3} \varrho_\varepsilon(t) \ln(\varrho_\varepsilon(t)) \,\mathrm{d}x \leqq \int_{\mathbb{R}^3} \varrho(t) \ln(\varrho(t)) \,\mathrm{d}x$$

for any $t \in I$. This gives the claimed convergence $\varrho_\varepsilon \to \varrho$ in $L^1(I \times \mathbb{R}^3)$ by convexity of $z \mapsto z \ln z$. Consequently, we have $\tilde{p} = a\varrho^\gamma + \delta\varrho^\beta$ and the proof of Theorem 6.1 is complete.

7. The Vanishing Artificial Pressure Limit

A weak solution to (1.2)–(1.9) is a triple $(\eta, \mathbf{u}, \varrho) \in \times Y^I \times X_\eta^I \times W_\eta^I$, where

$$W_\eta^I = C_w(\overline{I}; L^\gamma(\Omega_\eta)),$$

which satisfies the following:

(O1) The momentum equation is satisfied in the sense that

$$\begin{aligned}
&\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho\mathbf{u}\cdot\boldsymbol{\varphi} \,\mathrm{d}x - \int_{\Omega_\eta} \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\varphi}\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_I \int_{\Omega_\eta} \Big(\mu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} + (\lambda+\mu) \operatorname{div}\mathbf{u} \operatorname{div}\boldsymbol{\varphi}\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \int_I \int_{\Omega_\eta} a\varrho^\gamma \operatorname{div}\boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_I \bigg(\frac{\mathrm{d}}{\mathrm{d}t} \int_M \partial_t\eta b \,\mathrm{d}\mathcal{H}^2 - \int_M \partial_t\eta \,\partial_t b \,\mathrm{d}\mathcal{H}^2 + \int_M K'(\eta)\, b \,\mathrm{d}\mathcal{H}^2\bigg) \,\mathrm{d}t \\
&\quad\ = \int_I \int_{\Omega_\eta} \varrho\mathbf{f}\cdot\boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t + \int_I \int_M g\, b \,\mathrm{d}x \,\mathrm{d}t
\end{aligned} \tag{7.1}$$

holds for all $(b, \boldsymbol{\varphi}) \in C_0^\infty(M) \times C^\infty(\overline{I} \times \mathbb{R}^3)$ with $\mathrm{tr}_\eta(\boldsymbol{\varphi}) = b\nu$. Moreover, we have $(\varrho\mathbf{u})(0) = \mathbf{q}_0$, $\eta(0) = \eta_0$ and $\partial_t\eta(0) = \eta_1$.

(O2) The continuity equation is satisfied in the sense that

$$\int_I \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\eta} \varrho\psi \,\mathrm{d}x \,\mathrm{d}t - \int_I \int_{\Omega_\eta} \Big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\Big) \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{7.2}$$

holds for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and we have $\varrho(0) = \varrho_0$.

(O3) The boundary condition $\mathrm{tr}_\eta \mathbf{u} = \partial_t\eta\nu$ holds in the sense of Lemma 2.4.

Theorem 7.1. *Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). There is a weak solution $(\eta, \mathbf{u}, \varrho) \in Y^I \times X^I_\eta \times W^I_\eta$ to (1.2)–(1.9) in the sense of (O1)–(O3). Here, we have $I = (0, T_*)$, with $T_* < T$ only in case $\Omega_\eta(s)$ approaches a self intersection with $s \to T_*$. The solution satisfies the energy estimate*

$$
\begin{aligned}
&\sup_{t\in I}\int_{\Omega_\eta}\varrho|\mathbf{u}|^2\,\mathrm{d}x+\sup_{t\in I}\int_{\Omega_\eta}a\varrho^\gamma\,\mathrm{d}x+\int_I\int_{\Omega_\eta}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sup_{t\in I}\int_M|\partial_t\eta|^2\,\mathrm{d}\mathcal{H}^2+\sup_{t\in I}K(\eta)\\
&\quad\leqq c\bigg(\int_\Omega\frac{|\mathbf{q}_0|^2}{\varrho_0}\,\mathrm{d}x+\int_\Omega a\varrho_0^\gamma\,\mathrm{d}x+\int_I\|\mathbf{f}\|^2_{L^\infty(\Omega_\eta)}\,\mathrm{d}t+\int_I\|g\|_{L^2(M)}\,\mathrm{d}t\bigg)\\
&\qquad+c\bigg(\int_M|\eta_0|^2\,\mathrm{d}\mathcal{H}^2+\int_M|\eta_1|^2\,\mathrm{d}\mathcal{H}^2+K(\eta_0)\bigg),
\end{aligned}
$$

provided that $\eta_0, \eta_1, \varrho_0, \mathbf{q}_0, \mathbf{f}$ and g are regular enough to give sense to the right-hand side, that $\varrho_0 \geqq 0$ a.e. and (1.10) is satisfied.

Lemma 7.2. *Under the assumptions of Theorem 7.1, the continuity equation holds in the renormalized sense that is*

$$
\begin{aligned}
&\int_I\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\eta}\theta(\varrho)\psi\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\Big(\theta(\varrho)\partial_t\psi+\theta(\varrho)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad=-\int_I\int_{\Omega_\eta}(\varrho\theta'(\varrho)-\theta(\varrho))\operatorname{div}\mathbf{u}\,\psi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{7.3}
$$

for all $\psi \in C^\infty(\overline{I} \times \mathbb{R}^3)$ and all $\theta \in C^1(\mathbb{R})$ with $\theta(0) = 0$ and $\theta'(z) = 0$ for $z \geqq M_\theta$.

For a given δ we gain a weak solutions $(\eta_\delta, \mathbf{u}_\delta, \varrho_\delta)$ to (6.1)–(6.2) by Theorem 6.1. It is defined in the interval $[0, T_*]$, where T_* is restricted by the data only. The estimate from Theorem 6.1 holds uniformly with respect to δ. Hence we may take a subsequence, such that for some $\alpha \in (0, 1)$ we have

$$\eta_\delta \rightharpoonup^* \eta \quad \text{in} \quad L^\infty(I; W^{2,2}_0(M)) \tag{7.4}$$

$$\eta_\delta \rightharpoonup^* \eta \quad \text{in} \quad W^{1,\infty}(I; L^2(M)), \tag{7.5}$$

$$\eta_\delta \to \eta \quad \text{in} \quad C^\alpha(\overline{I} \times M), \tag{7.6}$$

$$\mathbf{u}_\delta \rightharpoonup^\eta \mathbf{u} \quad \text{in} \quad L^2(I; W^{1,2}(\Omega_{\eta_\delta})), \tag{7.7}$$

$$\varrho_\delta \rightharpoonup^{*,\eta} \varrho \quad \text{in} \quad L^\infty(I; L^\gamma(\Omega_{\eta_\delta})). \tag{7.8}$$

By Lemma 2.8 we find for all $q \in (1, \frac{6\gamma}{\gamma+6})$ that

$$\varrho_\delta\mathbf{u}_\delta \rightharpoonup^\eta \varrho\mathbf{u} \quad \text{in} \quad L^2(I, L^q(\Omega_{\eta_\delta})) \tag{7.9}$$

$$\varrho_\delta\mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup^\eta \varrho\mathbf{u} \otimes \mathbf{u} \quad \text{in} \quad L^1(I; L^1(\Omega_{\eta_\delta})). \tag{7.10}$$

Also we have, as before in Proposition 6.3, higher integrability of the density.

Lemma 7.3. *Let $\gamma > \frac{3}{2}$ ($\gamma > 1$ in two dimensions). Let $Q = J \times B \Subset I \times \Omega_\eta$ be a parabolic cube and $0 < \Theta \leqq \frac{2}{3}\gamma - 1$. The following holds for any $\delta \leqq \delta_0(Q)$:*

$$\int_Q \left(a\varrho_\delta^{\gamma+\Theta} + \delta\varrho_\delta^{\beta+\Theta}\right) \mathrm{d}x\, \mathrm{d}t \leqq C(Q) \tag{7.11}$$

with constant independent of δ.

Proof. The proof follows the lines of Lemma 6.3 with the difference that we test with $\psi\nabla\Delta_{\tilde{B}}^{-1}\varrho_\delta^\Theta$. We only show how to handle the most critical integral

$$J = \int_Q \psi\varrho_\delta\mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla^2\Delta_{\tilde{B}}^{-1}\varrho_\delta^\Theta\, \mathrm{d}x\, \mathrm{d}t$$

arising from the convective term. The bound $\Theta \leqq \frac{2}{3}\gamma - 1$ is needed to estimate it. It can be estimated using the continuity of $\nabla^2\Delta_{\tilde{B}}^{-1}$ and Hölder's inequality by

$$|J| \leqq c\int_0^T \|\varrho_\delta\|_\gamma\|\mathbf{u}_\delta\|_6^2\|\varrho_\delta^\theta\|_r\, \mathrm{d}t,$$

where $r := \frac{3\gamma}{2\gamma-3}$. We proceed, using Sobolev's inequality (note that $\mathbf{u}_\delta = 0$ on Γ), by

$$|J| \leqq C\Big(\sup_{0\leqq t\leqq T}\|\varrho_\delta\|_\gamma\Big)\Big(\sup_{0\leqq t\leqq T}\|\varrho_\delta^\Theta\|_r\Big)\int_0^T \|\nabla\mathbf{u}_\delta\|_2^2\, \mathrm{d}t.$$

We need to choose r such that $\Theta r \leqq \gamma$ which is equivalent to $\Theta \leqq \frac{2}{3}\gamma - 1$. Now, the various a-priori bounds yield $|J| \leqq c$ uniformly in δ. $\square$

In a fashion similar to Lemma 6.4 we can exclude concentrations of the pressure at the moving boundary. However, we have to assume $\gamma > \frac{12}{7}$ for this.

Lemma 7.4. *Let $\gamma > \frac{12}{7}$ ($\gamma > 1$ in two dimensions). Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I \times \Omega_\eta$ such that we have for, all $\delta \leqq \delta_0$,*

$$\int_{I\times\mathbb{R}^3\setminus A_\kappa} \left(a\varrho_\delta^\gamma + \delta\varrho_\delta^\beta\right)\chi_{\Omega_{\eta_\delta}}\mathrm{d}x\, \mathrm{d}t \leqq \kappa. \tag{7.12}$$

Proof. We follow the approach of Proposition 6.4 replacing ε by δ, so we test with

$$\boldsymbol{\varphi}_\delta(t,x) = \varphi\min\{K(s(x) - \eta_\delta(t,q(x))), 1\}\nu(q(x)).$$

The critical term is again

$$\int_I\int_{\Omega_{\eta_\delta}} \varrho_\delta\mathbf{u}_\delta\,\partial_t\boldsymbol{\varphi}_\delta\, \mathrm{d}x\, \mathrm{d}t. \tag{7.13}$$

Following the proof of Proposition 6.4 this can be estimated provided $\gamma > 3$. We want to improve on this. In order to do so we write

$$\partial_t\boldsymbol{\varphi}_\delta = -K\chi_{\{K(s(x)-\eta_\delta(t,q(x)))\leqq 1\}}\partial_t\eta_\delta(t,q(x))\nu(q(x))$$

$$= -K\chi_{\{K(s(x)-\eta_\delta(t,q(x)))\leqq 1\}}\mathbf{u}_\delta \circ \mathbf{\Psi}_\delta(t,0,q(x)).$$

By Lemma 2.4 and since $\nabla\mathbf{u}_\delta$ is uniformly bounded in L^2 (recall (7.7)) we find that

$$\mathbf{u}_\delta \circ \mathbf{\Psi}_\delta|_{\partial\Omega} \in L^2(I; L^q(\partial\Omega)) \quad \forall q < 4 \tag{7.14}$$

uniformly in δ ($q < \infty$ in two dimensions). In a manner similar to (6.24) we obtain

$$\begin{aligned}
&\bigg(\int_I \bigg(\int_{\Omega_{\eta_\delta}} |\partial_t \boldsymbol{\varphi}_\delta|^r \,\mathrm{d}x\bigg)^{\frac{2}{r}}\bigg)^{\frac{1}{2}} \\
&\leqq cK\bigg(\int_I \bigg(\int_{\Omega_{\eta_\delta}} |\mathbf{u}_\delta \circ \mathbf{\Psi}_\delta(t,0,q(x))|^q \,\mathrm{d}x\bigg)^{\frac{2}{q}} \\
&\quad \big|\{x \in \Omega_{\eta_\delta} : K(s-\eta_\delta(t,q)) \leqq 1\}\big|^{\frac{2(q-r)}{qr}} \,\mathrm{d}t\bigg)^{\frac{1}{2}} \\
&\leqq cK\bigg(\int_I \bigg(\int_{\partial\Omega_{\eta_\delta}} |\mathbf{u}_\delta \circ \mathbf{\Psi}_\delta(t,0,q(x))|^q \,\mathrm{d}\mathcal{H}^2\bigg)^{\frac{2}{q}} \,\mathrm{d}t\bigg)^{\frac{1}{2}} \\
&\quad \sup_I \big|\{x \in \Omega_{\eta_\delta} : K(s-\eta_\delta(t,q)) \leqq 1\}\big|^{\frac{q-r}{qr}} \\
&\leqq c\,K^{1-\frac{q-r}{qr}}
\end{aligned} \tag{7.15}$$

for all $r < q < 4$ (all $r < q < \infty$ in two dimensions) uniformly in δ. Now, the proof can be finished as in Proposition 6.4. We take

$$\varrho_\delta \mathbf{u}_\delta \in L^2\Big(I; L^{\frac{6\gamma}{\gamma+6}}(\mathbb{R}^3)\Big) \tag{7.16}$$

into account (which follows from the uniform a-priori bounds) and $\gamma > \frac{12}{7}$ (which yields $\frac{6\gamma}{\gamma+6} > \frac{4}{3}$). We see that the integral in (7.13) is uniformly bounded by $K^{1-\lambda}$ for some $\lambda \in (0,1)$ using Hölder's inequality and (7.15) (choosing r and q appropriately). $\square$

Lemmas 7.3 and 7.4 imply equi-integrability of the sequence $\varrho_\delta^\gamma \chi_{\Omega_{\eta_\delta}}$. This yields the existence of a function $\overline{p}$ such that (for a subsequence)

$$a\varrho_\delta^\gamma + \delta\varrho_\delta^\beta \rightharpoonup \overline{p} \quad \text{in} \quad L^1(I\times\mathbb{R}^3), \tag{7.17}$$

$$\delta\varrho_\delta^\beta \to 0 \quad \text{in} \quad L^1(I\times\mathbb{R}^3). \tag{7.18}$$

Similarly to Corollary 6.5 we have

Corollary 7.5. *Let $\kappa > 0$ be arbitrary. There is a measurable set $A_\kappa \Subset I\times\Omega_\eta$ such that*

$$\int_{I\times\mathbb{R}^3\setminus A_\kappa} \overline{p}\,\mathrm{d}x\,\mathrm{d}t \leqq \kappa. \tag{7.19}$$

Using (7.17) and the convergences (7.4)–(7.10) we can pass to the limit in (6.1)–(6.28) and obtain

$$\begin{aligned}
&-\int_I\int_{\Omega_\eta}\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_M\varrho\partial_t\eta\,\partial_t\eta\,b\,\gamma(\eta)\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\
&\quad+\int_I\int_{\Omega_\eta}\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mu\int_I\int_{\Omega_\eta}\nabla\mathbf{u}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+(\lambda+\mu)\int_I\int_{\Omega_\eta}\operatorname{div}\mathbf{u}\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\overline{p}\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_I\int_M\partial_t\eta\,\partial_t b\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t+2\int_I\int_M K'(\eta)\,b\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\
&=\int_I\int_{\Omega_\eta}\varrho\mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t+\int_I\int_M g\,b\,\mathrm{d}\mathcal{H}^2\,\mathrm{d}t\\
&\quad+\int_{\Omega_{\eta_0}}\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x+\int_M\eta_0\,b\,\mathrm{d}\mathcal{H}^2
\end{aligned}\tag{7.20}$$

for all test-functions $(b,\boldsymbol{\varphi})$ with $\mathrm{tr}_\eta\boldsymbol{\varphi}=b\nu$, $\boldsymbol{\varphi}(T,\cdot)=0$ and $b(T,\cdot)=0$. Moreover, it holds that

$$\int_I\int_{\Omega_\eta}\varrho\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_I\int_{\Omega_\eta}\operatorname{div}(\varrho\,\mathbf{u})\,\psi\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_{\eta_0}}\varrho_0\,\psi(0,\cdot)\,\mathrm{d}x\tag{7.21}$$

for all $\psi\in C^\infty(\overline{I\times\Omega_\eta})$. It remains to show that $\overline{p}=a\varrho^\gamma$.

7.1. The Effective Viscous Flux

We define the L^∞-truncation

$$T_k(z):=k\,T\Big(\frac{z}{k}\Big)\quad z\in\mathbb{R},\;\;k\in\mathbb{N}.\tag{7.22}$$

Here T is a smooth concave function on $\mathbb{R}$ such that $T(z)=z$ for $z\leqq 1$ and $T(z)=2$ for $z\geqq 3$. It is the aim of this subsection to show that

$$\begin{aligned}
&\int_{I\times\Omega_{\eta_\delta}}\big(a\varrho_\delta^\gamma+\delta\varrho_\delta^\beta-(\lambda+2\mu)\operatorname{div}\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\longrightarrow\int_{I\times\Omega_\eta}\big(\overline{p}-(\lambda+2\mu)\operatorname{div}\mathbf{u}\big)\,T^{1,k}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{7.23}$$

For this step we are able to use the theory established in [37] on a local level. We fix a small δ_0 and consider an arbitrary cube $\tilde{Q}=\tilde{J}\times\tilde{B}\Subset I\times\bigcup_{\delta\in[0,\delta_0]}\Omega_{\eta_\delta}$. To this end, we can choose $\theta=T_k$ in the renormalized continuity equation for $\tilde{\varrho}_\delta$, cf. Lemma 6.2. Hence, we find

$$\partial_t T_k(\varrho_\delta)+\operatorname{div}\big(T_k(\varrho_\delta)\mathbf{u}_\delta\big)+\big(T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\big)\operatorname{div}\mathbf{u}_\delta=0,\tag{7.24}$$

in the sense of distributions in $I\times\mathbb{R}^3$. In order to pass to the limit in this equation, let $T^{1,k}$ denote the weak limit of $T_k(\varrho_\delta)$ and let $T^{2,k}$ denote the weak limit of

$\big(T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)\big)\,\mathrm{div}\,\mathbf{u}_\delta$ (here it might be necessary to pass to a subsequence). To be more precise, the following holds:

$$T_k(\varrho_\delta) \rightharpoonup T^{1,k} \quad \text{in} \quad C_w(\overline{I}; L^p(\mathbb{R}^3)) \quad \forall p \in [1,\infty), \tag{7.25}$$

$$\big(T_k'(\varrho_\delta)\varrho_\delta - T_k(\varrho_\delta)\big)\,\mathrm{div}\,\mathbf{u}_\delta \rightharpoonup T^{2,k} \quad \text{in} \quad L^2(I\times\mathbb{R}^3). \tag{7.26}$$

Thus, letting $\delta \to 0$ in (7.24) yields

$$\partial_t T^{1,k} + \mathrm{div}\,\big(T^{1,k}\mathbf{u}\big) + T^{2,k} = 0 \tag{7.27}$$

in the sense of distributions on $I\times\mathbb{R}^3$. Note that we used

$$T_k(\varrho_\delta)\mathbf{u}_\delta \rightharpoonup T^{1,k}\mathbf{u} \quad \text{in} \quad L^2(I\times\mathbb{R}^3).$$

This, in turn, is a consequence of the convergences

$$\begin{aligned} T_k(\varrho_\delta) \to T^{1,k} \quad &\text{in} \quad L^2(I; W^{-1,2}(\mathbb{R}^3)),\\ \mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad &\text{in} \quad L^2(I; W^{1,2}(\mathbb{R}^3)). \end{aligned} \tag{7.28}$$

We remark that the former one follows from the compactness of the embedding $C_w(\overline{I}; L^p(\mathcal{O})) \hookrightarrow L^2(I; W^{-1,2}(\mathcal{O}))$ for $\mathcal{O} \Subset \mathbb{R}^3$ and (7.25) (with $p > \frac{6}{5}$). Note that $\mathbf{u}_\delta$ is extended to $\mathbb{R}^3$ by means of Lemma 2.5.

Next, we take Q with $Q \Subset \tilde{Q} \Subset I\times\Omega_{\eta_n}\delta$ and a cut off function $\psi \in C_0^\infty(\tilde{Q})$ with $0 \leqq \psi \leqq 1$ and $\psi \equiv 1$ in $Q = J\times B$. Now, we test (6.1) with $\psi\nabla\Delta^{-1}(\psi T^k(\varrho_\delta))$ and (7.20) with $\psi\nabla\Delta^{-1}(\psi T^{1,k})$. Using similar arguments as in Section 6.2 we find that

$$\begin{aligned} &\int_{I\times\mathbb{R}^3} \psi^2\big(a\varrho_\delta^\gamma - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\longrightarrow \int_{I\times\mathbb{R}^3} \psi^2\big(\overline{p} - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}\big)\,T^{1,k}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.29}$$

We have to remove ψ in order to conclude. For some given $\kappa > 0$ we choose a measurable set in accordance to Lemma 7.4 and Corollary 7.5 for δ_0 small enough (using the fact that $\eta_\delta \to \eta$ uniformly, cf. (7.6)). Without loss of generality we can assume that ∂A_κ is regular. Hence we can cover A_κ with parabolic cubes $Q_i = J_i \times B_i$ such that

$$A_\kappa \subset \bigcup_i Q_i \Subset \bigcap_{\delta\in[0,\delta_0]} \big(I\times\Omega_{\eta_\delta}\big).$$

They can be chosen in a way that we find a partition of unity (ψ_i) with respect to the family Q_i such that $\psi_i \in C_0^\infty(Q_i)$ and

$$\sum \psi_i = 1 \text{ in } A_\kappa.$$

In particular, (7.29) holds with $\psi = \psi_i$. We gain

$$\begin{aligned}\int_{I\times\Omega_{\eta_\delta}} &\big(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_{I\times\Omega_{\eta_\delta}} \bigg(\Big(1-\sum_i\psi_i\Big)\varrho_\delta^\gamma + \delta\varrho_\delta^\beta - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\bigg)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\sum_i\int_{Q_i}\psi_i\big(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using (7.7) and (7.12) the first integral on the right-hand side is bounded in terms of κ. Using (7.29) and (7.19) we find that

$$\begin{aligned}\lim_{\delta\to0}\bigg|&\int_{I\times\Omega_{\eta_\delta}}\big(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta - (\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}_\delta\big)\,T_k(\varrho_\delta)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_{I\times\Omega_\eta}\big(\overline{p}(\lambda+2\mu)\,\mathrm{div}\,\mathbf{u}\big)\,T^{1,k}\,\mathrm{d}x\,\mathrm{d}t\bigg|\end{aligned}$$

is bounded bin terms of κ. As κ is arbitrary we finally conclude that (7.23) holds.

7.2. Renormalized Solutions

The aim of this section is to prove Lemma 7.2. In order to do so it suffices to use the continuity equation and (7.23) again on the whole space.

First, we observe that ϱ_δ is renormalized solution to the continuity equation by Lemma 6.2, i.e. we have

$$\partial_t\theta(\varrho_\delta) + \mathrm{div}\,\big(\theta(\varrho_\delta)\mathbf{u}_\delta\big) + \big(\theta'(\varrho_\delta)\varrho_\delta - \theta(\varrho_\delta)\big)\,\mathrm{div}\,\mathbf{u}_\delta = 0 \tag{7.30}$$

in the sense of distributions on $I\times\mathbb{R}^3$. Note that (7.30) holds in particular for $\theta(z) = z$, which implies that the continuity equation can be regarded as a PDE on the whole-space. We are interested in the particular choice $\theta = T_k$, where the cut-off functions T_k are given by (7.22).

We have to show that, similar to (7.30), equation (7.27) actually holds globally. Thus, choosing $\theta = T_k$ in (7.30) letting $\delta\to0$ yields

$$\begin{aligned}0 =&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}T^{1,k}\psi\,\mathrm{d}x - \int_{\mathbb{R}^3}\big(T^{1,k}\partial_t\psi + T^{1,k}\mathbf{u}\cdot\nabla\psi\big)\,\mathrm{d}x\\ &+\int_{\mathbb{R}^3}T^{2,k}\psi\,\mathrm{d}x\end{aligned} \tag{7.31}$$

for all $\psi\in C^\infty(\overline{I}\times\mathbb{R}^3)$. This means that we have

$$\partial_t T^{1,k} + \mathrm{div}\,\big(T^{1,k}\mathbf{u}\big) + T^{2,k} = 0 \tag{7.32}$$

in the sense of distributions on $I \times \mathbb{R}^3$. Note that we extended ϱ by zero to $\mathbb{R}^3$. The next step is to show

$$\limsup_{\delta\to 0} \int_{I\times\mathbb{R}^3} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t \leqq C, \tag{7.33}$$

where C does not depend on k. The proof of (7.33) follows exactly the arguments from the classical setting with fixed boundary (see [21, Lemma 4.4] and [17]) using (7.23) and the uniform bounds on $\mathbf{u}$. We explain the details for the convenience of the reader. First, note that we have

$$\begin{aligned}
&\lim_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \Big((\varrho_\delta^\gamma + \delta\varrho_\delta^\beta) T_k(\varrho_\delta) - \overline{p} T^{1,k} \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \lim_{\delta\to 0} \bigg(\int_I \int_{\mathbb{R}^3} \Big(\varrho_\delta^\gamma + \delta\varrho_\delta^\beta - (\varrho^\gamma + \delta\varrho^\beta) \Big) \big(T_k(\varrho_\delta) - T_k(\varrho) \big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_I \int_{\mathbb{R}^3} \big(\overline{p} - (\varrho^\gamma + \delta\varrho^\beta) \big) \big(T_k(\varrho) - T^{1,k} \big) \,\mathrm{d}x\,\mathrm{d}t \bigg).
\end{aligned}$$

By convexity of $z \mapsto z^\gamma + \delta z^\beta$ we conclude that

$$\begin{aligned}
&\lim_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \big((\varrho_\delta^\gamma + \delta\varrho_\delta^\beta) T_k(\varrho_\delta) - \overline{p} T^{1,k} \big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \geqq \lim_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \big(\varrho_\delta^\gamma - \varrho^\gamma \big) \big(T_k(\varrho_\delta) - T^{1,k} \big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \geqq \int_I \int_{\mathbb{R}^3} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{7.34}$$

Moreover, we have

$$\begin{aligned}
&\limsup_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \big(\operatorname{div} \mathbf{u}_\delta \, T_k(\varrho_\delta) - \operatorname{div} \mathbf{u} \, T^{1,k} \big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \limsup_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \big(T_k(\varrho_\delta) - T^{1,k} \big) \operatorname{div} \mathbf{u}_\delta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqq c \limsup_{\delta\to 0} \|T_k(\varrho_\delta) - T^{1,k}\|_{L^2(I\times\mathbb{R}^3)} \\
&\quad \leqq c \limsup_{\delta\to 0} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^2(I\times\mathbb{R}^3)}.
\end{aligned} \tag{7.35}$$

Now we combine (7.34) end (7.35) with (7.23) to conclude (7.33).

By a standard smoothing procedure we can consider "renormalized solutions" for $T^{1,k}$ and deduce from (7.32) that

$$\begin{aligned}
\partial_t \theta(T^{1,k}) + \operatorname{div} \big(\theta(T^{1,k}) \mathbf{u} \big) + \big(\theta'(T^{1,k}) T^{1,k} - \theta(T^{1,k}) \big) \operatorname{div} \mathbf{u} \\
+ \theta'(T^{1,k}) T^{2,k} = 0
\end{aligned} \tag{7.36}$$

in the sense of distributions $I \times \mathbb{R}^3$. Here, we use that $\theta'(z) = 0$ for $z \geqq M_\theta$. We want to pass to the limit $k \to \infty$. On account of (7.8), we have, for all $p \in (1, \gamma)$,

$$\begin{aligned}\|T^{1,k}-\varrho\|^p_{L^p(I\times\mathbb{R}^3)} &\leqq \liminf_{\delta\to 0}\|T_k(\varrho_\delta)-\varrho_\delta\|^p_{L^p(I\times\mathbb{R}^3)}\\ &\leqq 2^p \liminf_{\delta\to 0}\int_{[|\varrho_\delta|\geqq k]}|\varrho_\delta|^p\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq 2^p k^{p-\gamma}\liminf_{\delta\to 0}\int_{I\times\mathbb{R}^3}|\varrho_\delta|^\gamma\,\mathrm{d}x\,\mathrm{d}t \longrightarrow 0,\quad k\to\infty,\end{aligned}$$

so we have

$$T^{1,k}\to\varrho \quad\text{in}\quad L^p(I\times\mathbb{R}^3) \tag{7.37}$$

as $k\to\infty$. Therefore, we are left to show that

$$\theta'(T^{1,k})T^{2,k}\to 0 \quad\text{in}\quad L^1(I\times\mathbb{R}^3)\text{ with }k\to\infty. \tag{7.38}$$

Recall that θ has to satisfy $\theta'(z)=0$ for all $z\geqq M$ for some $M=M_\theta$. We define

$$Q_{k,M}:=\left\{(t,x)\in I\times\mathbb{R}^3;\ T^{1,k}\leqq M\right\}$$

and gain by weak lower semicontinuety that

$$\begin{aligned}\int_{I\times\mathbb{R}^3}|\theta'(T^{1,k})T^{2,k}|\,\mathrm{d}x\,\mathrm{d}t &\leqq \sup_{z\leqq M}|\theta'(z)|\int_Q \chi_{Q_{k,M}}|T^{2,k}|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq C\liminf_{\delta\to 0}\int_{I\times\mathbb{R}^3}\chi_{Q_{k,M}}\Big|(T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta))\operatorname{div}\mathbf{u}_\delta\Big|\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq C\sup_\delta\|\operatorname{div}\mathbf{u}_\delta\|_{L^2(I\times\mathbb{R}^3)}\liminf_{\delta\to 0}\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|_{L^2(Q_{k,M})}.\end{aligned}$$

It follows from interpolation that

$$\begin{aligned}&\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|^2_{L^2(Q_{k,M})}\\ &\quad\leqq \|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|^\alpha_{L^1(I\times\mathbb{R}^3)}\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|^{(1-\alpha)(\gamma+1)}_{L^{\gamma+1}(Q_{k,M})},\end{aligned} \tag{7.39}$$

where $\alpha=\frac{\gamma-1}{\gamma}$. Moreover, we can show similarly to the proof of (7.37) that

$$\begin{aligned}\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|_{L^1(I\times\mathbb{R}^3)} &\leqq C\,k^{1-\gamma}\sup_\delta\int_{I\times\mathbb{R}^3}|\varrho_\delta|^\gamma\,\mathrm{d}x\,\mathrm{d}t\\ &\longrightarrow 0,\quad k\to\infty.\end{aligned} \tag{7.40}$$

Thus, it is enough to prove

$$\sup_\delta\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|_{L^{\gamma+1}(Q_{k,M})}\leqq C \tag{7.41}$$

independently of k. As $T_k'(z)z\leqq T_k(z)$ it holds by the definition of $Q_{k,M}$ that

$$\begin{aligned}&\|T_k'(\varrho_\delta)\varrho_\delta-T_k(\varrho_\delta)\|_{L^{\gamma+1}(Q_{k,M})}\\ &\quad\leqq 2\Big(\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^{\gamma+1}(I\times\mathbb{R}^3)}+\|T_k(\varrho_\delta)\|_{L^{\gamma+1}(Q_{k,M})}\Big)\\ &\quad\leqq 2\Big(\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^{\gamma+1}(I\times\mathbb{R}^3)}+\|T_k(\varrho_\delta)-T^{1,k}\|_{L^{\gamma+1}(I\times\mathbb{R}^3)}\end{aligned}$$

$$+\|T^{1,k}\|_{L^{\gamma+1}(Q_{k,M})}\Big).$$
$$\leqq 2\Big(\|T_k(\varrho_\delta)-T_k(\varrho)\|_{L^{\gamma+1}(I\times\mathbb{R}^3)}+\|T_k(\varrho_\delta)-T^{1,k}\|_{L^{\gamma+1}(I\times\mathbb{R}^3)}\Big)+CM.$$

Firstly we find that (7.33) and (7.25) imply (7.41). Secondly, (7.39)–(7.41) imply (7.38), so we can pass to the limit in (7.36) and gain

$$\partial_t\theta(\varrho)+\operatorname{div}\big(\theta(\varrho)\mathbf{u}\big)+\big(\theta'(\varrho)\varrho-\theta(\varrho)\big)\operatorname{div}\mathbf{u}=0 \tag{7.42}$$

in the sense of distributions on $I\times\mathbb{R}^3$. The proof of Lemma 7.2 is complete.

7.3. Strong Convergence of the Density

We introduce the functions L_k by

$$L_k(z)=\begin{cases} z\ln z, & 0\leqq z<k\\ z\ln k+z\int_k^z T_k(s)/s^2\,ds, & z\geqq k.\end{cases}$$

We can choose $\theta=L_k$ in (7.42) such that

$$\partial_t L_k(\varrho)+\operatorname{div}\big(L_k(\varrho)\mathbf{u}\big)+T_k(\varrho)\operatorname{div}\mathbf{u}=0 \tag{7.43}$$

in the sense of distributions on $I\times\mathbb{R}^3$. We also have that

$$\partial_t L_k(\varrho_\delta)+\operatorname{div}\big(L_k(\varrho_\delta)\mathbf{u}_\delta\big)+T_k(\varrho_\delta)\operatorname{div}\mathbf{u}_\delta=0$$

in the sense of distributions, cf. Lemma 6.2.

Using the testfunction $\psi\equiv 1$ in both equations implies

$$\begin{aligned}\int_{\mathbb{R}^3}L_k(\varrho_\delta)\,\mathrm{d}x-\int_{\mathbb{R}^3}T_k(\varrho_\delta(0))\varphi(0)\,\mathrm{d}x&\\ -\int_0^t\int_{\mathbb{R}^3}T_k(\varrho_\delta)\operatorname{div}\mathbf{u}_\delta\,\,\mathrm{d}x\,\mathrm{d}\sigma&\leqq 0\end{aligned} \tag{7.44}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}L_k(\varrho)\,\mathrm{d}x-\int_{\mathbb{R}^3}L_k(\varrho(0))\varphi(0)\,\mathrm{d}x&\\ -\int_0^t\int_{\mathbb{R}^3}T_k(\varrho)\operatorname{div}\mathbf{u}\,\,\mathrm{d}x\,\mathrm{d}\sigma&=0.\end{aligned} \tag{7.45}$$

The difference of both equations reads as

$$\begin{aligned}\int_{\mathbb{R}^3}\big(L_k(\varrho_\delta)(t)-L_k(\varrho)(t)\big)\,\mathrm{d}x&\leqq\int_{\mathbb{R}^3}\big(L_k(\varrho_\delta)(0)-L_k(\varrho)(0)\big)\,\mathrm{d}x\\ &\quad+\int_0^t\int_{\mathbb{R}^3}\big(T_k(\varrho)\operatorname{div}\mathbf{u}-T_k(\varrho)\operatorname{div}\mathbf{u}\big)\,\mathrm{d}x\,\mathrm{d}\sigma.\end{aligned}$$

We have the following convergences for all $p\in(1,\gamma)$:

$$L_k(\varrho_\delta)\to L^{1,k}\quad\text{in}\quad C_w(\overline{I};L^p(\mathbb{R}^3)),\quad\delta\to 0,$$

$$\varrho_\delta \ln(\varrho_\delta) \to L^{2,k} \quad \text{in} \quad C_w(\overline{I}; L^p(\mathbb{R}^3)), \quad \delta \to 0,$$

which is a consequence of the fundamental theorem on Young measures (see, for instance, [40, Thm. 4.2.1, Cor. 4.2.19]) and the convergence of ϱ_δ in $C_w(\overline{I}; L^\beta(\mathbb{R}^3))$. The latter one follows from the a-priori information on ϱ from (7.8) in combination with the control of the distributional time derivative of ϱ_δ coming from the continuity equation (considered on the whole-space), so we gain (using also the fact, that $\varrho_\delta(0) = \varrho_0 = \varrho(0)$)

$$\begin{aligned}\int_{\mathbb{R}^3} \big(L^{1,k}(t) - L_k(\varrho)(t)\big)\,\mathrm{d}x &\leqq \limsup_{\delta\to 0} \int_0^t \int_{\mathbb{R}^3} \big(T_k(\varrho)\operatorname{div}\mathbf{u} - T_k(\varrho_\delta)\operatorname{div}\mathbf{u}_\delta\big)\,\mathrm{d}x\,\mathrm{d}\sigma \\ &\leqq \int_0^t \int_{\mathbb{R}^3} \big(T_k(\varrho) - T^{1,k}\big)\operatorname{div}\mathbf{u}\;\mathrm{d}x\,\mathrm{d}\sigma, \end{aligned} \tag{7.46}$$

using (7.23) together with the monotonicity of the pressure. Due to (7.37) the right-hand side tends to zero if $k \to \infty$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} \big(L^{1,k}(t) - L_k(\varrho)(t)\big)\,\mathrm{d}x \leqq 0.$$

Thus, we have shown

$$\lim_{\delta\to 0} \int_I \int_{\mathbb{R}^3} \varrho_\delta \ln \varrho_\delta \,\mathrm{d}x\,\mathrm{d}t \leqq \int_I \int_{\mathbb{R}^3} \varrho \ln \varrho \,\mathrm{d}x\,\mathrm{d}t.$$

By weak lower semi-continuity for convex functionals the converse inequality holds as well. This finally means that

$$\int_I \int_{\mathbb{R}^3} \varrho_\delta \ln \varrho_\delta \,\mathrm{d}x\,\mathrm{d}t \longrightarrow \int_I \int_{\mathbb{R}^3} \varrho \ln \varrho \,\mathrm{d}x\,\mathrm{d}t.$$

Convexity of $z \mapsto z \ln z$ yields strong convergence of ϱ_δ. Hence, due to (7.20), the proof of Theorem 7.1 is shown, for the time interval $[0, T_*]$, with T_* depending on the data only (such that $\|\eta(t)\|_\infty < \frac{L}{2}$ in $(0, T^*)$). In the next section we will show how the interval of existence can be prolongated by a change of coordinates.

7.4. Maximal Interval of Existence

The interval of existence in Theorem 7.1 is restricted by the quantities of the given data, as well as the geometry of $\partial\Omega$. By our assumption on the initial geometry we find that $\Omega_{\eta(T_*)}$ has no self intersections. We define $\eta^* = (\eta(T_*))_\kappa$, where κ is a convolution operator in space. We define $\tilde{\Omega} = \Omega_{\eta^*} \in C^4$. If κ is conveniently small, then also $\tilde{\Omega}$ has no self intersection either. In particular, there exists some $\tilde{L} > 0$ such that on

$$\tilde{S}_{\tilde{L}} := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \partial\Omega_{\eta^*}) \leqq \tilde{L}\},$$

the function (see the beginning of Section 2.2)

$$\tilde{\Lambda} : \partial\tilde{\Omega} \times (-\tilde{L}, \tilde{L}) \to \tilde{S}_{\tilde{L}}, \quad \tilde{\Lambda}(\tilde{q}, \tilde{s}) = \tilde{q} + \tilde{s}\nu(q)$$

is well defined. Here we have $(q, s) = \Lambda^{-1}(\tilde{q})$ and ν is the outer normal of the initial geometry $\partial\Omega$. This implies that for $\tilde{\zeta} : \partial\tilde{\Omega} \to [-\tilde{L}, \tilde{L}]$, we may associate

$$\Omega_{\tilde{\zeta}} := \tilde{\Omega} \setminus \tilde{S}_{\tilde{L}} \cup \{x \in \tilde{S}_{\tilde{L}} :\ \tilde{s}(x) < \tilde{\zeta}(\tilde{q}(x))\}.$$

By the definition of $\tilde{L}$, there exists a diffeomorphism

$$\mathbf{\Psi}_{\tilde{\zeta}} : \tilde{\Omega} \to \Omega_{\tilde{\zeta}},$$

cf. Lemma 2.2. In particular, we may define the function

$$\zeta : \partial\Omega \to \mathbb{R}, \quad q \mapsto \tilde{\zeta}(q + \nu\eta^*(q)) + \eta^*(q)$$

as satisfying

$$\zeta(q) \in [\eta^*(q) - \tilde{L}, \eta^*(q) + \tilde{L}] \text{ for all } q \in \partial\Omega,$$

where $(q, s) = \Lambda^{-1}(\tilde{q})$. This implies that the mapping

$$\mathbf{\Psi}_{\zeta} := \mathbf{\Psi}_{\tilde{\zeta}} \circ \mathbf{\Psi}_{\eta^*} : \Omega \to \Omega_{\zeta} = \Omega_{\tilde{\zeta}}$$

is a well-defined diffeomorphism. The transformation can also be inverted. For any $\eta : \partial\Omega \to \mathbb{R}$ with

$$\eta(q) \in [\eta^*(q) - \tilde{L}, \eta^*(q) + \tilde{L}] \text{ for all } q \in \partial\Omega,$$

we may define

$$\tilde{\eta} : \partial\tilde{\Omega} \to [-\tilde{L}, \tilde{L}], \quad \tilde{q} \mapsto \eta(\tilde{q} - \nu(q)\eta^*(q)) - \eta^*(q).$$

By this construction we have changed the coordinate set Ω_η since

$$\mathbf{\Psi}_{\eta} = \mathbf{\Psi}_{\tilde{\eta}} \circ \mathbf{\Psi}_{\eta^*} : \Omega \to \Omega_{\eta} = \Omega_{\tilde{\eta}}.$$

This transformation can be used to extend the solution. To be precise, we set:

- $\tilde{\eta}_0 = \eta(T_*)$,
- $\tilde{\eta}_1 = \partial_t \eta(T^*)$,
- $\tilde{\varrho}_0 = \varrho(T^*)$,
- $\tilde{\mathbf{q}}_0 = \varrho(T^*)\mathbf{u}(T^*)$.

By the construction above, we can associate to any $\tilde{\zeta} \in C\big([T^*, T^{**}] \times \partial\tilde{\Omega}, [-\frac{\tilde{L}}{2}, \frac{\tilde{L}}{2}]\big)$ a function $\zeta \in C([T^*, T^{**}] \times \partial\Omega)$ such that

$$\eta(q) \in \left[\eta^*(q) - \frac{\tilde{L}}{2}, \eta^*(q) + \frac{\tilde{L}}{2}\right] \text{ for all } q \in \partial\Omega.$$

Moreover, the mappings $\mathbf{\Psi}_{\zeta} : \Omega \to \Omega_{\zeta}$ and $\mathbf{\Psi}_{\mathscr{R}_\kappa \zeta} : \Omega \to \Omega_{\mathscr{R}_\kappa \zeta}$ are both well defined, provided we choose κ small enough. Now, first Theorem 4.4 provides a solution $(\eta, \mathbf{u})$ to any given pair

$$(\zeta, \mathbf{v}) \in C([T^*, T^{**}] \times \partial\Omega, \mathbb{R}) \times L^2([T^*, T^{**}] \times \mathbb{R}^3).$$

Second, we wish to get a fixpoint by applying Theorem 4.2. The only modification is that the fixpoint mapping has to be adjusted slightly. Indeed, the fixed point has to be found in the set

$$D := \Big\{(\tilde{\zeta}, \mathbf{v}) \in C([T^*, T^{**}] \times \partial\tilde{\Omega}) \times L^2([T^*, T^{**}] \times \mathbb{R}^3)) : \\ \tilde{\zeta}(0) = \tilde{\eta}(T^*),\ \|\tilde{\zeta}\|_{L^\infty} \leqq \frac{\tilde{L}}{2},\ \|\mathbf{v}\|_{L^2([T^*, T^{**}]\times\mathbb{R}^3)} \leqq K\Big\}.$$

Here K has to be adjusted to T^{**} in accordance with the proof of Theorem 4.2. Finally, we set $F : D \to \mathfrak{P}(D)$,

$$F : (\tilde{\zeta}, \mathbf{v}) \mapsto \Big\{(\tilde{\eta}, \mathbf{u}) :\ (\eta, \mathbf{u}) \text{ solves (4.8)} \\ \text{with } (\zeta, \mathbf{v}) \text{ and satisfies the energy bounds}\Big\},$$

where $\tilde{\eta}$ is defined via the solution η by $\tilde{\eta} = \eta(\tilde{q} - \nu(q)\eta^*(q)) - \eta^*(q)$ as introduced above. The rest of the argument of Theorem 4.2 does not change, since the L^∞ bounds of η, ζ (which are critical for the fixed point argument) do not change by coordinate transformations. Once the fixed point is established, we may pass to the limit with κ, ε and δ as before. Observe, that in Section 6.3 one has to use the extension operator from Lemma 2.5 with respect to the coordinate transformation $\boldsymbol{\Psi}_{\tilde{\eta}}$ as it satisfies $\|\tilde{\eta}\|_\infty < \frac{\tilde{L}}{2}$ (here we use the fact that $\Omega_{\tilde{\eta}} = \Omega_\eta$ by our construction).

We remark that the solution η and $\Omega_{\mathscr{R}_\kappa \eta}$ are defined via the same reference coordinates $\partial\Omega$. This means it truly extends the solution and we can extend the interval of existence. Finally, the above procedure can be iterated until a self intersection is approached. This finishes the proof of Theorem 7.1.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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