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# NEW $G_2$ -HOLONOMY CONES AND EXOTIC NEARLY KÄHLER STRUCTURES ON $S^6$ AND $S^3 \times S^3$

LORENZO FOSCOLO AND MARK HASKINS

ABSTRACT. There is a rich theory of so-called (strict) nearly Kähler manifolds, almost-Hermitian manifolds generalising the famous almost complex structure on the 6-sphere induced by octonionic multiplication. Nearly Kähler 6-manifolds play a distinguished role both in the general structure theory and also because of their connection with singular spaces with holonomy group the compact exceptional Lie group  $G_2$ : the metric cone over a Riemannian 6-manifold  $M$  has holonomy contained in  $G_2$  if and only if  $M$  is a nearly Kähler 6-manifold.

A central problem in the field has been the absence of any complete inhomogeneous examples. We prove the existence of the first complete inhomogeneous nearly Kähler 6-manifolds by proving the existence of at least one cohomogeneity one nearly Kähler structure on the 6-sphere and on the product of a pair of 3-spheres. We conjecture that these are the only simply connected (inhomogeneous) cohomogeneity one nearly Kähler structures in six dimensions.

## 1. INTRODUCTION

At least since the early 1950s (see Steenrod's 1951 book [45, 41.22]) it has been well known that viewing  $\mathbb{S}^6$  as the unit sphere in  $\text{Im } \mathbb{O}$  endows it with a natural nonintegrable almost complex structure  $J$  defined via octonionic multiplication. Since  $J$  is compatible with the round metric  $g_{\text{rd}}$ , the triple  $(g_{\text{rd}}, J, \omega)$ , where  $\omega(\cdot, \cdot) = g_{\text{rd}}(J\cdot, \cdot)$ , defines an almost-Hermitian structure on  $\mathbb{S}^6$ . Its torsion has very special properties: in particular,  $d\omega$  is the real part of a complex volume form  $\Omega$ . Appropriately normalised, the pair  $(\omega, \Omega)$  defines an  $\text{SU}(3)$ -structure on  $\mathbb{S}^6$  which by construction is invariant under the exceptional compact Lie group  $G_2 \simeq \text{Aut}(\mathbb{O})$ .

Octonionic multiplication also defines a  $G_2$ -invariant 3-form  $\varphi$  on  $\text{Im } \mathbb{O}$  by

$$\varphi(u, v, w) = uv \cdot w.$$

We call this the standard  $G_2$ -structure on  $\mathbb{R}^7$ . Regarding  $\mathbb{R}^7$  as the Riemannian cone over  $(\mathbb{S}^6, g_{\text{rd}})$ ,  $\varphi$  and its Hodge dual are given in terms of  $(\omega, \Omega)$ :

$$(1.1) \quad \varphi = r^2 dr \wedge \omega + r^3 \text{Re } \Omega, \quad * \varphi = -r^3 dr \wedge \text{Im } \Omega + \frac{1}{2} r^4 \omega^2.$$

Conversely, viewing  $\mathbb{S}^6$  as the level set  $r = 1$  in  $\mathbb{R}^7$ , the  $\text{SU}(3)$ -structure  $(\omega, \Omega)$  is recovered from  $\varphi$  and  $*\varphi$  by restriction and contraction by the scaling vector field  $\frac{\partial}{\partial r}$ .

More generally, consider a 7-dimensional Riemannian cone  $C = C(M)$  over a smooth compact manifold  $(M^6, g)$ . Suppose that the holonomy of the cone is contained in  $G_2$ . Then there exists a pair of closed (in fact parallel) differential forms  $\varphi$  and  $*\varphi$ , pointwise equivalent to the model forms on  $\mathbb{R}^7$  and homogeneous with respect to scalings on  $C$ . Just as above, viewing  $M$  as the level set  $r = 1$  in  $C$ , the restriction and contraction by  $\frac{\partial}{\partial r}$  of  $\varphi$  and  $*\varphi$  define an  $\text{SU}(3)$ -structure  $(\omega, \Omega)$  on  $M$  satisfying (1.1). In particular, the closedness of  $\varphi$  and  $*\varphi$  is equivalent to

$$(1.2) \quad \begin{cases} d\omega = 3 \text{Re } \Omega, \\ d \text{Im } \Omega = -2\omega^2. \end{cases}$$

**Definition 1.3.** A *nearly Kähler 6–manifold* is a manifold  $M^6$  endowed with an  $SU(3)$ –structure  $(\omega, \Omega)$  satisfying (1.2). We call (1.2) the *nearly Kähler structure equations*.

There are other possible equivalent definitions of a nearly Kähler 6–manifold. By relating the holonomy reduction of the cone  $C(M)$  to the existence of a parallel spinor instead of a pair of distinguished parallel forms, nearly Kähler 6–manifolds can be characterised as those 6–manifolds admitting a real Killing spinor [34]. Alternatively, one could give a definition in terms of Gray–Hervella torsion classes of almost Hermitian manifolds [33]. Our definition corresponds then to what are usually called *strict* nearly Kähler 6–manifolds, to distinguish them from Kähler manifolds which, having vanishing torsion, belong to every torsion class.

*Remark.* The latter point of view allows one to define nearly Kähler manifolds in every even dimension. See [10, §14.3.2] for some references and an early history of the subject, including early lesser known contributions from the Japanese school before the terminology nearly Kähler had been adopted. Even in this more general context, nearly Kähler 6–manifolds play a distinguished role, see [39, Theorem 1.1].

Since every manifold with holonomy contained in  $G_2$  is Ricci-flat, nearly Kähler 6–manifolds are necessarily Einstein with scalar curvature 30 (in fact, Gray proved this fact before the connection with  $G_2$  holonomy had been noticed [32, Theorem 5.2]). In particular, a complete nearly Kähler 6–manifold is compact with finite fundamental group and its universal cover is also a complete nearly Kähler manifold. In the rest of the paper we will therefore restrict to the case of simply connected nearly Kähler 6–manifolds.

Besides the  $G_2$ –invariant nearly Kähler structure on the 6–sphere  $S^6 = G_2/SU(3)$ , there are only three known examples of complete simply connected nearly Kähler 6–manifolds, all of which are homogeneous:  $S^3 \times S^3 = SU(2)^3/\Delta SU(2)$ ,  $CP^3 = Sp(2)/U(1) \times Sp(1)$  and the flag manifold  $F_3 = SU(3)/T^2$ . They were first constructed in 1968 by Gray and Wolf [48] in their work on 3–symmetric spaces (which also yielded many homogeneous nearly Kähler manifolds in higher dimensions). Recently Cortés–Vásquez [24] have constructed and partially classified locally homogeneous nearly Kähler 6–manifolds by considering finite quotients of these homogeneous nearly Kähler structures.

Although as early as 1958 Calabi [16] exploited the fact that any oriented hypersurface of  $\mathbb{R}^7$  admits an  $SU(3)$ –structure thanks to the inclusion  $SU(3) \subset G_2$ , the connection between nearly Kähler geometry in 6 dimensions and  $G_2$  holonomy that we have emphasised seems to have been noticed only in the 1980s; the homogeneous nearly Kähler 6–manifolds described above then played an important role in the early development of metrics with holonomy  $G_2$ . The first explicit example of a local metric with full holonomy  $G_2$  given by Bryant in [12, Section 5] is the  $G_2$ –cone over the flag manifold  $F_3$ ; this appears to be the first appearance of the nearly Kähler equations (1.2). Furthermore, the first examples of complete  $G_2$ –metrics constructed by Bryant–Salamon [14] are asymptotically conical manifolds modelled at infinity on the cones over nearly Kähler  $S^3 \times S^3$ ,  $CP^3$  and  $F_3$ . In each case, Bryant–Salamon construct a 1–parameter family of (cohomogeneity one) complete metrics with holonomy  $G_2$  on the total space of a vector bundle over  $S^3$ ,  $S^4$  and  $CP^2$ , respectively. The parameter measures the size of the zero section and as it goes to zero the manifold develops an isolated singularity and converges in the Gromov–Hausdorff sense to the corresponding  $G_2$ –cone.

The behaviour of these solutions exemplifies the reason for the interest in nearly Kähler 6–manifolds from the point of view of  $G_2$  geometry: Riemannian cones over 6–dimensional nearly Kähler manifolds and complete metrics with holonomy  $G_2$  asymptotic to them provide local models for the simplest type of singularities and for singularity formation in families of smooth  $G_2$ –manifolds.

The obvious natural question now is whether there are any other nearly Kähler 6-manifolds. Thanks to the recent work of Butruille [15], no such manifold can be homogeneous. The local existence of nearly Kähler structures in dimension 6 was studied by Reyes Carrion [42, §4.5] and later by Bryant [13], in both cases using Cartan–Kähler theory; the conclusion is that nearly Kähler structures have the same local generality as Calabi–Yau structures and, in particular, there are many local inhomogeneous nearly Kähler structures. The outstanding question is therefore to find new complete examples.

An apparently promising source of nearly Kähler manifolds of any dimension is that of twistor spaces of quaternionic Kähler manifolds with positive scalar curvature [43]. In 6 dimensions the connection with nearly Kähler geometry was first noticed by Eells–Salamon [25] (later generalised to higher dimensions in [3]). Unfortunately, while this construction provides many incomplete examples, *eg* starting from quaternionic Kähler orbifolds, so far the only complete examples it has yielded are homogeneous. By a result of Hitchin [35] the only quaternionic Kähler (equivalently, self-dual Einstein) 4-manifolds with positive scalar curvature are  $S^4$  and  $CP^2$  endowed with their standard metrics, in which case the nearly Kähler structures on the twistor spaces coincide with the homogeneous ones on  $CP^3$  and  $F_3$ , respectively. In higher dimensions the only known quaternionic Kähler manifolds with positive scalar curvature are Wolf’s quaternionic symmetric spaces [47]; an influential conjecture of LeBrun–Salamon [37], known to hold up to dimension 8, asserts the non-existence of inhomogeneous examples.

On the other hand, the scarcity of nearly Kähler 6-manifolds, or equivalently of  $G_2$ -holonomy cones, is surprising when compared to geometries related to other special holonomy groups: there are infinitely many Calabi–Yau cones [44] and infinitely many hyperkähler and  $Spin(7)$ -cones [10, §13.7, §14.3]. Indeed, in this paper we show that complete nearly Kähler 6-manifolds need not be homogeneous.

**Main Theorem.** *There exists an inhomogeneous nearly Kähler structure on  $S^6$  and  $S^3 \times S^3$ .*

There are two very natural approaches to attempt to construct nearly Kähler 6-manifolds. On the one hand, from the perspective of symmetries, the next step beyond the homogeneous setting would be to consider cohomogeneity one nearly Kähler 6-manifolds, *ie* those that admit an isometric action by a compact Lie group whose generic orbit is of codimension one. A completely different starting point would be to exploit the existence of a large number of 6-dimensional *singular* nearly Kähler spaces and to attempt to produce smooth nearly Kähler manifolds from these by singular perturbation methods. As we will explain, both points of view will play an important role in the proof of the Main Theorem.

Apart from orbifolds, the simplest singular Einstein spaces with positive scalar curvature are those obtained by the *sine-cone* construction (also called spherical suspension): given a smooth compact Einstein manifold  $(N, g_N)$  with positive scalar curvature appropriately normalised, the sine-cone  $SC(N)$  over  $N$  is the product  $[0, \pi] \times N$  endowed with the metric  $g^{sc} = dr^2 + \sin^2 r g_N$ . This has two isolated singularities at  $r = 0$  and  $\pi$  modelled on the cone over  $N$ . The sine-cone construction has a very simple geometric interpretation in terms of cones: the cone over  $SC(N)$  is Ricci-flat and is isometric to the product of the Ricci-flat metric cone  $C(N)$  over  $N$  and the real line. Specialising to seven dimensions, the further requirement that the holonomy of the cone over  $SC(N)$  be contained in  $G_2$  forces  $C(N)$  to be a 6-dimensional Calabi–Yau cone. The induced geometric structure on the cross-section  $N$ , analogous to the nearly Kähler condition for a  $G_2$ -cone, is called a Sasaki–Einstein structure. Hence the existence of infinitely many Sasaki–Einstein 5-manifolds, see for example [44], yields infinitely many 6-dimensional nearly Kähler sine-cones.

Not every nearly Kähler sine-cone is a good starting point for constructing smooth nearly Kähler 6-manifolds via the desingularisation method. From the viewpoint of singular perturbation methods it is natural to consider only Calabi–Yau cones that admit asymptotically conical Calabi–Yau desingularisations. Many such examples are now known [19]. The simplest such cone is the conifold, which is well known to admit two Calabi–Yau desingularisations: the Candelas–de la Ossa structure on the small resolution [17] and the structure on the smoothing due to Candelas–de la Ossa and Stenzel [17, 46]. The Calabi–Yau structures on the conifold itself and on both of its desingularisations are cohomogeneity one. The group acting is  $SU(2) \times SU(2)$  and the generic (principal) orbit is diffeomorphic to  $SU(2) \times SU(2)/\Delta U(1) \simeq S^2 \times S^3$  in all three cases. The singularity of the conifold is replaced by a round totally geodesic holomorphic  $S^2$  in the small resolution and by a round totally geodesic special Lagrangian  $S^3$  in the smoothing. In both cases these totally geodesic spheres are the unique lower-dimensional (singular) orbits.

The cross-section of the conifold is  $S^2 \times S^3$  endowed with its homogeneous Sasaki–Einstein structure. The sine-cone over it is a cohomogeneity one nearly Kähler space with two isolated singularities modelled on the conifold. One could try to construct an approximate solution to (1.2) on manifolds obtained by gluing rescaled copies of either desingularisation of the conifold into neighbourhoods of each singular point. Since both the singular background and the “bubbles” we glue in are of cohomogeneity one, this raises the question of whether complete cohomogeneity one nearly Kähler structures exist on such manifolds.

Podestà and Spiro initiated the study of cohomogeneity one nearly Kähler 6-manifolds. In [40] they classified the possible group actions, principal and singular orbits and diffeomorphism-types of complete simply connected cohomogeneity one nearly Kähler 6-manifolds. In fact, the only case of possible interest is exactly the case described above: the principal orbit type is  $SU(2) \times SU(2)/\Delta U(1)$ ; there are two singular orbits, which are spheres of either 2 or 3 dimensions; the compact 6-manifold is obtained by identifying neighbourhoods of the two singular orbits along their boundary  $S^2 \times S^3$ ; these neighbourhoods are  $SU(2) \times SU(2)$ -equivariantly diffeomorphic to the small resolution or to the smoothing of the conifold; the four resulting manifolds are  $S^6$ ,  $S^3 \times S^3$ ,  $CP^3$  and  $S^2 \times S^4$ . In a second paper [41] they studied this case in more detail, but were unable to establish the existence of new complete nearly Kähler structures.

The nearly Kähler structures on  $S^6$  and  $S^3 \times S^3$  that we construct in the Main Theorem are of cohomogeneity one and in fact we conjecture these are the unique (inhomogeneous) complete cohomogeneity one nearly Kähler 6-manifolds. In particular, we conjecture that  $S^2 \times S^4$  carries no cohomogeneity one nearly Kähler structure and  $CP^3$  only its homogeneous one.

In the rest of the Introduction we give the plan of the paper, which serves at the same time as a detailed outline of the proof of the Main Theorem. The techniques we use in the paper are cohomogeneity one methods. Unlike the construction of, say, cohomogeneity one Sasaki–Einstein metrics [21, 30] we do not find explicit closed-form expressions for our new nearly Kähler structures and in this sense the theorem is an abstract existence result (see however the final section of the paper). Instead the paper is more in the spirit of Böhm [8, 9]. The desingularisation intuition, however, provides crucial geometric insight when applying the cohomogeneity one methods, in particular in the consideration of certain geometrically-motivated singular limits and rescalings.

**Plan of the paper.** In the same way that an oriented hypersurface of a 7-manifold with a  $G_2$ -structure admits an induced  $SU(3)$ -structure, any oriented hypersurface of a 6-manifold endowed with an  $SU(3)$ -structure  $(\omega, \Omega)$  comes naturally equipped with an  $SU(2)$ -structure. When  $(\omega, \Omega)$  satisfies differential equations such as (1.2), so does the induced  $SU(2)$ -structure.

The  $SU(2)$ -structures induced on oriented hypersurfaces of Calabi–Yau and nearly Kähler 6-manifolds are called *hypo* and *nearly hypo* structures, respectively. In the spirit of Hitchin [36], away from the singular orbits we regard a cohomogeneity one nearly Kähler (Calabi–Yau) 6-manifold as a curve in the space of invariant nearly hypo (hypo) structures on the principal orbit. This curve must satisfy a system of first order ODEs.

In Section 2 we parametrise the space of nearly hypo structures on  $S^2 \times S^3$  invariant under  $SU(2) \times SU(2)$ , showing that it is a smooth connected 4-manifold. In Section 3 we derive the ODE system (3.10) whose solutions are cohomogeneity one nearly Kähler structures. We note the existence of continuous and discrete symmetries of this system; the latter will play an important role in the proof of the Main Theorem. Up to the action of these symmetries, we find a 2-parameter family of  $SU(2) \times SU(2)$ -invariant cohomogeneity one nearly Kähler structures on the product of some interval with  $S^2 \times S^3$ . This gives an alternative more geometric derivation of results contained in [41]. We have been unable to find a closed form for the general solution of these ODEs, but there are four explicit solutions which correspond to the homogeneous nearly Kähler structures on  $S^6$ ,  $CP^3$  and  $S^3 \times S^3$  and to the sine-cone over the invariant Sasaki–Einstein structure on  $S^2 \times S^3$ . The latter two play a role in the proof of the Main Theorem.

The generic solution in this 2-parameter family does not extend to a complete nearly Kähler structure on a closed 6-manifold. In Section 4 we understand necessary conditions for such an extension to be possible. Based on the desingularisation philosophy, close to the sine-cone we might expect to find two 1-parameter families of local cohomogeneity one smooth nearly Kähler structures modelled on the two different Calabi–Yau desingularisations of the conifold. We prove that this is indeed the case; the proof consists of two steps. In Section 4, studying singular initial value problems for the ODE system (3.10), we prove the existence of two 1-parameter families of solutions  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  that extend smoothly over a singular orbit  $S^2$  or  $S^3$ , respectively. In both cases the parameter  $a$  or  $b$  measures the size of the singular orbit, but, unlike the Calabi–Yau case, the parameter does not arise from an overall rescaling and instead represents a nontrivial deformation. Any cohomogeneity one nearly Kähler structure that extends to a complete manifold must belong to (at least) one of these families. In the first part of Section 6 we then confirm the expectation that these families are nearly Kähler deformations of the Calabi–Yau desingularisations of the conifold. More precisely, in the limit where the size of the singular orbit tends to zero, suitably rescaled,  $\Psi_a$  and  $\Psi_b$  converge to the Calabi–Yau structures on the small resolution and the smoothing, respectively.

Since any complete cohomogeneity one nearly Kähler manifold has two singular orbits, it would necessarily contain a (unique) principal orbit of maximal volume. This gives a further necessary condition for a member of  $\{\Psi_a\}_{a>0}$  or  $\{\Psi_b\}_{b>0}$  to admit a smooth nearly Kähler completion. The space of invariant nearly hypo structures on  $S^2 \times S^3$  that could potentially arise as such a maximal volume orbit is a smooth submanifold of codimension 1 in the space of all invariant nearly hypo structures. The main result of Section 5 is that in fact every member of both families  $\{\Psi_a\}_{a>0}$ ,  $\{\Psi_b\}_{b>0}$  admits a unique maximal volume orbit. The proof of this uses a continuity argument relying on a compactness result for the space of maximal volume orbits with a given upper bound on volume. The discrete symmetries of the ODE system (3.10) play an important role in the matching construction described below. To this end we also determine the fixed locus of these symmetries in the space of maximal volume orbits.

The existence of maximal volume orbits now becomes a tool to detect which solutions in the families  $\{\Psi_a\}_{a>0}$ ,  $\{\Psi_b\}_{b>0}$  extend to a complete cohomogeneity one nearly Kähler structure. Topologically the resulting closed 6-manifold is described as the union of neighbourhoods of

the two singular orbits identified along their boundary. Hence we consider pairs of solutions in the two 1-parameter families and try to match them across a principal orbit using the discrete symmetries of the ODE system (3.10). The maximal volume orbit provides a geometrically preferred slice to carry out this “gluing”. The necessary matching conditions are stated in the Doubling and Matching Lemmas 5.19 and 5.20 at the end of Section 5. These are best formulated in terms of two continuous curves  $\alpha$  and  $\beta$  parametrising the maximal volume orbits of  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$ , respectively. Complete cohomogeneity one nearly Kähler manifolds are in one-to-one correspondence with intersection points of the two curves, self-intersection points of either curve and points on the curves lying in the fixed locus of the discrete symmetries in the space of invariant maximal volume orbits. The diffeomorphism type of the corresponding closed cohomogeneity one nearly Kähler 6-manifold is determined by the pair of singular orbits and the discrete symmetry used to identify the pair of solutions to (3.10) across their maximal volume orbit.

The proof of the Main Theorem is now reduced to the problem of describing the behaviour of the curves  $\alpha$  and  $\beta$ . The explicit solutions to (3.10) corresponding to the homogeneous nearly Kähler structure on  $S^6$ ,  $CP^3$  and  $S^3 \times S^3$  already yield distinguished points on the curves. In Section 6 we describe the limit of  $\alpha$  and  $\beta$  for small values of the parameter: as expected from the desingularisation philosophy,  $\Psi_a$  and  $\Psi_b$  converge to the sine-cone as  $a$  and  $b$  tend to zero. The proof of this fact makes use of a functional  $\mathcal{B}$  introduced by Böhm in [9] as a Lyapunov function for the ODE system describing cohomogeneity one Einstein metrics. Since the space of invariant metrics in our setting does not reduce to relative rescalings along distinct subspaces (there are isomorphic components in the isotropy representation of the principal orbit and therefore “non-diagonal” terms in the metric), it is not clear that  $\mathcal{B}$  is a Lyapunov function for the system (3.10). However, we establish that the functional  $\mathcal{B}$  restricted to the space of invariant maximal volume orbits has an absolute minimum at the homogeneous Sasaki–Einstein structure on  $S^2 \times S^3$ . This is enough to establish the convergence of  $\Psi_a$  and  $\Psi_b$  to the sine-cone as  $a$  and  $b$  tend to zero using the convergence of the bubbles to the Calabi–Yau desingularisations we have already proved.

Using a comparison argument for solutions of Sturm–Liouville equations, in Section 7 we compare  $\Psi_a$  and  $\Psi_b$  for small  $a$  and  $b$  with a solution of the linearisation of (1.2) on the sine-cone. This comparison argument yields enough information on the curve  $\beta$  to prove the existence of the complete nearly Kähler structure on  $S^3 \times S^3$  given in the Main Theorem. The fact that this is inhomogeneous follows from a curvature computation.

The existence of at least two complete cohomogeneity one nearly Kähler structures on  $S^3 \times S^3$  has the following consequence: an arc of the curve  $\beta$  together with its image under the discrete symmetries form the boundary of a bounded region in the space of invariant maximal volume orbits containing the homogeneous Sasaki–Einstein structure on  $S^2 \times S^3$ . By the convergence of  $\Psi_a$  to the sine-cone as  $a \rightarrow 0$ , the curve  $\alpha$  starts inside this region. In Section 8 we prove that  $\alpha$  is unbounded as the parameter  $a \rightarrow \infty$ . The proof is based on a less geometric ad hoc rescaling argument suggested by the actual shape of the Taylor series of  $\Psi_a$ . We conclude that up to discrete symmetries the curves  $\alpha$  and  $\beta$  must intersect in at least two points; one of these corresponds to the homogeneous nearly Kähler structure on  $S^6$ . The second intersection point yields a complete cohomogeneity one nearly Kähler structure on  $S^6$  which is shown to be inhomogeneous by a curvature computation.

The proof of the Main Theorem is now complete. In fact we conjecture that the theorem yields all (inhomogeneous) complete cohomogeneity one nearly Kähler structures. In Section 9 we provide some numerical evidence for this conjecture and some further information about

the new complete cohomogeneity one solutions that we have obtained as part of a systematic numerical study.

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## 2. COHOMOGENEITY ONE $SU(3)$ -STRUCTURES AND HOMOGENEOUS $SU(2)$ -STRUCTURES

As explained in the Introduction we adopt Hitchin's approach in our study of cohomogeneity one nearly Kähler 6-manifolds, *ie* we concentrate on the geometry induced on (invariant) hypersurfaces. In the context of 6-manifolds with an  $SU(3)$ -structure this approach has been pursued by Conti and Salamon [23] and by Fernandez *et al* [28] in the Calabi–Yau and nearly Kähler settings, respectively. This section thus starts by recalling the basic definitions of  $SU(2)$ -structures concentrating on those that arise as hypersurfaces in Calabi–Yau and nearly Kähler 6-manifolds: *hypo* and *nearly hypo* structures respectively. For applications to the construction of complete compact cohomogeneity one nearly Kähler 6-manifolds it follows from the results of Podestà and Spiro [40] (recalled in more detail at the beginning of Section 3) that the only interesting case is that of nearly hypo structures on  $S^2 \times S^3$  invariant under the action of  $SU(2) \times SU(2)$  with isotropy group  $\Delta U(1)$ . The main results of the section are therefore those about  $SU(2) \times SU(2)$ -invariant nearly hypo structures on  $S^2 \times S^3$ , especially Proposition 2.41, and related results about invariant hypo structures (Proposition 2.23 and Theorem 2.27).

**Hypersurfaces of 6-manifolds with an  $SU(3)$ -structure.** We recall from Conti–Salamon [23, §1] some basic facts about the geometry of orientable hypersurfaces of a 6-manifold endowed with an  $SU(3)$ -structure.

Let  $M^6$  be a 6-manifold endowed with an  $SU(3)$ -structure  $(\omega, \Omega)$ . An orientable hypersurface  $N^5 \hookrightarrow M$  is naturally endowed with an  $SU(2)$ -structure, *ie* an  $SU(2)$ -reduction of the frame bundle of  $N$ . This is equivalent to the existence of a quadruple  $(\eta, \omega_1, \omega_2, \omega_3)$  where  $\eta$  is a nowhere-vanishing 1-form and  $\omega_i$  are 2-forms on  $N$  satisfying

- (i)  $\omega_i \wedge \omega_j = \delta_{ij} v$ , where  $v$  is a fixed 4-form such that  $\eta \wedge v \neq 0$ ;
- (ii)  $X \lrcorner \omega_1 = Y \lrcorner \omega_2 \Rightarrow \omega_3(X, Y) \geq 0$ .

The quadruple  $(\eta, \omega_1, \omega_2, \omega_3)$  is given in terms of the  $SU(3)$ -structure  $(\omega, \Omega)$  and the unit normal  $\nu$  to  $N$  via

$$(2.1) \quad \eta = -\nu \lrcorner \omega, \quad \omega_1 = \omega|_N, \quad \omega_2 + i\omega_3 = -i\nu \lrcorner \Omega.$$

Conversely, given an  $SU(2)$ -structure on  $N$  we can define an  $SU(3)$ -structure  $(\omega, \Omega)$  on  $N \times \mathbb{R}$  by

$$\omega = \omega_1 + \eta \wedge dt, \quad \Omega = (\omega_2 + i\omega_3) \wedge (\eta + idt),$$

where  $t$  is a coordinate on  $\mathbb{R}$ .

Since  $SU(2) < SO(4) = SU(2)^+ \cdot SU(2)^- < SO(5)$  there is a unique metric  $g$  and orientation on  $N$  compatible with any  $SU(2)$ -structure. At each point  $x \in N$  the nowhere-zero 1-form  $\eta$  defines a splitting  $T_x N \simeq \mathbb{R} \oplus \ker \eta_x$ . The metric  $g$  and orientation on  $N$  determine a metric and orientation on the 4-plane field  $H := \ker \eta$  and hence the space of “horizontal” 2-forms



$\Lambda^2 H^*$  splits as a direct sum of self-dual and anti-self-dual horizontal forms:  $\Lambda^2 H^* = \Lambda^+ \oplus \Lambda^-$ . A triple  $(\omega_1, \omega_2, \omega_3)$  satisfying (i) determines a trivialisation of  $\Lambda^+$  and therefore a reduction of the structure group from  $\mathrm{SO}(4) = \mathrm{SU}(2)^+ \cdot \mathrm{SU}(2)^-$  to  $\mathrm{SU}(2)^-$ . In fact, we can always assume that  $(\omega_1, \omega_2, \omega_3)$  is an oriented basis of  $\Lambda^+$  with respect to the natural orientation induced from the orientation of  $\ker \eta$ ; this gives condition (ii) above.

*Remark 2.2.* For future reference we make the elementary observations that if  $(\eta, \omega_1, \omega_2, \omega_3)$  is an  $\mathrm{SU}(2)$ -structure on  $N$  then so are the quadruples

$$(2.3a) \quad \tau_1(\eta, \omega_1, \omega_2, \omega_3) := (-\eta, \omega_1, -\omega_2, -\omega_3),$$

and

$$(2.3b) \quad \tau_2(\eta, \omega_1, \omega_2, \omega_3) := (-\eta, -\omega_1, \omega_2, -\omega_3).$$

The involutions  $\tau_1$  and  $\tau_2$  have the following interpretations at the level of  $\mathrm{SU}(3)$ -structures. First note that if  $(\omega, \Omega)$  is an  $\mathrm{SU}(3)$ -structure then so is  $(-\omega, -\overline{\Omega})$ . Given an oriented hypersurface  $(N, \nu)$  of the  $\mathrm{SU}(3)$ -manifold  $(M, \omega, \Omega)$  besides the  $\mathrm{SU}(2)$ -structure  $(\eta, \omega_1, \omega_2, \omega_3)$  defined by (2.1) we can consider two alternative  $\mathrm{SU}(2)$ -structures on  $N$ : one in which we endow  $N$  with the opposite orientation  $-\nu$  and  $M$  with its original  $\mathrm{SU}(3)$ -structure and the other in which we endow  $N$  with its original orientation  $\nu$  and  $M$  with the  $\mathrm{SU}(3)$ -structure  $(-\omega, -\overline{\Omega})$ . Notice however that both symmetries change the orientation of the hypersurface  $N \subset M$ . Indeed, in the latter case changing  $\omega$  into  $-\omega$  changes the orientation on  $M$  and therefore, since  $\nu$  is kept fixed, the one on  $N$ . Using (2.1) we see that these two  $\mathrm{SU}(2)$ -structures differ from the original one by the action of the involutions  $\tau_1$  and  $\tau_2$  respectively. All three  $\mathrm{SU}(2)$ -structures clearly induce the same Riemannian metric on  $N$ .

*Remark 2.4.* Conti–Salamon also explain how to understand  $\mathrm{SU}(2)$ -structures on 5-manifolds in terms of spin geometry. Underlying this is the low dimensional isomorphism  $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$  and the fact that the spinor representation of  $\mathrm{Spin}(5)$  is isomorphic to the fundamental representation of  $\mathrm{Sp}(2)$  on  $\mathbb{H}^2$ . Hence the isotropy subgroup of a nonzero spinor  $\psi$  in five dimensions is isomorphic to  $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ . It follows that an  $\mathrm{SU}(2)$ -structure on a 5-manifold  $N$  is equivalent to the choice of a spin structure on  $N$  and a unit spinor. Further aspects of the geometry of  $\mathrm{SU}(2)$ -structures, including their intrinsic torsion also have formulations in terms of spin geometry. In this paper we find it most convenient to phrase everything in terms of differential forms rather than spinors. However sometimes for a compact notation it will be convenient to refer to an  $\mathrm{SU}(2)$ -structure on a 5-manifold  $N$  by  $\psi$ , the corresponding  $\mathrm{SU}(3)$ -structure on  $N \times \mathbb{R}$  by  $\Psi$  and the restriction of  $\Psi$  to  $N \times \{t\}$  by  $\psi_t$ .

*Conical  $\mathrm{SU}(3)$ -structures.* We will be interested in conical, asymptotically conical and conically singular  $\mathrm{SU}(3)$ -structures, mainly those of Calabi–Yau or nearly Kähler type. Consideration of conical Calabi–Yau structures leads us to the first important special class of  $\mathrm{SU}(2)$ -structures: *Sasaki–Einstein* structures.

The (smooth) metric cone  $C(N)$  over a smooth Riemannian manifold  $(N, g_N)$  is the non-compact manifold  $\mathbb{R}^+ \times N$  endowed with the incomplete Riemannian metric  $g_C = dr^2 + r^2 g_N$ , where  $r > 0$  denotes the (radial) coordinate on  $\mathbb{R}^+$ . Smooth metric cones provide the local models for the simplest isolated singularities of Riemannian metrics. Any smooth metric cone  $C(N)$  admits a 1-parameter family of dilations preserving the cone metric  $g_C$ . If  $N$  possesses additional geometric structure, eg a  $G$ -structure, we can make the additional demand that the 1-parameter family of dilations of the metric cone  $C(N)$  act on the extra geometric structure in the obvious way.

Motivated by this, given an  $SU(2)$ -structure  $(\eta, \omega_i)$  on a 5-manifold  $N$  we define a conical  $SU(3)$ -structure on  $C(N)$  via

$$(2.5) \quad \omega_C = r\eta \wedge dr + r^2\omega_1, \quad \Omega_C = r^2(\omega_2 + i\omega_3) \wedge (r\eta + idr).$$

The metric induced by the conical  $SU(3)$ -structure  $(\omega_C, \Omega_C)$  is the cone metric  $g_C$  associated with the Riemannian metric  $g_N$  induced by the  $SU(2)$ -structure on  $N$ .

**Definition 2.6.** An  $SU(2)$ -structure  $(\eta, \omega_1, \omega_2, \omega_3)$  on a 5-manifold  $N$  is called *Sasaki–Einstein* if it satisfies

$$(2.7) \quad d\eta = -2\omega_1, \quad d\omega_2 = 3\eta \wedge \omega_3, \quad d\omega_3 = -3\eta \wedge \omega_2.$$

Equation (2.7) is equivalent to requiring that the conical  $SU(3)$ -structure defined by (2.5) be Calabi–Yau, *ie*  $d\omega_C = d\Omega_C = 0$ .

*Remark.* The involution  $\tau_2$  defined in (2.3b) preserves the Sasaki–Einstein equations while the involution  $\tau_1$  defined in (2.3a) reverses the signs in all three equations in (2.7). Furthermore, the complex 2-form  $\omega_2 + i\omega_3$  can be multiplied by any complex number of unit norm while preserving (2.7). In other words,  $(\omega_C, e^{i\theta}\Omega_C)$  for  $e^{i\theta} \in \mathbb{S}^1$  define different conical Calabi–Yau structures on  $C(N)$  inducing the same cone metric.

**$SU(2) \times SU(2)$ -invariant  $SU(2)$ -structures on  $S^2 \times S^3$ .** Consider now a (not necessarily complete or compact) 6-manifold  $M$  with an  $SU(3)$ -structure preserved by a cohomogeneity one isometric action of a compact Lie group  $G$ . For reasons explained at the beginning of Section 3, we will assume  $G = SU(2) \times SU(2)$  and that a dense open set  $M^* \subset M$  is diffeomorphic to the product of an interval with the 5-dimensional homogeneous space  $N_{1,1} = SU(2) \times SU(2) / \Delta U(1) \simeq S^2 \times S^3$ . In the spirit of Hitchin’s work [36], we will regard the  $SU(3)$ -structure on  $M^*$  as a 1-parameter family of  $SU(2)$ -structures on  $N_{1,1}$  invariant under the action of  $SU(2) \times SU(2)$ .

Fix a basis  $H, E, V$  of  $\mathfrak{su}(2)$  with Lie brackets

$$[H, E] = V, \quad [H, V] = -E, \quad [E, V] = \frac{1}{2}H.$$

Let  $U^+ = (H, H)$  be the generator of the Lie algebra of  $\Delta U(1)$ . The vectors

$$(2.8) \quad U^- = (H, -H), \quad E_1 = (E, 0), \quad E_2 = (0, E), \quad V_1 = (V, 0), \quad V_2 = (0, V),$$

on  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  define left-invariant vector fields on  $N_{1,1}$ . Let  $u^-, e_1, e_2, v_1, v_2$  be the corresponding co-frame.

There is a distinguished  $SU(2) \times SU(2)$ -invariant  $SU(2)$ -structure on  $N_{1,1}$ : the Sasaki–Einstein structure on  $S^2 \times S^3$  that gives rise to the conical Calabi–Yau metric on the *conifold*  $\{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}$ . In terms of the above basis the standard Sasaki–Einstein structure on  $N_{1,1}$  is

$$(2.9) \quad \eta^{\text{se}} = \frac{2}{3}u^-, \quad \omega_1^{\text{se}} = \frac{1}{12}(e_1 \wedge v_1 - e_2 \wedge v_2), \quad \omega_2^{\text{se}} = \frac{1}{12}(e_1 \wedge v_2 + e_2 \wedge v_1), \quad \omega_3^{\text{se}} = \frac{1}{12}(e_1 \wedge e_2 + v_1 \wedge v_2).$$

Observe that (2.7) are not scale-invariant, so the numerical factors here are forced on us once we fix a basis of  $\mathfrak{su}(2)$ .

There is in fact a circle of invariant Sasaki–Einstein structures inducing the same metric due to the freedom of changing the phase of the complex 2-form  $\omega_2^{\text{se}} + i\omega_3^{\text{se}}$ . These are all equivalent because the flow of the Reeb vector field  $U^-$  preserves  $\eta^{\text{se}}$  and  $\omega_1^{\text{se}}$  and acts as a rotation in the  $(\omega_2^{\text{se}}, \omega_3^{\text{se}})$ -plane.

**Lemma 2.10.**  $\eta^{\text{se}}$  is the unique (up to scale) invariant 1-form on  $N_{1,1}$ . In particular, the distribution  $\ker \eta$  is independent of the choice of invariant  $SU(2)$ -structure on  $N_{1,1}$ .

Fix the volume form  $v = \omega_1^{\text{se}} \wedge \omega_2^{\text{se}} \wedge \omega_3^{\text{se}}$  on  $\ker \eta^{\text{se}}$ . The space of invariant self-dual 2-forms on  $\ker \eta^{\text{se}}$  is 3-dimensional, spanned by  $\omega_1^{\text{se}}, \omega_2^{\text{se}}, \omega_3^{\text{se}}$ , and there exists a unique invariant anti-self-dual 2-form on  $\ker \eta^{\text{se}}$  up to scale,

$$\omega_0^{\text{se}} = \frac{1}{12}(e_1 \wedge v_1 + e_2 \wedge v_2).$$

Furthermore,  $\omega_0^{\text{se}}$  is closed.

*Proof.* Write  $\text{Span}(U^-, E_1, E_2, V_1, V_2)$  as  $\mathbb{R}U^- \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2$ , where  $\mathfrak{n}_i = \text{Span}(E_i, V_i)$ . Then  $\mathfrak{n}_i \simeq \mathfrak{n}$  as  $\Delta\text{U}(1)$ -representations, where  $\mathfrak{n}$  is the complex 1-dimensional representation of  $\text{U}(1)$  with weight 1. Therefore as  $\text{U}(1)$ -representations  $\Lambda^2 \mathfrak{n}_i^* \simeq \mathbb{R}(e_i \wedge v_i)$  and  $\mathfrak{n}_1 \otimes \mathfrak{n}_2 \simeq \Lambda^2 \mathfrak{n}^* \oplus \mathbb{R}\text{id} \oplus \text{Sym}_0(\mathfrak{n})$ , where  $\text{Sym}_0(\mathfrak{n})$  is the complex 1-dimensional representation of weight 2.

$N_{1,1}$  is a circle bundle over  $S^2 \times S^2$ . The anti-self-dual 2-form  $\omega_0^{\text{se}}$  is a multiple of the pull-back of the Kähler-Einstein metric on  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  and is therefore closed.  $\square$

Endow the 4-dimensional vector space of invariant 2-forms on  $\ker \eta^{\text{se}}$  with the inner product  $\langle \omega, \omega' \rangle v = \omega \wedge \omega'$  of signature  $(1, 3)$ .

**Proposition 2.11.** *The space of invariant  $\text{SU}(2)$ -structures on  $N_{1,1}$  inducing the same orientation as the standard Sasaki-Einstein structure can be identified with  $\mathbb{R}^+ \times \mathbb{R}^+ \times \text{SO}_0(1, 3)$ .*

*Proof.* If  $(\eta, \omega_i)$  is an invariant  $\text{SU}(2)$ -structure on  $N_{1,1}$  then

$$(2.12) \quad \eta = \lambda \eta^{\text{se}}$$

for some  $\lambda \neq 0$ . The choice of orientation implies that  $\lambda > 0$ . The 2-forms  $\omega_1, \omega_2, \omega_3$  yield an oriented orthogonal basis of the space-like 3-dimensional subspace  $\text{Span}(\omega_1, \omega_2, \omega_3)$  of  $\Lambda^2(\ker \eta^{\text{se}})^*$ . Moreover  $|\omega_i|$  is independent of  $i$ . Therefore up to scale the triple  $(\omega_1, \omega_2, \omega_3)$  represents a point in the Lorentzian Stiefel manifold  $V_0(3; \mathbb{R}^{1,3}) \simeq \text{SO}_0(1, 3)$ . Indeed, we can always complete  $(\omega_1, \omega_2, \omega_3)$  to a basis of  $\mathbb{R}^{1,3}$  with the unique invariant 2-form  $\omega_0$  satisfying  $\langle \omega_0, \omega_i \rangle = 0$ ,  $|\omega_0|^2 = -|\omega_i|^2$  and  $\langle \omega_0, \omega_0^{\text{se}} \rangle > 0$ . Then

$$(2.13) \quad (\omega_0, \omega_1, \omega_2, \omega_3) = \mu A (\omega_0^{\text{se}}, \omega_1^{\text{se}}, \omega_2^{\text{se}}, \omega_3^{\text{se}})$$

for some  $\mu > 0$  and  $A \in \text{SO}_0(1, 3)$ . Here  $A \in \text{SO}_0(1, 3)$  is interpreted as an endomorphism of the space of invariant 2-forms on  $\ker \eta^{\text{se}}$  with respect to the basis  $\{\omega_0^{\text{se}}, \omega_1^{\text{se}}, \omega_2^{\text{se}}, \omega_3^{\text{se}}\}$ .  $\square$

*Remark 2.14.* Given any  $(\lambda, \mu, A) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \text{SO}_0(1, 3)$  we will denote the invariant  $\text{SU}(2)$ -structure  $(\eta, \omega_1, \omega_2, \omega_3)$  satisfying (2.12) and (2.13) by  $\psi_{\lambda, \mu, A}$ . The choice of notation  $\psi$  here is motivated by the spinor reformulation of  $\text{SU}(2)$ -structures alluded to in Remark 2.4 and the desire for a compact notation. From now on we make the standing assumption that our invariant  $\text{SU}(2)$ -structures satisfy  $\lambda > 0$ , ie  $\psi_{\lambda, \mu, A}$  induces the same orientation as the Sasaki-Einstein structure.

*Remark 2.15.* The involutions  $\tau_1$  and  $\tau_2$  defined in (2.3) act on the set of invariant  $\text{SU}(2)$ -structures. Adopting the notation of the previous remark  $\tau_1$  and  $\tau_2$  act as follows:

$$\tau_1^* \psi_{\lambda, \mu, A} = \psi_{-\lambda, \mu, AT_1}, \quad \tau_2^* \psi_{\lambda, \mu, A} = \psi_{-\lambda, \mu, AT_2},$$

where  $T_1, T_2 \in \text{SO}_0(1, 3)$  are defined by  $T_1 := \text{diag}(1, -1, 1, -1)$  and  $T_2 := \text{diag}(1, 1, -1, -1)$ . Observe that neither  $\tau_1$  nor  $\tau_2$  preserves the normalisation  $\lambda > 0$ , but their composition does.

The left-invariant vector field  $U^-$  generates the group of inner automorphisms of  $\text{SU}(2) \times \text{SU}(2)$  that fix  $\Delta\text{U}(1)$ .  $U^-$  induces a circle action on the space of invariant  $\text{SU}(2)$ -structures given by a rotation in the  $(\omega_2^{\text{se}}, \omega_3^{\text{se}})$ -plane. There are also discrete symmetries, cf Proposition 3.11 below.

Any invariant  $SU(2)$ -structure on  $N_{1,1}$  determines uniquely an invariant metric. The next result describes this map explicitly in terms of the parametrisation of invariant  $SU(2)$ -structures given in Proposition 2.11.

**Proposition 2.16** (cf [2, Lemma 4.1]).

(i) The set of invariant metrics on  $N_{1,1}$  is parametrised by  $\mathbb{R}^+ \times \mathbb{R}^+ \times S^+$  where

$$S^+ := \{w \in \mathbb{R}^{1,3} : |w|^2 = -1, w_0 > 0\}$$

is the upper hyperboloid in  $\mathbb{R}^{1,3}$ . We denote the corresponding invariant metric  $g_{\lambda,\mu,w}$ .

(ii) The invariant metrics corresponding to  $g_{\lambda,\mu,w}$  and  $g_{\lambda,\mu,w'}$ , where

$$w' = (w_0, w_1, \cos \theta w_2 - \sin \theta w_3, \sin \theta w_2 + \cos \theta w_3),$$

are isometric.

(iii) The map from the invariant  $SU(2)$ -structure  $\psi_{\lambda,\mu,A}$  to its invariant metric  $g_{\lambda,\mu,w}$  is given by

$$(\lambda, \mu, A) \mapsto (\lambda, \mu, pr_1(A)),$$

where  $pr_1(A) \in S^+$  is the projection of the matrix  $A \in SO_0(1, 3)$  onto its first column. In particular the set of invariant  $SU(2)$ -structures is a principal  $SO(3)$ -bundle over the space of invariant metrics.

*Proof.* Denote by  $\Delta\mathfrak{u}(1)^\perp$  the orthogonal complement of  $\Delta\mathfrak{u}(1)$  in  $\mathfrak{su}_2 \oplus \mathfrak{su}_2$ . In the proof of Lemma 2.10 we have already observed that

$$\Delta\mathfrak{u}(1)^\perp = \text{Span}(U^-, E_1, V_1, E_2, V_2) \simeq \mathbb{R} \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2$$

where  $\mathfrak{n}_i$  is isomorphic to the standard representation  $\mathfrak{n}$  of  $U(1)$ . An invariant metric  $g$  on  $N_{1,1}$  can be written as

$$g = \lambda^2 \eta^{\text{se}} \otimes \eta^{\text{se}} + g^T,$$

where the transverse metric  $g^T$  can be thought of as a  $U(1)$ -invariant inner product on  $\mathfrak{n}_1 \oplus \mathfrak{n}_2$ . When  $g$  is induced by an invariant  $SU(2)$ -structure  $\psi_{\lambda,\mu,A}$  then  $g(U^-, U^-) = \eta(U^-)^2$ , which motivates the notation. Furthermore, in this case we also define transverse almost complex structures  $J_i$  such that  $g^T(u, v) = \omega_i(u, J_i v)$ .

The main observation is that the  $U(1)$ -invariance of  $g^T$  forces additional structure on the transverse geometry. Indeed,  $J_0^{\text{se}} = [U^+, \cdot]$  defines a complex structure on  $\mathfrak{n}_1 \oplus \mathfrak{n}_2$  with  $J_0^{\text{se}} E_i = V_i$ . Since  $U^+$  generates the action of  $\Delta U(1)$ ,  $U(1)$ -invariant endomorphisms of  $\mathfrak{n}_1 \oplus \mathfrak{n}_2$  are precisely those commuting with  $J_0^{\text{se}}$ . In particular,  $J_0^{\text{se}} J_i = J_i J_0^{\text{se}}$  and  $g^T$  is an almost-Hermitian metric with respect to  $J_0^{\text{se}}$ . Therefore we can define the associated Hermitian form  $\omega_0(X, Y) = g^T(J_0^{\text{se}} X, Y)$ . Observe also that  $J_0^{\text{se}}$  induces the opposite orientation with respect to the volume form  $v = \omega_1^{\text{se}} \wedge \omega_1^{\text{se}}$  and therefore  $\omega_0$  is an anti-self-dual form. It follows that the map from invariant  $SU(2)$ -structures to invariant metrics is surjective and is a principal  $SO(3)$ -bundle as claimed. The explicit expression for  $\omega_0$  in terms of the parametrisation  $(\lambda, \mu, A)$  is

$$\omega_0 = \mu A \omega_0^{\text{se}} = \mu (w_0 \omega_0^{\text{se}} + w_1 \omega_1^{\text{se}} + w_2 \omega_2^{\text{se}} + w_3 \omega_3^{\text{se}}),$$

where  $w = (w_i)$  is the first column of the matrix  $A \in SO_0(1, 3)$ .

The circle action in (ii) is the induced action of the flow of the Reeb vector field  $U^-$  on invariant metrics.  $\square$

We are interested in  $SU(2)$ -structures induced on a hypersurface of a Calabi–Yau or nearly Kähler 6-manifold; these have been dubbed *hypo* and *nearly hypo* structures, respectively. We will study the  $SU(2) \times SU(2)$ -invariant ones on  $N_{1,1}$ . As we will see in a moment, hypo and nearly hypo structures can be defined by a set of constraints on the exterior differentials of

the forms defining the  $SU(2)$ -structure. By (2.7) and Lemma 2.10, for the invariant  $SU(2)$ -structure  $\psi_{\lambda,\mu,A}$  we have

$$(2.17) \quad d\eta = -2\lambda\omega_1^{\text{se}}, \quad d\omega_i = -2\frac{\mu}{\lambda}\eta \wedge T A \omega_i^{\text{se}}, \quad d(\eta \wedge \omega_i) = -2\lambda\mu \langle A \omega_i^{\text{se}}, \omega_i^{\text{se}} \rangle v,$$

where  $T \in \text{End}(\mathbb{R}^{1,3})$  is

$$(2.18) \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

*Remark 2.19.* As an immediate consequence we see that the invariant Sasaki–Einstein structures on  $N_{1,1}$  are those  $\psi_{\lambda,\mu,A}$  with  $\lambda = \mu = 1$  and  $A \in \text{SO}(2) \subset \text{SO}_0(1,1) \times \text{SO}(2) \subset \text{SO}_0(1,3)$ . Indeed, the equations (2.7) are not scale invariant and the only geometric degree of freedom is to rotate the form  $\omega_2 + i\omega_3$  by a phase  $e^{i\theta}$ . This freedom of rotation in the plane spanned by  $\omega_2^{\text{se}}$  and  $\omega_3^{\text{se}}$ , as already noted, is nothing but the action of the flow of the Reeb vector field on invariant 2-forms; in general the flow of the Reeb vector field on invariant metrics on  $N_{1,1}$  is nontrivial but in the case of the Sasaki–Einstein metric  $g^{\text{se}}$  the Reeb vector field is an additional Killing field. In particular  $g^{\text{se}}$  is invariant under the larger group  $U(1) \times SU(2) \times SU(2)$ .

**Hypo structures.** In this section, following Conti–Salamon [23, Definition 1.5], we consider the class of  $SU(2)$ -structures that arise on oriented hypersurfaces in Calabi–Yau 3-folds. To this end consider a 1-parameter family of  $SU(2)$ -structures  $(\eta, \omega_i)(t)$  such that

$$(2.20) \quad \omega = \eta \wedge dt + \omega_1, \quad \Omega = (\omega_2 + i\omega_3) \wedge (\eta + idt),$$

is a Calabi–Yau structure on  $N \times I$  for some interval  $I \subset \mathbb{R}$ ,  $ie$  an  $SU(3)$ -structure such that  $\omega$  and  $\Omega$  are both closed. The condition  $d\omega = 0 = d\Omega$  is equivalent to

$$(2.21a) \quad d\omega_1 = 0, \quad d(\eta \wedge \omega_2) = 0, \quad d(\eta \wedge \omega_3) = 0,$$

together with the evolution equations

$$(2.21b) \quad \partial_t \omega_1 = -d\eta, \quad \partial_t(\eta \wedge \omega_2) = -d\omega_3, \quad \partial_t(\eta \wedge \omega_3) = d\omega_2.$$

**Definition 2.22.** An  $SU(2)$ -structure  $(\eta, \omega_1, \omega_2, \omega_3)$  on a 5-manifold  $N$  satisfying (2.21a) is called a *hypo structure*. We call equations (2.21b) the *hypo evolution equations*.

*Remark.* The involutions  $\tau_1$  and  $\tau_2$  defined in Remark 2.2 both preserve the hypo equations. Moreover,  $(\eta, \omega_1, \omega_2, \omega_3)(t)$  solves the hypo evolution equations if and only if  $\tau_2(\eta, \omega_1, \omega_2, \omega_3)(t)$  does (and similarly for  $\tau_1(\eta, \omega_1, \omega_2, \omega_3)(-t)$ ). Furthermore, (2.21a) and (2.21b) are both invariant under changing the phase of the complex 2-form  $\omega_2 + i\omega_3$ .

We now describe the space of *invariant* hypo structures on  $N_{1,1}$ . Recall again the notation  $\psi_{\lambda,\mu,A}$  where  $(\lambda, \mu, A) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \text{SO}_0(1,3)$  adopted in Remark 2.14 to describe invariant  $SU(2)$ -structures on  $N_{1,1}$ .

**Proposition 2.23.** *For any  $(f, g) \in \mathbb{R}^+ \times \mathbb{R}^+$  the invariant hypo structures on  $N_{1,1}$  are invariant under the rescalings*

$$\eta \mapsto f\eta, \quad \omega_i \mapsto g\omega_i \quad \text{for } i = 1, 2, 3.$$

*In particular by rescaling it suffices to describe the invariant hypo structures with  $\lambda = \mu = 1$ .*

*The set of invariant hypo structures on  $N_{1,1}$  with  $\lambda = \mu = 1$  is the union of two smooth manifolds:*

- (i) *A two-dimensional manifold diffeomorphic to  $S(\text{O}^+(1,1) \times \text{O}(2)) \subset \text{SO}_0(1,3)$ .*

- (ii) A three-dimensional manifold diffeomorphic to  $O^+(1, 2)$  embedded in  $SO_0(1, 3)$  as the subgroup that fixes the line spanned by  $(0, 1, 0, 0)$ .

In each case there are two connected components, which are interchanged by  $\tau_1 \circ \tau_2$ . The two manifolds intersect in two circles, the one parametrising the 1-parameter family of invariant Sasaki–Einstein structures on  $N_{1,1}$  described in Remark 2.19 and its image under  $\tau_1 \circ \tau_2$ .

*Proof.* Let  $\psi_{\lambda, \mu, A}$  be an invariant hypo structure on  $N_{1,1}$ . Observe first that the hypo equations (2.21a) are invariant under  $\eta \mapsto f\eta$  and  $\omega_i \mapsto g\omega_i$  for any  $f > 0, g > 0$ . Therefore without loss of generality we now assume that  $\lambda = \mu = 1$ . Writing  $\omega_i = A\omega_i^{\text{se}}$  and using (2.17) we see that (2.21a) is satisfied if and only if  $A$  is of the form

$$(2.24) \quad A = \begin{pmatrix} w_0 & x_0 & y_0 & z_0 \\ w_1 & x_1 & 0 & 0 \\ w_2 & 0 & y_2 & z_2 \\ w_3 & 0 & y_3 & z_3 \end{pmatrix}.$$

Imposing the condition that  $A \in SO_0(1, 3)$  leads us to distinguish two cases:  $y_0 = z_0 = 0$  or otherwise. In the former case, up to the action of  $\tau_1 \circ \tau_2$ , (2.24) together with  $A \in SO_0(1, 3)$  forces

$$(2.25) \quad A = \begin{pmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix},$$

while in the latter we must have

$$(2.26) \quad A = \begin{pmatrix} w_0 & 0 & y_0 & z_0 \\ 0 & 1 & 0 & 0 \\ w_2 & 0 & y_2 & z_2 \\ w_3 & 0 & y_3 & z_3 \end{pmatrix} \in SO_0(1, 2).$$

The final statement about the intersection of the two components follows immediately.  $\square$

We are now going to solve the hypo evolution equations (2.21b) in the two components (i) and (ii) of invariant hypo structures on  $N_{1,1}$  given in the previous Proposition. The invariance assumption reduces these evolution equations to ODEs on the space of invariant hypo structures that we will solve explicitly. The limiting case corresponding to the intersection of the manifolds of Proposition 2.23 is of course the conifold, the Calabi–Yau cone over the Sasaki–Einstein structure (2.9). Up to scale there exist two  $SU(2) \times SU(2)$ -invariant complete Calabi–Yau metrics asymptotic to the conifold: one on the small resolution of the conifold [17], the total space of the vector bundle  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  over  $\mathbb{P}^1$ , and one on the smoothing of the conifold [17, 46], diffeomorphic to  $T^*S^3$ . These have been recently proven to be the unique complete asymptotically conical Calabi–Yau metrics with tangent cone at infinity the conifold [20].

**Theorem 2.27** (Candelas–de la Ossa [17]; Stenzel [46] only for part (ii)).

- (i) Up to scale there exists a unique smooth invariant Calabi–Yau structure on the total space of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  over  $\mathbb{P}^1$ .  
 (ii) Up to scale there exists a unique smooth invariant Calabi–Yau structure on  $T^*S^3$ .

Both Calabi–Yau structures are complete and asymptotic to the conifold in the sense made precise below.

We give a detailed outline of the proof of the theorem using the language of invariant hypo structures, partly as a warm-up for the more complicated analysis in the nearly Kähler case, and because these asymptotically conical Calabi–Yau manifolds will play a role in that analysis as limiting objects.

*Proof of Theorem 2.27(i).* We begin with the complete invariant Calabi–Yau structures that arise from the invariant hypo structures described in Proposition 2.23(i), *ie* where  $y_0 = z_0 = 0$ .

By acting with the flow of the Reeb vector field, *ie* choosing  $\theta = -\frac{\pi}{2}$  in equation (2.25), we can always assume that a hypo structure with  $y_0 = 0 = z_0$  satisfies

$$(2.28) \quad \eta = \lambda \eta^{\text{se}}, \quad \omega_1 = u_0 \omega_0^{\text{se}} + u_1 \omega_1^{\text{se}}, \quad \omega_2 = -\mu \omega_3^{\text{se}}, \quad \omega_3 = \mu \omega_2^{\text{se}},$$

with

$$-u_0^2 + u_1^2 = \mu^2.$$

Then the flow equations (2.21b) are equivalent to

$$(2.29) \quad \dot{u}_0 = 0, \quad \dot{u}_1 = 2\lambda, \quad \partial_t(\lambda\mu) = 3\mu.$$

The solution with  $u_0 = 0$  is the conifold. When  $u_0 \neq 0$ , by scaling and the discrete symmetry  $\tau_4$  of Proposition 3.11 that exchanges the two factors of  $\text{SU}(2) \times \text{SU}(2)$ , we can assume without loss of generality that  $u_0 = 1$ . It is convenient to introduce a new variable  $r$  such that  $u_1 = r^2$ . Here up to the action of  $\tau_1 \circ \tau_2$  we can always assume  $u_1 \geq 0$ . Hence  $\mu^2 = -u_0^2 + u_1^2 = r^4 - 1$  and  $\lambda = r\dot{r}$ . In terms of the new variable  $r$  from (2.29) we obtain

$$\frac{d}{dr}(\lambda\mu)^2 = 2\lambda\mu \frac{d}{dr}(\lambda\mu) = \frac{6\lambda\mu^2}{\dot{r}} = 6r\mu^2 = 6r^5 - 6r.$$

In Section 4 we will discuss necessary conditions for a cohomogeneity one  $\text{SU}(3)$ –structure to extend smoothly over a singular orbit. In view of Lemma 4.1, we require that, with respect to the variable  $t$ ,  $u_1(0) = 1$  and  $\lambda(0) = 0$ . Thus  $r \geq 1$  and

$$(2.30) \quad \mu = \sqrt{r^4 - 1}, \quad \lambda = \sqrt{\frac{r^6 - 3r^2 + 2}{r^4 - 1}}.$$

In particular as  $r \rightarrow \infty$  we have

$$(2.31a) \quad \lambda\mu = r^3 \sqrt{1 - 3r^{-4} + 2r^{-6}} = r^3 + O(r^{-1}),$$

$$(2.31b) \quad \frac{r\mu}{\lambda} = (r^2 - r^{-2})(1 - 3r^{-4} + 2r^{-6})^{-1/2} = r^2 + O(r^{-2}).$$

For  $r \sim 1$  using  $\lambda = r\dot{r}$  to transform back to the variable  $t$  yields  $t \sim \frac{2}{\sqrt{3}}\sqrt{r-1}$  and therefore

$$u_0 - u_1 = 1 - \frac{3}{2}t^2 + O(t^4), \quad \lambda = \frac{3}{2}t + O(t^3).$$

Lemma 4.1(i) guarantees that the resulting cohomogeneity one Calabi–Yau structure  $(\omega, \Omega)$  extends smoothly at  $r = 1$  over a 2–sphere  $\text{SU}(2) \times \text{SU}(2) / \text{U}(1) \times \text{SU}(2)$ . Moreover  $(\omega, \Omega)$  is asymptotic as  $r \rightarrow \infty$  to the conifold. To see this explicitly recall from (2.20) how the  $\text{SU}(3)$ –structure  $(\omega, \Omega)$  is obtained from the 1–parameter family of  $\text{SU}(2)$ –structures  $(\eta(t), \omega_i(t))$ . Converting from  $t$  to  $r$  using  $\lambda dt = \lambda \frac{dr}{\dot{r}} = r dr$  we obtain

$$(2.32a) \quad \omega = \omega_1 + \eta \wedge dt = u_0 \omega_0^{\text{se}} + u_1 \omega_1^{\text{se}} + \lambda \eta^{\text{se}} \wedge dt = \omega_0^{\text{se}} + r^2 \omega_1^{\text{se}} + r \eta^{\text{se}} \wedge dr,$$

$$(2.32b) \quad \text{Re } \Omega = \omega_2 \wedge \eta - \omega_3 \wedge dt = -\lambda \mu \omega_3^{\text{se}} \wedge \eta^{\text{se}} - \frac{\mu r}{\lambda} \omega_2^{\text{se}} \wedge dr,$$

$$(2.32c) \quad \text{Im } \Omega = \omega_3 \wedge \eta + \omega_2 \wedge dt = \lambda \mu \omega_2^{\text{se}} \wedge \eta^{\text{se}} - \frac{\mu r}{\lambda} \omega_3^{\text{se}} \wedge dr.$$

Substituting the expansions from (2.31) into (2.32) and comparing with (2.5) we obtain

$$(2.33a) \quad \omega = \omega_C + O(r^{-2}),$$

$$(2.33b) \quad \operatorname{Re} \Omega = \operatorname{Re} \Omega_C + O(r^{-4}),$$

$$(2.33c) \quad \operatorname{Im} \Omega = \operatorname{Im} \Omega_C + O(r^{-4}),$$

where we used the cone metric to compute norms. We stress that up to scaling and discrete symmetries, (2.32) is the unique solution to (2.21b) that extends smoothly over a singular orbit  $S^2$ .  $\square$

*Proof of Theorem 2.27(ii).* Consider instead the evolution of an invariant hypo structure with  $(y_0, z_0) \neq (0, 0)$ . By acting with the flow of the Reeb vector field and by changing the phase of  $\omega_2 + i\omega_3$ , we can always assume that  $y_0 = y_2 = z_3 = 0$  in (2.26) and hence

$$\eta = \lambda \eta^{\text{se}}, \quad \omega_1 = \mu \omega_1^{\text{se}}, \quad \omega_2 = -\mu \omega_3^{\text{se}}, \quad \omega_3 = v_0 \omega_0^{\text{se}} + v_2 \omega_2^{\text{se}},$$

with

$$-v_0^2 + v_2^2 = \mu^2.$$

Then the nearly hypo evolution equations (2.21b) become

$$(2.34) \quad \dot{\mu} = 2\lambda, \quad \partial_t(\lambda\mu) = 3v_2, \quad \partial_t(\lambda v_0) = 0, \quad \partial_t(\lambda v_2) = 3\mu.$$

If we introduce a new dependent variable  $s$  defined by

$$\frac{ds}{dt} = \frac{1}{\lambda} > 0,$$

then we can integrate the resulting system of ODEs explicitly as follows. In terms of  $s$  the ODE system (2.34) is equivalent to

$$(2.35) \quad \frac{d}{ds}(\mu^3) = 6(\mu\lambda)^2, \quad \frac{d}{ds}(\mu\lambda) = 3\lambda v_2, \quad \frac{d}{ds}(\lambda v_0) = 0, \quad \frac{d}{ds}(\lambda v_2) = 3\mu\lambda.$$

In particular both  $\mu\lambda$  and  $\lambda v_2$  satisfy the second order ODE  $f'' = 9f$ . Applying Lemma 4.2 one can determine the various constants of integration that ensure that the resulting cohomogeneity one  $SU(3)$ -structure extends smoothly over the exceptional orbit (a totally geodesic round  $S^3$ ) that without loss of generality we assume occurs at  $s = 0$ . Up to the action of the discrete symmetries of Proposition 3.11, the solution of (2.35) takes the form

$$\lambda v_0 = -\kappa, \quad \lambda\mu = \kappa \sinh 3s, \quad \lambda v_2 = \kappa \cosh 3s, \quad \mu^3 = \kappa^2(\sinh 3s \cosh 3s - 3s),$$

where  $\kappa$  is a positive real parameter and  $s \in [0, +\infty)$ . The parameter  $\kappa > 0$  can be changed by scaling: the choice  $\kappa = \frac{2}{3}$  is equivalent to the normalisation  $\lambda(0) = 1$ ; geometrically this corresponds to the exceptional orbit at  $s = 0$  being a unit 3-sphere.

Solving for the coefficients  $\lambda, \mu, v_1$  and  $v_2$  we obtain

$$(2.36) \quad \begin{aligned} \lambda &= \left(\frac{2}{3}\right)^{\frac{1}{3}} \frac{\sinh 3s}{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}}, & \mu &= \left(\frac{2}{3}\right)^{\frac{2}{3}} (\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}, \\ v_0 &= -\left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}}{\sinh 3s}, & v_2 &= \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{(\sinh 3s \cosh 3s - 3s)^{\frac{1}{3}}}{\tanh 3s}. \end{aligned}$$

In order to analyse the asymptotics of the resulting Calabi–Yau structure as  $s \rightarrow \infty$ , consider the change of variable  $r^2 = \mu(s)$  which, since  $\omega = d(-\frac{\mu}{2}\eta^{\text{se}})$ , defines a symplectomorphism between  $(0, \infty) \times N_{1,1}$  and the conifold. Thus

$$r^2 \sim \frac{e^{2s}}{3^{\frac{2}{3}}} (1 + O(se^{-6s}))^{\frac{1}{3}},$$



and therefore

$$\omega = \omega_C, \quad \Omega = \Omega_C + \xi + O\left(\frac{\log r}{r^6}\right),$$

where  $\xi = \frac{2}{3r}\omega_0^{\text{se}} \wedge (dr - ir\eta^{\text{se}}) = O(r^{-3})$  and we used the cone metric to compute norms.  $\square$

**Nearly Hypo structures.** In the previous theorem we described cohomogeneity one Calabi–Yau 3–folds as the evolution of invariant hypo structures under (2.21b). Following the same philosophy, in this section we consider the nearly Kähler analogue of hypo  $\text{SU}(2)$ –structures: in order to understand cohomogeneity one nearly Kähler manifolds we study the class of  $\text{SU}(2)$ –structures induced on hypersurfaces in a nearly Kähler 6–manifold. Fernandez *et al* [28, Definition 3.1] named these *nearly hypo* structures.

To this end consider a family of  $\text{SU}(2)$ –structures on  $N$  induced by a nearly Kähler structure on  $N \times \mathbb{R}$ . By (2.20) and the definition of a nearly Kähler structure in terms of  $(\omega, \Omega)$ , the  $\text{SU}(2)$ –structures on  $N$  must then satisfy

$$(2.37a) \quad d\omega_1 = 3\eta \wedge \omega_2, \quad d(\eta \wedge \omega_3) = -2\omega_1^2,$$

and evolve according to the evolution equations

$$(2.37b) \quad \partial_t \omega_1 = -3\omega_3 - d\eta, \quad \partial_t(\eta \wedge \omega_2) = -d\omega_3, \quad \partial_t(\eta \wedge \omega_3) = d\omega_2 + 4\eta \wedge \omega_1.$$

**Definition 2.38.** An  $\text{SU}(2)$ –structure  $(\eta, \omega_1, \omega_2, \omega_3)$  satisfying (2.37a) is called *nearly hypo*. We call equations (2.37b) the *nearly hypo evolution equations*.

*Remark 2.39.* Observe that the involutions  $\tau_1$  and  $\tau_2$  defined in Remark 2.2 both preserve the nearly hypo equations. In the latter case this corresponds to the fact that  $(\omega, \Omega)$  satisfies the nearly Kähler equations (1.2) if and only if  $(-\omega, -\overline{\Omega})$  does. Moreover,  $(\eta, \omega_1, \omega_2, \omega_3)(t)$  solves the nearly hypo evolution equations if and only if  $\tau_2(\eta, \omega_1, \omega_2, \omega_3)(t)$  does (and similarly for  $\tau_1(\eta, \omega_1, \omega_2, \omega_3)(-t)$ ). Unlike the hypo case, we are not free to change the phase of the holomorphic volume form.

As a first example, we now give a simple but fundamental class of mildly singular nearly Kähler structures associated with any Sasaki–Einstein structure on a compact smooth 5–manifold  $N$  via the so-called *sine-cone* construction and explain its relation both to  $\text{G}_2$  geometry in dimension 7 and to nearly hypo structures in dimension 5.

*The sine-cone construction of singular nearly Kähler spaces.* The metric cone  $C(N)$  over  $N$  endowed with an  $\text{SU}(2)$ –structure equipped with the conical  $\text{SU}(3)$ –structure  $(\omega_C, \Omega_C)$  defined in (2.5) is Calabi–Yau if and only if the  $\text{SU}(2)$ –structure is Sasaki–Einstein. Hence the metric product  $\mathbb{R} \times C(N)$  is a (non smooth) metric cone  $C'(N)$  whose holonomy is contained in  $\text{SU}(3) \subset \text{G}_2$ . Because of the  $\mathbb{R}$ –invariance of  $C'(N)$  its cross-section is not smooth but is instead the sine-cone (or metric suspension) over  $N$ , *ie*  $SC(N) := [0, \pi] \times N$  endowed with the Riemannian metric  $g^{\text{sc}} = dr^2 + \sin^2 r g_N$ .

Unless the cone  $C(N)$  is isometric to  $\mathbb{C}^3$  the sine-cone  $SC(N)$  is a compact but singular metric space with two isolated conical singularities at  $r = 0$  and  $r = \pi$  both modelled on  $C(N)$ . Since  $C'(N)$  is a (singular) Ricci-flat cone, its cross-section  $SC(N)$  with the metric  $g^{\text{sc}}$  is a singular Einstein space with scalar curvature 30. Moreover, since  $C'(N)$  has holonomy contained in  $\text{G}_2$ , its cross-section  $SC(N)$  admits a nearly Kähler structure compatible with the sine-cone metric  $g^{\text{sc}}$  and therefore every orientable hypersurface of  $SC(N)$  must admit a nearly hypo structure. In particular, the (totally geodesic) hypersurface  $\{\frac{\pi}{2}\} \times N$  admits a nearly hypo structure whose induced metric is isometric to the Sasaki–Einstein metric  $g_N$ .

In fact we notice as an immediate consequence of the Sasaki–Einstein and nearly hypo structure equations (2.7) and (2.37a) respectively that if  $(\eta, \omega_1, \omega_2, \omega_3)$  is Sasaki–Einstein then the “rotated”  $\text{SU}(2)$ –structure  $(\eta, -\omega_3, \omega_2, \omega_1)$  is in fact nearly hypo and they both induce

the same Riemannian metric. More generally, we find that the following explicit 1-parameter family of nearly hypo structures solves the nearly hypo evolution equations (2.37b) and induces the “rotated Sasaki–Einstein” nearly hypo structure when  $t = \pi/2$ :

$$(2.40) \quad \begin{aligned} \eta &= \sin t \eta^{\text{se}}, & \omega_1 &= \sin^2 t (\cos t \omega_1^{\text{se}} - \sin t \omega_3^{\text{se}}), \\ \omega_2 &= \sin^2 t \omega_2^{\text{se}}, & \omega_3 &= \sin^2 t (\sin t \omega_1^{\text{se}} + \cos t \omega_3^{\text{se}}), \end{aligned}$$

for  $t \in [0, \pi]$ . Clearly the metric induced on  $\{t\} \times N$  by this nearly hypo structure is  $\sin^2 t g_N$  as required for the nearly hypo structure induced by the nearly Kähler structure on the sine-cone. Nearly Kähler sine-cones were introduced in [28] generalising the construction of (non-smooth) Spin(7)–cones that had appeared in the physics literature [1, 6]; see also [10, §14.4] for further references.

Sine-cones (also called spherical or metric suspensions) have also played a key role in the structure theory for spaces with lower Ricci curvature bounds. They provide non-smooth metric spaces that have extremal properties analogous to the round metric on spheres (which is of course the sine-cone over a lower-dimensional round sphere of the appropriate size) and therefore appear in several “almost rigidity” statements, *eg* Cheeger and Colding’s Almost Maximal Diameter Theorem [18, Theorems 5.12 & 5.14] that generalises the classical Maximal Diameter Theorem of Cheng to (singular) limit spaces.

*Invariant nearly hypo structures on  $N_{1,1}$ .* We now specialise to the case of invariant nearly hypo structures on  $N_{1,1}$ . Since we will construct cohomogeneity one nearly Kähler manifolds by studying the nearly hypo evolution equations (2.37b) restricted to invariant nearly hypo structures, the following result will play a crucial role in the rest of the paper.

**Proposition 2.41.** *Invariant nearly hypo  $SU(2)$ –structures on  $N_{1,1}$  are parametrised by the product of a circle with the open set  $\mathcal{U}$  in  $SO_0(1, 2)$  defined by (2.43). Here the embedding of  $SO_0(1, 2) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times SO_0(1, 3)$  is given by equations (2.42) and (2.44) and the  $SO(2)$  factor corresponds to the orbits of the action of the Reeb vector field  $U^-$ . Moreover,  $\mathcal{U}$  is a (trivial) real line bundle over  $\mathbb{R}^2$ . In particular, the space of invariant nearly hypo structures is a smooth connected 4-manifold.*

*Proof.* For an invariant  $SU(2)$ –structure  $\psi_{\lambda, \mu, A}$  on  $N_{1,1}$  the defining equations (2.37a) are equivalent to

$$TA\omega_1^{\text{se}} = 3\lambda A\omega_2^{\text{se}}, \quad \langle A\omega_3^{\text{se}}, \omega_1^{\text{se}} \rangle = \frac{\mu}{\lambda},$$

where  $T$  is the matrix defined in (2.18). Together with the requirements  $\langle A\omega_i^{\text{se}}, A\omega_2^{\text{se}} \rangle = 0$  for  $i = 1, 3$  and  $|A\omega_2^{\text{se}}| = 1$  this implies that  $A$  is of the form

$$(2.42) \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} w_0 & x_0 & 0 & y_0 \\ w_1 & x_1 & 0 & \frac{\mu}{\lambda} \\ w_2 & -\lambda & 0 & y_2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \in SO(2) \times SO_0(1, 2).$$

The matrix

$$\begin{pmatrix} w_0 & x_0 & y_0 \\ w_1 & x_1 & \frac{\mu}{\lambda} \\ w_2 & -\lambda & y_2 \end{pmatrix}$$

lies in the open set  $\mathcal{U}$  of  $SO_0(1, 2)$  defined by

$$(2.43) \quad \mathcal{U} = \{B \in SO_0(1, 2) \mid B_{32} < 0, B_{23} > 0\},$$

where  $B_{ij}$  denotes the  $(i, j)$  entry of the matrix  $B$ . The map  $\mathrm{SO}(2) \times \mathcal{U} \rightarrow \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{SO}_0(1, 3)$  defined by (2.42) and

$$(2.44) \quad \mathcal{U} \ni B \longmapsto (-B_{32}, -B_{23}B_{32}) \in \mathbb{R}^+ \times \mathbb{R}^+$$

is clearly injective.

We conclude by describing the open set  $\mathcal{U}$  more precisely. Identify  $\mathrm{SO}_0(1, 2)$  with the unit tangent bundle of the hyperbolic plane  $\mathrm{SO}_0(1, 2)/\mathrm{SO}(2)$ . The projection of a matrix  $B \in \mathrm{SO}_0(1, 2)$  to its first column is the bundle projection and the fibrewise circle action is given by

$$(2.45) \quad B \longmapsto B \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix}.$$

The inequalities  $B_{32} < 0$  and  $B_{23} > 0$  define two open intervals of length  $\pi$  in each circle fibre. These two intervals must intersect either in a connected open subinterval or a pair of points. In the latter case, however, these two points correspond to matrices of the form

$$B = \begin{pmatrix} w_0 & x_0 & y_0 \\ w_1 & x_1 & 0 \\ w_2 & 0 & y_2 \end{pmatrix} \in \mathrm{SO}_0(1, 2).$$

Since  $x_1 y_2 = w_0 > 0$ , rotating by  $\pm \frac{\pi}{2}$  in the circle containing  $B$  we can arrange that both inequalities are satisfied. Thus the first case occurs and  $\mathcal{U}$  is an interval-bundle over  $\mathrm{SO}_0(1, 2)/\mathrm{SO}(2) \simeq \mathbb{R}^2$  as claimed.  $\square$

As an immediate corollary we obtain a characterisation of the invariant nearly hypo structures embedded as hypersurfaces of the sine-cone.

**Corollary 2.46.** *Let  $\psi_{\lambda, \mu, A}$  be an invariant nearly hypo structure on  $N_{1,1}$  such that  $x_0 = 0 = y_0$  (equivalently,  $w_1 = 0 = w_2$ ) in (2.42). Then  $(N_{1,1}, \psi_{\lambda, \mu, A})$  is an invariant hypersurface of the sine-cone over an invariant Sasaki–Einstein structure on  $N_{1,1}$ .*

### 3. COHOMOGENEITY ONE NEARLY KÄHLER MANIFOLDS

In this section we begin the study of cohomogeneity one nearly Kähler 6–manifolds proper. After quickly reviewing basic facts about smooth compact simply connected cohomogeneity one spaces we recall Podestà and Spiro’s classification of possible compact simply connected cohomogeneity one nearly Kähler 6–manifolds. The only potentially interesting cases all turn out to have principal orbit  $S^2 \times S^3$  invariant under  $\mathrm{SU}(2) \times \mathrm{SU}(2)$  with isotropy group  $\Delta \mathrm{U}(1)$ . Using our results on  $\mathrm{SU}(2) \times \mathrm{SU}(2)$ –invariant nearly hypo structures on  $S^2 \times S^3$  we specialise the nearly hypo evolution equations (2.37b) to this invariant setting and derive the fundamental ODEs (3.10) satisfied by any  $\mathrm{SU}(2) \times \mathrm{SU}(2)$ –invariant nearly Kähler structure on the (open dense) subset of principal orbits. We note the continuous and discrete symmetries of the fundamental ODEs, explaining their geometric origins. The discrete symmetries in particular play an important role in our construction of new complete cohomogeneity one nearly Kähler structures. We have been unable to find a closed form for the general solution to (3.10); however, four explicit solutions are described and their geometric significance explained. Two of these four solutions also play important roles in the proof of the Main Theorem.

Let  $M$  be a complete nearly Kähler 6–manifold (as always in the sense of Definition 1.3) acted upon isometrically by a connected compact Lie group  $G$  with cohomogeneity one, *ie* the orbit space  $M/G$  is 1–dimensional. Since  $M$  is Einstein with positive Einstein constant,  $M$  is compact with finite fundamental group. By [11, Theorem 9.3, Chapter I] the universal cover

of  $M$  is also a cohomogeneity one nearly Kähler manifold; hence there is no loss of generality in assuming  $M$  simply connected, and we will do so throughout the rest of this paper.

From the general theory of cohomogeneity one spaces [11, Theorem 8.2, Chapter IV],  $M/G$  is then a connected closed interval  $[0, T] \subset \mathbb{R}$ . The open set  $M^* \subset M$  corresponding to the interior of  $M/G$  is diffeomorphic to  $(0, T) \times G/K$ . We call  $K \subset G$  the principal isotropy subgroup and  $G/K$  the *principal orbit*. Corresponding to the boundary points of  $M/G$  there are two lower-dimensional *singular orbits* with isotropy subgroups  $K_1$  and  $K_2$  respectively. We call  $K_1$  and  $K_2$  the singular isotropy subgroups. They both contain the principal isotropy subgroup  $K$  and the coset  $K_i/K$  is diffeomorphic to a sphere. Moreover, there are representations  $K_i \rightarrow O(V_i)$  on Euclidean spaces  $V_i$  such that a neighbourhood of the singular orbit  $G/K_i$  in  $M$  is  $G$ -equivariantly diffeomorphic to a neighbourhood of the zero section of the vector bundle  $G \times_{K_i} V_i \rightarrow G/K_i$ . In fact,  $M$  is obtained by identifying the two disc bundles  $G \times_{K_i} D_i$ ,  $D_i \subset V_i$ , along their common boundary  $G/K$ .

The set of inclusions  $K \subset K_1, K_2 \subset G$  is called the *group diagram* of  $M$ . Two cohomogeneity one manifolds are  $G$ -equivariantly diffeomorphic if their group diagrams can be obtained one from the other with the following operations:

- (i) interchanging  $K_1$  and  $K_2$ ;
- (ii) conjugating  $K, K_1, K_2$  by the same element of  $G$ ;
- (iii) replacing  $K_1$  with  $hK_1h^{-1}$ , where  $h$  is an element of the connected component of the normaliser of  $K$  in  $G$ .

In [40, Theorem 1.1] Podestà and Spiro classified all possible group diagrams of cohomogeneity one nearly Kähler 6-manifolds; the list is given in Table 1. The last case is the sine-cone

$G$	$K$	$K_1$	$K_2$	$M$
$SU(2) \times SU(2)$	$\Delta U(1)$	$\Delta SU(2)$	$\Delta SU(2)$	$S^3 \times S^3$
$SU(2) \times SU(2)$	$\Delta U(1)$	$\Delta SU(2)$	$U(1) \times SU(2)$	$S^6$
$SU(2) \times SU(2)$	$\Delta U(1)$	$U(1) \times SU(2)$	$SU(2) \times U(1)$	$CP^3$
$SU(2) \times SU(2)$	$\Delta U(1)$	$U(1) \times SU(2)$	$U(1) \times SU(2)$	$S^2 \times S^4$
$SU(3)$	$SU(2)$	$SU(3)$	$SU(3)$	$S^6$

TABLE 1. Group diagrams of cohomogeneity one nearly Kähler 6-manifolds

over the round Sasaki–Einstein structure on  $\mathbb{S}^5 \simeq SU(3)/SU(2)$ . Since the space of invariant metrics on  $\mathbb{S}^5$  is 1-dimensional, it is clear that this is the unique nearly Kähler structure arising in that case.

The interesting case is therefore  $G = SU(2) \times SU(2)$  with principal orbit  $N_{1,1} = SU(2) \times SU(2)/\Delta U(1)$ , which motivates our interest in invariant nearly hypo structures on this homogeneous space. We will see later in the section that the first three group diagrams in the list are realised by known homogeneous nearly Kähler manifolds; on the other hand, no nearly Kähler structure is known to exist on  $S^2 \times S^4$ . In fact, this case is overlooked in [40].

*Remark.* Böhm [8] constructed infinitely many cohomogeneity one Einstein metrics on some of the manifolds of Table 1. By [40, Lemma 3.1] these cannot be induced by a nearly Kähler structure since on a nearly Kähler 6-manifold not isometric to the round 6-sphere any isometry must also preserve the almost complex structure. In particular, the isotropy group of the principal orbit must be contained in  $SU(2)$ . For example, the metrics constructed by Böhm on  $S^3 \times S^3$  are invariant under the action of  $SO(3) \times SO(4)$  with principal isotropy group  $SO(2) \times SO(3) \subset SO(5)$ .

**The fundamental ODE system.** In order to study the nearly hypo evolution equations (2.37b) restricted to invariant nearly hypo structures it is convenient to introduce an alternative parametrisation to the one of Proposition 2.41, whose main advantage is to reduce the problem to the study of a first order ODE system, rather than the mixed differential and algebraic system given by (2.37a) and (2.37b).

Given an invariant nearly hypo structure  $\psi_{\lambda,\mu,A}$  with  $A$  as in (2.42) we write

$$(3.1a) \quad \eta = \lambda\eta^{\text{se}}, \quad \omega_1 = u_0\omega_0^{\text{se}} + u_1\omega_1^{\text{se}} + u_2\cos\theta\omega_2^{\text{se}} + u_2\sin\theta\omega_3^{\text{se}}.$$

The first equation in (2.37a) and (2.37b), respectively, is equivalent to

$$(3.1b) \quad \omega_2 = -\frac{u_2}{\lambda}\sin\theta\omega_2^{\text{se}} + \frac{u_2}{\lambda}\cos\theta\omega_3^{\text{se}}, \quad \omega_3 = \frac{v_0}{\lambda}\omega_0^{\text{se}} + \frac{v_1}{\lambda}\omega_1^{\text{se}} + \frac{v_2}{\lambda}\cos\theta\omega_2^{\text{se}} + \frac{v_2}{\lambda}\sin\theta\omega_3^{\text{se}},$$

where  $v_0, v_1, v_2$  are determined by  $\lambda$  and  $u_0, u_1, u_2$  via

$$(3.2a) \quad \lambda\dot{u}_0 + 3v_0 = 0,$$

$$(3.2b) \quad \lambda\dot{u}_1 + 3v_1 - 2\lambda^2 = 0,$$

$$(3.2c) \quad \lambda\dot{u}_2 + 3v_2 = 0.$$

Here  $\dot{\cdot}$  denotes differentiation with respect to the arc length parameter  $t$  along a geodesic orthogonal to the principal orbits. Thus if  $B \in \mathcal{U}$  is the matrix

$$B = \begin{pmatrix} w_0 & x_0 & y_0 \\ w_1 & x_1 & y_1 \\ w_2 & x_2 & y_2 \end{pmatrix}$$

then the change of variables from the parametrisation of invariant nearly hypo structures in Proposition 2.41 to the one in (3.1) is

$$(3.3) \quad u_i = \mu x_i, \quad v_i = \lambda\mu y_i, \quad i = 0, 1, 2, \quad \text{where } \lambda = -x_2, \quad \mu = -x_2y_1.$$

The second equation of (2.37b) implies

$$\dot{\theta} = 0, \quad \lambda\dot{u}_2 + 3v_2 = 0.$$

Since we are free to change  $\theta$  by acting by the flow of the Reeb vector field  $U^-$ , we assume without loss of generality that  $\theta = 0$ . Then the last equation of (2.37b) is

$$(3.4a) \quad \dot{v}_0 - 4\lambda u_0 = 0,$$

$$(3.4b) \quad \dot{v}_1 - 4\lambda u_1 = 0,$$

$$(3.4c) \quad \lambda\dot{v}_2 - 4\lambda^2 u_2 + 3u_2 = 0.$$

Besides (3.2) and (3.4), necessary conditions for  $\psi_{\lambda,\mu,A}$  to define a nearly hypo structure are the algebraic constraints

$$(3.5a) \quad I_1 = \langle u, v \rangle = 0,$$

$$(3.5b) \quad I_2 = \lambda^2|u|^2 - u_2^2 = 0,$$

$$(3.5c) \quad I_3 = \lambda^2|u|^2 - |v|^2 = 0,$$

$$(3.5d) \quad I_4 = v_1 - |u|^2 = 0,$$

which correspond to  $\omega_1 \wedge \omega_3 = 0$ ,  $\omega_1^2 = \omega_2^2$ ,  $\omega_1^2 = \omega_3^2$  and the second equation of (2.37a), respectively;  $\omega_2 \wedge \omega_1 = 0 = \omega_2 \wedge \omega_3$  follow immediately from (3.1). Here  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$  denote the metric of signature  $(-+++)$  on  $\mathbb{R}^{1,2}$ . Furthermore, the pair of vectors  $u, v \in \mathbb{R}^{1,2}$  has to satisfy the sign constraint

$$(3.6) \quad u_1v_2 - u_2v_1 > 0,$$

which corresponds to the requirement that the matrix  $A$  in (2.42) lies in the restricted Lorentz group.

Following [41, Proposition 5.1], we now derive a differential equation for  $\lambda$  which will imply that equations (3.5) have to be imposed only at the initial time and will then be conserved for all time. Differentiating (3.5) using (3.2) and (3.4) we obtain:

$$(3.7a) \quad \lambda \dot{I}_1 = 3I_2 + 3I_3 + 2\lambda^2 I_4,$$

$$(3.7b) \quad \lambda \dot{I}_2 = -6\lambda^2 I_1 + 2\lambda^2 |u|^2 \dot{\lambda} + 4\lambda^4 u_1 + 6u_2 v_2,$$

$$(3.7c) \quad \lambda \dot{I}_3 = -14\lambda^2 I_1,$$

$$(3.7d) \quad \lambda \dot{I}_4 = 6I_2.$$

Thus if  $\lambda$  satisfies the first order differential equation

$$(3.8) \quad \lambda^2 |u|^2 \dot{\lambda} + 2\lambda^4 u_1 + 3u_2 v_2 = 0,$$

then  $I = (I_1, \dots, I_4)$  satisfies a homogeneous linear system (with coefficients depending on  $\lambda$ ) and is therefore uniquely determined by the initial conditions.

The following proposition follows immediately from this discussion.

**Proposition 3.9.** *Let  $(\lambda, u, v)$  be a solution of the ODE system*

$$(3.10a) \quad \lambda \dot{u}_0 + 3v_0 = 0,$$

$$(3.10b) \quad \lambda \dot{u}_1 + 3v_1 - 2\lambda^2 = 0,$$

$$(3.10c) \quad \lambda \dot{u}_2 + 3v_2 = 0,$$

$$(3.10d) \quad \dot{v}_0 - 4\lambda u_0 = 0,$$

$$(3.10e) \quad \dot{v}_1 - 4\lambda u_1 = 0,$$

$$(3.10f) \quad \lambda \dot{v}_2 - 4\lambda^2 u_2 + 3u_2 = 0,$$

$$(3.10g) \quad \lambda^2 |u|^2 \dot{\lambda} + 2\lambda^4 u_1 + 3u_2 v_2 = 0.$$

defined on an interval  $(a, b) \subset \mathbb{R}$  on which  $u_2 < 0$ ,  $\lambda, \mu^2 := |u|^2 > 0$  and (3.6) is satisfied. Assume also that  $I_1(t_0) = \dots = I_4(t_0) = 0$  for some  $a < t_0 < b$ . Then (3.1) defines an invariant nearly Kähler structure on  $(a, b) \times N_{1,1}$ . Conversely, any nearly Kähler structure on  $(a, b) \times N_{1,1}$  invariant under the action of  $SU(2) \times SU(2)$  on  $N_{1,1}$  takes the form (3.1) for a solution  $(\lambda, u, v)$  of (3.10) satisfying the given sign constraints and with  $I_1 = I_2 = I_3 = I_4 = 0$  for all time.

Moreover, given an invariant nearly hypo structure  $\psi_{\lambda, \mu, A}$  on  $N_{1,1}$  there exists a unique solution of (3.10) with initial condition  $\psi_{\lambda, \mu, A}$  (in particular,  $I_1(0) = \dots = I_4(0) = 0$ ). In particular, up to the action of the flow of the Reeb vector field there exists a 2-parameter family of local invariant nearly Kähler structures on  $(a, b) \times N_{1,1}$ .

*Remark.* In [22, Theorem 5] Conti shows that any compact real analytic nearly hypo 5-manifold can be embedded into a (local) real analytic nearly Kähler 6-manifold. The final statement of the Proposition, which follows from standard ODE theory, is a specialisation of Conti's result to the case of invariant nearly hypo structures on  $N_{1,1}$ .

**Symmetries of the fundamental ODE system.** In the rest of the paper we will make repeated use of various symmetries of the fundamental ODE system (3.10); the discrete symmetries of (3.10) in particular will turn out to play a crucial role in our construction of new complete nearly Kähler metrics on  $S^3 \times S^3$  and  $S^6$ .

To facilitate the description of these symmetries we introduce the alternative notation  $(\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t)$  for a solution  $\Psi = (\lambda(t), u_0(t), u_1(t), u_2(t), v_0(t), v_1(t), v_2(t))$  of (3.10).

**Proposition 3.11** (cf [41, Proposition 4.2]). *The system (3.10) is invariant under the following symmetries.*

- (i) *Time translation*  $t \mapsto t + t_0$ ,  $t_0 \in \mathbb{R}$ .
- (ii) *Time reversal*

$$\tau_1: (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (-\lambda, u_0, u_1, u_2, v_0, v_1, v_2, -t).$$

- (iii) *The involutions*  $\tau_2, \tau_3, \tau_4$  *defined by*

$$\tau_2: (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (-\lambda, -u_0, -u_1, -u_2, v_0, v_1, v_2, t),$$

$$\tau_3: (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (\lambda, u_0, u_1, -u_2, v_0, v_1, -v_2, t),$$

$$\tau_4: (\lambda, u_0, u_1, u_2, v_0, v_1, v_2, t) \mapsto (-\lambda, u_0, -u_1, u_2, -v_0, v_1, -v_2, t).$$

$\tau_2$  preserves the constraint  $u_2 = -\lambda\mu$ , while the remaining  $\tau_i$  send this constraint into  $u_2 = \lambda\mu$ . All the symmetries preserve the constraints (3.5).

*Proof.* The fact that these transformations are symmetries of (3.10) is straightforward to verify by direct computation. Instead we concentrate on explaining the geometric origin of each of these symmetries. The involutions  $\tau_1$  and  $\tau_2$  are simply the specialisation to the invariant case of the involutions  $\tau_1$  and  $\tau_2$  defined in Remark 2.2 and we already noted in Remark 2.39 that both involutions preserve the nearly hypo condition. The existence of the remaining involutions  $\tau_3$  and  $\tau_4$  is specific to the case of  $SU(2) \times SU(2)$ -invariant  $SU(2)$ -structures:  $\tau_3$  and  $\tau_4$  are induced by automorphisms of  $SU(2) \times SU(2)$  that fix  $\Delta U(1)$ . More precisely,  $\tau_4$  is the action on invariant  $SU(2)$ -structures of the outer automorphism of  $SU(2) \times SU(2)$  that exchanges the two factors. The Reeb vector field  $U^-$  generates the group of inner automorphisms of  $SU(2) \times SU(2)$  fixing  $\Delta U(1)$ . By normalising nearly hypo structures so that  $w_3 = 0$ , ie  $\theta = 0$  or  $\pi$  in (2.42), we quotient out this action except for a residual  $\mathbb{Z}_2$ -action generated by  $\tau_3$ .  $\square$

**Special solutions of the fundamental ODE system.** There are four distinguished solutions to the ODE system (3.10): the sine-cone over the homogeneous Sasaki–Einstein metric on  $N_{1,1}$  and the three homogeneous nearly Kähler solutions for which there is a subgroup of the full isometry group isomorphic to  $SU(2) \times SU(2)$ , namely nearly Kähler  $S^6$ ,  $CP^3$  and  $S^3 \times S^3$ . These special solutions will play a role in our analysis of general cohomogeneity one  $SU(2) \times SU(2)$ -invariant nearly Kähler metrics and provide explicit solutions of (3.10) that we will record below. It is immediate to verify that the given expressions define solutions of (3.10) but we also refer to [41, §4.2] where these expressions are derived.

**Example 3.12** (The sine-cone). The sine-cone over the standard Sasaki–Einstein structure on  $N_{1,1}$  has already been discussed in (2.40). In terms of (3.1), we have

$$\begin{aligned} \lambda &= \sin t, & u_0 &= 0, & u_1 &= \sin^2 t \cos t, & u_2 &= -\sin^3 t, \\ v_0 &= 0, & v_1 &= \sin^4 t, & v_2 &= \sin^3 t \cos t, \end{aligned}$$

for  $t \in [0, \pi]$ . The two conical singularities of the sine-cone occur at  $t = 0$  and  $t = \pi$ .

**Example 3.13** (The round sphere). In this case

$$\begin{aligned} \lambda &= \frac{3}{2} \cos t, & u_0 &= -\frac{3}{2} \sin t (2 - 5 \cos^2 t), & u_1 &= -3 \sin t (1 - 2 \cos^2 t), & u_2 &= -\frac{9}{2} \sin t \cos^2 t, \\ v_0 &= \frac{9}{4} \cos^2 t (4 - 5 \cos^2 t), & v_1 &= 9 \sin^2 t \cos^2 t, & v_2 &= \frac{9}{4} \cos^2 t (3 \cos^2 t - 2), \end{aligned}$$

for  $t \in [0, \frac{\pi}{2}]$ . The open set of principal orbits is compactified by adding a 3-sphere  $SU(2) \times SU(2)/\Delta SU(2)$  at  $t = 0$  and a 2-sphere  $SU(2) \times SU(2)/SU(2) \times U(1)$  at  $t = \frac{\pi}{2}$ .

**Example 3.14** (Homogeneous nearly Kähler structure on  $S^3 \times S^3$ ). The homogeneous nearly Kähler structure on  $S^3 \times S^3$  corresponds to the solution

$$\begin{aligned} \lambda &= 1, & u_0 &= u_1 = \frac{1}{\sqrt{3}} \sin(2\sqrt{3}t), & u_2 &= -\frac{2}{\sqrt{3}} \sin(\sqrt{3}t), \\ v_0 &= -\frac{2}{3} \cos(2\sqrt{3}t), & v_1 &= \frac{2}{3} (1 - \cos(2\sqrt{3}t)), & v_2 &= \frac{2}{3} \cos(\sqrt{3}t), \end{aligned}$$

for  $t \in [0, \frac{\pi}{\sqrt{3}}]$ . (Notice that there is a typo in [41, §4.2.3]: the range of  $t$  with their normalisations should be  $t \in [0, \frac{\pi}{\sqrt{6}}]$ .) The singular orbits are two 3–spheres,  $SU(2) \times SU(2)/\Delta SU(2)$  at  $t = 0$  and  $SU(2) \times SU(2)/\phi_3(\Delta SU(2))$  at  $t = \frac{\pi}{\sqrt{3}}$ , where  $\phi_3$  is the inner automorphism of  $SU(2) \times SU(2)$  generated by  $\frac{\pi}{4}U^-$ .

**Example 3.15** (Homogeneous nearly Kähler structure on  $CP^3$ ). In this case,

$$\begin{aligned} \lambda &= \frac{3\sqrt{2}}{4} \sin(\sqrt{2}t), & u_0 &= \frac{3}{8} (3 \cos^2(\sqrt{2}t) - 1), & u_1 &= \frac{3}{4} \cos(\sqrt{2}t), & u_2 &= -\frac{9}{8} \sin^2(\sqrt{2}t), \\ v_0 &= -\frac{9}{8} \cos(\sqrt{2}t) \sin^2(\sqrt{2}t), & v_1 &= \frac{9}{8} \sin^2(\sqrt{2}t), & v_2 &= \frac{9}{8} \cos(\sqrt{2}t) \sin^2(\sqrt{2}t), \end{aligned}$$

for  $t \in [0, \frac{\pi}{\sqrt{2}}]$ . The singular orbits are both 2–spheres,  $SU(2) \times SU(2)/U(1) \times SU(2)$  at  $t = 0$  and  $SU(2) \times SU(2)/SU(2) \times U(1)$  at  $t = \frac{\pi}{\sqrt{2}}$ .

#### 4. SOLUTIONS THAT EXTEND SMOOTHLY OVER THE SINGULAR ORBITS

In the previous sections we have described the subset consisting of principal orbits of a cohomogeneity one nearly Kähler manifold as a 1–parameter family of nearly hypo structures. In this section we will discuss (singular) boundary conditions for the ODE system (3.10), *ie* study conditions under which a cohomogeneity one nearly Kähler structure extends smoothly across a singular orbit. Recall from Table 1 that in the  $SU(2) \times SU(2)$ –invariant case there are only three types of singular orbits, a 3–sphere  $SU(2) \times SU(2)/\Delta SU(2)$  and 2–spheres  $SU(2) \times SU(2)/U(1) \times SU(2)$  and  $SU(2) \times SU(2)/SU(2) \times U(1)$ . The latter two are exchanged by the outer automorphism of  $SU(2) \times SU(2)$ . The main results of the section are Theorems 4.4 and 4.5: these establish the existence of two 1–parameter families of local cohomogeneity one nearly Kähler 6–manifolds that close smoothly on a singular orbit that is a round sphere of dimension two or dimension three, respectively. In both cases the parameter is the size of the singular orbit. In subsequent sections, by studying the behaviour of these two 1–parameter families as the size of the singular orbit shrinks to zero, we will show that they should be viewed as nearly Kähler deformations of the Calabi–Yau structures on the small resolution and on the smoothing of the conifold respectively.

To understand the conditions under which invariant tensors on a cohomogeneity one manifold extend smoothly across a singular orbit we will appeal to a method due to Eschenburg and Wang [27, §1]. For the convenience of the reader, we describe it briefly here.

Let  $M^n$  be a smooth manifold with a cohomogeneity one isometric action of a compact Lie group  $G$ . Let  $Q = G/K'$  be a singular orbit. Set  $V = T_q M/T_q Q$  and recall that a neighbourhood of  $Q$  in  $M$  is  $G$ –equivariantly diffeomorphic to a neighbourhood of the zero section of the normal bundle  $E = G \times_{K'} V \rightarrow Q$ .

In view of our applications and for concreteness, we only discuss conditions under which a  $G$ –invariant section  $h \in \Gamma(E; \text{End}(TE))$  extends smoothly over the zero section, but the method generalises to arbitrary tensors.  $G$ –invariance implies that  $h$  is determined by its restriction to  $V \simeq E_q$ . The choice of a complement  $\mathfrak{p}$  of the Lie algebra of  $K'$  in the Lie algebra of  $G$



determines a trivialisation  $TE|_V = V \oplus \mathfrak{p}$ . Fix a point  $v_0 \in V$  with  $|v_0| = 1$  (having fixed an invariant inner product on  $V$ ) and denote by  $K$  its stabiliser in  $G$ . The principal orbits of  $M$  are diffeomorphic to  $G/K = G \times_{K'} S^{d-1} \rightarrow Q$ , where  $S^{d-1}$  is the unit sphere in  $V$ .

Introduce polar coordinates  $(t, \sigma) \in [0, \infty) \times S^{d-1} \simeq V$ . Then  $h \in \Gamma(V; \text{End}(V \oplus \mathfrak{p}))$  can be thought of as a 1-parameter family of maps  $h_t \in \Gamma(S^{d-1}; \text{End}(V \oplus \mathfrak{p}))$ . Since  $K'$  acts transitively on  $S^{d-1}$ , the space  $\mathcal{W}$  of  $K'$ -equivariant maps  $h_t: S^{d-1} \rightarrow \text{End}(V \oplus \mathfrak{p})$  is isomorphic to  $\text{End}(V \oplus \mathfrak{p})^K$  via the evaluation at  $v_0$ . Denote by  $\mathcal{W}_p$  the subspace of  $\mathcal{W}$  consisting of the restriction of homogeneous polynomials of degree  $p$ . Notice that we can always increase the degree of  $h \in \mathcal{W}_p$  by 2 by multiplying  $h$  by  $v \mapsto |v|^2$ . However, by finite dimensionality of  $\mathcal{W}$  and polynomial approximation, we can find a basis of the vector space  $\text{End}(V \oplus \mathfrak{p})^K$  such that every element corresponds to a  $K'$ -equivariant homogeneous polynomial of minimum degree  $p \geq 0$ . Then a curve  $h_t \in \mathcal{W}$  defined for  $t \in [0, T)$  represents a smooth section  $h \in C^\infty(E; \text{End}(TE))$  if and only if it has Taylor series at  $t = 0$ ,  $h_t = \sum_{p \geq 0} h_p t^p$ , with  $h_p \in \mathcal{W}_p$  for all  $p$  [27, Lemma 1.1]. In this way the problem is reduced to a representation-theoretic computation.

We now specialise our discussion to the case  $n = 6$ ,  $G = \text{SU}(2) \times \text{SU}(2)$ ,  $K = \Delta\text{U}(1)$  and  $K' = \Delta\text{SU}(2), \text{U}(1) \times \text{SU}(2)$  or  $\text{SU}(2) \times \text{U}(1)$ .

**Closing smoothly on an  $S^2$ .** We first consider the case  $K' = \text{U}(1) \times \text{SU}(2)$  and  $Q \simeq S^2$ . Then  $\mathfrak{p} = \mathfrak{n}_1 = \text{Span}\{E_1, V_1\}$  in the notation of (2.8) and  $V \simeq \mathbb{H}$ , where  $\text{U}(1) \times \text{SU}(2)$  acts on  $V$  via  $(e^{i\theta}, q) \cdot x = qxe^{-i\theta}$ . In particular, the vector bundle  $E = (\text{SU}(2) \times \text{SU}(2)) \times_{\text{U}(1) \times \text{SU}(2)} V$  is isomorphic to  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow S^2$ .

As a  $\Delta\text{U}(1)$ -representation  $V = \mathbb{R}^2 \oplus \mathfrak{n}_2$ , where  $\mathfrak{n}_2 \simeq \mathfrak{n}$  is the standard representation of  $\text{U}(1)$  and  $\mathbb{R}^2$  is a trivial 2-dimensional representation. It follows that  $\text{End}(V \oplus \mathfrak{p})^{\text{U}(1)} = \text{End}(\mathbb{R}^2) \oplus 4\text{End}(\mathfrak{n})^{\text{U}(1)}$ . Moreover,  $\text{End}(\mathfrak{n})^{\text{U}(1)}$  is 2-dimensional, spanned by the identity and the complex structure.

Since  $\text{SU}(2)$  already acts transitively on the unit sphere in  $V$ , any  $\text{U}(1) \times \text{SU}(2)$ -equivariant polynomial  $h$  on  $\mathbb{S}^3 \subset V$  with values in  $\text{End}(V \oplus \mathfrak{p})$  must satisfy  $h(q) = M^{-1}h(1)M$ , where  $h(1) \in \text{End}(V \oplus \mathfrak{p})^{\text{U}(1)}$ ,  $M = (1, q) \in \text{U}(1) \times \text{SU}(2)$  and we identified  $\mathbb{S}^3$  with  $\text{SU}(2)$ . A simple computation then shows that

- (i)  $\text{id}_{\mathfrak{n}_1}, j_{\mathfrak{n}_1}, \text{id}_{\mathbb{R}^2} + \text{id}_{\mathfrak{n}_2}, j_{\mathbb{R}^2} + j_{\mathfrak{n}_2}$  correspond to constant polynomials (they are preserved by  $\text{U}(1) \times \text{SU}(2)$ );
- (ii)  $\text{End}(\mathfrak{n}_i, \mathfrak{n}_j)^{\text{U}(1)}$  with  $i \neq j$  correspond to polynomials of degree 1;
- (iii)  $\text{id}_{\mathbb{R}^2} - \text{id}_{\mathfrak{n}_2}, j_{\mathbb{R}^2} - j_{\mathfrak{n}_2}$  and  $\text{Sym}_0^2(\mathbb{R}^2)$  correspond to degree 2 polynomials.

Here  $j_X$  denotes the standard complex structure on  $X$ .

The general theory of [27] now gives conditions for the smooth extension of any  $\text{U}(1) \times \text{SU}(2)$ -equivariant  $\text{End}(V \oplus \mathfrak{p})$ -valued polynomial on  $V$ . However, in order to interpret these conditions as initial data for the ODE system (3.10) it is necessary to change coordinates from this description of a neighbourhood of the singular orbit to the coframe  $dt, u^-, e_i, v_i$  that we introduced on the set of principal orbits. To this end embed  $S^2 \times V$  in  $\text{Im } \mathbb{H} \times \mathbb{H}$  and let  $\text{SU}(2) \times \text{SU}(2)$  act via  $(q_1, q_2) \cdot (x, y) = (q_1 x q_1^*, q_2 y q_2^*)$ . Along the ray  $\gamma(t) = (i, t)$  the vector fields  $U^-, E_i, V_i$  of (2.8) are

$$\begin{aligned} U^- &= (0, -it), & E_1 &= \frac{1}{2\sqrt{2}}(-2k, -jt), & V_1 &= \frac{1}{2\sqrt{2}}(2j, -kt), \\ E_2 &= \frac{1}{2\sqrt{2}}(0, jt), & V_2 &= \frac{1}{2\sqrt{2}}(0, kt). \end{aligned}$$

Thus if  $(t, q_1, q_2, q_3)$  are coordinates on  $V = \mathbb{H}$ , along  $\gamma$  we have  $dt = dq_0$ ,  $tu^- = dq_1$ ,  $te_2 = 2\sqrt{2}dq_2$  and  $tv_2 = 2\sqrt{2}dq_3$ , while  $e_1, v_1$  are 1-forms along  $S^2$ .

In the notation of (3.1a) we write an invariant 2-form on  $(0, \epsilon) \times N_{1,1}$  as

$$\omega = \lambda \eta^{\text{se}} \wedge dt + u_0 \omega_0^{\text{se}} + u_1 \omega_1^{\text{se}} + u_2 \omega_2^{\text{se}} + u_3 \omega_3^{\text{se}},$$

where  $\lambda, u_0, \dots, u_3$  are functions depending on time only. Recalling (2.9), the change of variables above yields

$$\begin{aligned} \omega = & -\frac{2\lambda}{3t} dq_0 \wedge dq_1 + \frac{u_0 + u_1}{12} e_1 \wedge v_1 + \frac{2(u_0 - u_1)}{3t^2} dq_2 \wedge dq_3 + \\ & \frac{\sqrt{2}u_2}{6t} (e_1 \wedge dq_3 + dq_2 \wedge v_1) + \frac{\sqrt{2}u_3}{6t} (e_1 \wedge dq_2 - dq_3 \wedge v_1). \end{aligned}$$

It is now straightforward to use the representation-theoretic computation to deduce conditions on the coefficients of  $\omega$  so that it extends smoothly at  $t = 0$  along the singular orbit.

**Lemma 4.1.** *Let  $\omega = \lambda dt \wedge \eta^{\text{se}} + u_0 \omega_0^{\text{se}} + u_1 \omega_1^{\text{se}} + u_2 \omega_2^{\text{se}} + u_3 \omega_3^{\text{se}}$  be an invariant 2-form on  $(0, T) \times N_{1,1}$ . Then:*

- (i)  $\omega$  extends smoothly over a singular orbit  $SU(2) \times SU(2)/U(1) \times SU(2)$  at  $t = 0$  if and only if
  - (a)  $u_0, u_1, u_2, u_3$  are even and  $\lambda$  is odd;
  - (b)  $u_2(0) = 0 = u_3(0)$  and  $u_0(t) - u_1(t) = -\dot{\lambda}(0)t^2 + O(t^4)$ , where  $\dot{\lambda}$  denotes the derivative of  $\lambda$  with respect to  $t$ .
- (ii)  $\omega$  extends smoothly over a singular orbit  $SU(2) \times SU(2)/SU(2) \times U(1)$  at  $t = 0$  if and only if
  - (a)  $u_0, u_1, u_2, u_3$  are even and  $\lambda$  is odd;
  - (b)  $u_2(0) = 0 = u_3(0)$  and  $u_0(t) + u_1(t) = \dot{\lambda}(0)t^2 + O(t^4)$ .

*Proof.* The statement in (i) follows immediately from the representation-theoretic computation, since  $u_0 + u_1$  must be even,  $\frac{u_2}{t}$  and  $\frac{u_3}{t}$  odd,  $-\frac{2\lambda}{3t}$  and  $\frac{2(u_0 - u_1)}{3t^2}$  are both even and well-defined at  $t = 0$  where they must take the same value. Since  $U(1) \times SU(2)$  and  $SU(2) \times U(1)$  are exchanged by the outer automorphism of  $SU(2) \times SU(2)$ , the statement in (ii) follows immediately from (i) after applying the discrete symmetry  $\tau_4$  of Proposition 3.11.  $\square$

**Closing smoothly on an  $S^3$ .** When  $K' = \Delta SU(2)$  and  $Q \simeq S^3$ ,  $V$  and  $\mathfrak{p}$  are both isomorphic to the adjoint representation of  $SU(2)$ . The vector bundle  $E$  is trivial and we can identify it with  $T^*S^3$ .

Since  $\mathfrak{su}_2 = \mathbb{R} \oplus \mathfrak{n}$  as a  $U(1)$ -representation,  $\text{End}(V \oplus \mathfrak{p})^{U(1)} = 4 \text{End}(\mathbb{R} \oplus \mathfrak{n})^{U(1)}$ . It is easy to see that  $\text{End}(\mathbb{R} \oplus \mathfrak{n})^{U(1)}$  is a 3-dimensional vector space, spanned by  $\text{id}_{\mathbb{R}}$ ,  $\text{id}_{\mathfrak{n}}$  and  $j_{\mathfrak{n}}$ . Under the identification of the action of  $SU(2)$  on its Lie algebra with the standard action of  $SO(3)$  on  $\mathbb{R}^3$ ,  $SO(3)$ -equivariant polynomials on  $\mathbb{R}^3$  with values in  $\text{End}(\mathbb{R}^3)$  are generated by the constant polynomial  $\text{id}$ , the degree two polynomial  $v \mapsto \langle \cdot, v \rangle v$  and the degree one polynomial  $v \mapsto \cdot \times v$ , which by evaluation at  $(1, 0, 0)$  correspond to  $\text{id}_{\mathbb{R}} + \text{id}_{\mathfrak{n}}$ ,  $\text{id}_{\mathbb{R}}$  and  $j_{\mathfrak{n}}$ , respectively.

In order to understand how to change coordinates from this description of the neighbourhood of the singular orbit to the coframe  $dt, u^-, e_i, v_i$  on the set of principal orbits, think of  $S^3 \times \mathbb{R}^3$  as embedded in  $\mathbb{H} \times \text{Im } \mathbb{H}$  and let  $SU(2) \times SU(2)$  act via  $(q_1, q_2) \cdot (x, y) = (q_2 x q_1^*, q_1 y q_1^*)$ . Along the ray  $\gamma(t) = (1, it)$  the vector fields  $U^-, E_i, V_i$  of (2.8) are

$$\begin{aligned} U^- = (-i, 0), \quad E_1 = \frac{1}{2\sqrt{2}}(-j, -2kt), \quad V_1 = \frac{1}{2\sqrt{2}}(-k, 2jt), \\ E_2 = \frac{1}{2\sqrt{2}}(j, 0), \quad V_2 = \frac{1}{2\sqrt{2}}(k, 0). \end{aligned}$$

Thus if  $(t, x, y)$  are coordinates on  $\mathbb{R}^3 = \text{Im } \mathbb{H}$ , along  $\gamma$  we have  $e_+ = -\frac{\sqrt{2}}{t}dy$  and  $v_+ = \frac{\sqrt{2}}{t}dx$ , where  $e_\pm, v_\pm$  and are the 1-forms dual to the vector fields  $E_1 \pm E_2$  and  $V_1 \pm V_2$ , respectively. On the other hand,  $u^-, e_-, v_-$  gives instead a coframe on the singular orbit  $S^3$ .

**Lemma 4.2** (cf [41, Proposition 6.1]). *An invariant 2-form  $\omega = \lambda dt \wedge \eta^{\text{se}} + u_0 \omega_0^{\text{se}} + u_1 \omega_1^{\text{se}} + u_2 \omega_2^{\text{se}} + u_3 \omega_3^{\text{se}}$  on  $(0, T) \times N_{1,1}$  extends smoothly over a singular orbit  $\text{SU}(2) \times \text{SU}(2)/\Delta\text{SU}(2)$  if and only if*

- (i)  $u_0, u_1, u_2$  are odd and  $\lambda, u_3$  are even;
- (ii)  $u_0 + u_2 = O(t^3)$ ,  $u_3 = O(t^2)$  and  $u_1(t) = 2\lambda(0)t + O(t^3)$ .

*Remark.* Lemmas 4.1 and 4.2 give conditions for an invariant 2-form  $\omega$  to extend smoothly along a singular orbit. Suppose now we have a solution  $\Psi = (\lambda, u, v)$  of the fundamental ODE system (3.10) such that  $\lambda, u_0, u_1, u_2$  satisfy the conditions of either of these lemmas (with  $u_3 = 0$ ) at  $t = 0$  and the constraints  $I_1 = \dots = I_4 = 0$  of (3.5) hold for all time. Thus  $\Psi$  defines an invariant nearly Kähler structure  $(\omega, \Omega)$  on  $(0, T) \times N_{1,1}$  and  $\omega$  extends smoothly at  $t = 0$ . Since  $3\text{Re } \Omega = d\omega$  and  $\text{Im } \Omega$  is determined algebraically by  $\text{Re } \Omega$  [36, §2], it follows that the whole  $\text{SU}(3)$ -structure  $(\omega, \Omega)$  extends smoothly at  $t = 0$ .

**Extension of symmetries over a singular orbit.** Before dealing with the existence of solutions of the fundamental ODE system (3.10) closing smoothly on a singular orbit, we discuss here which symmetries of Proposition 3.11 extend over the singular orbits of Table 1.

**Lemma 4.3.** *Let  $\Psi(t)$  be a solution of the fundamental ODE system (3.10) defined on  $[0, T)$ .*

- (i) *If  $\Psi$  extends smoothly over a singular orbit  $S^2 = \text{SU}(2) \times \text{SU}(2)/\text{U}(1) \times \text{SU}(2)$  at  $t = 0$  then so does  $\tau_i(\Psi)$  for  $i = 1, 2, 3$ , while  $\tau_4(\Psi)$  extends smoothly over a singular orbit  $\text{SU}(2) \times \text{SU}(2)/\text{SU}(2) \times \text{U}(1)$ .*
- (ii) *If  $\Psi$  extends smoothly over a singular orbit  $S^3 = \text{SU}(2) \times \text{SU}(2)/\Delta\text{SU}(2)$  at  $t = 0$  then so does  $\tau_i(\Psi)$   $i = 1, 2, 4$ , while  $\tau_3(\Psi)$  extends smoothly over a singular orbit  $\text{SU}(2) \times \text{SU}(2)/\phi_3(\Delta\text{SU}(2))$ , where  $\phi_3 = \text{Ad}(\exp(\frac{\pi}{4}U^-))$ .*

*Proof.* It is obvious that  $\tau_1$  and  $\tau_2$  preserve the boundary conditions for closing smoothly on a singular orbit of any type. It remains to check whether the automorphisms  $\phi_3$  and  $\phi_4$  corresponding to  $\tau_3$  and  $\tau_4$ , respectively, fix not only  $\Delta\text{U}(1)$  but also the stabiliser  $K'$  of a point on the singular orbit.

Denote by  $(\phi_j)_*$  the infinitesimal action of  $\phi_j$ ,  $j = 3, 4$ , on  $\mathbb{R}U^- \oplus \mathfrak{n}_1 \oplus \mathfrak{n}_2$ , identified with the tangent space of  $N_{1,1}$  at a point. Then

$$(\phi_3)_* = \text{Ad}\left(\exp\left(\frac{\pi}{4}U^-\right)\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad (\phi_4)_* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Lie algebras of  $K_1 = \text{U}(1) \times \text{SU}(2)$  and  $K_2 = \text{SU}(2) \times \text{U}(1)$  are  $\mathbb{R}U^+ \oplus \mathbb{R}U^- \oplus \mathfrak{n}_1$  and  $\mathbb{R}U^+ \oplus \mathbb{R}U^- \oplus \mathfrak{n}_2$ , respectively. Thus  $\phi_3$  preserves both  $K_1$  and  $K_2$ , while  $\phi_4$  exchanges them. Similarly,  $(\phi_3)_*$  exchanges  $\mathfrak{n}_+$  and  $\mathfrak{n}_-$ , where  $\mathfrak{n}_\pm$  is the subspace generated by  $E_\pm = E_1 \pm E_2$  and  $V_\pm = V_1 \pm V_2$ , while  $\phi_4$  preserves  $\Delta\text{SU}(2)$ .  $\square$

**Existence of solutions extending smoothly over a singular orbit.** The main results of this section are the existence of two 1-parameter families of solutions to the fundamental ODE system (3.10) closing smoothly on a singular orbit.

**Theorem 4.4** (Invariant nearly Kähler metrics extending smoothly over a singular orbit  $S^2$ ). *For every  $a > 0$  there exists a unique solution  $\Psi_a$  to (3.10) satisfying the initial conditions*

$$\lambda(0) = u_2(0) = 0, \quad u_0(0) = u_1(0) = a^2, \quad v_0(0) = v_1(0) = v_2(0) = 0,$$

and such that  $\lambda, u_0, u_1, u_2$  satisfy the conditions of Lemma 4.1(i). In other words, for every  $a > 0$  there exists a unique  $SU(2) \times SU(2)$ -invariant smooth nearly Kähler structure defined in a sufficiently small neighbourhood of the zero section of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow S^2$  and such that the zero section has volume  $a^2$  with respect to the induced metric. Moreover,  $\Psi_a$  depends continuously on  $a \in (0, \infty)$ .

The singular orbit  $S^2$  here is  $SU(2) \times SU(2)/U(1) \times SU(2)$ . The composition  $\tau_2 \circ \tau_3 \circ \tau_4$  preserves the sign constraints  $\lambda > 0$ ,  $u_2 < 0$ ,  $v_1 > 0$  and by Lemma 4.3 it sends  $\Psi_a$  to a solution of (3.10) which extends smoothly across a singular orbit  $SU(2) \times SU(2)/SU(2) \times U(1)$ .

**Theorem 4.5** (Invariant nearly Kähler metrics extending smoothly over a singular orbit  $S^3$ ). *For every  $b > 0$  there exists a unique solution  $\Psi_b$  to (3.10) satisfying the initial conditions*

$$\lambda(0) = b, \quad v_0(0) = -v_2(0) = -\frac{2}{3}b^3, \quad v_1(0) = 0, \quad u_i(0) = 0, \quad i = 0, 1, 2.$$

*In other words, for every  $b > 0$  there exists a unique  $SU(2) \times SU(2)$ -invariant smooth nearly Kähler structure defined in a sufficiently small neighbourhood of the zero section of  $T^*S^3$  such that the zero section has volume  $b^3$  with respect to the induced metric. Moreover,  $\Psi_b$  depends continuously on  $b \in (0, \infty)$ .*

*Remark 4.6.* The solutions of Examples 3.13, 3.14 and 3.15 of course belong to the two 1-parameter families of the theorems. More precisely, the round nearly Kähler structure on  $S^6$ , Example 3.13, is  $\Psi_a$  with  $a = \sqrt{3}$  and  $\Psi_b$  with  $b = \frac{3}{2}$ , the homogeneous nearly Kähler structure on  $S^3 \times S^3$ , Example 3.14, is  $\Psi_b$  with  $b = 1$  and the homogeneous nearly Kähler structure on  $CP^3$ , Example 3.15, is  $\Psi_a$  with  $a = \frac{\sqrt{3}}{2}$ .

The main technical tool for proving these theorems is the following general result about first order singular initial value problems. We will appeal to it repeatedly in the paper.

**Theorem 4.7.** *Consider the singular initial value problem*

$$(4.8) \quad \dot{y} = \frac{1}{t}M_{-1}(y) + M(t, y), \quad y(0) = y_0,$$

where  $y$  takes values in  $\mathbb{R}^k$ ,  $M_{-1}: \mathbb{R}^k \rightarrow \mathbb{R}^k$  is a smooth function of  $y$  in a neighbourhood of  $y_0$  and  $M: \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}^k$  is smooth in  $t, y$  in a neighbourhood of  $(0, y_0)$ . Assume that

- (i)  $M_{-1}(y_0) = 0$ ;
- (ii)  $hId - d_{y_0}M_{-1}$  is invertible for all  $h \in \mathbb{N}$ ,  $h \geq 1$ .

*Then there exists a unique solution  $y(t)$  of (4.8). Furthermore  $y$  depends continuously on  $y_0$  satisfying (i) and (ii).*

The condition (ii) guarantees the existence of a unique formal power series solution  $y(t)$  to (4.8). Once a formal power series solution has been shown to exist, one can follow the arguments of [38, Theorem 7.1], [27, §5] and [29, §4]: use a truncation of the power series of sufficiently high degree as an approximate solution to (4.8) and deform it to a genuine solution by applying a contraction mapping fixed point argument. As for the continuous dependence on the initial conditions, one argues as in [29, §4]: the coefficients of the formal power series solution  $y(t)$  depend differentiably on  $y_0$  satisfying (i) and (ii) and the operator used in the fixed point argument is uniformly contracting with respect to the initial conditions.

*Proof of Theorem 4.4.* We first reformulate the problem in the form (4.8).

By Lemma 4.1.(i) if  $\Psi$  is a solution to (3.10) that extends smoothly along a singular orbit  $S^2 = \text{SU}(2) \times \text{SU}(2)/\text{U}(1) \times \text{SU}(2)$  we can write

$$\begin{aligned} u_0(t) &= a^2 + t^2 y_1(t), & u_1(t) &= a^2 + t^2 y_2(t), & u_2(t) &= t^2 y_3(t), \\ v_0(t) &= t^2 y_4(t), & v_1(t) &= t^2 y_5(t), & v_2(t) &= t^2 y_6(t), & \lambda(t) &= t y_7(t), \end{aligned}$$

for some  $a > 0$ .

In terms of  $y = (y_1, \dots, y_7)$  the system (3.10) becomes

$$\begin{aligned} \dot{y}_1 &= -\frac{1}{t} \left( 2y_1 + 3\frac{y_4}{y_7} \right), & \dot{y}_4 &= -\frac{1}{t} (2y_4 - 4a^2 y_7) + 4ty_1 y_7, \\ \dot{y}_2 &= -\frac{1}{t} \left( 2y_2 + 3\frac{y_5}{y_7} - 2y_7 \right), & \dot{y}_5 &= -\frac{1}{t} (2y_5 - 4a^2 y_7) + 4ty_2 y_7, \\ \dot{y}_3 &= -\frac{1}{t} \left( 2y_3 + 3\frac{y_6}{y_7} \right), & \dot{y}_6 &= -\frac{1}{t} \left( 2y_6 + 3\frac{y_3}{y_7} \right) + 4ty_3 y_7, \\ \dot{y}_7 &= -\frac{1}{t} \left( y_7 + \frac{y_7^2}{y_2 - y_1} + \frac{3y_3 y_6}{2a^2 y_7^2 (y_2 - y_1)} \right) + M_7(t, y), \end{aligned}$$

where

$$\begin{aligned} tM_7(t, y) &= \frac{y_7^2}{y_2 - y_1} + \frac{3y_3 y_6}{2a^2 y_7^2 (y_2 - y_1)} - \\ &= \frac{2y_7^2 (a^2 + t^2 y_2)}{2a^2 (y_2 - y_1) + t^2 (-y_1^2 + y_2^2 + y_3^2)} - \frac{3y_3 y_6}{y_7^2 (2a^2 (y_2 - y_1) + t^2 (-y_1^2 + y_2^2 + y_3^2))}. \end{aligned}$$

Hence  $y$  solves an ODE system as in (4.8).

It remains to check the hypotheses of Theorem 4.7. First, the requirement  $M_{-1}(y_0) = 0$  uniquely determines the initial condition  $y_0$  in terms of  $a$ :

$$(4.9) \quad y_0 = \left( -3a^2, -3a^2 + \frac{3}{2}, -\frac{3\sqrt{3}}{2}a, 3a^2, 3a^2, \frac{3\sqrt{3}}{2}a, \frac{3}{2} \right),$$

where we assumed  $u_2 < 0$  without loss of generality.

For  $y$  sufficiently close to  $y_0$ ,  $a$  bounded away from 0 and  $t$  sufficiently small we see that  $M_{-1}$  and  $M$  are real analytic functions of all of their arguments and depend smoothly on  $a > 0$ .

Finally, the linearisation of  $M_{-1}$  at  $y_0$  is

$$d_{y_0} M_{-1} = \begin{pmatrix} -2 & 0 & 0 & -2 & 0 & 0 & 4a \\ 0 & -2 & 0 & 0 & -2 & 0 & 2 + 4a \\ 0 & 0 & -2 & 0 & 0 & -2 & 2\sqrt{3}a \\ 0 & 0 & 0 & -2 & 0 & 0 & 4a \\ 0 & 0 & 0 & 0 & -2 & 0 & 4a \\ 0 & 0 & -2 & 0 & 0 & -2 & -2\sqrt{3}a \\ 1 & -1 & -\frac{2}{\sqrt{3}a} & 0 & 0 & \frac{2}{\sqrt{3}a} & -7 \end{pmatrix}.$$

The determinant of  $h\text{Id} - d_{y_0} M_{-1}$  is

$$(h+1)(h+2)^3(h+4)^3 \geq 512 > 0$$

for all integer  $h \geq 0$ . □

*Remark 4.10.* The first few terms of the Taylor series of  $\Psi_a$  at  $t = 0$  are:

$$\begin{aligned}\lambda(t) &= \frac{3}{2}t - \frac{2a^2 + 3}{12a^2}t^3 + \frac{116a^4 - 381a^2 + 261}{1440a^4}t^5 + \frac{5500a^6 - 26523a^4 + 34209a^2 - 13149}{90720a^6}t^7 + \dots, \\ u_0(t) &= a^2 - 3a^2t^2 + \frac{52a^2 - 3}{24}t^4 - \frac{172a^4 + 3a^2 - 18}{270a^2}t^6 + \dots, \\ u_1(t) &= a^2 - \frac{3}{2}(2a^2 - 1)t^2 + \frac{52a^4 - 32a^2 - 3}{24a^2}t^4 - \frac{2752a^6 - 1688a^4 + 93a^2 - 261}{4320a^4}t^6 + \dots, \\ u_2(t) &= \frac{-3\sqrt{3}}{2}at^2 + \frac{\sqrt{3}(16a^2 - 3)}{12a}t^4 + \frac{\sqrt{3}(-3412a^4 + 267a^2 + 423)}{8640a^3}t^6 + \dots, \\ v_0(t) &= 3a^2t^2 - \left(\frac{1}{4} + \frac{14a^2}{3}\right)t^4 + \frac{5516a^4 + 429a^2 + 261}{2160a^2}t^6 + \dots, \\ v_1(t) &= 3a^2t^2 + \left(2 - \frac{14a^2}{3}\right)t^4 + \frac{5516a^4 - 2541a^2 - 549}{2160a^2}t^6 + \dots, \\ v_2(t) &= \frac{3}{2}\sqrt{3}at^2 - \frac{\sqrt{3}(34a^2 - 3)}{12a}t^4 + \sqrt{3}\frac{13492a^4 + 273a^2 - 423}{8640a^3}t^6 + \dots\end{aligned}$$

The particular form of the coefficients will play an important role in Proposition 8.8.

The proof of Theorem 4.5 is similar, *cf* also [41, Theorem 6.4] and Proposition 6.3.

## 5. THE ORBITAL VOLUME FUNCTION AND MAXIMAL VOLUME ORBITS

It follows from previous work that any complete  $SU(2) \times SU(2)$ -invariant nearly Kähler 6-manifold must arise as the completion of some element (or possibly an element from both) of the two 1-parameter families  $\{\Psi_a\}_{a>0}, \{\Psi_b\}_{b>0}$  constructed in the previous section. Our strategy to understand whether any of these solutions closes smoothly along a *second* singular orbit at some time  $T > 0$  will be to consider pairs of solutions from these two families and to try to match them across a principal orbit. In this section we find a geometrically preferred slice to use as a tool in this matching.

To this end we study the properties of the orbital volume function  $V(t)$ , *ie* the volume of the hypersurfaces  $\{t\} \times N_{1,1}$ , for these two 1-parameter families of solutions. The main result of this section, Proposition 5.15, establishes that the orbital volume function of every solution constructed in Theorems 4.4 and 4.5 has a unique (strict) maximum. An important ingredient of the proof of Proposition 5.15 is Proposition 5.9 which establishes key properties of the space of possible maximal volume orbits  $\mathcal{V}$ , foremost of which are: (i) the orbital volume restricted to  $\mathcal{V}$  is bounded below by the volume of the invariant Sasaki–Einstein structure on  $N_{1,1}$  and is achieved only by a “rotated” invariant Sasaki–Einstein structure; (ii) for any  $C \geq 1$  the subset of maximal volume orbits with volume bounded above by  $C$  is compact and nonempty.

Proposition 5.15 allows us to define two curves  $\alpha$  and  $\beta : (0, \infty) \rightarrow \mathcal{V}$  that parametrise the maximal volume orbits of the two 1-parameter families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  respectively inside the space of all maximal volume orbits. The maximal volume orbit is the preferred principal orbit on which to investigate matching conditions; such matching conditions are then described in Lemmas 5.19 and 5.20 (which we call the Doubling and Matching Lemmas respectively) in terms of properties of the curves  $\alpha$  and  $\beta$  (or a certain projection of them). Establishing enough information about the curves  $\alpha$  and  $\beta$  to prove that they satisfy the conditions of the Doubling or Matching Lemmas in some cases will occupy the rest of the paper.

According to Proposition 2.16, the volume of  $N_{1,1}$  with respect to the metric induced by an invariant  $SU(2)$ -structure  $\psi_{\lambda,\mu,A}$  has the simple form  $V = V_0 \lambda \mu^2$ , where  $V_0$  is the volume of  $N_{1,1}$  with respect to the standard Sasaki–Einstein metric. For notational convenience in the rest of the paper we will write  $V$  for  $(V/V_0)$ . While it is possible to derive evolution equations for the orbital volume function  $V$  and related quantities directly from the fundamental ODE system (3.10), for some purposes it is more convenient to consider the system of ODEs describing arbitrary cohomogeneity one Einstein metric: computations will be minimised and, more importantly, we will be able to recognise often complicated algebraic expressions involving the  $SU(2)$ -structure  $\psi$  as basic geometric quantities.

**The Einstein equations for families of equidistant hypersurfaces.** Let  $g_t$  be a family of Riemannian metrics on an  $n$ -dimensional oriented manifold  $N$  and consider the metric  $\hat{g}$  on  $\mathbb{R}_t \times N$  defined by  $\hat{g} = dt^2 + g_t$ . We recall the derivation of the Einstein equations for  $\hat{g}$  given by Eschenburg–Wang in [27, Proposition 2.1].

Denote by  $\nu$  the unit normal of the family of hypersurfaces  $N_t := \{t\} \times N$  and by  $L$  the Weingarten operator given by

$$L(X) = \hat{\nabla}_X \nu, \quad \text{for every } X \in TN,$$

where  $\hat{\nabla}$  is the Levi-Civita connection of the metric  $\hat{g}$ . Then  $L = \frac{1}{2}g^{-1}g'$  and

$$(5.1) \quad L' + L^2 + \hat{R}_\nu = 0$$

is the Riccati equation. Here  $\hat{R}_\nu$  is the normal-tangential component of the curvature of  $\hat{g}$ , ie  $\hat{R}_\nu(X) = \hat{R}(X, \nu)\nu$  for every  $X \in TN$ , and  $'$  denotes differentiation with respect to  $t$ .

Assume now that  $\hat{g}$  is Einstein with scalar curvature  $(n+1)\Lambda$ . Then the Gauss equation for the hypersurface  $N_t$  implies that

$$(5.2) \quad L' + lL - r + \Lambda \text{id} = 0,$$

where  $l = \text{Tr } L$  is the mean curvature of the hypersurface  $N_t$  and  $r$  is the Ricci-endomorphism of the metric  $g_t$ . Moreover, if we regard  $L$  as a  $TN$ -valued 1-form and  $d^\nabla$  is the exterior differential induced by the Levi-Civita connection of  $g_t$  then the vanishing of the Ricci curvature in normal-tangential directions can be expressed using the Codazzi equation as

$$(5.3) \quad \text{Tr}(X \lrcorner d^\nabla L) = 0, \quad \text{for all } X \in TN.$$

Taking the trace of both (5.1) and (5.2) yields

$$(5.4a) \quad l' + |L|^2 + \Lambda = 0,$$

$$(5.4b) \quad \text{Scal} - (n-1)\Lambda = l^2 - |L|^2,$$

where  $\text{Scal}$  is the scalar curvature of  $g_t$  and  $|L|^2 = \text{Tr } L^2$ .

The Einstein equations are therefore a first order system for a pair  $(g, L)$  consisting of a Riemannian metric  $g$  on  $N$  and a symmetric endomorphism  $L$  of  $TN$  subject to the additional constraints (5.3) and (5.4a). (5.4b) is then a conserved quantity of this system.

Consider now a solution  $\Psi = (\psi_t)_{0 < t < T}$  of the fundamental ODE system (3.10). Each invariant nearly hypo structure  $\psi_t$  on  $N_{1,1}$  determines an invariant metric  $g$ . Furthermore, differentiating the map from invariant nearly hypo structures to invariant metrics along the direction of the vector field (3.10) defines the Weingarten operator  $L$ . The pair  $(g, L)$  then satisfies the system above with  $n = 5 = \Lambda$ .

The positivity of  $\Lambda$  has a number of immediate consequences.

**Proposition 5.5.** *Let  $\Psi$  be a solution of the fundamental ODE system (3.10) defined on  $(-T_1, T_2)$  for  $T_1, T_2 > 0$ . Moreover, let  $V$  be the orbital volume function  $V = \lambda\mu^2$ , where  $\mu^2 = |u|^2$  in the parametrisation of (3.1), and  $l$  be the mean curvature  $l = \text{Tr } L = \frac{\dot{V}}{V}$ .*

- (i) *Every critical point of  $V$  is a strict maximum.*
- (ii)  *$T_1 + T_2 \leq \pi$ .*
- (iii) *Fix  $t_0 \in (0, \pi)$  such that  $l(0) = 5 \cot(t_0)$ . Then*

$$l(t) \leq 5 \cot(t + t_0), \quad \text{for } t > 0;$$

$$l(t) \geq 5 \cot(t + t_0), \quad \text{for } t < 0.$$

- (iv) *For the same  $t_0$  as above and any  $t \in (-T_1, T_2)$*

$$V(t) \leq V(0) \frac{\sin^5(t + t_0)}{\sin^5(t_0)}.$$

*Proof.* Since  $V' = lV$  and therefore  $V'' = l'V$  at any critical point of  $V$ , (5.4a) implies (i).

Decomposing  $L$  into its trace  $l$  and its traceless part  $\mathring{L}$ , we write  $|L|^2 = \frac{1}{5}l^2 + |\mathring{L}|^2$ . Then (5.4) implies that  $l$  satisfies the inequality

$$l' + \frac{1}{5}l^2 + 5 \leq 0.$$

Comparing  $l$  with a solution of the scalar Riccati-type equation  $l' + \frac{1}{5}l^2 + 5 = 0$  we obtain (ii) and (iii) by [26, Theorem 4.1]. Direct integration of  $V' = lV$  then yields (iv).  $\square$

We will now write down explicit formulae for the geometric quantities appearing in (5.4b) in terms of an invariant nearly hypo structure  $\psi_{\lambda, \mu, A}$  as an illustration of how complicated these algebraic expressions can be. Two of these explicit formulae will be useful in the next section.

**Lemma 5.6.** *Let  $\psi_{\lambda, \mu, A}$  be one of the invariant nearly hypo structures on  $N_{1,1}$  parametrised in Proposition 2.41. Let  $x_i, y_i$  and  $w_i$  be given by (2.42) and, acting with the flow of the Reeb vector field if necessary, assume that  $\theta = 0$ .*

- (i) *The scalar curvature of the induced invariant metric  $g$  on  $N_{1,1}$  is*

$$\text{Scal} = 20 - 4 \frac{\lambda^2 x_1^2}{\mu^2} + 24 \frac{x_1 y_2}{\mu} - 4 \frac{\lambda^2 w_1^2}{\mu^2} - 9 \frac{w_2^2}{\lambda^2}.$$

- (ii) *The mean curvature  $l$  is*

$$l = 2 \frac{x_1}{y_1} - 3 \frac{y_2}{x_2}.$$

- (iii) *The norm squared of the traceless part of the Weingarten operator  $L$  is*

$$|\mathring{L}|^2 = \frac{36}{5} \left( \frac{\lambda x_1}{\mu} - \frac{y_2}{\lambda} \right)^2 + 4 \frac{\lambda^2 w_1^2}{\mu^2} + 9 \frac{w_2^2}{\lambda^2}.$$

*Proof.* First we give an expression for the scalar curvature of a general invariant SU(2)–structure on  $N_{1,1}$  and then specialise to the case of invariant nearly hypo structures. Bedulli and Vezzoni [5, Theorem 3.4] give a formula for the scalar curvature of the metric induced by an SU(2)–structure on a 5–manifold. Since  $\eta^{\text{se}}$  spans the space of invariant 1–forms on  $N_{1,1}$  and invariant functions on  $N_{1,1}$  are constant, this formula simplifies considerably in the case of invariant SU(2)–structures:

$$(5.7) \quad \text{Scal} = -5\phi_{11}^2 - \sum_{i=1}^3 \phi_i^2 - 4\phi_1\phi_{23} - 4\phi_2\phi_{31} - 4\phi_3\phi_{12} - \sum_{i=0}^3 \frac{1}{2} |\sigma_i|^2,$$



where  $\phi_i$ ,  $\phi_{ij}$  and  $\sigma_i$  are the  $SU(2)$ -irreducible components of the intrinsic torsion  $\Theta$  of the  $SU(2)$ -structure as described in Conti–Salamon [23, Proposition 2.3]. These irreducible components of  $\Theta$  are determined by the exterior derivatives of  $\eta$  and the triple  $\omega_i$  via

$$(5.8) \quad d\omega_i = \alpha_i \wedge \omega_i + \sum_{j=1}^3 \phi_{ij} \eta \wedge \omega_j + \sigma_i, \quad d\eta = \eta \wedge \alpha_4 + \sum_{i=1}^3 \phi_i \omega_i + \sigma_4.$$

To apply these two formulae we need to express the components of the intrinsic torsion of the invariant  $SU(2)$ -structure  $\psi_{\lambda,\mu,A}$  in terms of the parameters  $(\lambda, \mu, A) \in \mathbb{R}^+ \times \mathbb{R}^+ \times SO_0(1, 3)$ . Recall that from equations (2.12) and (2.13) we have  $\eta = \lambda \eta^{se}$ ,  $\omega_i = \mu A \omega_i^{se}$ . From (2.17) we obtain

$$d\eta = -2 \frac{\lambda}{\mu} \omega_1, \quad d\omega_i = \frac{1}{\lambda} \eta \wedge (A^{-1} T A) \omega_i,$$

where  $T$  is defined in (2.18).

We now specialise to the case of invariant nearly hypo  $SU(2)$ -structures. An explicit computation of  $A^{-1}$  and  $A^{-1} T A$  yields

$$\begin{aligned} d\eta &= -2 \frac{\lambda}{\mu} \left( -w_1 \omega_0 + x_1 \omega_1 + \frac{\mu}{\lambda} \omega_3 \right), & d\omega_1 &= 3\eta \wedge \omega_2, \\ d\omega_2 &= \eta \wedge \frac{1}{\lambda} (-3w_2 \omega_0 - 3\lambda \omega_1 + 3y_2 \omega_3), & d\omega_3 &= -3 \frac{y_2}{\lambda} \eta \wedge \omega_2. \end{aligned}$$

The first part of the lemma follows from (5.7) and (5.8) using the fact that  $|\omega_0|^2 = 2$  with respect to  $g$ .

The expression for the mean curvature  $l$  follows from (3.10) and the change of variables (3.1) since  $l = \frac{\dot{V}}{V} = \frac{\dot{\lambda}}{\lambda} + 2 \frac{\dot{\mu}}{\mu}$ . Once (i) and (ii) are known, (iii) follows from (5.4b).  $\square$

Formulae in the parametrisation  $(\lambda, u, v)$  of (3.1) follow immediately by applying the change of variables (3.3).

**The space of invariant maximal volume orbits.** If a solution of (3.10) gives rise to a *complete* invariant nearly Kähler metric on a closed 6-manifold  $M$  then clearly it would contain a unique orbit of maximal volume. In general the two 1-parameter families of invariant nearly Kähler metrics provided by Theorems 4.4 and 4.5 give rise to *incomplete* invariant nearly Kähler metrics. Nevertheless, in this section we show that every solution  $\Psi$  in the two 1-parameter families of Theorems 4.4 and 4.5 has a unique *maximal volume orbit*, ie the orbital volume function  $V = \lambda \mu^2$  reaches a strict maximum before  $\Psi$  blows up.

Let  $l$  be the mean curvature of an invariant nearly hypo structure  $\psi \in \mathcal{U} \times SO(2)$ , where  $\mathcal{U}$  is the open set of  $SO_0(1, 2)$  defined in Proposition 2.41. Here for any  $\delta > 0$  sufficiently small we extend  $\psi$  to an invariant nearly Kähler structure  $\Psi$  on  $(-\delta, \delta) \times N_{1,1}$  by Proposition 3.9 and define  $l$  by  $\dot{V}(0) = lV(0)$ . In this manner, we regard  $l$  as a smooth function on the space of invariant nearly hypo structures and denote by  $\mathcal{V}$  its zero level set.  $\mathcal{V}$  is clearly invariant under the circle action generated by the Reeb vector field and therefore, in the parametrisation of Proposition 2.41,  $\mathcal{V} = \mathcal{V}_0 \times SO(2)$  for a subset  $\mathcal{V}_0$  of  $\mathcal{U}$ .

**Proposition 5.9.** *The space  $\mathcal{V} = \mathcal{V}_0 \times SO(2)$  of invariant nearly hypo structures on  $N_{1,1}$  that satisfy the additional constraint  $l = 0$  is a smooth manifold.*

Let  $\pi: \mathcal{V}_0 \rightarrow \mathbb{R}^+ \times \mathbb{R}^+$  be the map given by  $\psi_{\lambda,\mu,A} \mapsto (\lambda, \mu)$ . Then  $\pi(\psi) = \pi(\psi')$  if and only if  $\psi'$  lies in the orbit of  $\psi$  under the group of discrete symmetries generated by the involutions  $\tau_1, \tau_2, \tau_3, \tau_4$  of Proposition 3.11. The image of  $\mathcal{V}_0$  under  $\pi$  is the wedge

$$W = \{(\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid \mu \geq \lambda \geq 1\}$$

and the projection  $\pi: \mathcal{V}_0 \rightarrow W$  is a 4-fold cover branched along the boundary of  $W$ .

In particular, the volume  $V = \lambda\mu^2$  of an invariant nearly hypo structure  $\psi \in \mathcal{V}$  is bounded below by 1 with equality if and only if  $\psi$  is an invariant Sasaki–Einstein structure on  $N_{1,1}$  and the set  $\mathcal{V} \cap \{V \leq C\}$  is compact for any  $C \geq 1$ .

*Proof.* By the first equation in (5.4) at any point the differential of  $l$  evaluated on the tangent vector corresponding to the evolution equations (2.37b) is strictly negative and therefore  $\mathcal{V}$  is a smooth submanifold of  $\mathcal{U} \times \text{SO}(2)$ . We will now describe  $\mathcal{V}$  in detail.

Let  $\psi$  be an invariant nearly hypo structure written in the form (3.1) with  $\theta = 0$ . By Lemma 5.6(ii) and the change of variables (3.3),  $\psi$  is a critical point of  $V$  if and only if

$$(5.10) \quad 2\lambda^4 u_1 = 3u_2 v_2.$$

We now use (5.10) to rewrite the constraints (3.5) in terms of  $\lambda$ ,  $\mu$  and  $u_1$ . First of all, substituting  $u_2 = -\lambda\mu$  in (5.10) yields

$$(5.11) \quad v_2 = -\frac{2\lambda^3 u_1}{3\mu}.$$

Notice that the inequality (3.6) then reads

$$(5.12) \quad u_1^2 < \frac{3\mu^4}{2\lambda^2}.$$

We now substitute the expression (5.11) for  $v_2$  along with those for  $u_2$  and  $v_1$  in terms of  $\lambda$  and  $\mu$  into the constraints (3.5). We obtain

$$(5.13a) \quad u_0^2 = u_1^2 + \mu^2(\lambda^2 - 1),$$

$$(5.13b) \quad \frac{9\mu^2}{4\lambda^6} v_0^2 = u_1^2 + \frac{9\mu^4}{4\lambda^6} (\mu^2 - \lambda^2),$$

$$(5.13c) \quad \frac{3\mu}{2\lambda^3} u_0 v_0 = \frac{\mu}{2\lambda^3} u_1 (3\mu^2 + 2\lambda^4).$$

Equating the product of the first two equations with the square of the third we see that  $x = u_1^2$  satisfies the quadratic equation

$$(5.14) \quad x^2 - cx + d = 0,$$

where  $c = \mu^2 \left(1 + \frac{3\mu^2}{\lambda^2} + \frac{9\mu^2}{4\lambda^4}\right) > 0$  and  $d = \frac{9\mu^6}{4\lambda^6} (\lambda^2 - 1)(\mu^2 - \lambda^2)$ . The discriminant  $\Delta = c^2 - 4d$  of the quadratic (5.14) is

$$\Delta = \frac{\mu^4}{\lambda^8} \left( \frac{45}{2} \mu^4 \lambda^2 + 15\mu^2 \lambda^6 + \frac{81}{16} \mu^4 + \lambda^8 - \frac{9}{2} \mu^2 \lambda^4 \right) \geq \frac{\mu^4}{\lambda^8} \left( \frac{45}{2} \mu^4 \lambda^2 + 15\mu^2 \lambda^6 \right) \geq 0,$$

where equality holds if and only if  $\lambda = \mu = 0$ . Hence (5.14) always has two distinct real roots.

The larger root  $x_+$  always has to be discarded because it contradicts (5.12). When  $d < 0$  the smallest root  $x_- = \frac{1}{2}(c - \sqrt{\Delta}) < 0$ . Assume then that  $d \geq 0$ . We distinguish two cases: (i)  $\mu < \lambda < 1$ ; (ii)  $\mu \geq \lambda \geq 1$ .

In case (i) we show that the choice  $x = x_-$  is incompatible with (5.13). Indeed, if the first two equalities of (5.13) were satisfied with this choice of  $u_1^2$ , then we must have  $c + 2\mu^2(\lambda^2 - 1) \geq \sqrt{\Delta}$  and  $c + \frac{9\mu^4}{2\lambda^6}(\mu^2 - \lambda^2) \geq \sqrt{\Delta}$ . However, the squares of the expressions on the left hand side of these inequalities are  $\Delta + \frac{\mu^4}{\lambda^6}(3\mu^2 + 2\lambda^4)^2(\lambda^2 - 1)$  and  $\Delta + \frac{9\mu^6}{4\lambda^{12}}(3\mu^2 + 2\lambda^4)^2(\mu^2 - \lambda^2)$ , both strictly less than  $\Delta$  by hypothesis.

In case (ii) the solution  $x = x_-$  to (5.14) is both non-negative and compatible with (5.12) and (5.13). Thus an admissible solution to the quadratic equation (5.14) only exists (and is

unique) when  $(\lambda, \mu) \in W$ . In this case  $u_1$  is determined up to sign by the pair  $(\lambda, \mu)$ . The first two equations in (5.13) therefore determine  $u_0$  and  $v_0$  up to sign in terms of  $(\lambda, \mu)$ . Acting by  $\tau_4 \circ \tau_1$  if necessary we can assume without loss of generality that  $u_1 \geq 0$ . Then  $u_1, u_2 = -\lambda\mu, v_1 = \mu^2, v_2, |u_0|, |v_0|$  are completely determined by  $(\lambda, \mu)$ . Moreover, the third equation of (5.13) implies that  $u_0v_0 \geq 0$ . The involution  $\tau_2 \circ \tau_3 \circ \tau_4$  maps  $(u_0, v_0) \mapsto (-u_0, -v_0)$  while keeping  $\lambda, u_1, u_2, v_1, v_2$  fixed and therefore can be used to remove the remaining ambiguity in the choice of sign of  $u_0, v_0$ .  $\square$

Thanks to the compactness statement we deduce the following crucial proposition.

**Proposition 5.15.** *Let  $\Psi$  be one of the solutions of (3.10) given by Theorems 4.4 and 4.5. Then  $\Psi$  intersects  $\mathcal{V}$  in a unique point.*

*Proof.* The proof is identical in the two cases and we therefore give it only for the family  $\Psi_a$ . Let  $S$  be the set of  $a \in (0, \infty)$  such that  $\Psi_a$  intersects  $\mathcal{V}$ . The solution of (3.10) of Example 3.13, ie the standard nearly Kähler structure on  $S^6$ , admits a maximal volume orbit and a singular orbit  $S^2$  (and  $S^3$ ). Hence  $S$  is non-empty.

$S$  is open: by Proposition 5.5(i) every nearly hypo structure with  $l = 0$  is a non-degenerate maximum of the orbital volume function. Moreover, by Proposition 5.9,  $l = 0$  defines a smooth hypersurface in the space of invariant nearly hypo structures and therefore if  $\Psi_a$  has a maximal volume orbit, so does  $\Psi_{a'}$  for any  $a'$  sufficiently close to  $a$ .

$S$  is closed: suppose that a sequence  $a_i$  in  $S$  converges to some  $a \in (0, \infty)$ . By the continuous dependence on the initial conditions, we can find some time  $t > 0$  sufficiently small such that the orbital volume  $V(t)$  and the mean curvature  $l(t)$  remain uniformly bounded for all  $a_i$ . By Proposition 5.5(iii), the maximal volume orbits of  $\Psi_{a_i}$  have uniformly bounded volume. By Proposition 5.9 the set of maximal volume orbits with an upper bound on the volume is compact and therefore  $\Psi_a$  must also contain a maximal volume orbit.

We conclude that  $S = (0, \infty)$ . The intersection point is unique by Proposition 5.5(i).  $\square$

Since the solutions  $\Psi_a, \Psi_b$  of Theorems 4.4 and 4.5 depend continuously on  $a, b > 0$ , respectively, Proposition 5.15 yields two continuous curves  $\alpha, \beta: (0, \infty) \rightarrow \mathcal{V}_0$ . Using Proposition 5.9 we project these two curves onto the wedge  $W$ .

**Definition 5.16.** Let  $\alpha_W, \beta_W: (0, \infty) \rightarrow W$  be the two continuous curves which parametrise the maximal volume orbit of the solutions  $\Psi_a, a > 0$ , and  $\Psi_b, b > 0$ , respectively, up to the action of discrete symmetries.

The fact that  $\alpha_W$  and  $\beta_W$  parametrise maximal volume orbits up to discrete symmetries follows from Proposition 5.9.

We now use the curves  $\alpha_W$  and  $\beta_W$  to state conditions for pairs of solutions  $\Psi_\epsilon, \Psi_{\epsilon'}$  to match across their maximal volume orbit and therefore define a complete invariant nearly Kähler structure on closed manifold. The task of the rest of paper is to study the behaviour of the curves  $\alpha_W$  and  $\beta_W$  to establish whether these matching conditions are ever satisfied.

**Doubling and matching.** Suppose that  $\Psi_1, \Psi_2$  are two of the solutions of (3.10) given by Theorems 4.4 and 4.5. In the parametrisation of (3.1) we write

$$\begin{aligned}\Psi_1(t) &= \left( \lambda(t), u_0(t), u_1(t), u_2(t), v_0(t), v_1(t), v_2(t) \right), \\ \Psi_2(t) &= \left( \hat{\lambda}(t), \hat{u}_0(t), \hat{u}_1(t), \hat{u}_2(t), \hat{v}_0(t), \hat{v}_1(t), \hat{v}_2(t) \right).\end{aligned}$$

Suppose that the maximal volume orbit of  $\Psi_1, \Psi_2$  occurs at  $t = T_1, T_2$ , respectively, and that  $(\lambda(T_1), \mu(T_1)) = (\hat{\lambda}(T_2), \hat{\mu}(T_2))$ , i.e. the two maximal volume orbits coincide up to the action of discrete symmetries. In particular,  $u_i(T_1), v_i(T_1)$  coincide with  $\hat{u}_i(T_2), \hat{v}_i(T_2)$  up to some sign.

Acting by  $\tau_2 \circ \tau_3 \circ \tau_4$  on  $\Psi_1$  we define

$$\tilde{\Psi}_1(t) = \left( \lambda(t), -u_0(t), u_1(t), u_2(t), -v_0(t), v_1(t), v_2(t) \right).$$

By the uniqueness of solutions of (3.10) with given initial conditions, if  $u_1(T_1) = \hat{u}_1(T_2)$  then necessarily  $\Psi_2 = \Psi_1$  or  $\Psi_2 = \tilde{\Psi}_1$  depending on the sign of  $u_0(T_1), \hat{u}_0(T_2)$ . Thus we assume without loss of generality that  $u_1(T_1) = -\hat{u}_1(T_2)$ .

Acting by a time translation and  $\tau_1 \circ \tau_2 \circ \tau_3$  or  $\tau_1 \circ \tau_4$ , respectively, define

$$\Psi_2^\pm(t) = \left( \hat{\lambda}(\tau), \mp \hat{u}_0(\tau), -\hat{u}_1(\tau), \hat{u}_2(\tau), \pm \hat{v}_0(\tau), \hat{v}_1(\tau), -\hat{v}_2(\tau) \right),$$

where  $\tau = T_1 + T_2 - t$  for  $T_1 \leq t \leq T_1 + T_2$ . By our assumptions either  $\Psi_1(T_1) = \Psi_2^+(T_1)$  and we define a smooth solution of (3.10)

$$(5.17) \quad \Psi(t) = \begin{cases} \Psi_1(t), & 0 \leq t \leq T_1, \\ \Psi_2^+(t), & T_1 \leq t \leq T_1 + T_2, \end{cases}$$

or  $\Psi_1(T_1) = \Psi_2^-(T_1)$  and we consider

$$(5.18) \quad \Psi(t) = \begin{cases} \Psi_1(t), & 0 \leq t \leq T_1, \\ \Psi_2^-(t), & T_1 \leq t \leq T_1 + T_2. \end{cases}$$

We deduce the following two lemmas.

**Lemma 5.19** (Doubling Lemma).

- (i) *Suppose that there exists  $a \in (0, \infty)$  such that  $\alpha_W(a)$  lies in the portion of the boundary of  $W$  with  $\lambda = 1, \mu > 1$ . Then (5.17) with  $\Psi_1 = \Psi_2 = \Psi_a$  defines a smooth invariant nearly Kähler structure on  $S^2 \times S^4$ .*
- (ii) *Suppose that there exists  $a \in (0, \infty)$  such that  $\alpha_W(a)$  lies in the portion of the boundary of  $W$  with  $\lambda = \mu > 1$ . Then (5.18) with  $\Psi_1 = \Psi_2 = \Psi_a$  defines a smooth invariant nearly Kähler structure on  $CP^3$ .*
- (iii) *Suppose that there exists  $b \in (0, \infty)$  such that  $\beta_W(b)$  lies on the boundary of  $W$  and  $\beta_W(b) \neq (1, 1)$ . Then (5.17) or (5.18) with  $\Psi_1 = \Psi_2 = \Psi_b$  defines a smooth invariant nearly Kähler structure on  $S^3 \times S^3$ .*

*Proof.* The point  $(1, 1) \in W$  corresponds to the standard Sasaki–Einstein structure on  $N_{1,1}$  and therefore has to be excluded.

In view of the previous discussion we have only to explain the topology of the underlying manifold  $M$  obtained by the gluing construction of (5.17) and (5.18).

Recall that  $\tau_3$  is induced by the inner automorphism  $\phi_3$  of  $SU(2) \times SU(2)$  generated by  $\frac{\pi}{4}U^-$ , i.e.  $\phi_3$  is conjugation by an element of the normaliser  $N$  of  $\Delta U(1)$  in  $SU(2) \times SU(2)$ . It is clear that  $N = U(1) \times U(1)$  is the maximal torus of  $SU(2) \times SU(2)$ . In particular  $N$  is connected and  $\phi_3$  can be deformed to the identity through a path in  $N$ . On the other hand,  $\tau_4$  is induced by the outer automorphism of  $SU(2) \times SU(2)$ , which fixes both  $\Delta U(1)$  and  $\Delta SU(2)$  but exchanges  $U(1) \times SU(2)$  and  $SU(2) \times U(1)$ . It is then straightforward to deduce the group diagrams of the resulting cohomogeneity one manifolds. The result follows from Table 1.  $\square$

*Remark.* Observe that the nearly Kähler structures of the lemma are obtained by “doubling” a solution  $\Psi_a$  or  $\Psi_b$  across its maximal volume orbit. In particular, the two singular orbits are spheres of the same dimension and have the same volume. Because of the non-trivial action

of the discrete symmetries, however, the maximal volume orbit is not totally geodesic. By [28, Lemma 2.1] this can only occur for the “rotated” Sasaki–Einstein structure in a nearly Kähler sine-cone.

**Lemma 5.20** (Matching Lemma).

- (i) *Suppose that there exists  $a < a' \in (0, \infty)$  such that  $\alpha_W(a) = \alpha_W(a')$ . Set  $\Psi_1 = \Psi_a$  and  $\Psi_2 = \Psi_{a'}$ . Then either (5.17) yields a smooth nearly Kähler structure on  $S^2 \times S^4$  or (5.18) defines a smooth invariant nearly Kähler structure on  $CP^3$ .*
- (ii) *Suppose that there exists  $b < b' \in (0, \infty)$  such that  $\beta_W(b) = \beta_W(b')$ . Set  $\Psi_1 = \Psi_b$  and  $\Psi_2 = \Psi_{b'}$ . Then either (5.17) or (5.18) defines a smooth invariant nearly Kähler structure on  $S^3 \times S^3$ .*
- (iii) *Suppose that there exist  $a, b \in (0, \infty)$  such that  $\alpha_W(a) = \beta_W(b)$ . Then either (5.17) or (5.18) with  $\Psi_1 = \Psi_a$  and  $\Psi_2 = \Psi_b$  defines a smooth invariant nearly Kähler structure on  $S^6$ .*

*Proof.* In all cases  $\Psi_2 \neq \Psi_1, \tilde{\Psi}_1$  and therefore one of the conditions of (5.17) or (5.18) are satisfied. The topology of the resulting 6–manifold follows from Table 1.  $\square$

## 6. LIMITS OF THE TWO 1–PARAMETER FAMILIES OF SMOOTHLY CLOSING NEARLY KÄHLER SOLUTIONS AS $a$ OR $b$ TEND TO 0

In order to be able to satisfy the conditions of the Doubling or Matching Lemmas 5.19 and 5.20 we will need information about the behaviour of the two curves  $\alpha$  and  $\beta$  parameterising the maximal volume orbits of the two 1–parameter families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  respectively. In this section we establish properties about the families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  in the limit where the size of the singular orbit tends to zero, *ie*  $a$  or  $b \rightarrow 0$ . This information will suffice for our applications to constructing a new complete cohomogeneity one nearly Kähler structure on  $S^3 \times S^3$  in Section 7; however, to prove the existence of a new complete cohomogeneity one nearly Kähler structure on  $S^6$  we will also need some understanding of the family  $\{\Psi_a\}_{a>0}$  in the limit where  $a \rightarrow \infty$ . We turn to this latter problem in Section 8.

The four compact  $SU(2) \times SU(2)$ –manifolds of Table 1 can each be thought of as desingularisation of the sine-cone of Example 3.12, where a neighbourhood of each conical singularity is replaced with a copy of either  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow S^2$  or of  $T^*S^3$ . By Theorem 2.27 both of these carry complete Calabi–Yau structures. Proposition 6.3 establishes that in the limit where the size of the singular orbit tends to zero, suitably rescaled  $\Psi_a$  and  $\Psi_b$  converge to the Calabi–Yau structures on the small resolution and the conifold respectively, and thus confirms our earlier expectation that the two 1–parameter families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  are nearly Kähler deformations of the two Calabi–Yau desingularisations of the conifold. Further confirmation of the desingularisation intuition is provided by the main result of this section, Theorem 6.9: this establishes that on every compact set of  $(0, \pi)$  the local nearly Kähler structures  $\Psi_a$  and  $\Psi_b$  converge to the sine-cone of Example 3.12 as  $a$  and  $b \rightarrow 0$ . The proof of this result uses the limiting behaviour of the rescalings of  $\Psi_a$  or  $\Psi_b$  as  $a$  or  $b \rightarrow 0$  as described above, along with the properties of the Böhm functional  $\mathcal{B}$  and results from the previous section about the space of maximal volume orbits  $\mathcal{V}$ .

**Bubbling-off of asymptotically conical Calabi–Yau 3–folds.** Let  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  be the two 1–parameter families of solutions to (3.10) given by Theorems 4.4 and 4.5, respectively. Set  $\epsilon = a$  or  $\epsilon = b$  and consider the scaling

$$(\omega, \Omega) \longmapsto (\epsilon^{-2}\omega, \epsilon^{-3}\Omega).$$

Under such a transformation, the two 1-parameter families define two families of local nearly Kähler structures with  $\text{Scal} = 30\epsilon^2$  defined in a neighbourhood of a singular orbit of fixed size. In terms of the parametrisation of (3.1), a solution  $\Psi_\epsilon(t) = (\lambda(t), u(t), v(t))$  of (3.10) defines a solution

$$\tilde{\Psi}_\epsilon = (\epsilon^{-1}\lambda(\epsilon t), \epsilon^{-2}u(\epsilon t), \epsilon^{-3}v(\epsilon t))$$

of the ODE system

$$\begin{aligned} (6.1a) \quad & \lambda\dot{u}_0 + 3\epsilon v_0 = 0, \\ (6.1b) \quad & \lambda\dot{u}_1 + 3\epsilon v_1 - 2\lambda^2 = 0, \\ (6.1c) \quad & \lambda\dot{u}_2 + 3\epsilon v_2 = 0, \\ (6.1d) \quad & \dot{v}_0 - 4\epsilon\lambda u_0 = 0, \\ (6.1e) \quad & \dot{v}_1 - 4\epsilon\lambda u_1 = 0, \\ (6.1f) \quad & \epsilon\lambda\dot{v}_2 - 4\epsilon^2\lambda^2 u_2 + 3u_2 = 0, \\ (6.1g) \quad & \epsilon\lambda^2|u|^2\dot{\lambda} + 2\epsilon\lambda^4 u_1 + 3u_2 v_2 = 0, \end{aligned}$$

together with the constraints

$$\begin{aligned} (6.2a) \quad & \langle u, v \rangle = 0, \\ (6.2b) \quad & u_2 = -\epsilon\lambda\mu \\ (6.2c) \quad & \lambda^2|u|^2 = |v|^2, \\ (6.2d) \quad & v_1 = \epsilon\mu^2, \end{aligned}$$

where  $\mu^2 := |u|^2$ .

**Proposition 6.3** (Rescaled nearly Kähler “bubbles” converge to Calabi–Yau structures).

- (i) For every  $a \geq 0$  there exists a unique smooth solution  $\tilde{\Psi}_a$  to (6.1) with  $\epsilon = a$  satisfying the initial conditions

$$\lambda(0) = 0 = u_2(0), \quad u_0(0) = 1 = u_1(0), \quad v_i(0) = 0, \quad i = 0, 1, 2.$$

- (ii) For every  $b \geq 0$  there exists a unique smooth solution  $\tilde{\Psi}_b$  to (6.1) with  $\epsilon = b$  satisfying the initial conditions

$$\lambda(0) = 1, \quad u_i(0) = 0, \quad i = 0, 1, 2, \quad v_0(0) = -\frac{2}{3} = -v_2(0), \quad v_1(0) = 0.$$

Moreover,  $\tilde{\Psi}_a$  and  $\tilde{\Psi}_b$  depend continuously on  $a, b \in [0, \infty)$  and  $\lim_{a \rightarrow 0} \tilde{\Psi}_a, \lim_{b \rightarrow 0} \tilde{\Psi}_b$  is the asymptotically conical Calabi–Yau structure of Theorem 2.27(i) and (ii), respectively.

*Proof.* The proof is analogous to the one of Theorems 4.4 and 4.5 and we only prove (ii).

For better comparison with Theorem 2.27(ii) and since  $\lambda(0) = 1$ , we introduce a new variable defined by  $\frac{ds}{dt} = \frac{1}{\lambda}$ . Write

$$\begin{aligned} u_0(s) &= sy_1(s), & u_1(s) &= sy_2(s), & u_2(s) &= \epsilon sy_3(s) \\ v_0(s) &= -\frac{2}{3} + s^2 y_4(s), & v_1(s) &= s^2 y_5(s), & v_2(s) &= \frac{2}{3} + s^2 y_6(s), & \lambda^2(s) &= y_7(s). \end{aligned}$$

In terms of the new variables the system (6.1) becomes

$$\begin{aligned} \dot{y}_1 &= -\frac{y_1 - 2b}{s} - 3bsy_4, & \dot{y}_4 &= -\frac{2y_4 - 4by_1y_7}{s}, \\ \dot{y}_2 &= -\frac{y_2 - 2y_7}{s} - 3bsy_5, & \dot{y}_5 &= -\frac{2y_5 - 4by_2y_7}{s}, \\ \dot{y}_3 &= -\frac{y_3 + 2}{s} - 3sy_6, & \dot{y}_6 &= -\frac{2y_6 - 4b^2y_3y_7 + 3y_3}{s}, \\ \dot{y}_7 &= -\frac{4(y_2y_7^2 + y_3)}{s(-y_1^2 + y_2^2 + b^2y_3^2)} - \frac{6sy_3y_6}{y_1^2 + y_2^2 + b^2y_3^2}, \end{aligned}$$

which has the form  $\dot{y} = \frac{1}{s}M_{-1}(y) + M(s, y)$  of (4.8). Condition (i) in Theorem 4.7 fixes the initial condition

$$y_0 = (2b, 2, -2, 4b^2, 4b, 3 - 4b^2, 1).$$

Then the linearisation of  $M_{-1}$  at  $y_0$  is

$$d_{y_0}M_{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 4b & 0 & 0 & -2 & 0 & 0 & 8b^2 \\ 0 & 4b & 0 & 0 & -2 & 0 & 8b \\ 0 & 0 & 4b^2 - 3 & 0 & 0 & -2 & -8b^2 \\ 0 & -1 & -1 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

Since

$$\det(h\text{Id} - d_{y(0)}A) = (h+1)^2(h+2)^4(h+3) \neq 0$$

for all  $h \geq 1$  the hypotheses of Theorem 4.7 are satisfied and the existence of a continuous 1-parameter family of solutions to (6.1) follows.

Taking the limit  $b \rightarrow 0$  in the equations and initial conditions we immediately obtain that  $\tilde{\Psi}_0$  satisfies  $u_0 = u_2 = v_1 = 0$  (and therefore  $\mu = u_1$ ),  $u_1' = 2\lambda^2$  and  $v_0 = -\frac{2}{3}$ . Moreover, the constraint (6.2b) implies that  $sy_3 = -\lambda\mu$  and therefore  $(\lambda\mu)' = 3v_2$  and  $v_2' = 3\lambda\mu$ . Here  $'$  denotes derivative with respect to  $s$ . Comparing the parametrisations (2.28) and (3.1), by Theorem 2.27(ii),  $\tilde{\Psi}_0$  is the unique invariant complete Calabi–Yau structure on  $T^*S^3$  normalised so that  $\lambda(0) = 1$ .  $\square$

*Remark 6.4.* With respect to the new variable  $s$  introduced in the proof of the proposition the Taylor series of  $\tilde{\Psi}_b$  at  $s = 0$  is

$$\begin{aligned} \lambda^2(s) &= 1 - \frac{9}{5}(b^2 - 1)s^2 + \frac{27}{35}(b^2 - 1)(2b^2 - 1)s^4 + \dots, \\ u_0(s) &= 2bs - 4b^3s^3 + \frac{6}{25}b^3(19b^2 - 9)s^5 + \dots, \\ u_1(s) &= 2s - \frac{2}{5}(13b^2 - 3)s^3 + \frac{6}{175}(172b^4 - 111b^2 + 9)s^5 + \dots, \\ u_2(s) &= -2bs + b(4b^2 - 3)s^3 - \frac{3}{100}b(152b^4 - 192b^2 + 45)s^5 + \dots, \\ v_0(s) &= -\frac{2}{3} + 4b^2s^2 - \frac{2}{5}b^2(19b^2 - 9)s^4 + \dots, \\ v_1(s) &= 4bs^2 - \frac{4}{5}b(11b^2 - 6)s^4 + \dots, \\ v_2(s) &= \frac{2}{3} - (4b^2 - 3)s^2 + \frac{1}{20}(152b^4 - 192b^2 + 45)s^4 + \dots \end{aligned}$$

One checks that as  $b \rightarrow 0$  these are exactly the first terms in the Taylor series at  $s = 0$  for the Calabi–Yau structure of Theorem 2.27(ii).

**Singular limit: the sine-cone.** Since the Calabi–Yau manifolds of Theorem 2.27 are asymptotic to the Calabi–Yau cone over the standard Sasaki–Einstein structure on  $N_{1,1}$ , Proposition 6.3 implies that as  $\epsilon = a, b$  converges to zero we can find some time  $t_\epsilon > 0$  such that the  $SU(2)$ -structure  $\Psi_\epsilon(t_\epsilon)$  is arbitrarily close to the standard invariant Sasaki–Einstein structure. We will now deduce from this that  $\Psi_\epsilon$  converges to the sine-cone over the standard Sasaki–Einstein structure on  $N_{1,1}$  away from the singular orbit(s) as  $\epsilon \rightarrow 0$ .

The main tool is a functional  $\mathcal{B}$ , introduced by Böhm in his work [9] on cohomogeneity one Einstein metrics, which plays the role of a Lyapunov function for the system (3.10).

Recall from Section 5 that the Einstein equations for a family of equidistant hypersurfaces  $\mathbb{R} \times N$  is a first-order system for a pair  $(g, L)$  of a Riemannian metric  $g$  on  $N$  and a symmetric endomorphism of  $TN$  satisfying, among other constraints, (5.4b). We restrict here to the case where the pair  $(g, L)$  is homogeneous with respect to the action of a compact Lie group  $G$ . In [9, Equation (8)] Böhm considers the function

$$\mathcal{B}(g, L) = V(g)^{\frac{2}{n}} \left( |\mathring{L}|^2 + \text{Scal}(g) \right),$$

where  $n$  is the dimension of  $N$ ,  $V(g)$  the volume of  $N$  with respect to the metric  $g$ ,  $\mathring{L}$  is the traceless part of  $L$  and  $|\mathring{L}|^2 = \text{Tr}(\mathring{L}^2)$ . Using (5.4b),  $\mathcal{B}(g, L)$  can be rewritten as

$$(6.6) \quad \mathcal{B}(g, L) = V(g)^{\frac{2}{n}} \left( (n-1)\Lambda + l^2 \right),$$

where  $l = \text{Tr} L$ . Observe that the power of  $V$  makes  $\mathcal{B}$  scale-invariant. Moreover, suppose that  $(g, L)$  is an integral curve of the ODE system defining an Einstein metric of cohomogeneity one on  $(-\delta, \delta) \times N$ , ie  $L = \frac{1}{2}g^{-1}g'$  and  $(g, L)$  satisfies (5.2) and (5.4). Then  $\mathcal{B}$  is decreasing in  $t$  whenever  $l \geq 0$ .

**Lemma 6.7** ([9, Proposition 2.2]). *If  $(g_t, L_t)$  is a 1-parameter family of  $G$ -invariant pairs defining an Einstein metric of cohomogeneity one on  $(-\delta, \delta) \times N$  with Einstein constant  $\Lambda$  then*

$$\frac{d}{dt} \mathcal{B}(g, L) = -2 \frac{n-1}{n} V^{\frac{2}{n}}(g) \text{Tr} L |\mathring{L}|^2.$$

*Proof.* Differentiate (6.6) using (5.4a) and  $V' = lV$ . □

We now specialise to the case of a homogeneous pair  $(g, L)$  on  $N_{1,1}$  induced by a hypo or nearly hypo structure. Thus  $n = 5$  and  $\Lambda = 0$  or  $\Lambda = 5$ , respectively. By abuse of notation we write  $\mathcal{B}(\psi)$  when the pair  $(g, L)$  is determined by a (nearly) hypo structure  $\psi$ .

**Lemma 6.8.** *Let  $\Psi_\epsilon$  be one of the solutions of Theorems 4.4 and 4.5 with  $\epsilon = a, b$ , respectively.*

- (i)  $\mathcal{B}|_{\Psi_\epsilon}$  attains its minimum on the maximal volume orbit;
- (ii)  $\mathcal{B}(\psi) \geq 20$  for every invariant nearly hypo structure  $\psi$  with  $l = 0$  and equality holds if and only if  $\psi$  is a rotated Sasaki–Einstein structure.

*Proof.* By Proposition 5.6(iii) if  $\mathring{L} = 0$ , then in particular  $w_1 = w_2 = 0$  and Corollary 2.46 implies that  $\psi$  is an invariant hypersurface of the sine-cone. Hence  $|\mathring{L}| > 0$  on the solutions of Theorems 4.4 and 4.5 and Lemma 6.7 implies (i). By (6.6), restricted to the space of invariant nearly hypo structure with a fixed value of  $l$ ,  $\mathcal{B}$  is proportional to a fixed power of the volume. The statement in (ii) then follows from Proposition 5.9. □

Observe that in view of part (ii) and the scale invariance of  $\mathcal{B}$ , we have  $\mathcal{B} \equiv 20$  both on the sine-cone and the conifold. We now deduce the main result of this section.



**Theorem 6.9.** *As  $a, b \rightarrow 0$ ,  $\Psi_a$  and  $\Psi_b$  converge to the sine-cone over the standard Sasaki–Einstein structure on  $N_{1,1}$  on every relatively compact neighbourhood of the maximal volume orbit  $t = \frac{\pi}{2}$  in  $(0, \pi) \times N_{1,1}$ .*

*Proof.* Consider first the functional  $\mathcal{B}$  restricted to one of the Calabi–Yau structures of Theorem 2.27. Since they are asymptotic to the conifold,  $\mathcal{B}$  approaches the value 20 asymptotically.

By the scale-invariance of  $\mathcal{B}$  and Proposition 6.3 for every  $\epsilon = a$  or  $b$  sufficiently small we can then find some time  $t_\epsilon > 0$  such that  $l(t_\epsilon) > 0$  and  $\mathcal{B}(\Psi_\epsilon(t_\epsilon))$  is arbitrarily close to 20.

Lemma 6.8 then implies that as  $\epsilon \rightarrow 0$  the maximal volume orbit  $\Psi_\epsilon(t_{\max, \epsilon})$  converges to the “rotated” standard Sasaki–Einstein structure. The theorem is now a consequence of the continuous dependence on the initial conditions for solutions of (3.10) (and their time reversals) starting from a principal orbit, cf Proposition 3.9.  $\square$

## 7. AN EXOTIC NEARLY KÄHLER STRUCTURE ON $S^3 \times S^3$

In this section we prove the existence of an inhomogeneous nearly Kähler structure on  $S^3 \times S^3$ . According to part (iii) of the Doubling Lemma 5.19 every point of intersection of the curve  $\beta_W$  of Definition 5.16 with the boundary of  $W$  (recall Proposition 5.9) corresponds to a smooth invariant nearly Kähler structure on  $S^3 \times S^3$ .

Our strategy is inspired by Böhm’s work in [8, §4]: we will consider a function  $f$  on the space of invariant nearly hypo structures such that every solution  $\Psi_b$  of Theorem 4.5 intersects the level set  $f^{-1}(0)$  transversally and such that the intersection of  $f^{-1}(0)$  with the space of maximal volume orbits  $\mathcal{V}_0$  lies in the preimage of the boundary of  $W$  under the projection of Proposition 5.9. Studying how the number of zeroes of  $f$  before the maximal volume orbit of  $\Psi_b$  varies as a function of  $b > 0$  will allow us to detect intersection points of  $\beta_W$  with the boundary of  $W$ .

There are in fact various possible choices for the function  $f$ . Our choice is motivated by the fact that we will need to know more than the existence of an intersection point of  $\beta_W$  with the boundary of  $W$ . Indeed  $\beta_W(1)$ , the maximal volume orbit of the homogeneous nearly Kähler structure on  $S^3 \times S^3$ , lies on the portion of  $\partial W$  where  $\lambda = 1$ . For our applications in the next section, we will need to know that  $\beta_W$  has at least one intersection point with the second line  $\lambda = \mu$  defining the boundary of  $W$ . By (5.13b),  $\lambda = \mu$  is equivalent to  $v_0 = 0$  on a maximal volume orbit. We will then count zeroes of  $v_0$  (equivalently, critical points of  $u_0$ ) and show that there exists  $b \in (0, 1)$  such that  $v_0 = 0$  on the maximal volume orbit of  $\Psi_b$ .

**Counting zeroes of  $v_0$ .** Let  $\Psi = (\lambda, u, v)$  be a solution of (3.10). The pair  $(u_0, v_0)$  then satisfies the system

$$\lambda \dot{u}_0 = -3v_0, \quad \dot{v}_0 = 4\lambda u_0.$$

An immediate consequence is that critical points of  $u_0$  (equivalently, zeroes of  $v_0$ ) are non-degenerate unless  $u_0 = 0 = v_0$ . By Corollary 2.46 this can only occur on an invariant hypersurface of the sine-cone of Example 3.12.

For all  $b \in (0, \infty)$  let  $T_b$  denote the time of the maximal volume orbit of  $\Psi_b$ .

**Definition 7.1.** For all  $b \in (0, \infty)$  such that  $v_0(T_b) \neq 0$  let  $\mathcal{C}(b)$  be the number of zeroes of  $v_0$  in  $(0, T_b)$ .

The following properties of  $\mathcal{C}(b)$  follow easily from the non-degeneracy of critical points of  $u_0$ , cf [8, Lemmas 4.4 and 4.5].

**Lemma 7.2.**

- (i) *Given  $0 < b' < b''$ , suppose that  $v_0(T_b) \neq 0$  for all  $b \in [b', b'']$ . Then  $\mathcal{C}(b)$  is constant on  $[b', b'']$ .*

(ii) Suppose that  $b_* > 0$  is the unique value of  $b \in [b_* - \delta, b_* + \delta]$  such that  $v_0(T_b) = 0$ . Then

$$|\mathcal{C}(b') - \mathcal{C}(b'')| \leq 1$$

for all  $b', b'' \in [b_* - \delta, b_* + \delta]$  with  $b' < b_* < b''$ .

It follows that for all  $0 < b' < b''$  there exist at least  $|\mathcal{C}(b') - \mathcal{C}(b'')|$  values of  $b \in [b', b'']$  such that  $v_0(T_b) = 0$ .

Recalling Example 3.14 and Remark 4.6, when  $b = 1$  then  $v_0 = -6 \cos(2\sqrt{3}t)$  and  $T_b = \frac{\pi}{2\sqrt{3}}$ . The existence of a second nearly Kähler structure on  $S^3 \times S^3$  is now a consequence of Lemma 7.2 and the following result.

**Proposition 7.3.** *For all  $b > 0$  sufficiently small  $\mathcal{C}(b) \geq 2$ .*

In the rest of the section we prove this proposition.

For every  $b > 0$  consider the solution  $\Psi_b$  of (3.10) given by Theorem 4.7 and observe that  $u_0$  is a solution of the second order ODE

$$(7.4) \quad (\lambda u_0')' + 12\lambda u_0 = 0,$$

with initial conditions

$$u_0(0) = 0, \quad u_0'(0) = 2b^2 > 0.$$

By Theorem 6.9,  $\lambda(t) \rightarrow \sin t$  uniformly on compact sets of  $(0, \pi)$  as  $b \rightarrow 0$ . Since we are interested in the behaviour of  $u_0$  for small  $b$ , it makes sense to compare  $u_0$ , which satisfies (7.4), with a solution of the simpler limiting equation

$$(7.5) \quad (\sin t \xi')' + 12 \sin t \xi = 0.$$

*Remark.* In fact (7.5) naturally arises when considering the linearisation of (3.10) on the sine-cone of Example 3.12. Indeed, since  $u_0 = 0 = v_0$  on the sine-cone it is not difficult to see that the space of solutions of the linearised equations consists of the vector field corresponding to time-translation along the sine-cone and the 2-parameter family of solutions to (7.5).

Equation (7.5) is one of Legendre's equations. Its solutions take the form

$$(7.6) \quad \xi_0(t) = C_1 \xi_{\text{reg}} + C_2 \xi_{\text{sing}}$$

for constants  $C_1, C_2 \in \mathbb{R}$  where

$$\xi_{\text{reg}} = 5 \cos^3 t - 3 \cos t, \quad \xi_{\text{sing}} = \frac{5}{2} \cos^2 t + \frac{1}{8} \cos t (4 \cos^2 t - 6 \sin^2 t) \log \frac{1 - \cos t}{1 + \cos t} - \frac{2}{3}.$$

**Lemma 7.7.** *There exists a solution  $\xi_0$  of (7.5) with the following property: there exists  $0 < t_1 < t_2 < t_3 < \frac{\pi}{2}$  such that  $\xi_0(t_1) = 0 = \xi_0(t_2)$ ,  $\xi_0 \geq 0$  on  $[t_1, t_2]$  and  $\xi_0$  has a negative minimum at  $t_3$ .*

*Proof.* The function  $\xi_0$  in (7.6) with  $C_1 = 0, C_2 = 1$  has all the required properties except that  $t_3 = \frac{\pi}{2}$ . Since the Legendre polynomial  $5 \cos^3 t - 3 \cos t$  vanishes at  $t = \frac{\pi}{2}$  and has strictly positive derivative there, by taking  $C_1 > 0$  small enough we ensure that the qualitative behaviour of  $\xi_0$  is unchanged and at the same time its minimum occurs at  $t_3 < \frac{\pi}{2}$ . See Figure 1.  $\square$

The solution  $\xi_0$  in the previous lemma is singular at  $t = 0$  and  $\pi$ , but in the following we will only consider its restriction to the interval  $[t_1, \pi - t_1]$ .

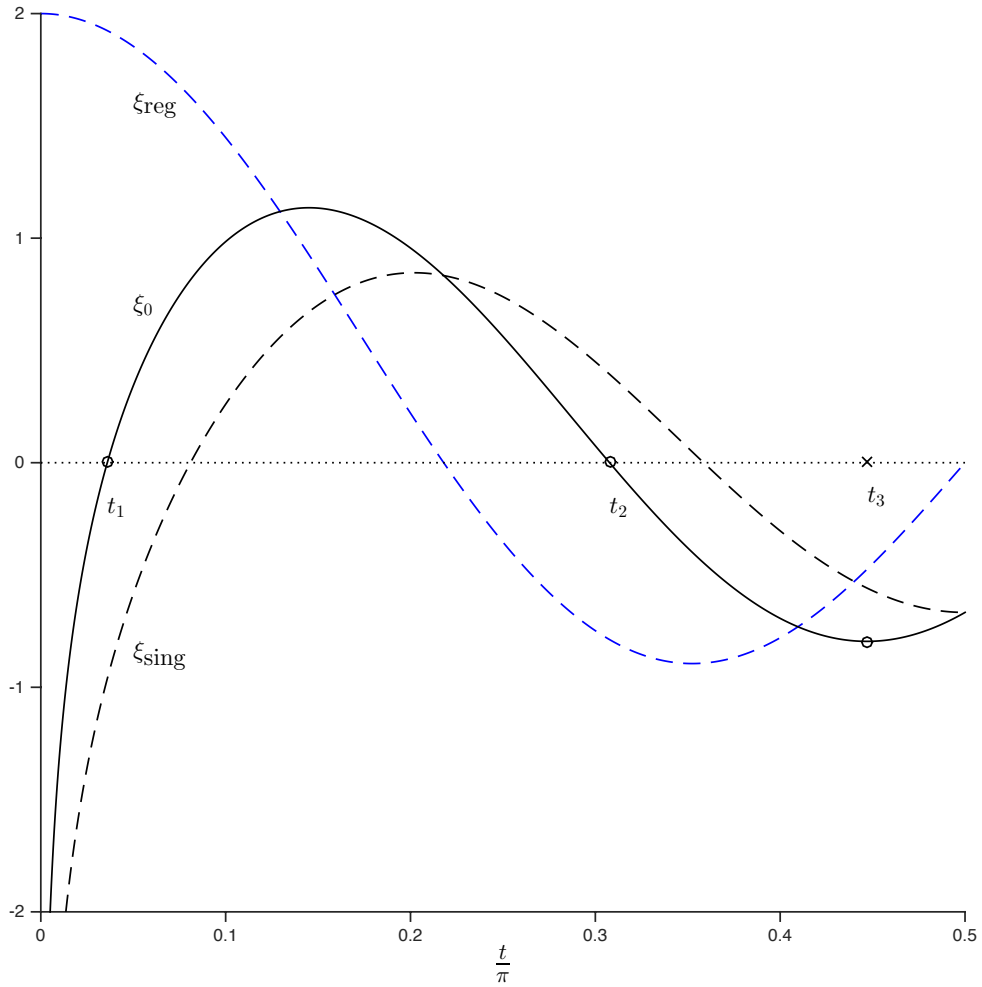


FIGURE 1. Solution  $\xi_0$  of (7.5) as in Lemma 7.7

**A Sturm comparison argument.** We are now going to compare the function  $u_0$  for a solution  $\Psi_b$  of (3.10) with  $b$  small enough, with the solution  $\xi_0$  of the Legendre equation (7.5) given in the previous lemma. The comparison result we need is the following generalisation of the Sturm comparison theorem for Sturm–Liouville equations.

**Lemma 7.8.** *Let  $\lambda_1, \lambda_2, q_1, q_2$  be continuous functions on  $[t_1, t_3] \subset \mathbb{R}$  such that*

$$\lambda_1 \geq \lambda_2 > 0, \quad q_2 \geq q_1,$$

*for all  $t \in [t_1, t_3]$ . Suppose that  $\xi$  is a solution of*

$$(\lambda_1 \xi')' + q_1 \xi = 0$$

such that  $\xi(t_1) = 0 = \xi(t_2)$  for some  $t_1 < t_2 < t_3$ ,  $\xi(t) \geq 0$  for all  $t \in [t_1, t_2]$  and  $\xi$  has a strict negative minimum at  $t_3$ . Then any solution  $u$  of

$$(\lambda_2 u')' + q_2 u = 0$$

has a strict negative minimum in  $(t_1, t_4)$  unless  $u(t_1) < 0$  and  $u'(t_1) \geq 0$ .

The proof is analogous to that of the classical Sturm comparison theorem, cf [7, Theorem 3, Chapter 10] and [8, Proposition 5.9].

**Proposition 7.9.** *There exists  $\epsilon > 0$  such that for all  $b < \epsilon$  the function  $u_0$  has a strict negative minimum before the maximal volume orbit.*

*Proof.* Fix  $0 < t_1 < t_3 < \frac{\pi}{2}$  as in Lemma 7.7.

By Theorem 6.9 for all  $\delta > 0$  there exists  $\epsilon > 0$  such that if  $b < \epsilon$  then  $|\lambda(t) - \sin t| < \delta$  for all  $t \in [t_1, \pi - t_1]$ .

Consider the Sturm–Liouville equation

$$(7.10) \quad ((\sin t + \delta) \xi')' + 12(\sin t - \delta) \xi = 0$$

on  $[t_1, \pi - t_1]$  with initial conditions  $\xi(t_1) = \xi_0(t_1)$ ,  $\xi'(t_1) = \xi'_0(t_1)$ , where  $\xi_0$  is the solution of (7.5) given by Lemma 7.7. Since the coefficients of (7.10) converge uniformly to the ones of (7.5) and  $\sin t + \delta \geq \sin t_1 > 0$ , by choosing  $\delta > 0$  small enough we can guarantee that  $\xi$  has the same behaviour as  $\xi_0$ , ie  $\xi$  has two zeroes  $t_1, t'_2$  and a negative minimum at  $t'_3 < \frac{\pi}{2}$ .

We can then apply Lemma 7.8 with  $\lambda_1 = \sin t + \delta$ ,  $q_1 = \sin t - \delta$ ,  $\lambda_2 = q_2 = \lambda$  to compare  $u_0$  to  $\xi$  on  $[t_1, t'_3]$ : either  $u_0$  has a strict negative minimum in  $(t_1, t'_3)$  or  $u_0(t_1) < 0$  and  $u'_0(t_1) \geq 0$ . In the former case, by taking  $\epsilon$  smaller if necessary we can make sure that the maximal volume orbit of  $\Psi_b$  occurs at some time  $t > t'_3$  for all  $b < \epsilon$ . In the latter case, because of the initial conditions,  $u_0$  must have a negative minimum in  $[0, t_1]$ .  $\square$

Because of the initial conditions,  $u_0$  must also have a positive maximum before achieving the minimum in the proposition. Thus Proposition 7.3 is proved.

*Remark 7.11.* The whole argument can also be carried out in the case of the solutions  $\Psi_a$  of Theorem 4.4: Definition 7.1, Lemma 7.2 and Proposition 7.9 still hold. However in this case  $u_0$  is a solution of (7.4) with initial conditions  $u_0(0) = a^2 > 0$ ,  $u'_0(0) = 0$ . We therefore conclude that  $\mathcal{C}(a) \geq 1$  for  $a > 0$  sufficiently small. When  $a^2 = 3$ , ie  $\Psi_a$  is the homogeneous nearly Kähler structure on  $S^6$  given in Example 3.13, one checks that  $\mathcal{C}(a) = 0$ . Thus there must exist at least one value of  $a \in (0, \sqrt{3})$  that satisfies the hypothesis of Lemma 5.19(ii). On the other hand, by Remark 4.6,  $a = \frac{\sqrt{3}}{2}$  already gives such a solution, the homogeneous nearly Kähler structure on  $CP^3$ .

**Theorem 7.12.** *There exists  $b \in (0, 1)$  such that  $\Psi_b$  defines an inhomogeneous nearly Kähler structure on  $S^3 \times S^3$ .*

*Proof.* By Lemma 7.2 and Proposition 7.3 there exists at least one  $b \in (0, 1)$  such that the maximal volume orbit of  $\Psi_b$  lies on the portion of the boundary of  $W$  with  $\lambda = \mu > 1$ . By part (iii) of the Doubling Lemma 5.19 this is enough to guarantee that  $\Psi_b$  defines a smooth nearly Kähler structure on  $S^3 \times S^3$ . It remains to show that this is not homogeneous and therefore defines a new nearly Kähler structure.

Consider the Riccati equation (5.1). Since  $L = \frac{1}{2}g^{-1}\dot{g}$ , the component in the direction of the Reeb vector field  $U^-$  gives

$$\frac{\ddot{\lambda}}{\lambda} + \hat{R}(U^-, U^-) = 0.$$

Suppose that the new nearly Kähler structure is homogeneous. We first show that  $\hat{R}(U^-, U^-)$  must be constant. Let  $\hat{g}$  be the metric on  $S^3 \times S^3$  induced by the new nearly Kähler structure and  $\nabla$  denote its Levi-Civita connection. Observe that  $\hat{R}(U^-, U^-) = \hat{R}(\partial_t, J\partial_t, \partial_t, J\partial_t)$  because  $J\partial_t$  is parallel to the Reeb vector field  $U^-$ . Since  $\partial_t$  is the unit tangent vector of a geodesic in  $(S^3 \times S^3, \hat{g})$  we have  $\nabla_{\partial_t} \partial_t = 0$ . Moreover, the nearly Kähler property implies that  $(\nabla_X J)X = 0$  for every tangent vector  $X$  (see [4, p. 517] for a simple derivation from the  $G_2$ -holonomy perspective). Therefore  $\nabla_{\partial_t}(J\partial_t) = 0$ . By Butruille's classification of homogeneous nearly Kähler 6-manifolds [15], if  $(S^3 \times S^3, \hat{g})$  is homogeneous it must be a 3-symmetric space. Then part (ii) in [31, Proposition 4.3] implies that

$$(\nabla_X R)(X, JX, X, JX) = 0$$

for every tangent vector  $X$ . We conclude that  $\hat{R}(U^-, U^-)$  is constant.

Now, since  $\lambda(0) = \lambda(T) > 0$ , where  $T$  is the maximal existence time of  $\Psi_b$ , and  $\lambda$  is even at  $t = 0$  and  $t = T$ , the only possibility would be  $\hat{R}(U^-, U^-) = 0$  and  $\lambda$  a constant. However, it is easy to check that the unique solution of (3.10) with  $\lambda = \text{const}$  is  $\Psi_b$  with  $b = 1$ , ie the homogeneous nearly Kähler structure on  $S^3 \times S^3$  of Example 3.14. For example, this follows from the power series expansions of Remark 6.4.  $\square$

*Remark 7.13.* A similar argument shows that the solutions of Theorems 4.4 and 4.5 are mutually non-isometric. Indeed, using the constancy of  $\hat{R}(U^-, U^-)$  on nearly Kähler 3-symmetric spaces and the Taylor series of Remarks 4.10 and 6.4 one can show that  $\Psi_a$  and  $\Psi_b$  cannot be homogeneous unless  $a = \frac{\sqrt{3}}{2}, \sqrt{3}$  and  $b = 1, \frac{3}{2}$ , the known homogeneous examples of Remark 4.6. On the other hand, assume that  $f$  is an isometry between, say, the metric induced by  $\Psi_a$  and  $\Psi_{a'}$  for  $a \neq a'$ . Since  $f$  cannot preserve the  $SU(2) \times SU(2)$ -orbits, the tangent space of a point is spanned by Killing fields and  $\Psi_a, \Psi_{a'}$  would then be homogeneous.

## 8. AN EXOTIC NEARLY KÄHLER STRUCTURE ON $S^6$

In this section we prove the existence of an inhomogeneous nearly Kähler structure on  $S^6$ . By part (iii) of the Matching Lemma 5.20 we have to show that there exist two values  $a, b \in (0, \infty)$  such that the two curves  $\alpha_W, \beta_W$  parametrising the maximal volume orbits of the solutions of Theorems 4.4 and 4.5 up to discrete symmetries intersect. One intersection point is already known to exist: by Remark 4.6 the choice  $a = \sqrt{3}$  and  $b = \frac{3}{2}$  yields the standard nearly Kähler structure on  $S^6$ . We will show that there exists a second intersection point. The key new ingredient is Proposition 8.8 which gives us some control over the solution  $\Psi_a$  as  $a \rightarrow \infty$ .

**Theorem 8.1.** *There exists  $a \neq \sqrt{3}$  and  $b \in (0, 1)$  such that  $\Psi_a$  and  $\Psi_b$  satisfy the conditions of part (iii) of the Matching Lemma 5.20 and therefore define an inhomogeneous nearly Kähler structure on  $S^6$ .*

The rest of the section contains the proof of the theorem, which consists of various steps.

We first give an alternative parametrisation of the space of maximal volume orbits  $\mathcal{V}$  to that of Proposition 5.9. Recall that  $\mathcal{V} = \mathcal{V}_0 \times SO(2)$ , where  $\mathcal{V}_0 = \mathcal{U} \cap l^{-1}(0) \subset SO_0(1, 2)$ . Denote by  $\pi$  the natural projection  $SO_0(1, 2) \rightarrow SO_0(1, 2)/SO(2)$  to  $\mathcal{V}_0$ . We identify  $SO_0(1, 2)/SO(2)$  with the upper hyperboloid

$$H = \{w = (w_0, w_1, w_2) \in \mathbb{R}^{1,2}, |w|^2 = -1, w_0 > 0\}$$

and take  $(w_1, w_2) \in \mathbb{R}^2$  as global coordinates on  $H$ .

**Lemma 8.2.** *The projection  $\pi: \mathcal{V}_0 \rightarrow H$  is a homeomorphism.*

*Proof.* Since  $H$  is endowed with the quotient topology we only have to show that  $\pi$  is a bijection.

In the notation of Proposition 2.41, let

$$B = \begin{pmatrix} w_0 & x_0 & y_0 \\ w_1 & x_1 & y_1 \\ w_2 & x_2 & y_2 \end{pmatrix} \in \mathcal{U}$$

parametrise an invariant nearly hypo structure with  $\theta = 0$  as usual. Thus  $x_2 < 0$ ,  $y_1 > 0$  and  $B \in \text{SO}_0(1, 2)$ . The projection  $\pi: \text{SO}_0(1, 2) \rightarrow H$  is then the map  $B \mapsto w \in H$ . Let  $\pi^{-1}(w)$  be the circle fibre (2.45) of  $\pi$  through  $B$ , parametrised by an angle variable  $\phi$ . Since  $\pi^{-1}(w) \cap \mathcal{U}$  is an interval containing 0 of length at most  $\pi$ , we take  $\tan \phi$  as a coordinate on  $\pi^{-1}(w) \cap \mathcal{U}$ . The range of  $\tan \phi$  is then the connected interval containing 0 where  $x_1 \tan \phi + y_1 > 0$  and  $x_2 - y_2 \tan \phi < 0$ .

By Lemma 5.6(ii) the restriction of  $l$  to  $\pi^{-1}(w) \cap \mathcal{U}$  is

$$l(\tan \phi) = 2 \frac{x_1 - y_1 \tan \phi}{x_1 \tan \phi + y_1} - 3 \frac{x_2 \tan \phi + y_2}{x_2 - y_2 \tan \phi}.$$

Now, on one hand

$$l'(\tan \phi) = -2 \frac{x_1^2 + y_1^2}{(x_1 \tan \phi + y_1)^2} - 3 \frac{x_2^2 + y_2^2}{(x_2 - y_2 \tan \phi)^2} < 0$$

and therefore  $\pi: \mathcal{V}_0 \rightarrow H$  is injective. On the other hand  $l(\tan \phi)$  approaches values of opposite sign as  $\tan \phi$  converges to the endpoints of its range, as can be easily checked by working out the precise range of  $\tan \phi$  according to the sign of  $x_1, y_2$ . More precisely,  $l(\tan \phi)$  approaches  $\pm\infty$  at the boundary of its range unless (i)  $x_1, y_2 > 0$  or (ii)  $x_1, y_2 < 0$ .

In case (i)  $\max\left(-\frac{y_1}{x_1}, \frac{x_2}{y_2}\right) < \tan \phi < \infty$  and  $-2\frac{y_1}{x_1} + 3\frac{x_2}{y_2} < l(\tan \phi) < \infty$ , while in case (ii)  $-\infty < \tan \phi < \min\left(-\frac{y_1}{x_1}, \frac{x_2}{y_2}\right)$  and  $-\infty < l(\tan \phi) < -2\frac{y_1}{x_1} + 3\frac{x_2}{y_2}$ .  $\square$

The lemma implies that the two continuous curves  $\alpha$  and  $\beta$  parametrising the maximal volume orbits of  $\Psi_a, \Psi_b$  can also be regarded as curves in  $H$ .

**Definition 8.3.** Let  $\alpha_H, \beta_H: (0, \infty) \rightarrow H$  be the two continuous curves in  $H \simeq \mathcal{V}_0$  parametrising the maximal volume orbits of the solutions  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  of Theorems 4.4 and 4.5.

We collect properties of  $\alpha_H, \beta_H$  that are readily deduced from results of the previous sections.

**Lemma 8.4.**

- (i) *The curves  $\alpha_H, \beta_H$  do not self-intersect.*
- (ii) *The curves  $\alpha_H$  and  $\beta_H$  cannot intersect for positive values of the parameters  $a, b > 0$ .*
- (iii)  *$\lim_{a \rightarrow 0^+} \alpha_H(a) = \lim_{b \rightarrow 0^+} \beta_H(b) = (0, 0)$  and  $\alpha_H(a), \beta_H(b)$  are distinct from the origin for  $a, b > 0$ .*

*Proof.* Parts (i) and (ii) follow from the uniqueness of solutions to (3.10) with given initial conditions since  $H \simeq \mathcal{V}_0$  by Lemma 8.2. Part (iii) follows from Theorem 6.9 and Corollary 2.46.  $\square$

It will be important to understand the induced action of the discrete symmetries of Proposition 3.11 on  $H$ . Observe that in terms of the parametrisation  $(\lambda, u, v)$  of (3.1)

$$(8.5) \quad w_0 = \frac{u_1 v_2 - u_2 v_1}{V}, \quad w_1 = \frac{u_0 v_2 - u_2 v_0}{V}, \quad w_2 = \frac{u_1 v_0 - u_0 v_1}{V},$$

where  $V = \lambda\mu^2$  and  $\mu^2 = |u|^2$ , since  $w$  is the Minkowski “cross-product” of the two orthogonal space-like vectors  $\frac{u}{|u|}, \frac{v}{|v|} \in \mathbb{R}^{1,2}$ . An immediate consequence of Proposition 3.11 and (8.5) is the following lemma.

**Lemma 8.6.** *Set  $\epsilon = a, b > 0$  and let  $\Psi_\epsilon$  be the solution to (3.10) given by Theorems 4.4 or 4.5, respectively. The image of  $\alpha_H$  or  $\beta_H$  under the involutions*

$$(w_1, w_2) \mapsto (-w_1, -w_2), \quad (w_1, w_2) \mapsto (-w_1, w_2), \quad (w_1, w_2) \mapsto (w_1, -w_2)$$

*parametrises the maximal volume orbit of, respectively,*

$$\tau_2 \circ \tau_3 \circ \tau_4(\Psi_\epsilon), \quad \tau_1 \circ \tau_2 \circ \tau_3(\Psi_\epsilon), \quad \tau_1 \circ \tau_4(\Psi_\epsilon).$$

Finally, the results of Section 7 allow us to deduce the following crucial property of the curve  $\beta_H$ .

**Lemma 8.7.** *There exists  $0 < b' < b'' \leq 1$  such that the arc  $\beta_H(b)$ ,  $b' \leq b \leq b''$ , and its image under the involutions of Lemma 8.6 form the boundary of a bounded closed set  $D \subset H$  which contains the origin in its interior.*

*Proof.* By the proof of Proposition 7.3 for  $b > 0$  sufficiently small the function  $u_0$  in  $\Psi_b$  has at least two critical points and one zero before the maximal volume orbit. On the other hand, by Remark 4.6 the solution  $\Psi_b$  with  $b = 1$  is the homogeneous nearly Kähler structure on  $S^3 \times S^3$ . Thus for  $b = 1$  the function  $u_0$  has a unique maximum before the maximal volume orbit and a unique zero, which occurs at the maximal volume orbit. We do not know whether the number of critical points or zeroes of  $u_0$  before the maximal volume orbits is monotone in  $b$ . However, the observations above guarantee the existence of an interval  $0 < b' < b'' \leq 1$  such that  $u_0$  has a unique maximum and a unique zero before the maximal volume orbit for all  $b' \leq b \leq b''$ , a minimum on the maximal volume orbit when  $b = b'$  and a zero on the maximal volume orbit when  $b = b''$ .

By (5.13) and (8.5), on a maximal volume orbit  $w_1 = 0$  if and only if  $v_0 = 0$  and similarly  $w_2 = 0$  if and only if  $u_0 = 0$ . Thus the arc  $\beta_H(b)$ ,  $b' < b < b''$  is contained in an *open* quadrant of the  $(w_1, w_2)$ -plane. We conclude that the arc  $\beta_H(b)$ ,  $b' \leq b \leq b''$  together with its image under the involutions of Lemma 8.6 form a continuous closed curve  $\gamma$  in  $H \simeq \mathbb{R}^2$ . This curve is simple by Lemma 8.4(i) and does not contain the origin by part (iii) of the same Lemma. The existence of the domain  $D$  follows from the Jordan curve theorem. By the construction of  $\gamma$  the origin is contained in the interior of  $D$ .  $\square$

By Lemma 8.4(ii) the boundary of  $D$  cannot contain the points  $\alpha_H(\frac{\sqrt{3}}{2})$ ,  $\alpha_H(\sqrt{3})$  (the maximal volume orbits of the homogeneous nearly Kähler structures on  $CP^3$  and  $S^6$  respectively by Remark 4.6) nor their image under the group generated by reflections along the axes. If  $\alpha_H(\frac{\sqrt{3}}{2})$  or  $\alpha_H(\sqrt{3})$  do not belong to  $D$  then the proof of Theorem 8.1 is complete, because the curve  $\alpha_H$  must intersect the boundary of  $D$ . The bad case is therefore when  $\alpha_H(\frac{\sqrt{3}}{2})$  and  $\alpha_H(\sqrt{3})$  both lie in the interior of  $D$ .

**Proposition 8.8.** *The curve  $\alpha_H$  exits any compact set of  $\mathcal{V}_0$  as  $a \rightarrow \infty$ .*

*Proof.* In order to understand the behaviour of  $\Psi_a$  as  $a \rightarrow \infty$ , we observe that the Taylor series of Remark 4.10 suggest that we consider the rescaling

$$(8.9) \quad \tilde{\Psi}_a(t) = \left( \lambda(t), \frac{u_0(t)}{a^2}, \frac{u_1(t)}{a^2}, \frac{u_2(t)}{a}, \frac{v_0(t)}{a^2}, \frac{v_1(t)}{a^2}, \frac{v_2(t)}{a} \right).$$

Observe that  $\tilde{\Psi}_a(t)$  does not satisfy the constraints (3.5) and therefore does not define an  $SU(2)$ -structure.

We now want to derive differential equations satisfied by  $\tilde{\Psi}_a$  which are well-behaved in the limit  $a \rightarrow +\infty$ . Using the conserved quantities (3.5) satisfied by  $\Psi_a$ , we can rewrite the last equation of (3.10) as

$$(8.10) \quad \dot{\lambda} = -\frac{2\lambda^2 u_1}{v_1} - \frac{3v_2}{u_2}.$$

Then  $\tilde{\Psi}_a$  is a solution of the ODE system

$$\begin{aligned} \lambda \dot{u}_0 + 3v_0 &= 0, & \dot{v}_0 - 4\lambda u_0 &= 0, \\ \lambda \dot{u}_1 + 3v_1 - 2\epsilon^2 \lambda^2 &= 0, & \dot{v}_1 - 4\lambda u_1 &= 0, \\ \lambda \dot{u}_2 + 3v_2 &= 0, & \lambda \dot{v}_2 - 4\lambda^2 u_2 + 3u_2 &= 0, \end{aligned}$$

where  $\epsilon = \frac{1}{a}$ , together with (8.10).

The conditions of Lemma 4.1 and of Theorem 4.4 suggest that we write

$$\begin{aligned} u_0(t) &= 1 + t^2 y_1(t), & u_1(t) &= 1 + t^2 y_2(t), & u_2(t) &= t^2 y_3(t), \\ v_0(t) &= t^2 y_4(t), & v_1(t) &= t^2 y_5(t), & v_2(t) &= t^2 y_6(t), & \lambda(t) &= t y_7(t). \end{aligned}$$

Then  $y = (y_1, \dots, y_7)$  satisfies

$$\begin{aligned} \dot{y}_1 &= -\frac{1}{t} \left( 2y_1 + 3\frac{y_4}{y_7} \right), & \dot{y}_4 &= -\frac{1}{t} (2y_4 - 4y_7) + 4ty_1 y_7, \\ \dot{y}_2 &= -\frac{1}{t} \left( 2y_2 + 3\frac{y_5}{y_7} - 2\epsilon^2 y_7 \right), & \dot{y}_5 &= -\frac{1}{t} (2y_5 - 4y_7) + 4ty_2 y_7, \\ \dot{y}_3 &= -\frac{1}{t} \left( 2y_3 + 3\frac{y_6}{y_7} \right), & \dot{y}_6 &= -\frac{1}{t} \left( 2y_6 + 3\frac{y_3}{y_7} \right) + 4ty_3 y_7, \\ \dot{y}_7 &= -\frac{1}{t} \left( y_7 + 2\frac{y_7^2}{y_5} + 3\frac{y_6}{y_3} \right) - 2t\frac{y_2 y_7^2}{y_5}, \end{aligned}$$

and the initial condition

$$y_0 = \left( -3, -3 + \frac{3}{2}\epsilon^2, -\frac{3\sqrt{3}}{2}, 3, 3, \frac{3\sqrt{3}}{2}, \frac{3}{2} \right).$$

Thus  $y$  is a solution of a singular initial value problem of the form  $\dot{y} = \frac{1}{t} M_{-1}(y) + M(t, y)$ . It is immediate to check that  $M_{-1}(y_0) = 0$  and that the linearisation of  $M_{-1}$  at  $y_0$

$$d_{y_0} M_{-1} = \begin{pmatrix} -2 & 0 & 0 & -2 & 0 & 0 & 4 \\ 0 & -2 & 0 & 0 & -2 & 0 & 4 + 2\epsilon^2 \\ 0 & 0 & -2 & 0 & 0 & -2 & 2\sqrt{3} \\ 0 & 0 & 0 & -2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & -2 & 0 & 0 & -2 & -2\sqrt{3} \\ 0 & 0 & \frac{2}{\sqrt{3}} & 0 & \frac{1}{2} & \frac{2}{\sqrt{3}} & -3 \end{pmatrix}$$

satisfies

$$\det(h\text{Id} - d_{y_0} M_{-1}) = h(h+1)(h+2)^3(h+4)^2 \neq 0$$

for all integer  $h \geq 1$ . Theorem 4.7 then implies the existence of a 1-parameter family  $\tilde{\Psi}_a$  depending continuously on  $\epsilon = \frac{1}{a} \geq 0$ .

Hence as  $a \rightarrow \infty$ ,  $\tilde{\Psi}_a$  approaches a well-defined smooth limit  $\tilde{\Psi}_\infty$  defined on an interval  $0 \leq t < T$ . In particular, reversing the scaling (8.9), there exist smooth functions  $l_\infty, V_\infty$  such



that the mean curvature  $l$  and the orbital volume function  $V$  of  $\Psi_a$  are  $C^0$ -close to  $l_\infty$  and  $a^4 V_\infty$ , respectively, for  $a$  sufficiently large. Fix  $0 < t_* < \min(T, \frac{\pi}{2})$  such that  $l_\infty(t_*) > 0$  (the initial conditions guarantee the existence of such  $t_*$ ). Then the comparison results of Proposition 5.5 imply that for  $a$  sufficiently large the volume  $V_{\max}$  of the maximal volume orbit of  $\Psi_a$  satisfies

$$V_{\max} \geq a^4 \frac{V_\infty(t_*)}{\sin^5(t_* + \frac{\pi}{2})} - \delta,$$

where  $\delta > 0$  can be chosen arbitrarily small as  $a \rightarrow \infty$ . Thus the maximal volume orbit of  $\Psi_a$  has unbounded volume as  $a \rightarrow \infty$ . Recalling that  $V = \lambda\mu^2$ , the result follows immediately from the parametrisation of  $\mathcal{V}_0$  as a branched 4-fold cover of the wedge  $W$  in the  $(\lambda, \mu)$ -plane given in Proposition 5.9.  $\square$

*Proof of Theorem 8.1.* By Proposition 8.8 the curve  $\alpha_H$  intersects the boundary of  $D$ . By Lemmas 8.4(ii) and 8.6 such an intersection point can only occur on the image of the arc  $\beta_H(b)$ ,  $b \in (b', b'')$ , under the symmetries  $(w_1, w_2) \mapsto (-w_1, w_2)$  or  $(w_1, w_2) \mapsto (w_1, -w_2)$ . Part (iii) of the Matching Lemma 5.20 then implies the existence of a smooth nearly Kähler structure on  $S^6$ . It remains only to show that this is not homogeneous.

As in the proof of Theorem 7.12 we look at the Riccati equation (5.1) in the direction of the Reeb vector field  $U^-$ . If the constructed nearly Kähler structure on  $S^6$  were homogeneous then it would have to be the standard nearly Kähler structure on  $S^6$ . In particular,  $\hat{R}(U^-, U^-) = 1$  and  $\lambda = C_1 \cos t + C_2 \sin t$  for some constants  $C_1$  and  $C_2$ . Without loss of generality assume that the singular orbit  $S^3$  occurs at  $t = 0$ . Since  $\lambda$  must be even in  $t$  we have  $C_2 = 0$ . The singular orbit  $S^2$  must then occur at  $t = \frac{\pi}{2}$  (the first zero of  $\lambda$ ) and the Taylor series of Remark 4.10 (or the condition  $y_7(0) = \frac{3}{2}$  in the proof of Theorem 4.4) imply that  $C_1 = \frac{3}{2}$ . This however is impossible since  $\lambda(0) = b < b'' \leq 1$  by assumption.  $\square$

## 9. CONJECTURES AND NUMERICAL RESULTS

Theorems 7.12 and 8.1 guarantee the existence of at least one complete inhomogeneous nearly Kähler structure both on  $S^3 \times S^3$  and on  $S^6$  (as stated in the Main Theorem). In fact we make the following:

*Conjecture.* The Main Theorem yields all (inhomogeneous) complete cohomogeneity one nearly Kähler structures on simply connected manifolds. In particular,  $S^2 \times S^4$  does not admit any cohomogeneity one nearly Kähler structure and  $CP^3$  admits only its homogeneous one.

This conjecture is motivated by a systematic numerical study of the ODE system (3.10). In this final less formal section we discuss numerical results in support of the Conjecture and provide some numerical information about the nearly Kähler structures of the Main Theorem. A more detailed account of the numerics may appear elsewhere.

**The numerical scheme.** The proofs of Theorems 4.4 and 4.5, where existence of the two 1-parameter families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  is established, can be turned into a constructive numerical scheme useful in the study of the system (3.10). These proofs showed the existence of recurrence relations that uniquely determine the coefficients of the Taylor series of  $\Psi_a$  and  $\Psi_b$  at  $t = 0$  once initial conditions are fixed. The initial conditions are uniquely determined by the choice of  $a$  or  $b$  respectively, *eg* see (4.9) for the initial conditions in terms of  $a$ . After computing the first several terms of the Taylor series by hand we made these recurrence relations explicit and then computed the first 50 nonzero terms in these Taylor series symbolically in MATLAB using its Symbolic Math Toolbox. The first few terms of these power series expansions are recorded in Remarks 4.10 and 6.4.

The main problem in using numerical methods to study the existence of new complete cohomogeneity one nearly Kähler structures is the inevitability of singularities in the coefficients of the equations (3.10). We overcome this problem as follows. First using the symbolic polynomials associated to the two families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  described above we find (very high order) approximations to regular initial conditions for the ODE system (3.10) simply by evaluating these polynomials at some sufficiently small positive value  $t_*$  of  $t$ . Now, by Proposition 5.15, we know that every solution  $\Psi_a$  or  $\Psi_b$  has a unique maximal volume orbit that it attains at some time  $t_{\text{vmax}}$  before the solution develops its second singularity. Moreover, the maximal volume orbit is characterised algebraically by the equality (5.10). Therefore for each positive  $a$  or  $b$  we approximate the solution  $\Psi_a$  or  $\Psi_b$  on the interval  $[t_*, t_{\text{vmax}}]$  by using one of the standard MATLAB ODE solvers (we found ODE45 to be suitable) to integrate numerically equation (3.10) beginning with nonsingular initial conditions at  $t = t_*$  and detecting  $t_{\text{vmax}}$  by evolving the solution numerically until a zero of  $2\lambda^4 u_1 - 3u_2 v_2$  occurs.

In particular, this numerical scheme allows us to obtain very accurate numerical approximations to the two curves  $\alpha$  and  $\beta : (0, \infty) \rightarrow \mathcal{V}$  parameterising the (unique) maximal volume orbits of the two 1-parameter families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$  respectively. We can therefore use these numerical approximations of  $\alpha$  and  $\beta$  to study when the conditions of the Doubling and the Matching Lemmas 5.19 and 5.20 can be satisfied. This has the great advantage that we never have to solve numerically toward an unknown final time when the solution becomes singular again (some numerical experimentation makes it clear that in practice that more naive strategy is very unstable.)

**Conjectures based on numerics.** The properties of the numerical approximations to the curves  $\alpha$  and  $\beta$  so obtained suggest a number of concrete conjectures on the behaviour of  $\Psi_a$  and  $\Psi_b$  as functions of the parameters, that one might hope to establish analytically.

*Conjecture 9.1.* The volume of the maximal volume orbit of  $\Psi_a$  and  $\Psi_b$  is strictly increasing in  $a$  and  $b$ , respectively.

*Remark.* The fact that the volume of the maximal volume orbit of  $\Psi_a$  is eventually strictly increasing, *ie* strictly increasing for all  $a$  sufficiently large, follows from the rescaling argument employed in the proof of Proposition 8.8. A different rescaling argument based on further contemplation of the power series expansions for  $\Psi_b$  might establish the same result for the family  $\{\Psi_b\}_{b>0}$ .

An immediate consequence of the verification of this conjecture would be: the curves  $\alpha_W$  and  $\beta_W$  of Definition 5.16 can never self-intersect and hence parts (i) and (ii) of the Matching Lemma 5.20 can never be applied. In particular, for any complete cohomogeneity one nearly Kähler structure on  $CP^3$ ,  $S^2 \times S^4$  and  $S^3 \times S^3$  both singular orbits must have the same volume, *ie* is obtained by “doubling” some member of one of the two families  $\{\Psi_a\}_{a>0}$  and  $\{\Psi_b\}_{b>0}$ .

We next consider how many complete nearly Kähler structures arise by applying the Doubling Lemma 5.19. By (5.13) the conditions of the lemma are satisfied if and only if either  $u_0$  or  $v_0$  has a zero on the maximal volume orbit. In Definition 7.1 we considered the number  $\mathcal{C}(b)$  of zeroes of  $v_0$  before the maximal volume orbit of  $\Psi_b$ . According to Remark 7.11 it is possible to extend this definition to the family  $\{\Psi_a\}_{a>0}$  as well, and we write this count as  $\mathcal{C}(a)$ .

*Conjecture 9.2.* The count  $\mathcal{C}$  of zeroes of  $v_0$  before the maximal volume orbit satisfies:

- (i)  $\mathcal{C}(a)$  and  $\mathcal{C}(b)$  are decreasing in  $a$  and  $b$ , respectively;
- (ii) for  $a > 0$  sufficiently small  $\mathcal{C}(a) = 1$ ; for  $b > 0$  sufficiently small  $\mathcal{C}(b) = 2$ ;
- (iii)  $\mathcal{C}(a) = 0$  for all  $a \geq \sqrt{3}$  and  $\mathcal{C}(b) = 1$  for all  $b \geq 1$ .

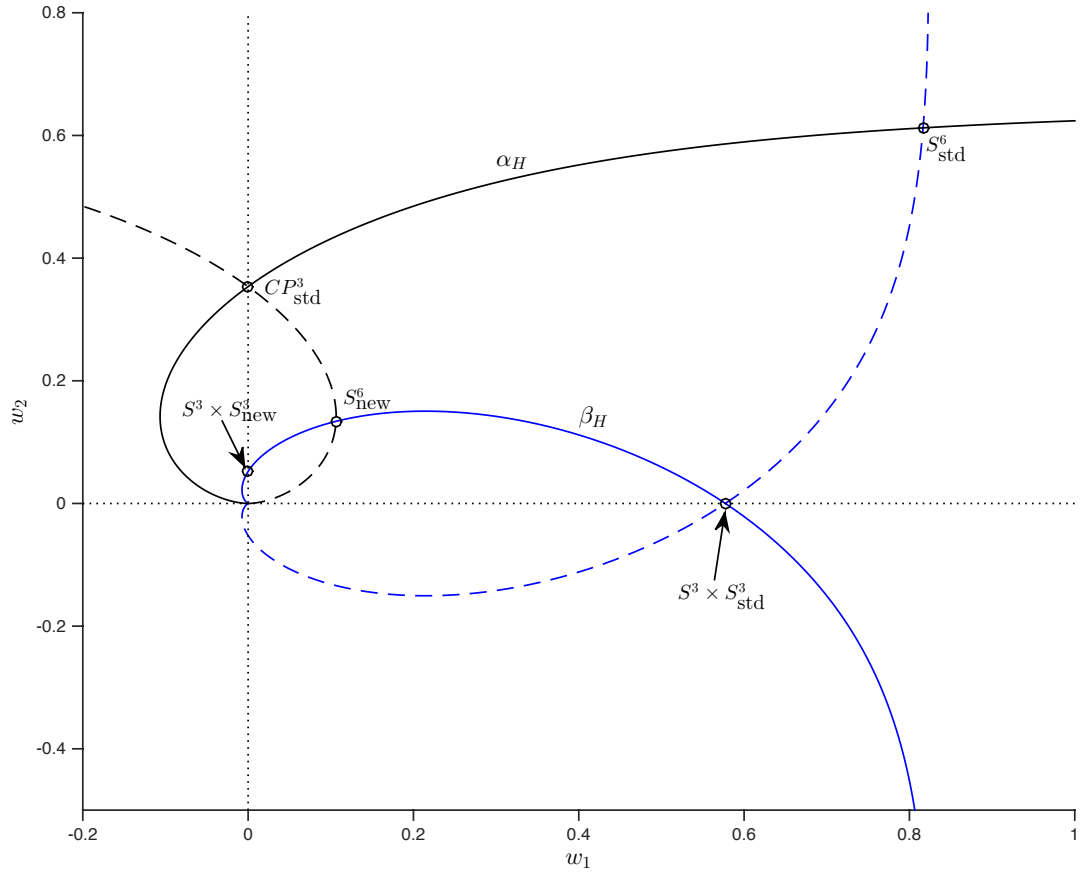


FIGURE 2. Plots of  $\alpha_H$  and  $\beta_H$  showing locations of the 5 complete cohomogeneity one nearly Kähler structures

Based on numerical evidence, we also conjecture analogous properties for the count of zeroes of  $u_0$  before the maximal volume orbit.

As a result, the conditions of the Doubling Lemma 5.19 are satisfied only three times: once in the family  $\{\Psi_a\}_{a>0}$ , corresponding to the homogeneous nearly Kähler structure on  $CP^3$ ; twice in the family  $\{\Psi_b\}_{b>0}$ , yielding the inhomogeneous nearly Kähler structure on  $S^3 \times S^3$  of the Main Theorem and the homogeneous nearly Kähler structure of Example 3.14.

Finally, Figure 2 plots the numerical approximations to the curves  $\alpha_H$  and  $\beta_H$  of Definition 8.3. This provides further numerical evidence that the Doubling Lemma 5.19 yields exactly three cohomogeneity one nearly Kähler structures; these correspond to the three points of intersection of the curves  $\alpha_H$  and  $\beta_H$  with the axes (the origin parametrises the circle of Sasaki–Einstein structures on  $N_{1,1}$  and must therefore be excluded). The plot also suggests that  $\alpha_H$  intersects the image of  $\beta_H$  under reflections in the two axes exactly twice; these two intersection points yield the inhomogeneous nearly Kähler structure on  $S^6$  given by the Main Theorem and the homogeneous one of Example 3.13.

**The volumes of the cohomogeneity one nearly Kähler structures.** According to the Conjecture above, there exist exactly two new complete cohomogeneity one nearly Kähler structures. The numerical analysis of the previous subsection provides more concrete information about these two new solutions than the abstract existence proofs of Theorems 7.12 and 8.1, and gives a way to compare them quantitatively with the known homogeneous nearly Kähler structures and with the sine-cone.

In Table 2 we consider various quantities describing the geometry of a complete cohomogeneity one nearly Kähler 6-manifold: the size of the two singular orbits  $O_1$  and  $O_2$  in terms of the parameters  $a$  or  $b$ ; the maximum  $V_{\max}$  of the orbital volume function; and the total volume  $\text{vol}$  normalised so that  $\text{vol}(S_{\text{std}}^6) = 1$ .

The values of these quantities on the two new inhomogeneous solutions are numerical approximations. For a more accurate result, we cut  $S^6$  and  $S^3 \times S^3$  along the maximal volume orbit of their inhomogeneous nearly Kähler structures and compute numerically the total volume on each half separately. Information about the sine-cone and the homogeneous nearly Kähler structures on  $S^6$ ,  $S^3 \times S^3$  and  $CP^3$  is computed analytically from the explicit solutions of Examples 3.12–3.15. The values in the tables are all obtained directly from those expressions. Since the orbital volume function is  $V = \lambda\mu^2$ , the total volume is easily deduced by integration. For example, for the homogeneous nearly Kähler structure on  $S^6$  we have  $V(t) = \frac{27}{2} \sin^2 t \cos^3 t$  and hence  $\text{Vol}(S_{\text{std}}^6) = \frac{9}{5}V_0$ , where  $V_0$  is the volume of  $N_{1,1}$  with respect to the standard Sasaki–Einstein metric. Since the volume of the 6-sphere with respect to the round metric of curvature 1 is  $\frac{16}{15}\pi^3$ , we must have  $V_0 = \frac{16}{27}\pi^3$ .

$M$	$O_1$	$O_2$	$V_{\max}$	$\text{vol}$
sine-cone	0	0	1	$\frac{16}{27} \approx 0.5926$
$S^3 \times S_{\text{new}}^3$	$b = 0.3736$	$b = 0.3736$	1.0041	0.5929
$S_{\text{new}}^6$	$a = 0.5646$	$b = 0.5985$	1.0385	0.5752
$CP^3$	$a = \frac{\sqrt{3}}{2}$	$a = \frac{\sqrt{3}}{2}$	$\frac{27\sqrt{2}}{32} \approx 1.1932$	$\frac{5}{8}$
$S^3 \times S_{\text{std}}^3$	$b = 1$	$b = 1$	$\frac{4}{3}$	$\frac{10\pi}{27\sqrt{3}} \approx 0.6718$
$S_{\text{std}}^6$	$a = \sqrt{3}$	$b = \frac{3}{2}$	$\frac{81\sqrt{3}}{25\sqrt{5}} \approx 2.5097$	1

TABLE 2. Cohomogeneity one nearly Kähler manifolds

The quantities in Table 2 all give a measure of how both inhomogeneous nearly Kähler structures, in particular the one on  $S^3 \times S^3$ , are much closer to the sine-cone than the homogeneous ones. Observe that the total volume  $\text{vol}$  is greater than the volume of the sine-cone in all cases except for the inhomogeneous nearly Kähler structure on  $S^6$ .

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