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Citation for published version:
Gruber, D, Martin, A & Steenbock, M 2015, 'Finite index subgroups without unique product in graphical small cancellation groups', *Bulletin of the London Mathematical Society*, vol. 47, no. 4, pp. 631–638. https://doi.org/10.1112/blms/bdv040

Digital Object Identifier (DOI):
10.1112/blms/bdv040

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Bulletin of the London Mathematical Society

FINITE INDEX SUBGROUPS WITHOUT UNIQUE PRODUCT IN GRAPHICAL SMALL CANCELLATION GROUPS

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ABSTRACT. For every integer $k \geqslant 1$, we construct a torsion-free hyperbolic group without unique product all of whose subgroups up to index k are themselves non-unique product groups. This is achieved by generalising a construction of Comerford to graphical small cancellation presentations, showing that for every subgroup H of a graphical small cancellation group there exists a free group F such that $H * F$ admits a graphical small cancellation presentation.

The unique product property was introduced as a way to prove Kaplansky's zero-divisor conjecture [Kap57] on the group ring of a torsion-free group [Coh74]. A group G is said to have the *unique product property* if every pair of non-empty finite subsets A and B of G admits a *unique product*, that is, if there exists $c \in G$ for which there exist unique elements $a \in A$ and $b \in B$ satisfying $c = ab$. Delzant showed [Del97] that a group acting with large enough injectivity radius by isometries on a hyperbolic space has the unique product property. Every *residually finite* group G has finite index subgroups acting on G with arbitrarily large injectivity radii. Therefore, as noted in [Hai03, Chi06], every residually finite hyperbolic group admits a finite index subgroup with the unique product property (and even a *diffuse* finite index subgroup [Bow00], see also [KR14]).

It is still unknown whether every hyperbolic group is residually finite. In light of the above, the existence of an infinite hyperbolic group *all* of whose finite index subgroups are non-unique product would provide an example of a non-residually finite hyperbolic group. In [AS14] Arzhantseva–Steenbock use a version of the Rips construction [Rip82] to produce, for every integer $k \geqslant 1$, an explicit torsion-free hyperbolic group that has a non-unique product subgroup of index k. They ask whether, given k, there exist torsion-free hyperbolic groups all of whose subgroups *up to* index k are non-unique product groups. In this note, we answer this question in the positive.

The first torsion-free groups without the unique product property are due to Rips–Segev [RS87]. Such groups can be realised as finitely presented graphical small cancellation groups over a free product of torsion-free hyperbolic groups [Ste13], thus providing the first examples of torsion-free *hyperbolic* non-unique product groups. Very little is known about their residual properties or the properties of their subgroups. We construct Rips–Segev groups which have many finite index subgroups without the unique product property. More precisely, we prove the following:

2010 *Mathematics Subject Classification.* 20F06, 20F67.

Key words and phrases. Graphical small cancellation theory, hyperbolic groups, unique product property.

Theorem. *Let $k \geqslant 1$ be an integer. There exists a torsion-free hyperbolic group G without the unique product property such that for all $1 \leqslant h \leqslant k$:*

(1) there exists a subgroup of index h;

(2) every subgroup of index h is a non-unique product group.

Our proof together with [Ste13] can be used to construct an explicit presentation of G. In the course of our proof, we provide a generalisation of a construction of Comerford [Com78] to graphical small cancellation presentations, which is of independent interest. Given a small cancellation presentation of a group G and an index h subgroup H, it provides an explicit small cancellation presentation for $H * F_{h-1}$, where F_{h-1} is the free group of rank $h-1$.

Acknowledgments. We thank Goulnara Arzhantseva for encouraging us to write this note. We also thank the referee for helpful suggestions. D. Gruber and A. Martin are supported by the ERC grant ANALYTIC no. 259527 of G. Arzhantseva. M. Steenbock is recipient of the DOC fellowship of the Austrian Academy of Sciences and is partially supported by the ERC grant ANALYTIC no. 259527 of G. Arzhantseva.

1. COMERFORD CONSTRUCTION FOR GRAPHICAL SMALL CANCELLATION

We extend the aforementioned construction of Comerford [Com78] for classical small cancellation presentations [LS77] to graphical small cancellation presentations as considered in [Oll06, Gru15, Ste13].

Let Γ be a graph. A *labelling* of Γ by a set S is a map ω that assigns to each edge an orientation and an element of S, called its *label*.

A *line graph* is a finite simplicial graph homeomorphic to the unit interval. A line graph is *pointed* if it comes with a choice of degree 1 vertex, called its *initial* vertex; the other degree 1 vertex is called the *terminal* vertex. Given a pointed line graph p with a labelling ω by a set S, the *label of* p, denoted $\omega(p)$, is the product in the free monoid on $S \sqcup S^{-1}$ of the labels of the successive edges of p, starting at the initial vertex. Here a letter is given exponent $+1$ if the direction in which the corresponding edge is traversed corresponds to the orientation defined by ω, and exponent -1 if it is traversed in the opposite direction.

A *labelled path on* Γ is a label-preserving graph homomorphism $\iota : p \to \Gamma$, where p is a labelled pointed line graph. The *label of* $\iota : p \to \Gamma$ is the label of p. A labelled path $\iota : p \to \Gamma$ is a *simple closed* path if ι is injective on the complement of the initial vertex of p and if the initial and terminal vertices of p are mapped to the same vertex.

A graph labelled by a set S defines a group $G(\Gamma)$ given by the following presentation:

$$\langle S \mid \Gamma\rangle := \langle S \mid \text{labels of simple closed labelled paths on } \Gamma\rangle.$$

A labelling is *reduced* if for every label-preserving immersion $\iota : p \to \Gamma$, where p is a a pointed line graph labelled by S, the label of p is freely reduced in the free monoid on $S \sqcup S^{-1}$. A *piece* with respect to a labelled graph Γ is a label-preserving *immersion* $\iota : p \to \Gamma$, where p is a labelled pointed line graph, such that there exists another label-preserving immersion $\iota' : p \to \Gamma$ that is essentially distinct. Here *essentially distinct* means that there does not exist a label-preserving automorphism $\phi : \Gamma \to \Gamma$ such that $\iota = \phi \circ \iota'$.

Two labelled paths $\iota : p \to \Gamma$ and $\iota' : p' \to \Gamma$, are *concatenable* if the terminal vertex of p and the initial vertex of p' are mapped to the same vertex. In such a case, we define the line graph pp' from the disjoint union of p and p' by identifying the terminal vertex of p and the initial vertex of p', and we choose the initial vertex of p as the initial vertex of pp'. The maps ι and ι' yield a label-preserving map $\iota\iota' : pp' \to \Gamma$, called the *concatenation* of the two labelled paths. Note that the concatenation of paths defines an associative law, which allows us to talk about the concatenation of an arbitrary finite number of paths. A labelled path $\iota : p \to \Gamma$ is a *subpath* of $\iota' : p' \to \Gamma$ if there exists a label-preserving immersion $i : p \to p'$ such that $\iota = \iota' \circ i$.

Definition. A labelled graph Γ satisfies the $Gr(n)$ *small cancellation condition* for $n \in \mathbb{N}$ if

- the labelling is reduced and
- no nontrivial simple closed labelled path on Γ is the concatenation of fewer than n pieces.

Let S be a set, and denote by $F(S)$ the free group on S. Let us denote by ℓ either the *word length* on $F(S)$, which counts the number of generators in a reduced word, or the *free product length* (or syllable length [LS77, Ch.V.9]) associated to a partition Π of S, which counts the number of factors in the normal form of an element with respect to the decomposition $F(S) = *_{P\in\Pi} F(P)$.

Definition. A labelled graph Γ satisfies the $Gr'_\ell(\lambda)$ *small cancellation condition* for $\lambda > 0$ if

- the labelling is reduced and
- every piece $p \to \Gamma$ that is subpath of a nontrivial simple closed labelled path $\gamma \to \Gamma$ satisfies $\ell(\omega(p)) < \lambda\ell(\omega(\gamma))$.

Denote by $Gr'(\lambda)$ the $Gr'_\ell(\lambda)$-condition where ℓ is the word length. Given a partition of S, denote by $Gr'_*(\lambda)$ the $Gr'_\ell(\lambda)$-condition where ℓ is the associated free product length.

If Γ satisfies the $Gr'_\ell(\lambda)$-condition for $\lambda \leqslant \frac{1}{n-1}$, then Γ satisfies the $Gr(n)$-condition.

Groups defined by graphs with metric small cancellation with respect to the word length of the free group were first studied in [Oll06]. Non-metric graphical small cancellation over free groups was first studied in [Gru15]. Metric graphical small cancellation over arbitrary free products was first studied in [Ste13].

Proposition 1. *Let $n \in \mathbb{N}$ and $\lambda > 0$. Let Γ be a graph labelled by a set S, and let H be a subgroup of index h (finite or infinite) in $G(\Gamma)$. Then there exists a graph Γ_H labelled by $S \times (G(\Gamma)/H)$ such that $G(\Gamma_H) = H * F_{h-1}$, where F_{h-1} is the free group of rank $h-1$, and such that:*

- *If Γ satisfies the $Gr(n)$-condition, then so does Γ_H.*
- *If Γ satisfies the $Gr'(\lambda)$-condition, then so does Γ_H.*

- *If* Γ *satisfies the* $Gr'_*(\lambda)$*-condition with respect to* $F(S) = *_{P\in\Pi} F(P)$*, where* Π *is a partition of* S*, then* Γ_H *satisfies the* $Gr'_*(\lambda)$*-condition with respect to* $F(S\times G(\Gamma)/H) = *_{P\in\Pi} F\big(P\times G(\Gamma)/H\big)$*.*

Proof. We extend the proof of Comerford [Com78]. Denote by K the labelled oriented graph that has a single vertex and for each $s\in S$ a single oriented edge labelled s. In each component Γ^i of Γ fix a basepoint. The labelling of Γ by S can be viewed as a basepoint-preserving graph homomorphism

$$\omega : \Gamma \to K.$$

We construct a space X with fundamental group $G := G(\Gamma)$ via the following mapping cone construction: For each component Γ^i of Γ, we attach the topological cone $C\Gamma^i$ over Γ^i onto K along the map ω. The fundamental group of X is the quotient of the fundamental group of K by the normal subgroup generated by the images of the fundamental groups of the Γ^i. Since the fundamental group of every Γ^i is normally generated by the simple closed paths in Γ^i, we have $\pi_1(X) = G(\Gamma)$.

Let H be a subgroup of index h (finite or infinite) in $G(\Gamma)$, and denote

$$S_H := S\times G(\Gamma)/H.$$

For simplicity, we write an ordered pair (s,v) as s_v. We now construct a graph Γ_H labelled by S_H such that $G(\Gamma_H) = H * F_{h-1}$.

Let $\pi_H : X_H \to X$ be a connected cover with $\pi_1(X_H) = H$. Then $\pi_H^{-1}(K)$ is a Schreier coset graph of $H \leqslant G(\Gamma)$, and, in particular, every vertex of $\pi_H^{-1}(K)$ is an element of $G(\Gamma)/H$. The map $\pi_H^{-1}(K) \to K$ is a labelling of $\pi_H^{-1}(K)$. We construct a new labelling of $\pi_H^{-1}(K)$ over S_H as follows: We do not change orientations of edges. We replace the label of every edge starting at a vertex v and labelled by s by the label s_v. Denote the resulting labelled graph by K_H and its labelling function by ω_H.

Recall that we fixed basepoints in the components Γ^i of Γ and that the topological cone over each Γ^i is simply connected. Thus, for each vertex $v\in K_H$, there exists a graph homomorphism $\omega_v : \Gamma \to K_H$ taking all basepoints to v. We interpret this homomorphism as labelling ω_v on Γ. Denote the graph Γ with the labelling ω_v by Γ_v and denote

$$\Gamma_H := \bigsqcup_{v\in G(\Gamma)/H} \Gamma_v.$$

In X_H, identify all vertices in $\pi_H^{-1}(K)$, and denote the resulting space by X_H^*. We compute the fundamental group of X_H^* in two ways to show:

$$G(\Gamma_H) = \pi_1(X_H^*) = G(\Gamma) * F_{h-1}.$$

Consider X_H^* with the labelling of edges induced from K_H. The image of K_H in X_H^* has a single vertex and for each $s_v \in S_H$ a single oriented edge labelled s_v. X_H^* is obtained by attaching the topological cone over each component of Γ_H along the labelling map. Thus, by the same reasoning as above, we have $\pi_1(X_H^*) = G(\Gamma_H)$.

Now consider the disjoint union of X_H and a space consisting of a single vertex b, and add edges connecting b to every vertex of $\pi_H^{-1}(K)$. The fundamental group of the resulting space is $H * F_{h-1}$, and the space is homotopy equivalent to X_H^*. Therefore, $\pi_1(X_H^*) = H * F_{h-1}$.

The (not label-preserving) maps of labelled graphs $\pi_v : \Gamma_v \to \Gamma$ induced by the identity on the underlying graphs are isometries with respect to the word length or the free product length respectively. The labelling of each Γ_v is reduced if the labelling of Γ is. We show that every piece in Γ_H maps to a piece in Γ via a map π_v. Since nontrivial simple closed paths map to nontrivial simple closed paths, this is sufficient to show that Γ_H satisfies the claimed small cancellation conditions if Γ does.

We start by an observation: Let e be an edge in a component Γ^i of Γ, and let $s = \omega(e)$. Let $v \in G/H$. By the unique lifting property of covering spaces, there exists a unique lift of the map $C\Gamma^i \to X$ (induced by $\omega : \Gamma \to K$) to X_H which sends e to the edge labelled s_v, and thus a unique lift of $\Gamma^i \to K$ to K_H sending e to the edge labelled s_v.

Now let $\iota_1 : p \to \Gamma^i \subset \Gamma$ and $\iota_2 : p \to \Gamma^j \subset \Gamma$ be two immersions of a non-trivial labelled line graph p into Γ and $v, w \in G/H$ such that the labellings $\omega_v \circ \iota_1, \omega_w \circ \iota_2 : p \to K_H$ by S_H coincide. Assume that there exists an ω-preserving automorphism ϕ of Γ such that $\phi \circ \iota_1 = \iota_2$. Let e be an edge of p. By construction, the maps $\omega_w \circ \phi$ and ω_v are two lifts of $\omega : \Gamma^i \to K$ to K_H that coincide on $\iota_1(e)$, hence they are equal by the above observation. Thus, ϕ induces an isomorphism from Γ^i to Γ^j that is compatible with labellings ω_v of Γ^j and ω_w of Γ^j. This can be extended to a label-preserving automorphism $\phi_H : \Gamma_H \to \Gamma_H$ by sending Γ^j with labelling ω_w to Γ^i with labelling ω_v by means of ϕ^{-1}, and by being the identity on every other labelled component.

Therefore, if p is a non-trivial line graph with labelling $\omega_p : p \to K_H$ by S_H and $\iota_1 : p \to \Gamma_v \subset \Gamma_H$ and $\iota_2 : p \to \Gamma_w \subset \Gamma_H$ are essentially distinct label-preserving immersions into Γ_H (where Γ_H is considered with the labelling ω_H), then $\pi_v \circ \iota_1 : p \to \Gamma$ and $\pi_w \circ \iota_2 : p \to \Gamma$ are essentially distinct label-preserving immersions into Γ (where Γ is considered with the labelling ω and p with the labelling $\pi_H \circ \omega_p$ by S). □

2. Groups without unique product

The first construction of torsion-free groups without the unique product property is due to [RS87]. We present here a generalisation of this construction, following [Ste13], which allows more flexibility in the choice of generators and relators in the presentations under consideration. This will be used to prove our main theorem.

Let $F(S)$ and $F(T)$ be free groups over non-empty disjoint sets S and T. We start by constructing a graph Γ labelled by $S \sqcup T$ which will be used to define non-unique product groups. This is done in three steps.

Choose non-trivial cyclically reduced elements $a \in F(S)$ and $b \in F(T)$. Let $N \geqslant 1$ be an integer and choose integers $C_1, \ldots, C_N \geqslant 1$. For each $1 \leqslant i \leqslant N$, let p_i be the pointed line graph labelled by S whose label in the free monoid on $S \sqcup S^{-1}$ is a^{C_i}. Denote by $u_{i,j}$ the terminal vertex of the initial subpath labelled a^j. Let p_b be the pointed line graph labelled by T whose label in the free monoid on $T \sqcup T^{-1}$ is b. Denote the initial vertex of p_b by v_0 and the terminal vertex by v_1.

For every $1 \leqslant i \leqslant N$, we now construct a new graph p_i' out of p_i as follows. Consider $C_i + 1$-many copies of p_b, denoted $(p_b)_{i,0}, \ldots, (p_b)_{i,C_i}$. We construct the graph p_i' from the disjoint union of p_i and the various $(p_b)_{i,j}, 0 \leqslant j \leqslant C_i$, by identifying the vertex $u_{i,j}$ of p_i with the vertex $(v_0)_{i,j}$ of $(p_b)_{i,j}$ for every $0 \leqslant j \leqslant C_i$. Each p_i' naturally comes with a labelling by $S \sqcup T$.

We now define the graph Γ from the disjoint union of the labelled graphs $p_i', 1 \leqslant i \leqslant N$ as follows. For each $1 \leqslant i \leqslant N$, choose four integers $1 \leqslant N_{i,1}, N_{i,2}, N_{i,3}, N_{i,4} \leqslant N$ and for each $1 \leqslant j \leqslant 4$, an integer $0 \leqslant P_{i,j} \leqslant C_{N_{i,j}}$. We identify the vertex $u_{i,0}$ (respectively $(v_1)_{i,0}$, u_{i,C_i}, $(v_1)_{i,C_i}$) with the vertex $(v_1)_{N_{i,1},P_{i,1}}$ (respectively $u_{N_{i,2},P_{i,2}}$, $(v_1)_{N_{i,3},P_{i,3}}$, $u_{N_{i,4},P_{i,4}}$). As before, Γ naturally inherits a labelling by $S \sqcup T$.

Note that Γ depends on the various choices of $a, b, N, (C_i), (N_{i,j})$ and $(P_{i,j})$. We will denote it $\Gamma\big(a, b, N, (C_i), (N_{i,j}), (P_{i,j})\big)$ when emphasising this dependence.

Definition. The graph $\Gamma = \Gamma\big(a, b, N, (C_i), (N_{i,j}), (P_{i,j})\big)$ is called the *Rips–Segev graph (over $F(S) * F(T)$)* associated to the *coefficient system* $\big(a, b, N, (C_i), (N_{i,j}), (P_{i,j})\big)$.

Combinatorial considerations of graphs with large girth yield the following existence result:

Proposition 2 ([Ste13]). *For all non-trivial cyclically reduced $a \in F(S)$ and $b \in F(T)$, there exists an explicit choice of coefficients such that the associated Rips–Segev graph is connected and satisfies the $Gr_*'(\frac{1}{6})$-condition with respect to the free product length on $F(S) * F(T)$.* □

Consider a connected Rips–Segev graph $\Gamma = \Gamma\big(a, b, N, (C_i), (N_{i,j}), (P_{i,j})\big)$. We now construct non-empty finite subsets of elements of $F(S) * F(T)$. For $1 \leqslant i \leqslant N$, choose a path γ_i in Γ from $u_{1,0}$ to $u_{i,0}$ and let w_i be the label of γ_i in $F(S) * F(T)$. For each $1 \leqslant i \leqslant N$, we define the following subsets of $F(S) * F(T)$:

$$A_i := \{w_i, w_i a, w_i a^2, \ldots, w_i a^{C_i - 1}\}.$$

Finally, let

$$A := \bigcup_{1 \leqslant i \leqslant N} A_i \quad \text{and} \quad B := \{1, a, b, ab\}.$$

In presence of graphical small cancellation conditions, the image of A and B in $G(\Gamma)$ define non-empty finite subsets without a unique product. More precisely, we have the following fundamental results about Rips–Segev graphs:

Proposition 3 ([Ste13]). *Let Γ be a finite labelled graph over $F(S) * F(T)$ which is a non-empty disjoint union of connected Rips–Segev graphs over $F(S) * F(T)$. If Γ satisfies the $Gr_*'(\frac{1}{6})$-condition with respect to the free product length on $F(S) * F(T)$, then $G(\Gamma)$ is torsion-free hyperbolic and does not have the unique product property.* □

The proof uses the following arguments: Results on $Gr_*'(\frac{1}{6})$-presentations over free products [Ste13], or, alternatively, $Gr(7)$-presentations over free groups [Gru15] yield that $G(\Gamma)$ is torsion-free hyperbolic and that every component of Γ injects into the Cayley graph of $G(\Gamma)$. Consider a component Γ^i of Γ. Since Γ^i injects into the

Cayley graph, the sets A and B associated to Γ^i inject into $G(\Gamma)$ under the projection $F(S) * F(T) \to G(\Gamma)$. The labelled paths on Γ^i give rise to more than one way of writing each element in AB as product of elements of A and B, therefore ensuring the non-unique product property. A direct proof that A and B embed can be found in [AS14], again using graphical small cancellation over free products.

We now move to the proof of our main theorem. Fix an integer $k \geqslant 1$. In the above notation let $S := \{s\}$ and $T := \{t\}$. Set

$$a := s^{k!}, b := t^{k!}.$$

By Proposition 2, we can find coefficients $\big(N, (C_i), (N_{i,j}), (P_{i,j})\big)$ such that the associated Rips–Segev graph $\Gamma := \Gamma\big(a, b, N, (C_i), (N_{i,j}), (P_{i,j})\big)$ is connected and satisfies the $Gr'_*(\frac{1}{6})$-condition with respect to the free product length on $F(\{s\}) * F(\{t\})$. We now show that $G := G(\Gamma)$ is a group for k as claimed in our main theorem.

Lemma. *Let Q be a 2-generated group of cardinality $h \leqslant k$. Then G admits a surjective homomorphism to Q.*

Proof. Let $\{s', t'\}$ be a generating set for Q. Since Q has cardinality h, s' and t' both have order dividing $k!$. By construction, every defining relator of G (that is, every label of a simple closed path on Γ) is a product of powers of $s^{k!}$ and $t^{k!}$. Thus, the surjective map $F(\{s\}) * F(\{t\}) \to Q$ sending s to s' and t to t' maps the defining relators of G to the identity. This yields a surjective homomorphism $G \to Q$. □

Proof of the main theorem. Let $h \leqslant k$ and H a subgroup of G of index h. We use the same notations as in the proof of Proposition 1. Recall that $\Gamma_H = \bigsqcup_{v \in G/H} \Gamma_v$, where G/H is the set of vertices of K_H, and each Γ_v is isomorphic to Γ as an unlabelled oriented graph.

For each $v \in G/H$, the connected component of the preimage under $\pi_H : K_H \to K$ of the oriented edge labelled s (respectively t) containing v is the image of a simple closed path α_v (respectively β_v) labelled by $\{s\} \times G/H$ (respectively $\{t\} \times G/H$). Since the cover $\pi_H : K_H \to K$ is of degree $h \leqslant k$, the simple closed paths α_v and β_v have at most k edges (see Figure 1 for an example). Define

$$a_v := \omega_H(\alpha_v)^{k!/|\alpha_v|} \text{ and } b_v := \omega_H(\beta_v)^{k!/|\beta_v|}.$$

Here $|\,.\,|$ denotes the edge-length of paths of K_H.

Thus, the map of labelled graphs $\Gamma \to \Gamma_v$ induced by the identity on the underlying graph sends every path on Γ with label a that starts at some $u_{i,j}$ to a path on Γ_v with label a_v, and every path with label b starting at some $(v_0)_{i,j}$ to a path on Γ_v with label b_v. Therefore, the graph Γ_v is the Rips–Segev graph over $F(\{s\} \times G/H) * F(\{t\} \times G/H)$ with coefficient system $\big(a_v, b_v, N, (C_i), (N_{i,j}), (P_{i,j})\big)$.

By Proposition 1, the labelling of $\Gamma_H = \bigsqcup_v \Gamma_v$ satisfies the $Gr'_*(\frac{1}{6})$-condition with respect to the free product length on $F(\{s\} \times G/H) * F(\{t\} \times G/H)$. Thus, $G(\Gamma_H) = H * F_{h-1}$ does not satisfy the unique product property by Proposition 3. The unique product property is stable under free products, and it is satisfied by free groups. Therefore, H does not have the unique product property. □

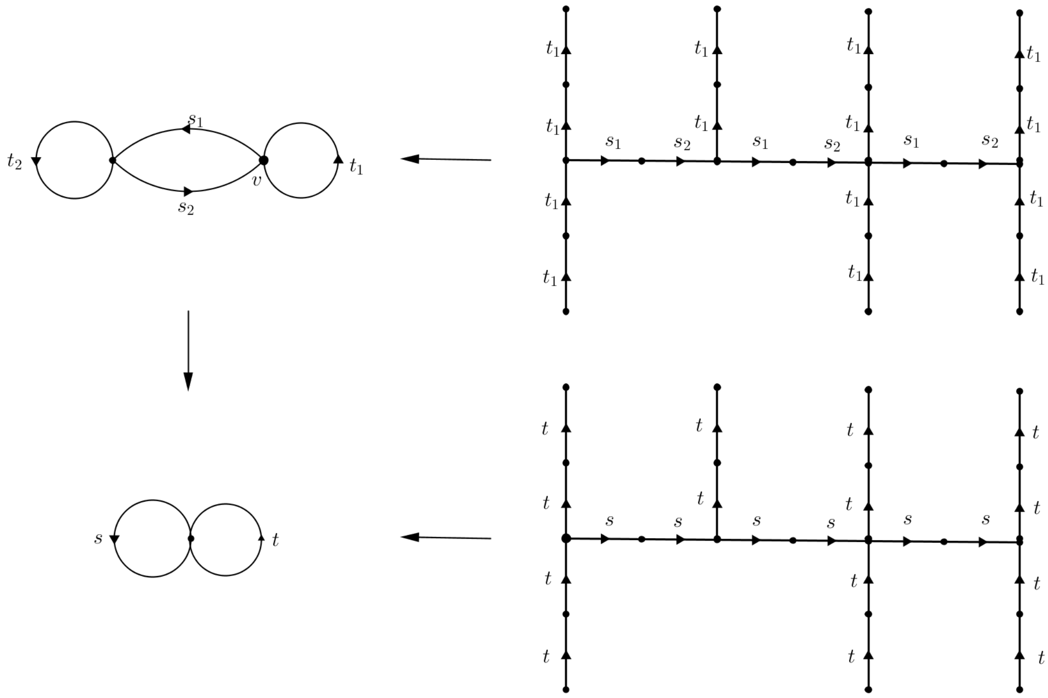

FIGURE 1. The situation for an index 2 subgroup in the case $a = s^2$, $b = t^2$. Upper left: K_H, upper right: a part of $\Gamma_v \subseteq \Gamma_H$, lower left: K, lower right: a part of Γ.

REFERENCES

[AS14] G. Arzhantseva and M. Steenbock, *Rips construction without unique product* (2014), available at `arXiv:1407.2441`.

[Bow00] B. Bowditch, *A variation on the unique product property*, J. London Math. Soc. (2) **62** (2000), no. 3, 813–826.

[Chi06] I. M. Chiswell, *Locally invariant orders on groups*, Internat. J. Algebra Comput. **16** (2006), no. 6, 1161–1179.

[Coh74] J. M. Cohen, *Zero divisors in group rings*, Comm. Algebra **2** (1974), 1–14.

[Com78] L. Comerford Jr., *Subgroups of small cancellation groups*, J. London Math. Soc. (2) **17** (1978), no. 3, 422–424.

[Del97] T. Delzant, *Sur l'anneau d'un groupe hyperbolique*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 4, 381–384.

[Gru15] D. Gruber, *Groups with graphical $C(6)$ and $C(7)$ small cancellation presentations*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 2051–2078.

[Hai03] S. Hair, *New methods for finding non-left-orderable and unique product groups*, M.Sc. thesis, Virginia Polytechnic Institute, 2003.

[Kap57] I. Kaplansky, *Problems in the theory of rings. Report of a conference on linear algebras, June, 1956, pp. 1-3*, National Academy of Sciences-National Research Council, Washington, Publ. 502, 1957.

[KR14] S. Kionke and J. Raimbault, *On geometric aspects of diffuse groups* (2014), available at `arXiv:1411.6449`. With an appendix by N. Dunfield.

[LS77] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[Oll06] Y. Ollivier, *On a small cancellation theorem of Gromov*, Bull. Belg. Math. Soc. Simon Stevin **13** (2006), no. 1, 75–89.

[Rip82] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14** (1982), no. 1, 45–47.

[RS87] E. Rips and Y. Segev, *Torsion-free group without unique product property*, J. Algebra **108** (1987), no. 1, 116–126.

[Ste13] M. Steenbock, *Rips-Segev torsion-free groups without unique product* (2013), available at arXiv:1307.0981.

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