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ORDERED GROUPOID QUOTIENTS AND CONGRUENCES ON INVERSE SEMIGROUPS

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ABSTRACT. We introduce a preorder on an inverse semigroup S associated to any normal inverse subsemigroup N , that lies between the natural partial order and Green's \mathcal{J} -relation. The corresponding equivalence relation \simeq_N is not necessarily a congruence on S , but the quotient set does inherit a natural ordered groupoid structure. We show that this construction permits the factorisation of any inverse semigroup homomorphism into a composition of a quotient map and a star-injective functor, and that this decomposition implies a classification of congruences on S . We give an application to the congruence and certain normal inverse subsemigroups associate to an inverse monoid presentation.

1. INTRODUCTION

Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Recall that the *natural partial order* on S is defined by

$$s \leq t \iff \text{there exists } e \in E(S) \text{ with } s = et.$$

The natural partial order may be characterized in a number of alternative ways, including:

- there exists $f \in E(S)$ with $s = tf$,
- $s = ss^{-1}t$,
- $s = ts^{-1}s$.

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(see [5, Proposition 5.2.1]). In this paper, we shall generalize the natural partial order by introducing a preorder \leq_N on S for any normal inverse subsemigroup N : the natural partial order then corresponds to the minimal normal inverse subsemigroup $E(S)$, and at the other extreme, the preorder associated to S itself is the \mathcal{J} -preorder. Symmetrizing the preorder \leq_N yields an equivalence relation \simeq_N (the identity when $N = E(S)$ and the \mathcal{J} -relation when $N = S$). However, this relation need not be a congruence, and so the set of equivalence classes S/\simeq_N need not be an inverse semigroup.

However, we may investigate \simeq_N further by exploiting the relationship between inverse semigroups and ordered groupoids. An ordered groupoid is a small category in which every morphism is invertible, equipped with a partial order on morphisms. (The definition is recalled in detail in section 2.) An inverse semigroup can be considered as an ordered groupoid in which the identities form a semilattice, and from any such ordered groupoid a corresponding inverse semigroup can be constructed. In the study of inverse semigroups, it is often fruitful to extend the point of view to ordered groupoids, and this is a major theme of [7]. By [1, Theorem 4.14] the quotient set S/\simeq_N always inherits a natural ordered groupoid structure. Moreover, to any homomorphism $\phi : S \rightarrow \Sigma$ of inverse semigroups, we associate its kernel $K = \{s \in S : s\phi \in E(\Sigma)\}$, and ϕ then factorises as

$$(1.1) \quad S \longrightarrow S/\simeq_K \longrightarrow \Sigma$$

with the map $S/\simeq_K \rightarrow \Sigma$ a star-injective functor from the ordered groupoid S/\simeq_K to Σ (considered as an ordered groupoid).

The factorisation given in (1.1) is dual to that given in Steinberg's *Fibration Theorem* for ordered groupoids [15, Theorem 5.1]. The Fibration Theorem shows that any mapping of ordered groupoids may also be factorized as the composition of an enlargement (an inclusion with additional favourable properties) followed by a star-surjective functor, analogous to the factorization of a continuous map of topological spaces as the composition of a homotopy equivalence followed by a fibration (see for example [16, Theorem I.7.30]).

Now any congruence ρ on an inverse semigroup determines a normal inverse subsemigroup K , its *kernel*, which consists of all elements of S that are ρ -equivalent to idempotents. We compare the structures of the ordered groupoid S/\simeq_K and the quotient inverse semigroup S/ρ , and we show that if \simeq_K is a congruence, then it is the minimal congruence with kernel K . We show how to classify congruences by the factorisation

$$S \longrightarrow S/\simeq_K \longrightarrow S/\rho$$

and look at certain congruences and their kernels associated with an inverse monoid presentation, and the relationships between them.

2. ORDERED GROUPOIDS AND INVERSE SEMIGROUPS

A *groupoid* G is a small category in which every morphism is invertible. We consider a groupoid as an algebraic structure following [4]: the elements are the morphisms, and composition is an associative partial binary operation. The set of identities in G is denoted $E(G)$, and an element $g \in G$ has domain gg^{-1} and range $g^{-1}g$.

An *ordered groupoid* (G, \leq) is a groupoid G with a partial order \leq satisfying the following axioms:

$$\text{OG1 for all } g, h \in G, \text{ if } g \leq h \text{ then } g^{-1} \leq h^{-1},$$

OG2 if $g_1 \leq g_2, h_1 \leq h_2$ and if the compositions g_1h_1 and g_2h_2 are defined, then $g_1h_1 \leq g_2h_2$,

OG3 if $g \in G$ and x is an identity of G with $x \leq g\mathbf{d}$, there exists a unique element $(x|g)$, called the *restriction* of g to x , such that $(x|g)(x|g)^{-1} = x$ and $(x|g) \leq g$,

As a consequence of [OG3] we also have:

OG3* if $g \in G$ and y is an identity of G with $y \leq g\mathbf{r}$, there exists a unique element $(g|y)$, called the *corestriction* of g to y , such that $(g|y)^{-1}(g|y) = y$ and $(g|y) \leq g$,

since the corestriction of g to y may be defined as $(y|g^{-1})^{-1}$.

Let G be an ordered groupoid and let $a, b \in G$. If $a^{-1}a$ and bb^{-1} have a greatest lower bound $\ell \in E(G)$, then we may define the *pseudoproduct* of a and b in G as $a \otimes b = (a|\ell)(\ell|b)$, where the right-hand side is now a composition defined in G . As Lawson shows in Lemma 4.1.6 of [7], this is a partially defined associative operation on G .

If $E(G)$ is a meet semilattice then G is called an *inductive* groupoid. The pseudoproduct is then everywhere defined and (G, \otimes) is an inverse semigroup. On the other hand, given an inverse semigroup S with semilattice of idempotents $E(S)$, then S is a poset under the natural partial order, and the restriction of its multiplication to the partial composition

$$s \cdot t = st \in S \text{ defined when } s^{-1}s = tt^{-1}$$

gives S the structure of an ordered groupoid, with set of identities $E(S)$. These constructions give an isomorphism between the categories of inverse semigroups and inductive groupoids: this is the *Ehresmann-Schein-Nambooripad Theorem* [7, Theorem 4.1.8]. We call a product $st \in S$ with $s^{-1}s = tt^{-1}$ a *trace product*. Any product in S can be expressed as a trace product, at the expense of changing the factors, since $st = stt^{-1} \cdot s^{-1}st$.

Let $e \in E(G)$. Then the *star* of e in G is the set $\text{star}_G(e) = \{g \in G : gg^{-1} = e\}$. A functor $\phi : G \rightarrow H$ is said to be *star-injective* if, for each $e \in E(G)$, the restriction $\phi : \text{star}_G(e) \rightarrow \text{star}_H(e\phi)$ is injective. A star-injective functor is also called an *immersion*. If G is inductive, then $\text{star}_G(e)$ is just the Green \mathcal{R} -class of e in the inverse semigroup (G, \otimes) .

3. NORMAL INVERSE SUBSEMIGROUPS AND QUOTIENTS

An inverse subsemigroup N of an inverse semigroup S is *normal* [12] if it is full – that is, if $E(N) = E(S)$ – and if, for all $s \in S$ and $n \in N$, we have $s^{-1}ns \in N$. A normal inverse subsemigroup N of S determines a relation \leq_N on S , defined using the natural partial order \leq on S , as follows:

$$(3.1) \quad s \leq_N t \iff \text{there exist } a, b \in N \text{ such that } a \cdot s \cdot b \leq t.$$

Note the requirement that trace products occur here. We define the relation \simeq_N by symmetrizing \leq_N :

$$(3.2) \quad s \simeq_N t \iff \text{there exist } a, b, c, d \in N \text{ such that } a \cdot s \cdot b \leq t \text{ and } c \cdot t \cdot d \leq s$$

Lemma 3.1.

- (a) *The relation $\leq_{E(S)}$ is the natural partial order \leq on S .*
- (b) *The relation \leq_S is the \mathcal{J} -preorder $\preceq_{\mathcal{J}}$ on S .*
- (c) *If $s \leq t$ in the natural partial order on S then $s \leq_N t$ for any normal inverse subsemigroup N of S .*

- (d) $s \leq_N e$ for some $e \in E(S)$ if and only if $s \in N$.
- (e) If $s \leq_N s^2$ then $s \in N$.
- (f) If $s \leq_N t$ then $st^{-1} \in N$.
- (g) The relation \leq_N is a preorder on S and hence \simeq_N is an equivalence relation on S .

Proof. (a) This is clear, since if $e, f \in E(S)$, a trace product $e \cdot s \cdot f$ is equal to s .
(b) If $a \cdot s \cdot b \leq t$ then $aa^{-1} \leq tt^{-1}$ and $a^{-1}a = ss^{-1}$. Hence $s \preceq_{\mathcal{J}} t$. On the other hand, if $s \preceq_{\mathcal{J}} t$ then there exists $p \in S$ with $pp^{-1} \leq tt^{-1}$ and $p^{-1}p = ss^{-1}$. Then

$$p \cdot s \cdot (s^{-1} \cdot p^{-1} \cdot pp^{-1}t) \leq t$$

and so $s \leq_S t$.

- (c) Since N is full, $ss^{-1}, s^{-1}s \in N$ and $s = ss^{-1} \cdot s \cdot s^{-1}s \leq t$.
- (d) Suppose that $a, b \in N$ with $a \cdot s \cdot b \leq e$. Therefore $a \cdot s \cdot b = f \leq e$ with $f \in E(S)$, and then $s = a^{-1}a \cdot s \cdot bb^{-1} = a^{-1}fb^{-1} \in N$. Conversely, if $n \in N$ then $nn^{-1} \cdot n \cdot n^{-1} = nn^{-1}$ and so $n \leq_N nn^{-1}$.
- (e) We have $a, b \in N$ with $a \cdot s \cdot b \leq s^2$ and so $s \leq a^{-1}s^2b^{-1}$. Therefore $s = ss^{-1}a^{-1}s^2b^{-1}$ and it follows that

$$s^{-1} = s^{-1}ss^{-1} = s^{-1}ss^{-1}a^{-1}s^2b^{-1}s^{-1} = (s^{-1}a^{-1}s)(sb^{-1}s^{-1}) \in N$$

and so $s \in N$.

- (f) If $a \cdot s \cdot b \leq t$ then $s = a^{-1}a \cdot s \cdot bb^{-1} \leq a^{-1}tb^{-1}$ and so $st^{-1} \leq a^{-1}(tb^{-1}t^{-1}) \in N$. Since N is full, we deduce that $st^{-1} \in N$.
- (g) It is clear that \leq is reflexive. Suppose that $s, t, u \in S$ and that $s \leq_N t \leq_N u$. There exist $a, b, p, q \in N$ such that $a \cdot s \cdot b \leq t$ and $p \cdot t \cdot q \leq u$. Then $(pa)s(bq) \leq u$, and $(pa)s(bq)$ is the trace product $(pa) \cdot s \cdot (bq)$ since

$$(pa)^{-1}(pa) = a^{-1}p^{-1}pa = a^{-1}tt^{-1}a = a^{-1}a = ss^{-1},$$

and $aa^{-1} \leq tt^{-1}$. Similarly $s^{-1}s = (bq)(bq)^{-1}$. Therefore $(pa) \cdot s \cdot (bq) \leq u$ and $s \leq_N u$. □

Corollary 3.2. *The normal inverse subsemigroup N is determined by the preorder \leq_N , and we obtain an order-preserving embedding of the poset of normal inverse subsemigroups of S into the poset of preorders on S that contain the natural partial order.*

Proof. Part (d) of Lemma 3.1 shows that

$$N = \{s \in S : \text{there exists } e \in E(S) \text{ with } s \leq_N e\}.$$

□

Remark 3.3. Not every preorder containing the natural partial order arises from a normal inverse subsemigroup. Consider the symmetric inverse monoid \mathcal{I}_n and define a preorder \preceq by $\alpha \preceq \beta \iff \mathbf{d}(\alpha) \subseteq \mathbf{d}(\beta)$ where $\mathbf{d}(\gamma)$ is the domain of $\gamma \in \mathcal{I}_n$. Then $\alpha \preceq \text{id}$ for all $\alpha \in \mathcal{I}_n$, and so the normal inverse subsemigroup associated to \preceq is I_n itself, but \preceq is not the \mathcal{J} -preorder on \mathcal{I}_n and so is not equal to $\leq_{\mathcal{I}_n}$.

We denote the \simeq -class of $s \in S$ by $[s]_N$.

Proposition 3.4.

- (a) If $n \in N$ then $nn^{-1} \simeq_N n \simeq_N n^{-1} \simeq_N n^{-1}n$,
- (b) The equivalence relation \simeq_N saturates N ,
- (c) The equivalence relation \simeq_N determines N as

$$N = \bigcup_{e \in E(S)} [e]_N.$$

- (d) If $s \simeq_N t$ then $ss^{-1} \simeq_N tt^{-1}$, $s^{-1}s \simeq_N t^{-1}t$, and $s^{-1} \simeq_N t^{-1}$.
- (e) The restriction of \simeq_N to $E(S)$ coincides with Green's \mathcal{J} -relation \mathcal{J}_N induced on $E(S) = E(N)$,
- (f) In the case $N = S$ the relation \simeq_S coincides with Green's \mathcal{J} -relation on S ,
- (g) In the case $N = E(S)$ the relation $\simeq_{E(S)}$ is the trivial relation on S .

Proof.

- (a) If $n \in N$ then $n^{-1} \cdot n \cdot n^{-1} = n^{-1}$ and $n \cdot n^{-1} \cdot n = n$, and hence $n \simeq_N n^{-1}$. Similarly $nn^{-1} \cdot n \cdot n^{-1} = nn^{-1}$ and $nn^{-1} \cdot nn^{-1} \cdot n = n$, whence $n \simeq_N nn^{-1}$.
- (b) Suppose that $s \in S$ and that for some $n \in N$ we have $s \simeq_N n$. By part (a) we may assume that $n \in E(S)$: then there exist $p, q \in N$ such that $p \cdot s \cdot q \leq n$. Hence for some $e \in E(S)$ we have $p \cdot s \cdot q = e$ and so $s = p^{-1} \cdot e \cdot q^{-1} = p^{-1}q^{-1} \in N$.
- (c) This follows from parts (a) and (b).
- (d) Suppose that $s \simeq_N t$, with $a, b, c, d \in N$ as in (3.2). Then $bb^{-1} = s^{-1}s$, $b^{-1}b \leq t^{-1}t$, $dd^{-1} = t^{-1}t$, $d^{-1}d \leq s^{-1}s$ and so $s^{-1}s \simeq_N t^{-1}t$. Similarly $ss^{-1} \simeq_N tt^{-1}$. Since $b^{-1} \cdot s^{-1} \cdot a^{-1} \leq t^{-1}$ and $d^{-1} \cdot t^{-1} \cdot c^{-1} \leq s^{-1}$, we also have $s^{-1} \simeq t^{-1}$.
- (e) If $e \simeq_N f$ then there exist $a, b, p, q \in N$ with $a \cdot e \cdot b \leq f$ and $p \cdot f \cdot q \leq e$. Therefore we have $a^{-1}a = e$ and $aa^{-1} \leq f$, and so $e \leq_{\mathcal{J}_N} f$. By symmetry $f \leq_{\mathcal{J}_N} e$ and so $e \mathcal{J}_N f$. Conversely, if $e \mathcal{J}_N f$ there exist $m, n \in N$ with $mm^{-1} \leq f$ and $m^{-1}m = e$, $nn^{-1} \leq e$ and $n^{-1}n = f$. Then $m \cdot e \cdot m^{-1} \leq f$ and $n \cdot f \cdot n^{-1} \leq e$, and so $e \simeq_N f$.
- (f) This follows from part (b) of Lemma 3.1.
- (g) By Lemma 3.1(a), $\leq_{E(S)}$ is the natural partial order, which is of course anti-symmetric.

□

However, \simeq_N need not be a congruence on S .

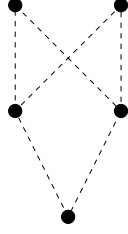
Example 3.5.

- (a) In the symmetric inverse monoid \mathcal{S}_4 , let $f : \{1\} \rightarrow \{2\}$, let S be the inverse subsemigroup

$$S = \{\text{id}_{\{1,3\}}, \text{id}_{\{1\}}, \text{id}_{\{2\}}, f, f^{-1}, 0\}$$

of \mathcal{S}_4 , and let $N = S$. Then $\text{id}_{\{1\}} = ff^{-1} \simeq_S f^{-1}f = \text{id}_{\{2\}}$. But $\text{id}_{\{1,3\}} \text{id}_{\{1\}} = \text{id}_{\{1\}}$ is not \simeq_S -related to $\text{id}_{\{1,3\}} \text{id}_{\{2\}} = 0$. In this example, the poset of \mathcal{J} -classes is just a chain, and so is a semilattice.

- (b) Now let $g : \{3\} \rightarrow \{4\}$ in \mathcal{S}_4 and let T be the inverse subsemigroup of \mathcal{S}_4 generated by $\{\text{id}_{\{1,3\}}, \text{id}_{\{2,4\}}, f, g\}$. Here the \mathcal{J} -classes do not form a semilattice, and so \simeq_T is not a congruence, and the quotient T / \simeq_T is not an inverse semigroup: it is the poset



Following the notation in [1], we shall denote the quotient S/ \simeq_N by $S//N$ and let $\pi : S \rightarrow S//N$ be the quotient map. Our next result sets out the ordered groupoid structure on $S//N$: it is a special case of [1, Theorem 4.14], but the description is simpler for quotients of inverse semigroups than for general quotients of ordered groupoids, and so seems worth stating in detail.

Theorem 3.6. *For any inverse semigroup S and normal inverse subsemigroup N , the quotient set $S//N$ is an ordered groupoid, with the following structure:*

- (a) *the identities are the classes $[e]_N$ where $e \in E(S)$, and a class $[s]_N$ has domain $[ss^{-1}]_N$, range $[s^{-1}s]_N$, and inverse $[s^{-1}]_N$.*
- (b) *as a poset, $E(S//N)$ is isomorphic to N/\mathcal{J}_N .*
- (c) *If $s, t \in S$ and $s^{-1}s \simeq_N tt^{-1}$, then there exists $a \in N$ with $aa^{-1} \leq s^{-1}s$ and $a^{-1}a = tt^{-1}$: the composition of $[s]_N$ and $[t]_N$ is then defined as $[sat]_N$.*
- (d) *The ordering \leq_N of \simeq_N -classes is given by*

$$[s]_N \leq_N [t]_N \iff \text{there exist } a, b \in N \text{ such that } a \cdot s \cdot b \leq t.$$

Proof. It follows from part (d) of Proposition 3.4 that the domain and range of $[s]_N$ and its inverse $[s]_N^{-1}$ are well-defined.

Suppose that $s, t \in S$ and $s^{-1}s \simeq_N tt^{-1}$ but that we choose another element $z \in N$ with $zz^{-1} \leq s^{-1}s$ and $z^{-1}z = tt^{-1}$. Then $az^{-1}z = att^{-1} = aa^{-1}a = a$ and $s^{-1}sz = (s^{-1}s)(zz^{-1})z = zz^{-1}z = z$. Hence

$$sat = saz^{-1}zt = saz^{-1}s^{-1}szt = (saz^{-1}s^{-1}) \cdot szt \simeq_N szt$$

and so for fixed s, t the \simeq_N -class of the element sat does not depend on the choice of a . We denote this class by $s \check{\circ} t$. Suppose that $s \simeq_N s_1$ and that we choose $a_1 \in N$ to form $s_1 \check{\circ} t = [s_1a_1t]_N$. There exist $u, v \in N$ with $u \cdot s_1 \cdot v \leq s$, and so $vv^{-1} = s_1^{-1}s_1 \geq a_1a_1^{-1}$. Then

$$(v^{-1}a_1)(v^{-1}a_1)^{-1} = v^{-1}a_1a_1^{-1}v \leq v^{-1}v \leq s^{-1}s$$

and

$$(v^{-1}a_1)^{-1}(v^{-1}a_1) = a_1^{-1}vv^{-1}a_1 = a_1^{-1}a_1 = tt^{-1}.$$

Therefore we can use the element $v^{-1}a_1$ to form the class $s \check{\circ} t = [sv^{-1}a_1t]_N$. Now $sv^{-1}s_1^{-1} = (sv^{-1}v)(v^{-1}s_1^{-1}) = (us_1v)(v^{-1}s_1^{-1}) = u$ and since $s^{-1}s \geq v^{-1}v$, we have

$$sv^{-1}a_1t = sv^{-1}vv^{-1}a_1t = sv^{-1}s_1^{-1}s_1a_1t = u \cdot (s_1at) \simeq_N s_1a_1t$$

and so $s \check{\circ} t \simeq_N s_1 \check{\circ} t$. Similarly, $s \check{\circ} t$ does not depend on the choice of the element t within its \simeq_N -class, and the product $[s]_N \cdot [t]_N = [sat]_N$ is well-defined.

Now the relation $s^{-1}s \simeq_N tt^{-1}$ furnishes not only $a \in N$ with $aa^{-1} \leq s^{-1}s$ and $a^{-1}a = tt^{-1}$ but also $p \in N$ with $pp^{-1} \leq s^{-1}s$ and $pp^{-1} = tt^{-1}$. Consider $saps^{-1} \in N$: we have

$$(saps^{-1})(saps^{-1})^{-1} = saps^{-1}sp^{-1}a^{-1}s^{-1} \leq sapp^{-1}a^{-1}s^{-1} = (sat)(sat)^{-1}$$

and

$$\begin{aligned} (saps^{-1})^{-1}(saps^{-1}) &= sp^{-1}a^{-1}s^{-1}saps^{-1} \\ &= sp^{-1}a^{-1}aps^{-1} \\ &= sp^{-1}ps^{-1} = ss^{-1} \end{aligned}$$

and, since $(sat)(sat)^{-1} \leq ss^{-1}$, we have $ss^{-1} \simeq_N (sat)(sat)^{-1}$. Therefore, $[sat]_N$ has domain $[ss^{-1}]_N$ and range $[t^{-1}t]_N$, and the composition $[s]_N \cdot [t]_N = [sat]_N$ does give a groupoid structure on $S//N$.

Now by Lemma 3.1, \leq_N induces the given partial order on the \simeq_N -classes, and it remains to show that this partial order makes $S//N$ into an ordered groupoid.

Now if $a \cdot s \cdot b \leq t$ then $b^{-1} \cdot s^{-1} \cdot a^{-1} \leq t^{-1}$ and so $[s]_N \leq_N [t]_N$ implies that $[s]_N^{-1} \leq_n [t]_N^{-1}$. Now suppose that $[s_1]_N \leq [s]_N$, $[t_1]_N \leq [t]_N$ and that the compositions $[s]_N \cdot [t]_N$ and $[s_1]_N \cdot [t_1]_N$ exist. There exist $m, n, u, v \in N$ such that $m \cdot s_1 \cdot m \leq s$ and $u \cdot t_1 \cdot v \leq t$: since $[m \cdot s_1 \cdot n]_N = [s_1]_N$ and $[u \cdot t_1 \cdot v]_N = [t_1]_N$ we may as well assume that $s_1 \leq s$ and $t_1 \leq t$.

We now have $a, p \in N$ with $aa^{-1} \leq s^{-1}s$, $a^{-1}a = tt^{-1}$ and $b, q \in N$ with $bb^{-1} \leq s_1^{-1}s_1$, $b^{-1}b = t_1t_1^{-1}$, $qq^{-1} \leq t_1t_1^{-1}$ and $q^{-1}q = s_1^{-1}s_1$. Now

$$s_1bt_1 \simeq_N sab^{-1}s^{-1} \cdot s_1bt_1 = sab^{-1}s_1^{-1}s_1bt_1 \leq sat_1 \leq sat$$

and so $[s_1]_N \cdot [t_1]_N = [s_1bt_1]_N \leq [sat]_N = [s]_N \cdot [t]_N$.

Finally we suppose that $[n]_N \leq_N [ss^{-1}]_N$ for some $n \in N$ and $s \in S$. By part (a) of Proposition 3.4 we may replace n by $e = nn^{-1}$. Then there exist $a, b \in N$ with $a \cdot e \cdot b \leq ss^{-1}$ and so $aa^{-1} \leq ss^{-1}$ and $a^{-1}a = e$. Then the class $[aa^{-1}s]_N$ has domain $[aa^{-1}]_N = [e]_N$ and $[aa^{-1}s]_N \leq_N [s]_N$. We wish to show that $[aa^{-1}s]_N$ is the unique \simeq_N -class with these properties.

Suppose that $[k]_N \leq_N [s]_N$ and that $[kk^{-1}]_N = [e]_N$. There exist $u, v \in N$ with $u \cdot k \cdot v \leq s$ and $b, q \in N$ with $bb^{-1} \leq kk^{-1}$, $b^{-1}b = e$, $qq^{-1} \leq e$, and $q^{-1}q = kk^{-1}$. We have $a \in N$ as in the previous paragraph. Then $kvs^{-1} = u^{-1}$ and $u^{-1}uq^{-1} = q^{-1}$: hence

$$\begin{aligned} k &\simeq_N k \cdot vs^{-1}uq^{-1}a^{-1}s = q^{-1}a^{-1}s = q^{-1}a^{-1}aa^{-1}s \\ &= q^{-1}a^{-1} \cdot aqq^{-1}a^{-1}s \simeq_N aqq^{-1}a^{-1}s \leq aa^{-1}s \end{aligned}$$

and so $[k]_N \leq_N [aa^{-1}s]_N$. By symmetry, they are equal. \square

Corollary 3.7. [1, Theorem 4.15] *Given a homomorphism $\phi : S \rightarrow \Sigma$ of inverse semigroups, let $K = \{s \in S : x\phi \in E(\Sigma)\}$. Then K is a normal inverse subsemigroup of S , and ϕ factorises as a composition*

$$S \xrightarrow{\pi} S//K \xrightarrow{\kappa} \Sigma$$

where κ , defined by $[s]_K\kappa = s\phi$, is a star-injective functor.

Proof. The map κ is well-defined, since if $[s]_K = [s']_K$ then, for some $a, b \in K$ we have $a \cdot s \cdot b \leq s'$. Now $a\phi = (a^{-1}a)\phi = (ss^{-1})\phi$ and similarly, $b\phi = (s^{-1}s)\phi$. It follows that $(a \cdot s \cdot b)\phi = s\phi \leq s'\phi$. By symmetry, $s'\phi \leq s\phi$.

To show that κ is a functor, suppose that $[s]_K$ and $[t]_K$ are composable in $S//K$. Then there exist $a, p \in K$ with

$$aa^{-1} \leq s^{-1}s, a^{-1}a = tt^{-1}, pp^{-1} \leq tt^{-1}, p^{-1}p = s^{-1}s$$

and the composition of $[s]_K$ and $[t]_K$ is defined, as in Theorem 3.6, by $[s]_K \cdot [t]_K = [sat]_K$. Then

$$\begin{aligned} ([sat]_K)\kappa &= (sat)\phi = (s\phi)(a\phi)(t\phi) \\ &= (s\phi)(a^{-1}a)\phi(t\phi) \text{ (since } a \in K) \\ &= (s\phi)(tt^{-1})\phi(t\phi) = (s\phi)(t\phi) \end{aligned}$$

Now $(s\phi)(t\phi)$ is a trace product $(s\phi) \cdot (t\phi)$ since

$$\begin{aligned} (s\phi)^{-1}(s\phi) &= (s^{-1}s)\phi = (p^{-1}p)\phi \\ &= (pp^{-1})\phi \text{ (since } p \in K) \\ &\leq (tt^{-1})\phi = (t\phi)(t\phi)^{-1} \end{aligned}$$

and similarly

$$\begin{aligned} (t\phi)(t\phi)^{-1} &= (tt^{-1})\phi = (a^{-1}a)\phi \\ &= (aa^{-1})\phi \text{ (since } a \in K) \\ &\leq (s^{-1}s)\phi = (s\phi)^{-1}(s\phi). \end{aligned}$$

Therefore $(s\phi)(t\phi)$ is a trace product defined in the inductive groupoid (Σ, \cdot) and κ is a functor.

To show that κ is star-injective, suppose that for some $u, v \in S$ we have $[uu^{-1}]_K = [vv^{-1}]_K$ and $u\phi = v\phi$. We claim that $u \simeq_K v$. By symmetry, it is sufficient to show that $u \leq_K v$. Now by part (e) Proposition 3.4 there exist $a, b \in K$ with $aa^{-1} \leq uu^{-1}, a^{-1}a = vv^{-1}, bb^{-1} \leq vv^{-1}$ and $b^{-1}b = uu^{-1}$. Then

$$b \cdot u \cdot u^{-1}b^{-1}v = (b uu^{-1} b^{-1})v \leq v$$

and

$$\begin{aligned} (u^{-1}b^{-1}v)\phi &= (u^{-1})\phi(b^{-1}b)\phi v\phi \text{ (since } b \in K) \\ &= (u^{-1}b^{-1}b)\phi v\phi \\ &= (u^{-1}\phi)v\phi = (u\phi)^{-1}v\phi \in E(\Sigma) \text{ since } u\phi = v\phi, \end{aligned}$$

and so $u^{-1}b^{-1}v \in K$ and $u \leq_K v$ as required. \square

Corollary 3.8. *The factorization of $\phi : S \rightarrow \Sigma$ is unique, in the sense that if ϕ also factorizes as $S \rightarrow S//N \xrightarrow{\nu} \Sigma$ with ν a star-injective functor, then $N = K$ (and hence $\nu = \kappa$.)*

Proof. If $n \in N$ then by part (a) of Proposition 3.4, we have $n \simeq_N nn^{-1}$ and so $n\phi \in E(\Sigma)$. Hence $N \subseteq K$.

Now if $k \in K$ then $k\phi \in E(\Sigma)$, and since ν is star-injective, then $[k]_n$ is an identity in $S//N$ and so, for some $e \in E(S)$ we have $k \simeq_N e$. Then there exists $a, b \in N$ such that $a \cdot k \cdot b \leq e$, and so $a \cdot k \cdot b = f \in E(S)$. But $a^{-1}a = kk^{-1}$ and $bb^{-1} = k^{-1}k$, so that

$$k = (kk^{-1})k(k^{-1}k) = a^{-1}akbb^{-1} = a^{-1}fb^{-1} \in N.$$

Hence $K \subseteq N$ and so $N = K$. \square

We note that this factorisation of an inverse semigroup homomorphism requires the use of an intermediate ordered groupoid. We shall apply it to the study of congruences in section 4. For the further study of inverse semigroups, it is clearly of interest to know when we can form a quotient *inverse semigroup* $S//N$. Since an inverse semigroup is equivalent to an ordered groupoid in which the poset of identities is a semilattice, we have the following.

Proposition 3.9. *Let S be an inverse semigroup and N a normal inverse subsemigroup of S . Then the quotient ordered groupoid $S//N$ is an inverse semigroup if and only if the poset of \mathcal{J}_N -classes of S is a semilattice. (This is certainly the case if \mathcal{J}_N is a congruence on $E(N)$.)*

Example 3.10. In the symmetric inverse monoid \mathcal{I}_n , let N be the subset of non-permutations together with the identity map id . Then N is a normal inverse subsemigroup. Since only $\text{id} \in N$ can form trace products with permutations, \leq_N restricts to the identity on the symmetric group $S_n \subset \mathcal{I}_n$. Moreover, for any $\text{id} \neq \nu \in N$ and $\sigma \in S_n$ we have $\nu^{-1} \cdot \nu \cdot \sigma|_{\text{r}(\nu)} \leq \sigma$ so that $\nu \leq_N \sigma$. On the elements of N , the relation \simeq_N is equal to the \mathcal{D} (and \mathcal{J}) relation, and so for non-identity $\alpha, \beta \in N$ we have $\alpha \simeq_N \beta \iff |\mathbf{d}(\alpha)| = |\mathbf{d}(\beta)|$. It follows that $\mathcal{I}_n//S_n$ consists of the group S_n as a set of $n!$ maximal elements, and a chain $e_{n-1} > e_{n-2} > \dots > e_1 > e_0$ of identities corresponding to the cardinalities of non-identity elements of N .

Example 3.11. Polycyclic and gauge monoids. Let $A = \{a_1, a_2, \dots, a_n\}$ with $n \geq 1$. The polycyclic monoid P_n (introduced in [11]) is the inverse hull of A^* : its underlying set is $(A^* \times A^*) \cup \{0\}$ and the multiplication of non-zero elements is given by:

$$(s, t)(u, v) = \begin{cases} (s, pv) & \text{if } t = pu \text{ for some } p \in A^*, \\ (ps, v) & \text{if } u = pt \text{ for some } p \in A^*, \\ 0 & \text{otherwise.} \end{cases}$$

The semilattice of idempotents is

$$E(P_n) = \{(p, p) : p \in A^*\} \cup \{0\}$$

and the natural partial order between non-zero elements is given by $(u, v) \leq (s, t)$ if and only if $u = ps, v = pt$ for some $p \in A^*$.

Full inverse subsemigroups of P_n have the form $Q \cup \{0\}$ where Q is a left congruence on A^* : see [8, Theorem 3.3], with a change to left congruence required by our differing conventions. Meakin and Sapir [10] established the first correspondence of this kind, showing that the lattice of congruences on A^* is isomorphic to the lattice of *positively self-conjugate* submonoids of P_n , where an inverse submonoid R is positively self-conjugate if $(w, 1)R(1, w) \subseteq R$ for every $w \in A^*$.

Lemma 3.12. *A full inverse semigroup $N = Q \cup \{0\}$ of P_n is normal if and only if Q is a right cancellative congruence on A^* .*

Proof. By [8, Theorem 3.3] Q must be a left congruence on A^* . Suppose that $(q_1, q_2) \in Q$ and that $(h_1, h_2), (k_1, k_2) \leq (w_1, w_2) \in P_n$. Then $h_i = uw_i$ and $k_i = vw_i$ for $i = 1, 2$ and some $u, v \in A^*$. Then N is normal if and only

$$\begin{aligned} (h_1, h_2)^{-1}(q_1, q_2)(k_1, k_2) &= (uw_2, uw_1)(q_1, q_2)(vw_1, vw_2) \\ &= (uw_2, vw_2) \in Q \end{aligned}$$

where $q_1 = uw_1$ and $q_2 = vw_1$. Therefore N is normal if and only if, for all $w_1, w_2 \in A^*$, we have that $(uw_1, vw_1) \in Q$ implies that $(uw_2, vw_2) \in Q$ and this is equivalent to Q being a right cancellative two-sided congruence on A^* . \square

The *gauge inverse monoid* G_n is defined by

$$G_n = \{(s, t) : |s| = |t|\} \cup \{0\}.$$

It was introduced in [6], and by Lemma 3.12 it is a normal inverse submonoid of P_n . By [6, Lemma 3.4] Green's relations \mathcal{D} and \mathcal{J} coincide in G_n , and clearly (s, t) and (u, v) are \mathcal{D} -related in G_n if and only if $|s| = |t| = |u| = |v|$. Thus the non-zero \mathcal{J} -classes are indexed by the non-negative integers and the \mathcal{J} -order on them is trivial. The \mathcal{J} -class of 0 is minimal and $E(P_n)/\mathcal{J}_{G_n}$ is a semilattice, and so $P_n//G_n$ is an inverse semigroup. To identify it, we note that $(u, v) \leq_{G_n} (s, t)$ if and only if there exist $h, k \in A^*$ such that $|h| = |u|, |k| = |v|$ and

$$(h, k) = (h, u)(u, v)(v, k) \leq (s, t).$$

It follows that $h = ps$ and $k = pt$ for some $p \in A^*$ and so

$$(u, v) \leq_{G_n} (s, t) \iff \text{there exists } p \in A^* \text{ such that } |u| - |s| = |v| - |t| = |p| \geq 0.$$

The relation \simeq_{G_n} is then given by

$$(u, v) \simeq_{G_n} (s, t) \iff |u| = |s| \text{ and } |v| = |t|$$

and the \simeq_{G_n} classes of the non-zero elements in P_n are thus parametrized by pairs of non-negative integers. Now in $P_n//G_n$ we have

$$[(u, v)]_{G_n} [(s, t)]_{G_n} = [(u, v)(v, s)(s, t)]_{G_n} = [(u, t)]_{G_n}$$

and so $P_n//G_n$ is isomorphic to the Brandt semigroup on the set of non-negative integers.

4. CONGRUENCES AND KERNELS

Let ρ be a relation on an inverse semigroup S . Following [12], the *trace* $\text{tr}(\rho)$ of ρ is its restriction to $E(S)$, and the *kernel* $\ker \rho$ is the set

$$\ker \rho = \{s \in S : s \rho e \text{ for some } e \in E(S)\}.$$

Proposition 4.1. *The kernel of the relation \simeq_N is N and its trace is Green's relation \mathcal{J}_N on $E(N) = E(S)$.*

Proof. This follows from Lemma 3.1(c), Proposition 3.4(a) and (d). \square

Recall from [12] that a congruence ρ on the semilattice of idempotents $E(S)$ of S is *normal* if, for all $s \in S, e \rho f$ implies that $s^{-1}es \rho s^{-1}fs$. Then [12, Definition 4.2] a *congruence pair* (K, ν) on S consists of a normal inverse semigroup K of S and a normal congruence ν on $E(S)$ such that

$$(4.1) \quad \text{if } e \in E(S) \text{ and } s \in S \text{ satisfy } se \in K \text{ and } s^{-1}s\nu e \text{ then } s \in K,$$

$$(4.2) \quad \text{if } u \in K \text{ then } uu^{-1} \nu u^{-1}u.$$

For any congruence ρ , its kernel and trace form a congruence pair. Conversely, given a congruence pair (K, ν) the relation $\rho_{(K, \nu)}$ defined by

$$(4.3) \quad s \rho_{(K, \nu)} t \iff st^{-1} \in K \text{ and } s^{-1} s \nu t^{-1} t$$

is a congruence with kernel K and trace ν . This correspondence is the basis of the characterization of congruences in [12, Theorem 4.4]. The lattice of all congruences on both regular and inverse semigroups was earlier studied by Reilly and Scheiblich [13].

If ρ is a congruence on S , let $\rho(s)$ be the class of $s \in S$ and let $\rho_* : S \rightarrow S/\rho$ be the quotient map, $s \mapsto \rho(s)$. Now S/ρ is an inverse semigroup and so is an inductive groupoid with its trace product. If $K = \ker \rho$ then \simeq_K is a relation on S , and as in [1] we have the quotient map $\pi : S \rightarrow S//K$, $s \mapsto [s]_K$ where $S//K$ is an ordered groupoid. Applying Corollary 3.7 to the homomorphism $S \rightarrow S/\rho$ we obtain:

Proposition 4.2. *If K is the kernel of the congruence ρ on an inverse semigroup S then $s \simeq_K t$ implies that $s \rho t$, and the induced mapping $\kappa : S//K \rightarrow S/\rho$ carrying $[s]_K \mapsto \rho(s)$ is a surjective star-injective functor.*

The converse of proposition 4.2 is the following.

Theorem 4.3. *Let N be a normal inverse subsemigroup of S and let $\psi : S//N \rightarrow Q$ be a surjective, star-injective functor to an inverse semigroup Q . Let ν be the congruence on $E(S)$ determined by the composition*

$$E(S) \rightarrow E(S//N) \rightarrow E(Q).$$

Then (N, ν) is a congruence pair, and the associated congruence $\rho_{(N, \nu)}$ on S is that determined by the composition

$$\phi : S \xrightarrow{\pi} S//N \xrightarrow{\psi} Q.$$

Proof. It is clear that ν is a normal congruence on $E(S)$. Suppose that $e \in E(S)$, $s \in S$, $se \in N$ and $(s^{-1}s)\phi = e\phi$. Then

$$s\phi = (ss^{-1}s)\phi = (s\phi)(s^{-1}s)\phi = (s\phi)(e\phi) = (se)\phi$$

and $(se)\phi \in E(Q)$ since $se \in N$. Hence $s\phi = s\pi\nu \in E(Q)$. Since ν is star-injective, then $s\pi \in E(S//N)$ and so $s \in N$ by part (c) of Proposition 3.4.

Now if $u \in N$ then $u\phi = u\pi\psi \in E(Q)$ and so

$$u\phi = u^{-1}\phi = (uu^{-1})\phi = (u^{-1}u)\phi$$

and therefore $uu^{-1} \nu u^{-1}u$. This confirms that (N, ν) is a congruence pair.

If $s, t \in S$ and $s\phi = t\phi$ then $(st^{-1})\phi = (st^{-1})\pi\psi \in E(Q)$. Since ψ is star-injective, then $(st^{-1})\pi \in E(S//N)$ and again $s \in N$ by part (c) of Proposition 3.4. Moreover, $(s^{-1}s)\phi = (s\phi)^{-1}(s\phi) = (t\phi)^{-1}(t\phi) = (t^{-1}t)\phi$ and so $s^{-1}s \nu t^{-1}t$. Therefore $s \rho_{(N, \nu)} t$. Conversely, if $s \rho_{(N, \nu)} t$ then $st^{-1} \in N$ and $s^{-1}s \nu t^{-1}t$. Then $(st^{-1})\phi = (st^{-1})\pi\psi \in E(Q)$, since $(st^{-1})\pi \in E(S//N)$, and so

$$t\phi \geq (st^{-1})\phi(t\phi) = (st^{-1}t)\phi = (ss^{-1}s)\phi = s\phi,$$

and by symmetry, $t\phi = s\phi$. □

Howie [5, Exercise 5.11.16] defines a full inverse semigroup N of an inverse semigroup S to have the *kernel property* if, whenever $s, t \in S$ with $st \in N$ and $n \in N$ then $snt \in N$. A full inverse subsemigroup with the kernel property is called *normal* in [3]. It is easy to see that an inverse subsemigroup with the kernel property is normal in the sense of [12] (the sense used in this paper), and that the kernel of any congruence has the kernel property. Moreover, an inverse subsemigroup with the kernel property is the kernel of its syntactic congruence, and so:

Theorem 4.4 ([3], Theorem 3.3). *An inverse subsemigroup N of an inverse semigroup S is the kernel of a congruence if and only if it is full and has the kernel property.*

Hence, if \simeq_N is a congruence, N must have the kernel property.

Theorem 4.5. *If N is a full inverse subsemigroup of an inverse semigroup S and N has the kernel property, then \simeq_N is a congruence on S if and only if \mathcal{J}_N is a normal congruence on $E(S)$, and \simeq_N is then the minimal congruence on S with kernel N .*

Proof. By Proposition 4.1, the relation \simeq_N is a congruence on S if and only if (N, \mathcal{J}_N) is a congruence pair. Now N is normal, and (4.2) holds by part (a) of Proposition 3.4

For (4.1), suppose that $se \in N$ and that $s^{-1}s \simeq_N e$. Then there exists $a \in N$ with $aa^{-1} \leq e$ and $a^{-1}a = s^{-1}s$. Now

$$se = ss^{-1}se = sess^{-1} \geq saa^{-1}s^{-1}$$

and so $saa^{-1}s^{-1}a \in N$. Now by two applications of the kernel property using $a^{-1}a \in N$,

$$sa^{-1}(aa^{-1})as^{-1}s = sa^{-1}as^{-1}s = ss^{-1}ss^{-1}s = s \in N.$$

Hence (4.1) holds, and therefore (N, \mathcal{J}_N) is a congruence pair if and only if \mathcal{J}_N is a normal congruence on $E(S)$.

Now if ρ is a congruence with kernel N we have, for $s, t \in S$,

$$s \simeq_N t \implies s \rho t,$$

and so \simeq_N is minimal. □

4.1. Idempotent separating congruences. A congruence ρ on an inverse semigroup S is *idempotent separating* if its trace is the identity relation on $E(S)$. The classification of congruences by congruence pairs [12, Theorem 4.4] shows that an idempotent separating congruence is entirely determined by its kernel K , a normal inverse subsemigroup of S which, by (4.2), must also satisfy the property that for all $a \in K$, $aa^{-1} = a^{-1}a$. (Hence K is a *Clifford* inverse semigroup). The congruence ρ is then defined, according to (4.3), by:

$$(4.4) \quad s \rho t \iff st^{-1} \in K \text{ and } s^{-1}s = t^{-1}t.$$

Proposition 4.6. *If ρ is an idempotent-separating congruence on S with kernel K then the relations ρ and \simeq_K are equal, and so $\kappa : S//K \rightarrow S/\rho$ is an isomorphism of inverse semigroups.*

Proof. If $s \rho t$ then $st^{-1} \in K$ and since $s^{-1}s = t^{-1}t$, we see that st^{-1} is a trace product in (S, \cdot) . Hence $s \cdot t^{-1} \cdot t$ is also a trace product in S , and $st^{-1}t \leq s$. Since $st^{-1} \in K$, this shows that $[t]_K \leq [s]_K$. By symmetry, they are equal (or we can repeat the argument using $ts^{-1} \in K$ and $ts^{-1}s \leq t$). So if ρ is idempotent-separating then $s \rho t$ implies that $s \simeq_K t$, and Proposition 4.2 gives the reverse implication. □

Remark 4.7. The converse of this result is not true: see Example 4.9 below.

4.2. Closed inverse subsemigroups. For a subset A of an inverse semigroup S , we denote by A^\uparrow the smallest closed subset of S containing A . If A is an inverse subsemigroup of S , then so is A^\uparrow .

Let N be a *closed* inverse subsemigroup of S : so if $n \in N$ and $n \leq s$ then $s \in N$. The relation $a \equiv_N b \iff ab^{-1} \in N$ is then an equivalence relation on the subset $\text{star}_S(N) = \{s \in S : ss^{-1} \in N\}$ and the equivalence classes are the *cosets* of N . This notion of coset was introduced by Schein [14]. If N is normal, then $\text{star}_S(N) = S$ and \equiv_N is an equivalence relation on S and it is easy to see that it is then also a congruence, with kernel N . If $s, t \in S$ and there exists $e \in E(S)$ with $es = et$ then $est^{-1}e \in E(S) \subseteq N$ and, since N is closed, $st^{-1} \in N$. It follows that \equiv_N contains the minimal group congruence σ on S , and if $\sigma_* : S \rightarrow S/\sigma$ then S/\equiv_N is isomorphic to the quotient group $(S/\sigma)/N\sigma_*$.

The relation \simeq_N is finer than \equiv_N :

Proposition 4.8. *Let N be a closed normal inverse subsemigroup of S . Then for all $s, t \in S$, $s \simeq_N t$ implies that $s \equiv_N t$, and $s \equiv_N t$ if and only if $st^{-1}t \simeq_N ts^{-1}s$.*

Proof. If $s \simeq_N t$ then $s \equiv_N t$ by part (f) of Lemma 3.1. Now if $st^{-1} \in N$ we have $st^{-1} \cdot ts^{-1}s = st^{-1}t$ and $ts^{-1} \cdot st^{-1}t = ts^{-1}s$, and so $st^{-1}t \simeq_N ts^{-1}s$. Conversely, if $st^{-1}t \simeq_N ts^{-1}s$, then by the first part $st^{-1} = (st^{-1}t)(s^{-1}st^{-1}) \in N$ and so $s \equiv_N t$. \square

Example 4.9. Let T be any inverse semigroup and let G be a group: set $S = T \times G$. We identify T with $T_1 = \{(t, 1_G) : t \in T\}$, which is a closed, normal, inverse subsemigroup of S . Then $(u, g) \equiv_T (v, h)$ if and only if $g = h$, so that S/\equiv_T is isomorphic to G . However, by part (f) of Proposition 3.4, we have

$$(u, g) \simeq_T (v, h) \iff u\mathcal{J}t \text{ and } g = h$$

and so $S//T \cong (T/\mathcal{J}) \times G$. Hence $\kappa : S//T \rightarrow S/\equiv_T$ is an isomorphism if and only if T is simple.

If T is not a group, then \equiv_T is not idempotent separating, and so the converse of Proposition 4.6 is not true in general.

5. INVERSE MONOID PRESENTATIONS

Let $\mathcal{P} = \langle X : R \rangle$ be a presentation of the inverse monoid M . We assume that R consists of a set of pairs (ℓ, r) with $\ell, r \in \text{FIM}(X)$, the free inverse monoid on X . The pairs in R generate a congruence $\simeq_{\mathcal{P}}$ on $\text{FIM}(X)$ with M isomorphic to the quotient $\text{FIM}(X)/\simeq_{\mathcal{P}}$. We let $\pi : \text{FIM}(X) \rightarrow M$ denote the quotient map.

Let $K(\mathcal{P})$ be the kernel of $\simeq_{\mathcal{P}}$. We note that $K(\mathcal{P})$ is the image in $\text{FIM}(X)$ of the idempotent problem (see [2]) of \mathcal{P} in $(X \sqcup X^{-1})^*$. By Theorem 4.4 $K(\mathcal{P})$ is a full inverse subsemigroup of $\text{FIM}(X)$ with the kernel property, and hence is normal.

Proposition 5.1. *Let $N(\mathcal{P})$ be the smallest normal inverse subsemigroup of $\text{FIM}(X)$ containing the set*

$$Q(R) = \{\ell^{-1}r, r\ell^{-1} : (\ell, r) \in R\}.$$

Then

$$N(\mathcal{P}) \subseteq K(\mathcal{P}) \subseteq N(\mathcal{P})^\uparrow.$$

Proof. Elements of $N(\mathcal{P})$ are products of conjugates of elements of $Q(R)$ and their inverses and idempotents in $\text{FIM}(X)$. Since each element of $Q(R)$ is mapped by π to an idempotent of M we therefore have $N(\mathcal{P}) \subseteq K(\mathcal{P})$.

Suppose now that $u \simeq_{\mathcal{P}} v$: then there exists $u = u_0, u_1, \dots, u_{k-1}, u_k = v$ such that, for all i with $0 \leq i \leq k-1$, there exist $p_i, q_i \in \text{FIM}(X)$ such that $u_i = p_i \ell q_i$ and $u_{i+1} = p_i r q_i$, or vice versa. Assume the former: then

$$p_i r q_i \geq p_i r \ell^{-1} p_i^{-1} p_i \ell q_i$$

with $p_i r \ell^{-1} p_i^{-1} \in N$. Hence $u_{i+1} \geq n_i u_i$ and so, for some $n \in N$, we have $v \geq n u$. Hence if for some $e \in E(\text{FIM}(X))$ we have $e \simeq_{\mathcal{P}} v$, that is if $v \in K(\mathcal{P})$, then $v \geq n e$ with $n e \in N$ and so $v \in N^\dagger$. \square

Proposition 5.2. *The monoid M is E -unitary if and only if $K(\mathcal{P})$ is a closed inverse submonoid of $\text{FIM}(X)$, in which case $K(\mathcal{P}) = N(\mathcal{P})^\dagger$.*

Proof. Suppose that M is E -unitary, that $u \geq v$ in $\text{FIM}(X)$ and that $v \in K(\mathcal{P})$. Then $v\pi \in E(M)$ and since $u\pi \geq v\pi$ we have $u\pi \in E(M)$. By Lallement's Lemma [5, Lemma 2.4.3], there exists $e \in E(\text{FIM}(X))$ with $e\pi = u\pi$, and so $u \in K(\mathcal{P})$.

Conversely, suppose that $K = K(\mathcal{P})$ is closed. Then by Proposition 5.1, we have $K = N(\mathcal{P})^\dagger$. As in section 4.2, the relation $u \equiv_K v \iff uv^{-1} \in K$ is a congruence on $\text{FIM}(X)$ with kernel K , and the quotient $\text{FIM}(X)/\equiv_K$ is isomorphic to the quotient group $F(X)/N(\mathcal{P})\sigma_*$, where $F(X)$ is the free group on X and σ_* is the canonical map $\text{FIM}(X) \rightarrow F(X)$. This quotient group is the maximal group image \widehat{M} of M . By Proposition 4.2 there are surjective star-injective functors $\text{FIM}(X)//K \rightarrow \text{FIM}(X)/\equiv_K$ and $\text{FIM}(X)//K \rightarrow M$ making the square

$$\begin{array}{ccc} \text{FIM}(X)//K & \longrightarrow & M \\ \downarrow & & \downarrow \sigma_* \\ \text{FIM}(X)/\equiv_K & \xrightarrow{\cong} & \widehat{M} \end{array}$$

commute. It follows that $\sigma_* : M \rightarrow \widehat{M}$ must also be star-injective, and this is equivalent to M being E -unitary (see, for example, [7, Theorem 2.4.6]). \square

Example 5.3.

- (a) Take $M = \mathcal{S}_2$ with $\tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\varepsilon = \begin{pmatrix} 1 & 2 \\ 1 & * \end{pmatrix}$. Let $X = \{t, e\}$, and let \mathcal{P} be a presentation of \mathcal{S}_2 with generating set X , with $t\pi = \tau$ and $e\pi = \varepsilon$. Since \mathcal{S}_2 is not E -unitary, K is not closed. Indeed, $(e^{-1}ete)\pi = 0$ but $(te)\pi = \begin{pmatrix} 1 & 2 \\ 2 & * \end{pmatrix}$, and so $e^{-1}ete \in K(\mathcal{P})$ but $te \notin K(\mathcal{P})$. In consequence, $N(\mathcal{P})$ here is not closed.
- (b) We note that the closure N^\dagger of $N = N(\mathcal{P})$ does not determine $\simeq_{\mathcal{P}}$. Consider the free inverse monoid M on two commuting generators [9], presented by $\mathcal{P} = \langle a, b : ab = ba \rangle$. Then $baba^{-1}b^{-1}b^{-1} \in N$ and so $u = baba^{-1}b^{-1}$ and $v = b$ lie in the same coset of N^\dagger in $\text{FIM}(X)$, but $u \neq v$ in M . This is verified by mapping

$M \rightarrow \mathcal{S}_2$ by

$$a \mapsto \begin{pmatrix} 1 & 2 \\ 1 & * \end{pmatrix} \text{ and } b \mapsto \begin{pmatrix} 1 & 2 \\ * & 2 \end{pmatrix}.$$

Then u maps to 0 but v does not. We note that in this case, M is E -unitary [9, Proposition 2.4], and so $K(\mathcal{P}) = N^\uparrow$.

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