On one-relator products induced by generalised triangle groups II

Citation for published version:
Chinyere, I & Howie, J 2018, 'On one-relator products induced by generalised triangle groups II', *Communications in Algebra*, vol. 46, no. 4, pp. 1464-1475. https://doi.org/10.1080/00927872.2017.1346772

Digital Object Identifier (DOI):
10.1080/00927872.2017.1346772

ON ONE-RELATOR PRODUCTS INDUCED BY GENERALISED TRIANGLE GROUPS II

I. CHINYERE AND J. HOWIE

Abstract. This is the second of two papers in which we study a group which is the quotient of a free product of groups by the normal closure of a single word that is contained in a subgroup which has the form of a free product of two cyclic groups. We use known properties of generalised triangle groups, together with detailed analysis of pictures and of words in free monoids, to prove a number of results such as a Freiheitssatz and the existence of Mayer-Vietoris sequences for such groups under suitable hypotheses. The results generalise those in an earlier article of the second author and Shwartz.

1. Introduction

This is the second of two papers, a sequel to [1], in which we study a group which is the quotient of a free product of groups by the normal closure of a single word that is contained in a subgroup which has the form of a free product of two cyclic groups. Specifically, suppose that R is a cyclically reduced word in a free product $G_1 * G_2$, of the form $x_1Uy_1U^{-1}x_2Uy_2U^{-1}\cdots x_mUy_mU^{-1}$, where the x_i, y_i are elements of $G_1 \cup G_2$ such that $x_1, \ldots, x_m$ generate a cyclic subgroup $A := \langle a \rangle$ of G_1 or G_2, and $y_1, \ldots, y_m$ generate a cyclic subgroup $U^{-1}BU := \langle b \rangle$ of G_1 or G_2. Let $n > 1$ be an integer. Then we consider the group $G = (G_1 * G_2)/N(R^n)$, where $N(R^n)$ denotes the normal closure of R^n.

Note that R is contained in the subgroup $A * B$ of $G_1 * G_2$. We say that G is *induced* from the generalised triangle group $H := (A * B)/N(R^n)$. This gives rise to a pushout diagram as in Figure 1 below, which we call a *generalised triangle group description* of G. Our results will require this description to be *maximal* in the sense that the subword U is as short as possible – see Section 3 for full details.

In [1] we proved a number of results, generalising those in [6], under a hypothesis which we called Hypothesis A. Here we prove similar results under a different hypothesis – also more general than those in [6], namely:

Hypothesis B. The generalised triangle description in Figure 1 is maximal, $n \geq 2$, R is a cyclically reduced word of free-product length at least 2 as a

Date: May 16, 2017.

Key words and phrases. One-relator group, generalised triangle groups, Pictures.

$$\begin{tikzcd} A * B \arrow[r] \arrow[d] & H \arrow[d] \\ G_1 * G_2 \arrow[r] & G \end{tikzcd}$$

FIGURE 1. *Push-out diagram.*

word in the free product $\langle a\rangle * \langle UbU^{-1}\rangle$, and no letter of R (that is, no letter of U and no power of a or b that occurs in R) has order 2 in G_1 or G_2.
Under the assumptions of Hypothesis B, we prove Theorems 1.1-1.5 below.

Theorem 1.1. *Let H be the generalised triangle group inducing the one relator product G. Then the maps $G_1 \to G$, $G_2 \to G$ and $H \to G$ are all injective.*

Theorem 1.2. *If a and b either both have finite order or both have infinite order, and the word problem is solvable for H, G_1 and G_2, then the word problem is solvable for G.*

Theorem 1.3. *Assume that the membership problems for $\langle a\rangle$ and $\langle b\rangle$ in $G_1 * G_2$ and for $\langle x\rangle$ and $\langle y\rangle$ in H are solvable. Then the word problem for G is also solvable.*

Theorem 1.4. *If a cyclic permutation of R^n has the form W_1W_2 with $0 < \ell(W_1) < \ell(R^n)$ as words in $G_1 \cup G_2$, then $W_1 \neq 1 \neq W_2$ as elements of G. In particular R has order n in G.*

Theorem 1.5. *The pushout of groups in Figure 1 is* geometrically Mayer-Vietoris *in the sense of* [2]. *In particular it gives rise to Mayer-Vietoris sequences*

$$\cdots \to H_{k+1}(G,M) \to H_k(A*B,M) \to$$
$$H_k(G_1 * G_2, M) \oplus H_k(H,M) \to H_k(G,M) \to \cdots$$

and

$$\cdots \to H^k(G,M) \to H^k(G_1 * G_2, M) \oplus H^k(H,M)$$
$$\to H^k(A*B,M) \to H^{k+1}(G,M) \to \cdots$$

for any $\mathbb{Z}G$-module M.

As in [1], these theorems will proven using the notion of *clique-picture* from [6] and Theorem B below.

Theorem B. If Hypothesis B above holds, then a minimal clique-picture over G satisfies the small-cancellation condition $C(6)$.

The rest of the paper is arranged in the following way. In Section 2 we define some of the terminologies that are used in the paper and present some preliminary results. In Section 3 we recall from [1] the concepts of refinements and clique-pictures. Section 4 contains the proof of Theorem B. Section 5 is the final section. There we give proofs of Theorems 1.1- 1.5.

We are grateful to an anonymous referee for useful and insightful comments on an earlier version of the paper, which have significantly improved the exposition. The first author was supported in this work by a Maxwell Institute Scholarship from Heriot-Watt University.

2. Preliminary results

In this section we recall or prove some results about periodicity of words in a free monoid, and about generalised triangle groups. These will be used later in the proof of Theorem B.

Recall that a word $w = z_1 z_2 \cdots z_n$ in the free monoid S^* on an alphabet S is said to have *period* $\mu < n$ if $z_i + \mu = z_i$ for all $i \in \{1, \ldots, n-\mu\}$, or equivalently if the initial and terminal segments of w of length $n-\mu$ are equal in S^*.

The following is a key fact about periodicity of words, which we use frequently.

Theorem 2.1. [3] *Let S be an alphabet, let $w = z_1 z_2 \cdots z_n \in S^*$ be a word having initial segment w_1 with period γ and terminal segment w_2 with period ρ. If w_1 and w_2 intersect in a segment u with $\ell(u) \geq \gamma + \rho - gcd(\gamma, \rho)$, then w has period $gcd(\gamma, \rho)$.*

We are particularly concerned with reduced words in a free product $G_1 * G_2$ of groups G_1, G_2, which we can regard as words in the free monoid on $(G_1 \cup G_2) \setminus \{1\}$.

Lemma 2.2. *Suppose that $X, W \in G_1 * G_2$ are reduced words such that W has a period $\gamma = 2\ell(X) < \ell(W)$. If W contains a segment identically equal to X and a segment identically equal to X^{-1}, then W contains a letter of order* 2.

Proof. Take V to be any subword of W of length γ. The periodicity of W implies that V is cyclically reduced, and that each of X, X^{-1} is identically equal to a cyclic subword of V. Since V is reduced, it cannot be a cyclic permutation of XX^{-1}, so the cyclic subwords X, X^{-1} of V intersect.

Hence there is an initial segment Y of X or X^{-1} that coincides with a terminal segment of X^{-1} or X respectively. Thus $Y \equiv Y^{-1}$ and so Y has an odd length and its middle letter has order 2. □

Lemma 2.3. [1, Lemma 3.2] *Suppose that $W \in G_1 * G_2$ is a word of the form*

$$W = x_1 V_1 y_1 V_1^{-1} = z_0 z_1 \cdots z_{2k-1}$$

for some letters x_1, y_1 and some word V_1, where $\ell(W) = 2k$. Suppose also that W has a cyclic permutation of the form $z_j z_{j+1} \cdots z_{2k-1} z_0 \cdots z_{j-1} = x_2 V_2 y_2 V_2^{-1}$, for some letters x_2, y_2 and some word V_2, where $j \not\equiv 0 \bmod k$. Then one of the following holds:

(1) $\{x_1, y_1\} = \{x_2, y_2\}$ *and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 y_1^{\beta(j)} V_3^{-1}]$$

for some odd integer $s > 1$ *and some word* V_3, *with* $\alpha(j), \beta(j) = \pm 1$ *for each* j.

(2) $y_i = x_i^{-1}$ *for* $i = 1, 2$, *and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 x_2^{\beta(j)} V_3^{-1}]$$

for some even integer $s > 0$ *and some word* V_3, *with* $\alpha(j), \beta(j) = \pm 1$ *for each* j.

Lemma 2.4. *For an integer* $m > 1$, *a non-trivial element* $X \in PSL_2(\mathbb{C})$ *has order* m *if and only if* $Tr(X) = 2\cos\frac{\delta\pi}{m}$ *for some* δ *satisfying* $gcd(\delta, m) = 1$.

Proof. Since $m > 1$, if $\gcd(\delta, m) = 1$ then $Tr(X) = 2\cos\frac{\delta}{m}$ if and only if the eigenvalues of X are $\exp(\pm i\delta\pi/m)$, which are distinct primitive $2m$'th roots of unity except in the case where δ is even and m is odd, when they are distinct primitive m'th roots of unity. In either case, X has order m in $PSL(2, \mathbb{C})$. Conversely, if X has finite order m then its eigenvalues are primitive m'th or $2m$'th roots of unity, so its trace has the stated form. □

Proposition 2.5. *Let*

$$H = \frac{\langle\ x\ \rangle * \langle\ y\ \rangle}{N(\ (xy)^2\ )}$$

be a one-relator product, where $\langle\ x\ \rangle$ *and* $\langle\ y\ \rangle$ *are cyclic groups of orders* $p, q \leq \infty$ *respectively. If* $V = V(x, y) = x^\alpha y^\beta x^\gamma y^\delta$ *is trivial in* H, *with* $\alpha, \gamma \in \{1, 2, \dots, p-1\}$ *if* $p < \infty$ *and* $\beta, \delta \in \{1, 2, \dots, q-1\}$ *if* $q < \infty$, *then one of the following holds:*

(1) $2 \in \{p, q\}$;
(2) $\alpha = \beta = \gamma = \delta = 1$;
(3) $\alpha = \gamma = p - 1$ (-1 *if* $p = \infty$) *and* $\beta = \delta = q - 1$ (-1 *if* $q = \infty$).

Proof. If $p = \infty$ then H is a free product of two cyclic groups, of order 2 and q, generated by xy and y respectively. The proof is an easy exercise in this case. A similar remark applies in the case where $q = \infty$. So we may restrict to the case where p, q are both finite.

Assume that $p \neq 2 \neq q$ and consider the elements

$$X = \begin{pmatrix} \exp(\pi i/p) & 0 \\ 1 & \exp(-\pi i/p) \end{pmatrix} \text{ and } Y = \begin{pmatrix} \exp(\pi i/q) & t \\ 0 & \exp(-\pi i/q) \end{pmatrix}$$

in $SL_2(\mathbb{C})$. By Lemma 2.4, X and Y have orders p and q respectively in $PSL_2(\mathbb{C})$. If we take $t = -2\cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right)$, then $Tr(XY) = 0$ and hence the map $x \mapsto X$, $y \mapsto Y$ extends to a faithful representation of H in $PSL_2(\mathbb{C})$.

Note that, for any integers α, β, X^α and Y^β have the forms

$$X^\alpha = \begin{pmatrix} \exp(\alpha\pi i/p) & 0 \\ \sin(\alpha\pi/p)/\sin(\pi/p) & \exp(-\alpha\pi i/p) \end{pmatrix},$$

$$Y^\beta = \begin{pmatrix} \exp(\beta\pi i/q) & t\sin(\beta\pi/q)/\sin(\pi/q) \\ 0 & \exp(-\beta\pi i/q) \end{pmatrix}$$

respectively. Suppose that $X^\alpha Y^\beta X^\gamma Y^\delta = 1$ in $PSL(2, \mathbb{C})$.
Then $X^\alpha Y^\beta = \pm Y^{-\delta} X^{-\gamma}$ in $SL(2, \mathbb{C})$. By comparing the left lower entries of both sides of this equation we have

$$\sin\frac{\alpha\pi}{p}\exp\frac{\beta\pi i}{q} = \pm\sin\frac{\gamma\pi}{p}\exp\frac{\delta\pi i}{q}.$$

Equating the arguments of each side gives $\beta = \delta$. A similar argument using the upper right entries of the matrices $X^\alpha Y^\beta$ and $\pm Y^{-\delta} X^{-\gamma}$ gives $\alpha = \gamma$. Hence $V = (X^\alpha Y^\beta)^2 = \pm I$. By comparing off-diagonal entries, we see that $X^\alpha \neq \pm Y^{-\beta}$, so $V \neq +I$. Hence $V = -I$, and so $Tr(X^\alpha Y^\beta) = 0$, i.e

$$2\cos\left(\frac{\alpha}{p} + \frac{\beta}{q}\right)\pi + t\frac{\sin\frac{\alpha\pi}{p}\sin\frac{\beta\pi}{q}}{\sin\frac{\pi}{p}\sin\frac{\pi}{q}} = 0.$$

Hence we obtain

$$\tan\frac{\alpha\pi}{p}\tan\frac{\beta\pi}{q} = \tan\frac{\pi}{p}\tan\frac{\pi}{q}$$

Since $p, q > 2$, the last equality holds if and only if either $\alpha = \beta = 1$ or $\alpha = p - 1$ and $\beta = q - 1$. □

We also use the following result from [5]. (See also [1, Remark 2.5] for the case when at least one of p, q is infinite.)

Theorem 2.6. [5] *Let*

$$H = \frac{\langle\ x\ \rangle * \langle\ y\ \rangle}{N(\ W(x, y)^r\ )}$$

be a generalised triangle group, where x has order $p \leq \infty$ and y has order $q \leq \infty$. Write $W(x, y) = \prod_{i=1}^k x^{\alpha_i} y^{\beta_i}$, where $k > 0$, $0 < \alpha_i < p$ if $p < \infty$, and $0 < \beta_i < q$ if $q < \infty$. Suppose that $V(x, y) = \prod_{i=1}^l x^{\gamma_i} y^{\delta_i}$ is trivial in H, with $l > 0$, $0 < \gamma_i < p$ if $p < \infty$, and $0 < \delta_i < q$ if $q < \infty$. Then $l \geq kr$.

3. Refinements and clique-pictures

We have $G = (G_1 * G_2)/N(R^n)$, where $R \in A * B = \langle a\rangle * \langle UbU^{-1}\rangle$. As in [1], we refer to the pushout diagram in Figure 1 as a *generalised triangle group description* of G. A *refinement* of this description is another generalised triangle group description involving cyclic conjugates A', B' of A and/or B, such that $A * B \subsetneq A' * B'$. Say $A' = \langle c\rangle$ and $B' = \langle VdV^{-1}\rangle$, where $c, d \in \{a, b\}$ and $\ell(V) < \ell(U)$. The description is *maximal* if it has no refinements.
For definitions of pictures and clique-pictures, see for example [1, 6]. Roughly speaking:

Definition 3.1. A *clique-picture* $\mathbf{P}$ on a compact oriented surface Σ over $G = (G_1 * G_2)/N(R^n)$ consists of

(1) a finite collection of pairwise disjoint compact sub-surfaces of Σ (called *cliques*);
(2) a collection of pairwise disjoint arcs in Σ, each of which which either is an embedded circle disjoint from all the cliques, or joins distinct points on the boundaries of cliques and/or of Σ;
(3) an assignment of a non-trivial *label* – an element of $G_1 \cup G_2$ – at each *corner* - that is, each component of the boundary of a clique or of Σ with the ends of arcs removed.

Reading clockwise around a boundary component of a clique or of Σ, the labels are required to alternate between elements of G_1 and G_2. The resulting cyclically reduced word in $G_1 * G_2$ is called a *boundary label* if we are reading around a component of $\partial\Sigma$, or a *clique-label* otherwise.
The labels associated to a clique of genus g are required to be (up to cyclic permutation) elements of $A * B$ which satisfy a genus g quadratic equation in H.
In a *region* of $\mathbf{P}$ (that is, a component of $\Sigma \setminus \{\text{cliques and arcs}\}$) of genus g, the labels are required all to belong to the same free factor G_1 or G_2. Moreover, the words obtained by reading anti-clockwise around each boundary component of the region are required to satisfy a genus g quadratic equation in that free factor.

A clique v in a clique-picture $\mathbf{P}$ is a *boundary clique* if it is connected to $\partial\Sigma$ by at least one arc. Otherwise it is an *interior clique.*
In practice, one reduces as quickly as possible to the simplest situation, where Σ is a disc or a sphere, and all cliques and all regions are discs (so $\mathbf{P}$ can be thought of as a connected planar graph). The solubility of the quadratic equations in the definition then reduces to the condition that the (unique) clique- or region- label be trivial in H or G_i respectively.
A clique-picture in which every clique is a disc with label a cyclic permutation of $R^{\pm n}$ is a *picture* in the usual sense.
Two arcs e, f of $\mathbf{P}$ are *parallel* if there is a sequence of arcs $e_0 = e, e_1, \dots, e_k = f$ such that each pair e_i, e_{i+1} cobounds (together with two corners) a disc-region of $\mathbf{P}$. Note that the two corner labels in this region are mutually inverse elements of G_1 or G_2, by the rules for a clique-picture. The relation of parallelism is an equivalence relation on the set of arcs of $\mathbf{P}$. A *zone* Z is an equivalence class of arcs under this relation, together with the disc-regions in the definition. If Z has size ω, then the $2(\omega - 1)$ corner labels of the disc-regions form two cyclic subwords s, t of clique-labels, such that $s \equiv t^{-1}$.
A clique-picture $\mathbf{P}$ satisfies the small-cancellation condition C(n) if each interior, simply-connected clique in $\mathbf{P}$ has at least n incident zones. If $\mathbf{P}$ is a clique-picture satisfying C(6) in which every clique is simply-connected,

then we may apply the methods of small-cancellation theory to deduce a number of interesting properties of $\mathbf{P}$.

Suppose that a zone Z of size $\omega \geq 1$ joins two cliques u, v (or a clique u to itself). Let D be a small regular neighbourhood of Z. Then D is a topological disc meeting each of u and v in a small disc at its boundary (resp. meeting u in two disjoint discs at ∂u). Replacing u, v by a single clique $u \cup v \cup D$ (resp. u by $u \cup D$) and deleting the corner-labels from $\mathrm{Int}(D)$, we may or may not obtain a new clique-picture $\mathbf{P}'$, depending on whether or not the rules are satisfied. (Specifically, this depends on whether or not the new clique-labels are also (up to cyclic permutation) words in $A * B$. If so, then the solubility of the new quadratic equations for $\mathbf{P}'$ follows from that for $\mathbf{P}$.) If $\mathbf{P}'$ is indeed a clique-picture, then we say that $\mathbf{P}'$ is obtained from $\mathbf{P}$ by *amalgamation of cliques*. Since amalgamation of cliques reduces the number of arcs in a clique-picture, it can be performed only finitely many times. A clique-picture for which amalgamation of cliques is impossible is said to be *reduced*.

4. Theorem B

In this section we prove Theorem B. We assume throughout that Figure 1 represents a maximal generalised triangle group description of G, where $A = \langle a \rangle$ and $B = \langle UbU^{-1} \rangle$. We also assume that v is a simply-connected interior clique in a reduced clique-picture over G, with 5 or fewer incident zones. We aim to derive a contradiction and hence prove the theorem.

Let $Z_1, \dots, Z_q$ (with $q \leq 5$) be the zones incident at v. Let u_i be the clique that is joined to v by Z_i, and let s_i, t_i be the corresponding cyclic subwords of the labels of v and u_i, so that $s_i \equiv t_i^{-1}$.

We write

$$\mathrm{label}(v) = z_0 z_1 \cdots z_{ml-1} = \prod_{k=1}^{m} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}].$$

Let $l := \ell(aUbU^{-1}) = 2\ell(U) + 2$. Then the letters z_j of $\mathrm{label}(v)$ with $j \equiv 0 \bmod l/2$ are called *special*. Note that special letters are always powers of a or of b. Precisely q letters of $\mathrm{label}(v)$ are not in any of the subwords s_i: we let $s(1), \dots, s(q)$ be the corresponding subscripts, so that

$$z_{s(1)} s_1 \cdots z_{s(q)} s_q$$

is a cylcic permutation of $\mathrm{label}(v)$.

Lemma 4.1. *For any zone Z_i, if $s(i)+j \equiv t(i)+r \bmod l$ where $1 \leq j, r < \omega_i$, then s_i has an element of order 2 in G_1 or G_2.*

Proof. Suppose by contradiction that $s(i) + j \equiv t(i) + r \bmod l$ where $1 \leq j, r < \omega_i$. Recall that s_i is the cyclic subword $s_i = z_{s(i)+1} \cdots z_{s(i+1)-1}$ of

$$\mathrm{label}(v) = z_0 z_1 \cdots z_{ml-1} = \prod_{k=1}^{m} [a^{\alpha(k)} U b^{\beta(k)} U^{-1}].$$

Similarly, we express t_i as the cyclic subword $t_i = y_{t(i)+1} \cdots y_{t(i+1)-1}$ of

$$\text{label}(u_i) = y_0 y_1 \cdots y_{dl-1} = \prod_{k=1}^{d} [a^{\gamma(k)} U b^{\delta(k)} U^{-1}].$$

By hypothesis, $y_{t(i)+r}$ is the i'th letter of t_i, which is inverse to some letter of s_i. That is, $y_{t(i)+r} = z_{s(i)+r'}^{-1}$ for some r' with $1 \le r' < \omega_i$. In particular $s(i) + j \equiv t(i) + r \equiv s(i) + r' \bmod 2$, since $y_{t(i)+r}$ and $z_{s(i)+r'}$ belong to the same free factor. Thus $j + r'$ is even. Moreover, $z_{s(i)+(j+r')/2} = y_{t(i)+(j+r')/2}^{-1}$. If $z_{s(i)+(j+r')/2}$ is a special letter, then so is $y_{t(i)+(j+r')/2}^{-1}$. But in this case an amalgamation is possible, contrary to the hypothesis. Otherwise $z_{s(i)+(j+r')/2} = y_{t(i)+(j+r')/2}$, so $z_{s(i)+(j+r')/2}$ has order 2 in G_1 or G_2. □

Lemma 4.2. *If no letter of $R(a, UbU^{-1})$ has order 2, then there is no zone Z_i with $\omega_i > l/2$.*

Proof. Suppose that $\omega_i > l/2$ for some zone Z_i. As in the proof of Lemma 4.1, we have $s_i = z_{s(i)+1} \cdots z_{s(i+1)-1}$ and $t_i = y_{t(i)+1} \cdots y_{t(i+1)-1}$. By Lemma 4.1 we may assume that $s(i)+j \not\equiv t(i)+r \bmod l$ for all $j, r \in \{1, 2, \ldots, \omega_i - 1\}$, so in particular $\omega_i \le l/2 + 1$. Moreover, if $\omega_i = l/2 + 1$ then we must have $t(i) + 1 \equiv s(i+1) \bmod l$. But since $z_{s(i+1)-1} = y_{t(i)+1}^{-1}$ belongs to the same free factor as $z_{t(i)+1} = z_{s(i+1)}$, this is a contradiction. Hence $\omega_i \le l/2$. □

Corollary 4.3. *If there is an interior clique v in a reduced clique-picture over G with at most 5 incident zones, then $R = aUbU^{-1}$ (up to cyclic permutation) and* $\text{label}(v) = (aUbU^{-1})^{\pm 2}$ *(up to cyclic permutation).*

Proof. The clique v has at most 5 incident zones, so by Lemma 4.2 its label has length at most $5l/2$. But this length is a multiple of l, so at most $2l$. By Theorem 2.6 the length must be exactly $2l$, so $\text{label}(v)$ has the form $a^{\alpha} U b^{\beta} U^{-1} a^{\gamma} U b^{\delta} U^{-1}$ for some integers $\alpha, \beta, \gamma, \delta$. Proposition 2.5 gives the required values for $\alpha, \beta, \gamma, \delta$. □

Corollary 4.3 enables us to strengthen Lemma 4.2 as follows.

Lemma 4.4. *Suppose that $\mathcal{P}$ is a reduced clique picture over G, containing an interior clique with at most 5 incident zones. If no letter of $R(a, UbU^{-1})$ has order 2, then there is no zone Z_i with $\omega_i \ge l/2$.*

Proof. By Lemma 4.2 we are reduced to the case where $\omega_i = l/2$. By Lemma 4.1 , we must have $t(i) \in \{s(i+1)-1, s(i+1), s(i+1)+1\} \bmod l$. The first possibility leads to a contradiction as in the proof of Lemma 4.2. The third possibility also leads to a contradiction for similar reasons, since it implies that $t(i+1) \equiv s(i) + 1 \bmod l$. Hence we may assume that $t(i) \equiv s(i+1) \bmod l$ and hence that $s(i) \equiv s(i+1) + l/2 \equiv t(i) + l/2 \equiv t(i+1) \bmod l$.
If $z_{s(i)}$ is special, then so is $y_{t(i+1)}$, and we may amalgamate cliques, contrary to the hypothesis. Hence $z_{s(i)}$ is not special. In other words $s(i) \not\equiv 0 \bmod l/2$. Thus s_i contains precisely one special letter – say a without loss

of generality. Hence also t contains precisely one special letter, which is necessarily a power of b. Thus $aUb^{\psi}U^{-1}$ is a proper cyclic permutation of $z_{s(i)}s_iz_{s(i+1)}t_i \equiv z_{s(i)}s_iz_{s(i+1)}s_i^{-1}$ for some ψ. Applying Lemma 2.3, we see that

$$aUb^{\psi}U^{-1} = \prod_{k=1}^{s}[a^{\alpha(k)}Vc^{\beta(k)}V^{-1}]$$

for some word V, some $\alpha(k), \beta(k) = \pm 1$ and some $c \in \{b^{\psi}, z_{s(i)}, z_{s(i+1)}\}$, where $s > 1$.
Moreover, from the proof of Lemma 2.3 in [1] we see that we may take $c = b^{\psi}$ if s is odd, while if s is even then $a = b^{\psi}$. In either case, $\langle a\rangle * \langle UbU^{-1}\rangle$ is a proper subgroup of $\langle a\rangle * \langle VbV^{-1}\rangle$ or $\langle a\rangle * \langle VcV^{-1}\rangle$ with $\ell(V) < \ell(U)$, giving a refinement of the generalised triangle group description of G. This contradiction completes the proof.

□

Proposition 4.5. *Let X be an initial (resp. terminal) segment of U of length $\ell(X) \geq \frac{l}{4}$ which is also a terminal (resp. initial) segment of $s_i^{\pm 1}$ for some i. Then one of the following holds:*

(1) there is an initial (resp. terminal) segment X^+ of U of length $\ell(X)+\gamma$ and period γ, for some $\gamma > 0$; or
(2) U has period $\gamma \leq 2(\ell(U) - \ell(X))$.
Furthermore if $\gamma = 2(\ell(U) - \ell(X))$, then $s(i+1) \equiv t(i)$ (respectively $t(i) + \omega_i$) mod l, and U has a terminal (resp. initial) segment of the form $xw^{-1}z_{s(i+1)}w$ (resp. $wz_{s(i)}w^{-1}x$) for some letter x and some word w with $\ell(w) = \gamma/2 - 1$.

Proof. By symmetry it suffices to prove the case where X is an initial segment of U and a terminal segment of s_i. Then $s_i = YaX$ for some terminal segment Y of U^{-1}. Write $U \equiv Xz_{s(i+1)}V$ for some terminal segment V of U. Then $t(i) \in \{s(i+1), s(i+1)+1, \ldots, s(i) - \omega_i\}$ mod l, by Lemma 4.1. Suppose first that $t(i) \in \{\frac{l}{2}, \frac{l}{2}+1, \ldots, s(i) - \omega_i\}$ mod l. Then t_i is a subword of U^{-1}. Thus $s_i \equiv t_i^{-1}$ is identically equal to the subword W of U of length $\omega_i - 1$ which ends in the $(l - t(i) - 1)$'th letter $z_{l-t(i)-1}$ of U. In particular, the terminal segment of W of length $\ell(X)$ is identically equal to X. Let $\gamma = l - t(i) - 1 - \ell(X)$. It follows that the initial segment X^+ of U of length $\ell(X) + \gamma$ has period γ, provided of course that $\gamma > 0$. But $\gamma = 0$ only if $t(i) = l - 1 - \ell(X) \equiv l - s(i+1)$ mod l, in which case $s(i) \equiv 0$ mod l, $s_i \equiv X$, and v amalgamates with its neighbour along zone Z_i, contradicting our underlying assumption that the clique-picture P is reduced.

Suppose next that $t(i) \in \{s(i+1), s(i+1)+1), \ldots, \frac{l}{2} - 1\}$ mod l. By Lemma 4.1, t_i is a subword of $Vb^{\psi}U^{-1}$ for some ψ. Therefore t_i has the form $w_1b^{\psi}w_2^{-1}$ for some terminal segments w_1 and w_2 of U with $\ell(w_1) \leq \ell(V) = \ell(U) - (\ell(X)+1) < \frac{l}{4} - 1$. Denote by S the terminal segment of U

of length

$$\begin{aligned}\ell(S) &= \ell(X) - \ell(w_1) - 1\\ &= \ell(t_i) - \ell(w_1) - \ell(Y) - 2\\ &= \ell(w_2) - \ell(Y) - 1.\end{aligned}$$

Then $w_2 \equiv YaS$ and $X \equiv Sb^{-\psi}w_1^{-1}$ and so S is identically equal to an initial segment of U. Thus U has period

$$\begin{aligned}\gamma &= \ell(U) - \ell(S)\\ &= \ell(U) - \ell(X) + \ell(w_1) + 1\\ &\leq 2(\ell(U) - \ell(X)).\end{aligned}$$

Finally if $\gamma = 2(\ell(U) - \ell(X))$, then $1 + \ell(X) + \ell(w_1) = \ell(U)$. Thus $V \equiv w_1$ and so t_i is an initial segment of $Vb^{\psi}U^{-1}$. Equivalently, t_i is an initial segment of $Vb^{\psi}U = w_2b^{\psi}U$, so $t(i) \equiv s(i+1) \bmod l$ as claimed.
Moreover $U \equiv Xz_{s(i+1)}V \equiv Sb^{-\psi}w_1^{-1}z_{s(i+1)}w_1$ and hence U has a terminal segment of the form claimed, taking $w = w_1$ and $x = b^{-\psi}$.

□

Proof of Theorem B. Assume by way of contradiction that v is an interior clique in a reduced clique-picture $\mathcal{P}$ over G, corresponding to a maximal generalised triangle group description of G, and that v has at most 5 incident zones. Then by Lemma 4.3 there are precisely 5 incident zones (say $Z_1, \dots, Z_5$ of sizes $\omega_1, \dots, \omega_5$ respectively) at v, and up to cyclic permutation and inversion we have

$$\mathrm{label}(v) = z_0z_1\cdots z_{2l-1} \equiv (aUbU^{-1})^2.$$

There is also a cyclic permutation of label(v) of the form

$$z_{s(1)}s_1z_{s(2)}s_2z_{s(3)}s_3z_{s(4)}s_4z_{s(5)}s_5,$$

where each s_i is the word of length $\omega_i - 1$ carried by the zone Z_i. Recall that Z_i identifies s_i with t_i^{-1} for some cyclic subword t_i of the label of a clique neighbouring v.
It follows easily from Lemma 4.4 that at least three consecutive s_i (with respect to their cyclic order) contain a special letter of label(v). Without loss of generality, we can assume that the four special letters of label(v), namely $z_0 = a$, $z_{l/2} = b$, $z_l = a$, $z_{3l/2} = b$ lie in s_5, s_1, s_2 and $s_3z_{s(3)}$ respectively.
We can write $UbU^{-1} \equiv X_5z_{s(1)}s_1z_{s(2)}Y_2$, where X_5 is a terminal segment of s_5 and Y_2 is an initial segment of s_2. Note also that X_5 and Y_2^{-1} are initial segments of U. Define M to be the longer of X_5, Y_2^{-1}. Since s_1 has length at most $l/2 - 2$, it follows that M has length at least $l/4$.
By Proposition 4.5 one of the following holds. Either (i) there is a proper initial segment M^+ of U, of length $\ell(M) + \mu$ and period μ, for some $\mu > 0$, or (ii) U has period μ for some μ with $\ell(U) - \ell(M) \leq \mu \leq 2(\ell(U) - \ell(M))$. Moreover, if (ii) holds and U has a period $\gamma < 2(\ell(U) - \ell(M))$, then there

is a segment of s_1 of the form VbV^{-1}, where V is the terminal segment of U of length $\gamma/2$. But then Lemma 2.2 implies that U contains a letter of order 2, contrary to Hypothesis B. Hence if (ii) holds we may assume that $\mu = 2(\ell(U) - \ell(M))$, and that U has no period less than μ.
Similarly, we can write $U^{-1}aU \equiv X_1 z_{s(2)} s_2 z_{s(3)} Y_3$ for some terminal segment X_1 of s_1 and initial segment Y_3 of s_3, where X_1^{-1} and Y_3 are both terminal segments of U. We define N to be the longest of X_1^{-1}, Y_3, note that $\ell(N) \geq l/4$, and apply Proposition 4.5 to N.
Either (i) there is a proper terminal segment N^+ of U, of length $\ell(N) + \nu$ and period ν, for some $\nu > 0$, or (ii) U has period ν, for some ν with $\ell(U) - \ell(N) \leq \nu \leq 2(\ell(U) - \ell(N))$. Moreover, in case (ii) we may assume that $\nu = 2(\ell(U) - \ell(N))$, and that U has no period less than ν.

We now split the remainder of the proof into four cases.

Case 1 $\mu \leq \ell(U) - \ell(M)$ and $\nu \leq \ell(U) - \ell(N)$.
The two segments M^+ and N^+ of U intersect in a segment S with $\ell(S) = \ell(M^+) + \ell(N^+) - \ell(U) > \mu + \nu$. Hence by Theorem 2.1 U is periodic with period $\rho := gcd(\mu, \nu) \leq \mu < 2(\ell(U) - \ell(M))$. It follows that s_1 contains a subword the form wx_1w^{-1} with $\ell(w) = \frac{\rho}{2}$. By Lemma 2.2 there is a letter of order 2 in U, contrary to Hypothesis B.

Case 2 $\mu = 2(\ell(U) - \ell(M))$ and $\nu \leq \ell(U) - \ell(N)$.
Then by Proposition 4.5, U has a terminal segment of the form $u_1wu_2w^{-1}$ for some letters u_1, u_2 and some word w with $\ell(w) = \frac{\mu}{2} - 1$.
If $\nu + 2 \leq \mu$, then N^+ has a subword of the form $\hat{w}u_2\hat{w}^{-1}$ where $\hat{w}$ is the terminal segment of w of length $\nu/2$. This contradicts Lemma 2.2, since U has no letter of order 2.
Finally if $\nu + 2 > \mu$, then

$$\nu + \mu \leq 2\nu + 1 \leq 1 + \ell(U) - \ell(N) + \nu \leq l/4 + \nu \leq \ell(N) + \nu = \ell(N^+).$$

It follows from Theorem 2.1 that U has period $\lambda = \gcd(\mu, \nu)$. Hence we can apply Lemma 2.2 to s_2 since

$$\frac{\lambda}{2} \leq \lambda - 1 \leq \nu - 1 \leq \ell(U) - \ell(N).$$

Again, there is a letter of order 2 in U, contrary to Hypothesis B.

Case 3 $\nu = 2(\ell(U) - \ell(N))$ and $\mu \leq \ell(U) - \ell(M)$.
The proof in this case is similar to that in Case 2.

Case 4 $\mu = 2(\ell(U) - \ell(M))$ and $\nu = 2(\ell(U) - \ell(N))$.
Then U has period μ and ν, and each of these is the shortest possible period of U. In particular $\mu = \nu$.

Hence $\ell(X_5) \leq \ell(M) = \ell(U) - \mu/2$. Also $s_1 \equiv Y_1 z_{l/2} X_1$ for some Y_1 such that $U \equiv X_5 z_{s(1)} X_1$. Hence

$$\ell(Y_1) = \ell(U) - 1 - \ell(X_5) \geq \mu/2 - 1$$

and

$$\ell(X_1) = \ell(s_1) - 1 - \ell(Y_1) < \ell(U) - 1 - \ell(Y_1) \leq \ell(U) - \mu/2 = \ell(N).$$

It follows that $N = Y_3$. By a similar argument, $M = X_5$.

If $s(4) = 3l/2$,in other words if there is no special letter in s_3, then $Y_3 = s_3$, and $X_4 := s_4$ is an initial segment of U^{-1}. In this case, we may use symmetry to repeat the entire argument up to this point, replacing N by the longer N' of Y_1 and X_4. If the pair M, N' falls into Case 1,2 or 3 of the proof, then we obtain a contradiction as before. In Case 4 we deduce as above that $N' = X_4$ and that $\ell(X_5) < \ell(N') = \ell(U) - \mu/2 = \ell(X_5)$, which is again a contradiction.

Hence we may assume that $z_{3l/2}$ is a letter of s_3. Since U has period $\mu = 2(\ell(U) - \ell(X_5)) = 2(\ell(U) - \ell(Y_3))$, it follows from Proposition 4.5 that U has an initial segment of the form $w_1 z_{s(3)} w_1^{-1} x_1$ and a terminal segment of the form $x_2 w_2 z_{s(1)}^{-1} w_2^{-1}$ for some letters $x_1 = z_\mu, x_2 = z_{\frac{l}{2}-\mu}$, and words w_1, w_2 satisfying $\ell(w_1) = \ell(w_2) = \frac{\mu}{2} - 1$.

We also have from Proposition 4.5 that $t(3) + \omega_3 \equiv s(3) \equiv \frac{\mu}{2} \mod l$. The $\frac{l-\mu}{2}$-th letter of s_3 is $z_{3l/2} = b$.

Now the zone Z_3 connects v to some clique u_3 with label of the form

$$\mathrm{label}(u_3) = y_0 y_1 \cdots y_{rl-1} = \prod_{k=1}^{r} [a^{\gamma(k)} U b^{\delta(k)} U^{-1}].$$

Thus Z_3 identifies $z_{l/2}$ with y_d^{-1} for some

$$d \equiv t(3) + \omega_3 - \left(\frac{l-\mu}{2}\right) \equiv \frac{\mu}{2} - \left(\frac{l-\mu}{2}\right) \equiv \frac{l}{2} + \mu \mod l.$$

Since $0 < \mu < l/2$, $d \not\equiv 0 \mod l/2$, so y_d is not a special letter of label(u_3). Hence

$$b = z_{l/2} = y_d^{-1} = z_d^{-1} = z_{(l/2)+\mu}^{-1} = z_{(l/2)-\mu} = x_2.$$

A similar argument shows that $x_1 = a$.

Note also that

$$w_1 z_{s(3)} w_1^{-1} a = z_1 \cdots z_\mu$$

is a cyclic conjugate of

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = z_{\frac{l}{2}-\mu} \cdots z_{\frac{l}{2}-1},$$

by the periodicity of U.

Now apply Lemma 2.3 to this pair of cyclically conjugate words to obtain

$$b w_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V x^{\alpha(t)} V^{-1} \tag{4.6}$$

for some letter x, some word V, some integer s and some $\alpha(t), \beta(t) \in \{\pm 1\}$. The integer s in (4.6) is defined to be k/m, where $k = \mu/2$ and $m = \gcd(j, k)$. (See the proof of Lemma 2.3 from [1].) Here j in turn is defined as the number of places by which one has to cyclically permute $bw_2 z_{s(1)}^{-1} w_2^{-1}$ to obtain $aw_1 z_{s(3)} w_1^{-1}$ – in other words

$$z_\mu z_1 \cdots z_{\mu-1} \equiv z_{\frac{l}{2}-\mu+j} \cdots z_{\frac{l}{2}-1} z_{\frac{l}{2}-\mu} \cdots z_{\frac{l}{2}-\mu+j-1}.$$

Suppose first that s is even in (4.6). Then Lemma 2.3 gives that $b = z_{s(1)}$, $a = z_{s(3)}^{-1}$, and $x = a$ in (4.6). Thus U has a terminal segment

$$bw_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V a^{\alpha(t)} V^{-1}$$

Similarly U has an initial segment of the form

$$w_1 z_{s(3)} w_1^{-1} a = \prod_{t=1}^{s} V^{-1} b^{\delta(t)} V a^{\gamma(t)}$$

Putting these together, using the periodicity of U, we obtain

$$U = V^{-1} \prod_{t=1}^{r} b^{\zeta(t)} V a^{\eta(t)} V^{-1}$$

for some integer r and some $\zeta(t), \eta(t) \in \{\pm 1\}$.
Thus $UbU^{-1} \in \langle a, V^{-1}bV \rangle$ and we have a refinement of our generalised triangle group description of G, contrary to our underlying hypotheses.

Next suppose that s is odd in (4.6). Then $\{b, z_{s(1)}^{-1}\} = \{a, z_{s(3)}\}$ by Lemma 2.3. There is therefore an expression

$$bw_2 z_{s(1)}^{-1} w_2^{-1} = \prod_{t=1}^{s} b^{\beta(t)} V z_{s(1)}^{\alpha(t)} V^{-1},$$

and an analogous expression

$$w_1 z_{s(3)} w_1^{-1} a = \prod_{t=1}^{s} V^{-1} z_{s(3)}^{\delta(t)} V a^{\gamma(t)}.$$

Again we can fit these together using the periodicity of U to get an expression for U.
If $b = z_{s(3)}$ and $a = z_{s(1)}^{-1}$ then again this has the form

$$U = V^{-1} \prod_{t=1}^{r} b^{\zeta(t)} V a^{\eta(t)} V^{-1}$$

which leads to a refinement, contrary to the hypothesis.
If on the other hand $b = a$ and $z_{s(3)} = z_{s(1)}^{-1}$ then we obtain an expression of the form

$$U = V z_{s(3)}^{\eta(0)} V^{-1} \prod_{t=1}^{r} a^{\zeta(t)} V z_{s(3)}^{\eta(t)} V^{-1} \in \langle a, V z_{s(3)} V^{-1} \rangle$$

As before, this yields a refinement, contrary to the hypothesis.

This completes the proof.

5. Applications

The proofs of Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 largely follow those in [1]. We give brief outlines below, going into more detail only where these details differ from [1]. Otherwise the reader is referred to [1] for the full details.
As in [1] we suppose that the triangle group description for G is maximal. By Theorem B, a reduced clique-picture over G satisfies the $C(6)$ property.

Proof of Theorem 1.1. Assuming the result is false, we choose a picture P on D^2 over G with fewest possible cliques, whose boundary label either belongs to $G_1 \cup G_2$ or is a word in $\langle a, UbU^{-1}\rangle$. Any non-simply connected clique has a boundary labelled by a word in $\langle a, UbU^{-1}\rangle$ which bounds a subpicture with fewer cliques, which can be replaced by a single-clique picture that amalgamates with our given clique. Hence we may assume that all cliques in P are simply connected, and obtain a clique-picture $\mathcal{P}$ with the same boundary label as our original picture. Our clique-picture may not be minimal, but can be replaced by a minimal clique-picture with the same boundary label, so we may assume it is minimal. The C(6) property ensures that the boundary label has length greater than 1, so does not belong to $G_1 \cup G_2$. Hence we assume that the boundary label is a word in $\langle a, UbU^{-1}\rangle$.

At this point our proof diverges slightly from that in [1]. The C(6) property ensures that some boundary clique v in $\mathcal{P}$ is joined to the boundary by a single zone, and other cliques by at most 3 zones. Each zone has size strictly less than $l/2$, while the clique-label of v has length at least $2l$ by the Spelling Theorem 2.6. Hence the zone joining v to the boundary has size greater than $l/2$, which allows us to amalgamate v with the boundary. In other words, removing v from $\mathcal{P}$ gives a counter-example to the theorem with fewer cliques than $\mathcal{P}$, contradicting our assumptions. □

Proof of Theorem 1.2. Suppose that W is a word in $G_1 * G_2$ that represents the identity element of G, and let $\mathcal{P}$ be a minimal clique-picture with boundary label W. Following [1, Lemma 6.1] we find an upper bound on the length of any clique-label of $\mathcal{P}$ as a linear function of $\ell(W)$. The difference here is that the sizes of interior zones are bounded above by $l/2$ rather than l, while the sizes of clique-labels are bounded below by $2l$ rather than $4l$. The resulting upper bound for lengths of clique-labels is $2l$ for interior cliques, and $3l$ for boundary cliques, giving an overall upper bound of $3l$.
Combining the linear upper bound for clique-label lengths with the quadratic upper bound [7, V, Theorem 3.1] for the number of cliques, we see that only finitely many finite graphs can arise as clique-pictures.
If a and b both have finite order, then each of these graphs in turn can be labelled in only finitely many ways as a clique-picture. So the word problem for G reduces to a finite search, modulo the word problems for G_1, G_2, H.

If a, b both have infinite order, then H is a one-relator group with torsion, and hence hyperbolic. Each clique can then be replaced by a picture over H whose number of vertices is bounded above by a linear function of $3\ell(W)$. Hence G satisfies a quadratic isoperimetric inequality, and hence has solvable word problem.

□

Proof of Theorem 1.3. The proof follows exactly that in [1]. Theorem 1.2 reduces us to the case where b (say) has finite order and a has infinite order. The argument in the proof of Theorem 1.2 reduces us to considering a single graph as a potential clique-picture, where the labels are all fixed except for some which are variable powers of a. Variable labels in regions of degree less than 4 can be identified using the membership problem in $G_1 * G_2$. A boundary vertex of low degree (whose existence is guaranteed by the C(6) property) is shown to have at most one variable label that cannot be identified using the above rule. This variable label can also be identified using the membership problem in H. The proof then follows by applying an inductive hypothesis to the graph with this boundary vertex (and incident edges) removed. □

Proof of Theorem 1.4. Given a clique-picture with boundary label W_1, say, we can obtain a new clique-picture with a single boundary clique and boundary-label W_2^{-1}, then amalgamate cliques if necessary to make it reduced. By [7, V, Corollary 3.4] and the C(6) property this new clique-picture consists of a single clique, and the result follows from the Spelling Theorem 2.6. □

Proof of Theorem 1.5. As in [1], we construct a pushout of connected CW-complexes realising Figure 1 on fundamental groups, in which the complexes representing $A * B$, H and $G_1 * G_2$ are Eilenberg-Maclane spaces. Using mapping cylinder constructions if necessary, we assume that all the maps in our pushout are injective, so that the three Eilenberg-Maclane spaces are subcomplexes X_0, X_1, X_2 of the complex X with $\pi_1(X) = G$. Using the C(6) property for reduced spherical clique-pictures, we deduce that $\pi_2(X)$ is generated as a $\mathbb{Z}G$-module by the images of $\pi_2(X_0)$, $\pi_2(X_1)$ and $\pi_2(X_2)$, which are all zero by construction. Hence $\pi_2(X) = 0$. It then follows from [4, Theorem 4.3] and Theorem 1.1 that X is also an Eilenberg-Maclane complex, and so the pushout of Figure 1 is geometrically Mayer-Vietoris, as claimed. □

References

[1] I. Chinyere and J. Howie, On one-relator products induced by generalised triangle groups I, *Comm. Algebra*, to appear.

[2] M. Edjvet and J. Howie, On the abstract groups $(3, n, p; 2)$, *J. London Math. Soc.* (2) **53** (1996) 271–288.

[3] N. J. Fine and H. S. Wilf, Uniqueness theorem for periodic functions, *Proc. Amer. Math. Soc.* **16** (1965) 109–114.

[4] J. Howie, The quotient of a free product of groups by a single high-powered relator. III. The word problem, *Proc. London Math. Soc.* (3) **62** (1991) 590–606.

[5] J. Howie and N. Kopteva, The Tits alternative for generalised tetrahedron groups, *J. Group Theory* **9** (2006) 173–189.

[6] J. Howie and R. Shwartz, One-relator products induced from generalised triangle groups, *Comm. Algebra* **32** (2004) 2505–2526.

[7] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, (1977).

Ihechukwu Chinyere, Department of Mathematics, Michael Okpara University of Agriculture, Umudike, P.M.B 7267 Umuahia, Abia State, Nigeria.
E-mail address: ihechukwu@aims.ac.za

James Howie, Department of Mathematics and Maxwell Institute for Mathematical Sciences, Heriot–Watt University, Edinburgh EH14 4AS
E-mail address: J.Howie@hw.ac.uk