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ON ONE-RELATOR PRODUCTS INDUCED BY GENERALISED TRIANGLE GROUPS I

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ABSTRACT. In this paper and a sequel, we study a group which is the quotient of a free product of groups by the normal closure of a single word that is contained in a subgroup which has the form of a free product of two cyclic groups. We use known properties of generalised triangle groups, together with detailed analysis of pictures and of words in free monoids, to prove a number of results such as a Freiheitssatz and the existence of Mayer-Vietoris sequences for such groups under suitable hypotheses. The results generalise those in an earlier article of the second author and Shwartz.

1. INTRODUCTION

A *one-relator product* G of groups G_1 and G_2 is the quotient of the free product $G_1 * G_2$ by the normal closure of a single element W which is cyclically reduced of free product length at least two. Such groups are natural generalisations of one-relator groups, and have been the subject of several articles generalising results from the rich theory of one-relator groups. In this paper we consider the case where $W = R^n$ is a proper power and R is contained in the free product $A * B$ of cyclic subgroups A, B of conjugates of G_1 or G_2. In other words $R = x_1 U y_1 U^{-1} x_2 U y_2 U^{-1} \cdots x_m U y_m U^{-1}$, where the x_i, y_i are elements of $G_1 \cup G_2$ such that $x_1, \dots, x_m$ generate a cyclic subgroup $A := \langle a \rangle$ of G_1 or G_2, and $y_1, \dots, y_m$ generate a cyclic subgroup $U^{-1} B U := \langle b \rangle$ of G_1 or G_2. Note that we will always assume that the first letter of U does not belong to the same free factor as a (and the last letter of U does not belong to the same free factor as b), so that reduced words in $A * B$ are reduced as written when regarded as words in $G_1 * G_2$. As in [9], we also require the technical condition that (a, b) be *admissible*: whenever both a and b belong to same factor, say G_1, then either the subgroup of G_1 generated by $\{a, b\}$ is cyclic or $\langle a \rangle \cap \langle b \rangle = 1$.

A *generalised triangle group* is a one-relator product of two cyclic groups in which the relator is a proper power. Such groups have been widely studied, but in the usual definition the two cyclic groups in the free product are assumed to be finite. For the purposes of this paper however, we will allow the cyclic groups to be infinite. If a one-relator product is in the form of G

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described above, then the relator $W = R^n$ is contained in $A * B$ with A, B cyclic. We can form the generalised triangle group $H := (A * B)/N(R^n)$. In this case we say that G is *induced* by the generalised triangle group H. Note that G can be realised as a push-out of groups as shown in Figure 1 below.

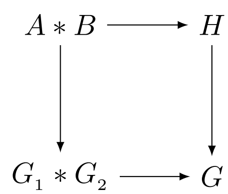

FIGURE 1. *Push-out diagram.*

We refer to this pushout representation of G as a *generalised triangle group description* of G, and we require it to be *maximal* in the sense that the word U is shortest among all such decompositions of R. (See Section 3 below for more details.)

The concept of one-relator product induced by a generalised triangle group was introduced in [9], where a number of results were proved under the hypotheses that $n \geq 3$ and that the pair (a, b) is admissible. In the present paper we prove similar results under a hypothesis that is in general weaker than those assumed in [9].

Hypothesis A. The generalised triangle description in Figure 1 is maximal, $n \geq 2$, R is a cyclically reduced word of free-product length at least 4 as a word in the free product $\langle a \rangle * \langle UbU^{-1} \rangle$, and the pair (a, b) is admissible.

In a subsequent paper [1], we will prove similar results under a different weakening of the hypotheses of [9], namely:

Hypothesis B. The generalised triangle description in Figure 1 is maximal, $n \geq 2$, R is a cyclically reduced word of free-product length at least 2 as a word in the free product $\langle a \rangle * \langle UbU^{-1} \rangle$, and no letter of R (that is, no letter of U and no power of a or b that occurs in R) has order 2 in G_1 or G_2.

Under either of the above hypotheses, we will prove the following:

Theorem 1.1. *Let H be the generalised triangle group inducing the one relator product G. Then the maps $G_1 \to G$, $G_2 \to G$ and $H \to G$ are all injective.*

Theorem 1.2. *If a and b either both have finite order or both have infinite order, and the word problem is solvable for H, G_1 and G_2, then the word problem is solvable for G.*

The conditions on the orders of a, b in Theorem 1.2 seem rather artificial, and may be unnecessary. But in our current proof they are necessary. However, they may be removed under slightly stronger hypotheses on G_1, G_2 and H:

Theorem 1.3. *Assume that the membership problems for* $\langle a\rangle$ *and* $\langle b\rangle$ *in* $G_1 * G_2$ *and for* $\langle x\rangle$ *and* $\langle y\rangle$ *in* H *are solvable. Then the word problem for* G *is solvable.*

Theorem 1.4. *If a cyclic permutation of* R^n *has the form* $W_1 W_2$ *with* $0 < \ell(W_1) < \ell(R^n)$ *as words in* $G_1 \cup G_2$*, then* $W_1 \neq 1 \neq W_2$ *as elements of* G*. In particular* R *has order* n *in* G*.*

Theorem 1.5. *The pushout of groups in Figure 1 is* geometrically Mayer-Vietoris *in the sense of* [3]*. In particular it gives rise to Mayer-Vietoris sequences*

$$\cdots \to H_{k+1}(G,M) \to H_k(A*B,M) \to$$
$$H_k(G_1 * G_2, M) \oplus H_k(H,M) \to H_k(G,M) \to \cdots$$

and

$$\cdots \to H^k(G,M) \to H^k(G_1 * G_2, M) \oplus H^k(H,M)$$
$$\to H^k(A*B,M) \to H^{k+1}(G,M) \to \cdots$$

for any $\mathbb{Z}G$*-module* M*.*

The first part of Theorem 1.1 is a generalisation of Magnus' Freiheitssatz for one-relator groups [11]. There are many generalisations of the Freiheitssatz to one-relator products of a special nature. Most relevant to the present paper, it was proved for arbitrary one-relator products in [5, 6], provided $n \geq 4$. Theorems 1.2 and 1.3 were proved by Magnus [12] for one-relator groups. Known versions for one-relator products include the case when $n \geq 4$ [7]. Theorem 1.4 is a version of a result of Weinbaum [13] for one-relator groups. Theorems 1.1, 1.2, 1.4 and 1.5 were proved in [9] under the hypotheses that $n \geq 3$ and that the pair (a,b) is admissible.

The above theorems will proven using the notion of *clique-picture* from [9] and Theorem A below.

Theorem A. If Hypothesis A above holds, then a minimal clique-picture over G satisfies the small-cancellation condition $C(6)$.

These results naturally lead one to speculate that similar results apply for any relator R^n with $n \geq 2$, without any of the restrictions in Hypotheses A or B. A less ambitious speculation might be that the condition of admissibility could be dropped from Hypothesis A. We are not aware of any counter-examples to such speculation, but the methods currently available to us do not permit us to weaken either of the hypotheses.

The rest of the paper is arranged in the following way. In Section 2 we define some of the terminologies that are used in the paper. In Section 3 we consider the various possible generalised triangle group descriptions and how they are related via push-out diagrams. In Section 4 we recall the idea of pictures and clique-pictures. Section 5 contains various lemmas that are used to prove Theorem A. Section 6 is the final section. There we give proofs of Theorems 1.1- 1.5.

It is worth mentioning at this point that many of the arguments in this paper are adaptations of the ones found in [6], [5] and [9]. The reader can consult those for better understanding.

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2. Preliminaries

In this section we give some basic definitions and results on periodic words in a free monoid.

Let I be an indexing set, $S = \{z_i\}_{i \in I}$ be a set and S^* the free monoid on S. Each z_i is called a *letter*. We assume S is equipped with an involution $z_i \mapsto {z_i}^{-1}$. A word in S^* is just a list, or ordered tuple, of letters. Let $w = z_1 z_2 \cdots z_n \in S^*$ for some integer $n \geq 0$. Then n is called the length of w and is denoted by $\ell(w)$. A *segment* or *subword* of w is a collection of consecutive letters in w. A segment is called *initial* if it has the form $z_1 z_2 \cdots z_k$ for some $k \leq n$ and terminal if it is of the form $z_j z_{j+1} \cdots z_n$ for $j \geq 1$. We call a segment of w *proper* if it misses at least one letter in w. Let $u = z_i z_{i+1} \cdots z_{i+t}$ and $v = z_j z_{j+1} \cdots z_{j+s}$ be segments of w. Suppose without loss of generality that $i + t \leq j + s$, we say u and v *intersect* if $j \leq i + t$. In this case the *intersection* is the segment u if $j \leq i$ and $z_j z_{j+1} \cdots z_{i+t}$ otherwise. We call w a *proper power* if it has the form $w = u^t$ for some proper initial segment u of w. A *cyclic permutation* of w is a word of the form $z_{\rho(1)} z_{\rho(2)} \cdots z_{\rho(n)}$ where ρ is some power of the permutation $(1\ 2 \cdots\ n)$. A *proper cyclic permutation* is one in which ρ is not the identity. Two words u and v are said to be *identically equal*, written $u \equiv v$, if they are equal in S^*.

In practice, we will usually be dealing with words in a free product $G_1 * G_2$ of two groups. In this context, the alphabet S is $G_1 \cup G_2$. Each word $w = z_1 z_2 \cdots z_n \in (G_1 \cup G_2)^*$ has an *inverse* $w^{-1} = {z_n}^{-1} {z_{n-1}}^{-1} \cdots {z_1}^{-1}$. Cyclic permutations of a word w represent conjugates in $G_1 * G_2$ of the element represented by w. Often the term *cyclic conjugate* will be used as a synonym for cyclic permutation. As usual, a word $w = z_1 z_2 \cdots z_n$ in $G_1 \cup G_2$ is said to be *reduced* if either $n \leq 1$ or if (i) $z_i \neq 1$ for each i, and (ii) z_i and z_{i+1} do not belong to the same factor group G_1 or G_2, for each $0 < i < n$. The word w is *cyclically reduced* if it and all its cyclic permutations are reduced. A *cyclic subword* of w is an initial segment of some cyclic conjugate of w.

Definition 2.1. A word $w = z_1 z_2 \cdots z_n \in S^*$ of length n has a *period* γ if $\gamma \leq n$ and $z_i = z_{i+\gamma}$ for all $i \leq n - \gamma$.

Remark 2.2. It follows immediately that if a word w has an initial segment u and a terminal segment $v \equiv u$, then w has period $\gamma = \ell(w) - \ell(u)$.

Theorem 2.3. [4] *Let* $w = z_1 z_2 \cdots z_n \in S^*$ *be a word having initial segment* w_1 *with period* γ *and terminal segment* w_2 *with period* ρ. *If* w_1 *and* w_2 *intersect in a segment* u *with* $\ell(u) \geq \gamma + \rho - gcd(\gamma, \rho)$, *then* w *has period* $gcd(\gamma, \rho)$.

In Section 5 below we will introduce a generalisation of the concept of periodicity, and prove an appropriate generalisation of Theorem 2.3, which we will use in the proof of Theorem A.
Another result which will be very useful in the analysis of clique-pictures is the following theorem which goes by the name *Spelling Theorem for generalised triangle groups.*

Theorem 2.4. [8] *Let*

$$H = \frac{\langle\ x\ \rangle * \langle\ y\ \rangle}{N(\ W(x,y)^r\ )}$$

be a generalised triangle group, where x *has order* $p \leq \infty$ *and* y *has order* $q \leq \infty$. *Write* $W(x,y) = \prod_{i=1}^{k} x^{\alpha_i} y^{\beta_i}$, *where* $k > 0$, $0 < \alpha_i < p$ *if* $p < \infty$, *and* $0 < \beta_i < q$ *if* $q < \infty$. *Suppose that* $V(x,y) = \prod_{i=1}^{l} x^{\gamma_i} y^{\delta_i}$ *is trivial in* H, *with* $l > 0$, $0 < \gamma_i < p$ *if* $p < \infty$, *and* $0 < \delta_i < q$ *if* $q < \infty$. *Then* $l \geq kr$.

Remark 2.5. The proof of Theorem 2.4 in [8] assumes that p, q are finite, but we require a stronger version in which one or both of x, y may have infinite order. However, the general case easily reduces to the case proven in [8] as follows. If for example x has infinite order (and the α_i, γ_i are arbitrary nonzero integers), choose a large positive integer P, greater than the absolute value of any of the exponents α_i or γ_i of x appearing in the words W and V, and factor out x^P from H. Similarly if y has infinite order, we may factor out a large power y^Q of y. The result is a quotient group of H to which the proof of Theorem 2.4 in [8] applies, and where the images of the words V, W have the same length as the originals. The general result follows.

3. Refinements

As mentioned in the Introduction, we refer to the pushout diagram in Figure 1 as a *generalised triangle group description* of G. A *refinement* of this description is another generalised triangle group description involving cyclic conjugates A', B' of A and/or B, such that $A * B \subsetneq A' * B'$. Say $A' = \langle c \rangle$ and $B' = \langle VdV^{-1} \rangle$, where $c, d \in \{a, b\}$ and $\ell(V) < \ell(U)$. The description is *maximal* if it has no refinements.
The two descriptions in the refinement are linked by the commutative diagram in Figure 2, in which both squares are push-outs. A generalised triangle group description for G is said to be *maximal* if no refinement is possible. It follows from the inequality condition $\ell(V) < \ell(U)$ that maximal refinements always exist (but not necessarily that they are unique). From now on we will assume that we are working with a maximal refinement.
We next note a useful consequence in this context of Lemma 3.2.

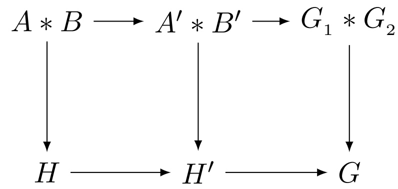

FIGURE 2. *Double push-out diagram.*

Lemma 3.1. *Suppose that, in the above, the generators S, T of A, B have the forms $S \equiv a$, $T \equiv UbU^{-1}$ for some letters a, b and some word U. Suppose that (a, b) is an admissible pair.*
If there are integers $\alpha, \beta, \gamma, \delta$ such that $a^\alpha U b^\beta U^{-1}$ and $a^\gamma U b^\delta U^{-1}$ are proper cyclic conjugates in $G_1 * G_2$, then a refinement is possible.

The proof of Lemma 3.1 will make use of the following technical result.

Lemma 3.2. *Suppose that $W \in S^*$ is a word of the form*

$$W \equiv x_1 V_1 y_1 V_1^{-1} = z_0 z_1 \cdots z_{2k-1}$$

for some letters x_1, y_1 and some word V_1, where $\ell(W) = 2k$. Suppose also that W has a cyclic permutation of the form $z_j z_{j+1} \cdots z_{2k-1} z_0 \cdots z_{j-1} \equiv x_2 V_2 y_2 V_2^{-1}$, for some letters x_2, y_2 and some word V_2, where $j \not\equiv 0 \bmod k$. Then one of the following holds:

(1) *$\{x_1, y_1\} = \{x_2, y_2\}$ and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 y_1^{\beta(j)} V_3^{-1}]$$

for some odd integer $s > 1$ and some word V_3, with $\alpha(j), \beta(j) = \pm 1$ for each j.

(2) *$y_i = x_i^{-1}$ for $i = 1, 2$, and*

$$W \equiv \prod_{j=1}^{s} [x_1^{\alpha(j)} V_3 x_2^{\beta(j)} V_3^{-1}]$$

for some even integer $s > 0$ and some word V_3, with $\alpha(j), \beta(j) = \pm 1$ for each j.

Proof. We have

$$x_1 V_1 y_1 V_1^{-1} \equiv z_0 z_1 \cdots z_{2k-1}$$

and

$$x_2 V_2 y_2 V_2^{-1} \equiv z_j z_{j+1} \cdots z_{2k-1} z_0 \cdots z_{j-1}$$

where $\ell(W) = 2k$ and $j \in \{1, \ldots, 2k-1\}$. Thus $z_i = z_{2k-i}^{-1}$ unless $i \equiv 0 \bmod k$, and $z_i = z_{2j-i}^{-1}$ unless $i \equiv j \bmod k$. (Here, and below, we interpret all subscripts modulo $2k$.) Let $m = \gcd(j, k)$ and let $V_3 = z_1 z_2 \cdots z_{m-1}$. For $i \not\equiv 0 \bmod m$, we have $z_i = z_{2j-i}^{-1} = z_{i-2j}$ and also $z_i = z_{i+2k}$, and so

$z_i = z_{i+2m}$. Hence also $z_i = z_{2k-i}^{-1} = z_{2m-i}^{-1}$ for all $i \not\equiv 0 \bmod m$. It follows that $x_1V_1y_1V_1^{-1} \equiv \prod_{t=1}^{s}[\xi_t V_3 \eta_t V_3^{-1}]$ for some letters ξ_t, η_t, where $s = k/m$. By hypothesis $j \not\equiv 0 \bmod k$, and so $s > 1$.
Suppose first that s is odd. Then, replacing j by $k + j$ if necessary, we may assume that j/m is also odd. We have a chain of equalities $z_0 = z_{2j}^{-1} = z_{2k-2j} = \cdots$ that continues until it reaches $z_d^{\pm 1}$ for some subscript $d \in \{j, k, j+k\}$. Since the equalities link letters whose subscripts differ by multiples of $2m$, and since j, k are both odd multiples of m, we must have $d = j + k$. Moreover, every z_e with $e \equiv 0 \bmod 2m$ appears in the chain. There are precisely s such letters, so an even number of equalities, and hence $\xi_1 = x_1 = z_0 = z_{j+k} = y_2$ and $\xi_t = x_1^{\pm 1} = y_2^{\pm 1}$ for each t. By a similar argument $\eta_t = y_1^{\pm 1} = x_2^{\pm 1}$ for each t.
Now suppose that s is even. Then j/m is odd. Arguing as above, we have a chain of $s - 1$ equalities $z_0 = z_{2j}^{-1} = \cdots$, which must end with z_k^{-1}, and a similar chain of equalities equating z_j with z_{j+k}^{-1}. Hence in this case $\xi_t = x_1^{\pm 1} = y_1^{\mp 1}$ for each t, and $\eta_t = x_2^{\pm 1} = y_2^{\mp 1}$ for each t. □

Proof of Lemma 3.1.
We may apply Lemma 3.2 to $a^\alpha U b^\beta U^{-1}$ and $a^\gamma U b^\delta U^{-1}$ except in the situation where $a^\alpha U b^\beta U^{-1} \equiv b^\delta U^{-1} a^\gamma U$. Let us first consider this exceptional situation. Then in particular $U \equiv U^{-1}$ and so U has the form VxV^{-1} for some word V with $\ell(V) < \ell(U)$ and some letter x of order 2. But we also have $a^\alpha = b^\delta$, and so by definition of admissibility a, b have a common root c say. But then $A * B$ is a proper subgroup of $C * D$, where C and D are the cyclic subgroups of $G_1 * G_2$ generated by c, U respectively. This is a refinement, as required.
Now apply Lemma 3.2 with $x_1 = a^\alpha$, $y_1 = b^\beta$, $x_2 = a^\gamma$, $y_2 = b^\delta$, and $V_1 = V_2 = U$. Consider first the case when s is odd in the conclusion of Lemma 3.2. In this case, $A * B$ is a proper subgroup of $A * B'$, where B' is the cyclic subgroup generated by $V_3 b V_3^{-1}$. Since $\ell(V_3) < \ell(U)$, we again have a refinement.
Finally, suppose that s is even in Lemma 3.2. Then $a^\alpha = b^{-\beta}$, so by admissibility a, b have a common root c. Then $A * B$ is a proper subgroup of $C * D$, where C, D are the cyclic subgroups generated by $c, V_3 c V_3^{-1}$ respectively. As above, we have a refinement.

□

4. Pictures and Clique-pictures

Pictures have been used widely to prove results for one relator groups. In this section we recall only the basic ideas about pictures and clique-pictures as can be found in [9].

4.1. **Pictures.** A *picture* P over G on an oriented surface Σ is made up of the following data:

- a finite collection of pairwise disjoint closed discs in the interior of Σ called *vertices*
- a finite collection of disjoint closed arcs called *edges*, each of which is either:
  - a simple closed arc in the interior of Σ meeting no vertex of P,
  - a simple arc joining two vertices (possibly same one) on P,
  - a simple arc joining a vertex to the boundary $\partial\Sigma$ of Σ,
  - a simple arc joining $\partial\Sigma$ to $\partial\Sigma$,
- a collection of *labels* (i.e words in $G_1 \cup G_2$), one for each corner of each *region* (i.e connected component of the complement in Σ of the union of vertices and arcs of P) at a vertex and one along each component of the intersection of the region with $\partial\Sigma$. For each vertex, the label around it spells out the word $R^{\pm n}$ (up to cyclic permutation) in the clockwise order as a cyclically reduced word in $G_1 * G_2$. We call a vertex *positive* or *negative* depending on whether the label around it is R^n or R^{-n} respectively.

For us Σ will either be the 2-sphere S^2 or 2-disc D^2. A picture on Σ is called *spherical* if either $\Sigma = S^2$ or $\Sigma = D^2$ but with no arcs connected to ∂D^2. If P is not spherical, ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that P contains at least one vertex or arc), which is called the *exterior*. All other regions are called *interior*.

We shall be interested mainly in *connected* pictures. A picture is *connected* if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed arc or joins two points of $\partial\Sigma$, unless the picture consists only of that arc. In a connected picture, all interior regions $\triangle$ of P are simply-connected, i.e topological discs. Just as in the case of vertices, the label around each region – read *anticlockwise* – gives a word which in a connected picture is required to be trivial in G_1 or G_2. Hence it makes sense to talk of G_1-regions or G_2-regions. Each arc is required to separate a G_1-region from a G_2-region. This is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from G_1 and G_2.

A vertex is called *exterior* if it is possible to join it to the *exterior* region by some arc without intersecting any arc of P, and *interior* otherwise. For simplicity we will indeed assume from this point that our Σ is either S^2 or D^2. It follows that reading the label round any *interior* region spells a word which is trivial in G_1 or G_2. The *boundary label* of P on D^2 is a word obtained by reading the *labels* on ∂D^2 in an *anticlockwise* direction. This word (which we may be assumed to cyclically reduced in $G_1 * G_2$) represents an identity element in G. In the case where P is spherical, the *boundary label* is an element in G_1 or G_2 determined by other labels in the *exterior* region.

Two distinct vertices of a picture are said to *cancel* along an arc e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $G_1 * G_2$. Such vertices can be removed from a picture via a sequence of *bridge moves* (see Figure 3 and [2] for more details), followed by deletion of a *dipole* without changing the boundary label. A *dipole* is a connected spherical sub-picture that contains precisely two vertices, does not meet $\partial\Sigma$, and such that none of its interior regions contain other components of P. This gives an alternative picture with the same boundary label and two fewer vertices.

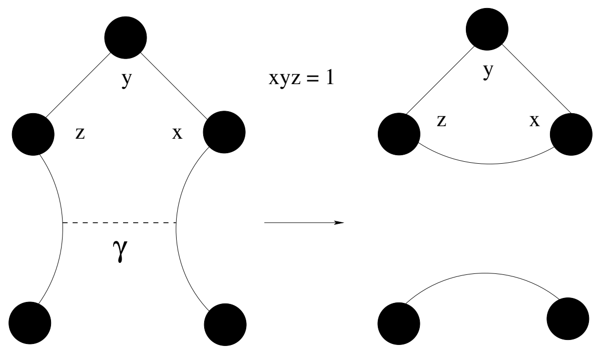

FIGURE 3. *Diagram showing bridge-move.*

We say that a picture P is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. A picture P on D^2 is *minimal* if it is non-empty and has the minimum number of vertices amongst all pictures over G with the same boundary label as P. Clearly minimal pictures are reduced. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of G occurs as the boundary label of some reduced picture on D^2.

4.2. **Clique-pictures.** Clique-pictures appeared in [9] and are modelled on generalised triangle groups. For the rest of this paper,

$$G = \frac{(G_1 * G_2)}{N(R^n)}$$

is a one-relator product induced by the generalised triangle group

$$H = \frac{A * B}{N(R^n)}$$

In other words R is a word in $A*B = \{a, UbU^{-1}\}$ for some word $U \in G_1 * G_2$ and letters a and b in $G_1 \cup G_2$. We will also assume throughout that this generalised triangle group description for G is maximal.

Definition 4.1. Two arcs in a picture P over G are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region $\triangle$ of Γ. We will also use the term *parallel* to denote the equivalence relation generated by this relation.

If u and v are two vertices in a picture over G that are joined by an arc e, then we may use the endpoints of e as the starting points for reading the labels L_u and L_v of u and v respectively. In each case the label is a cyclic permutation of $R'(a, UbU^{-1})^{\pm n}$. We may assume, without loss of generality, that the word $R'(x, y)$ begins with the letter x. Choose a cyclic permutation $R^*(x, y)$ of $R'(x, y)^{-1}$ that also starts with x.

Each of L_u and L_v^{-1} is a cyclic conjugate of $R'(a, UbU^{-1})^n$ or $R^*(a, UbU^{-1})^n$ – in particular an n'th power. We can write $L_u = (YZ)^n$, where $ZY = R'(a, UbU^{-1})$ or $ZY = R^*(a, UbU^{-1})$ and $L_v^{-1} = (Y'Z')^n$, where $Z'Y' = R'(a, UbU^{-1})$ or $Z'Y' = R^*(a, UbU^{-1})$.

We define $u \sim v$ if and only if $\ell(Y') \equiv \ell(Y) \bmod l$, where $l = \ell(aUbU^{-1})$. Since our generalised triangle group description of G is maximal, by definition no refinement is possible. It follows immediately from Lemma 3.1 that $\ell(Y')$ and $\ell(Y)$ are unique modulo l (for otherwise there are two words of the form $a^\alpha Ub^\beta U^{-1}$ that are proper cyclic conjugates of each other), and so the relation $\sim$ is well-defined. The point of the relation $\sim$ is that, when $u \sim v$, then the 2-vertex sub-picture consisting of u and v, joined by e and any arcs parallel to e, has boundary label a word in $\{a, UbU^{-1}\}$, after cyclic reduction and cyclic permutation. (Indeed, the cyclic reduction of the label can be achieved by performing bridge moves to make the number of edges parallel to e be a multiple of $l/2$.) Now let $\approx$ denote the transitive, reflexive closure of $\sim$. Then $\approx$ is an equivalence relation on vertices. After a sequence of bridge moves, we may assume that arcs joining equivalent vertices do so in parallel classes each containing a multiple of $l/2$ arcs. Define a *clique* to be the sub-picture consisting of any $\approx$-equivalence class of vertices, together with all arcs between vertices in that $\approx$-class (assumed to occur in parallel classes of multiples of $l/2$ arcs), and all regions that are enclosed entirely by such arcs.

Any clique in a picture P over G may be regarded as a picture over the underlying generalised triangle group $H = (A * B)/N(R')$. Its boundary label(s) are words in $A * B$. In the special case where the clique is simply connected, its unique boundary label represents the identity element of H.

Definition 4.2. Let G be a one-relator product induced by a generalised triangle group as above, and let P be a picture on a surface Σ, such that every clique of P is simply-connected. Then the *clique-quotient* of P is the picture formed from P by contracting each clique to a point, and regarding it as a vertex.

A clique-quotient is a special case of what we call a *clique-picture*. We define abstract clique-pictures as follows.

Definition 4.3. A *clique-picture* $\mathbf{P}$ over G is a picture over $G = (G_1 * G_2)/N(\mathcal{R})$, where $\mathcal{R}$ is the image in $G_1 * G_2$ of the normal closure in $A * B$ of R. Thus every vertex label in a clique-picture is the image in $G_1 * G_2$ of a word in $A * B$ that represents the identity element of H. Hence also every such vertex label represents the identity element of G. We will also refer to the vertices in a clique-picture as *cliques* and their labels as *clique-labels*.

In a clique-picture, as in an ordinary picture over G, the clique-labels are words in $\{a, UbU^{-1}\}$, up to cyclic permutation. If u, v are two cliques joined by an edge e, then the labels of u, v beginning at the end-points of e have the form YZ and $Y'Z'$ respectively, where ZY and $Z'Y'$ are words in $\{a, UbU^{-1}\}$. As before, the lengths of Y and Y' are unique modulo l by Lemma 3.1. If $\ell(Y) \equiv \ell(Y') \mod l$, then we can replace u and v by a new clique whose label is $YZY'Z'$, where $ZY'Z'Y$ is a word in $\{a, UbU^{-1}\}$. This process of joining two cliques in a clique-picture to form a single clique is called *amalgamation*. (Here we also include the possibility of amalgamating a clique with itself. By this we mean adding arcs from v to v and/or regions to an existing clique v, which could alter some properties of the clique such as simple-connectivity). If some boundary component C of a clique-picture has a label which is a word in $\{a, UbU^{-1}\}$, then we may also talk about amalgamating a clique v with C, as if C itself were a clique. The conditions for such an amalgamation are the same as for amalgamating two cliques as above. The result of such an amalgamation is effectively to remove v from the picture, changing the label on C to another word in $\{a, UbU^{-1}\}$. If it is possible to amalgamate two cliques (possibly after doing bridge-moves), we say that $\mathbf{P}$ is not *reduced*, and *reduced* otherwise. A clique-picture (on the disk D^2) is *minimal* if it has the fewest possible cliques among all clique-pictures with the same boundary label.

The idea of parallel arcs also arises in clique-pictures, as follows.

Definition 4.4. Let Γ be a picture or a clique-picture over G. Two arcs of Γ are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region $\triangle$ of Γ. We will also use the term *parallel* to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a *class of* ω *parallel arcs* or ω-*zone*. Given a ω-*zone* with $\omega > 1$ joining vertices u and v of Γ, consider the $\omega - 1$ two-sided regions separating these arcs. Each such region has a corner label x_u at u and a corner label x_v at v, and the picture axioms imply that $x_u x_v = 1$ in G_1 or G_2. The $\omega - 1$ corner labels at v spell a cyclic subword s of length $\omega - 1$ of the label of v. Similarly the corner labels at u spell out a cyclic subword t of length $\omega - 1$ of the label of u. Moreover, $s = t^{-1}$. If we assume that Γ is reduced, then u and v do not cancel or amalgamate. In the spirit of small-cancellation-theory, we refer to t and s as *pieces*.

As in graphs, the *degree* of a vertex in Γ is the number of *zones* incident on it. For a region, the *degree* is the number corners it has. We say that a vertex v of Γ satisfies the local $C(m)$ condition if it is joined to at least m *zones*. We say that Γ satisfies $C(m)$ if every interior vertex satisfies local $C(m)$.

One advantage clique-pictures have over ordinary pictures is that some cyclic permutation of the inverse of any clique-label can also be interpreted as a clique-label. Thus we may regard any clique as having either possible orientation, as convenient. We make the convention that all our cliques have the same (positive or clockwise) orientation.

Notations. Throughout the remainder of this paper we shall assume that our clique-picture is minimal and hence reduced. Recall that $l = \ell(aUbU^{-1})$. Thus, up to cyclic permutation, the clique-label of a clique v has the form

$$\text{label}(v) = \prod_{i=1}^{m} a^{\alpha_i} U b^{\beta_i} U^{-1} = z_0 z_1 \cdots z_{ml-1} \tag{4.5}$$

for $0 < \alpha_i < p$ and $0 < \beta_i < q$.

We call a letter z_j of the clique-label label(v) in (4.5) *special* if $j \equiv 0 \mod l/2$. Note that every special letter is equal to a power of a or of b.
Let v be a clique of degree k. This means that there are k zones incident at v, say $Z_1, Z_2, \ldots, Z_k$ labelled consecutively in clockwise order around v as shown in Figure 4.

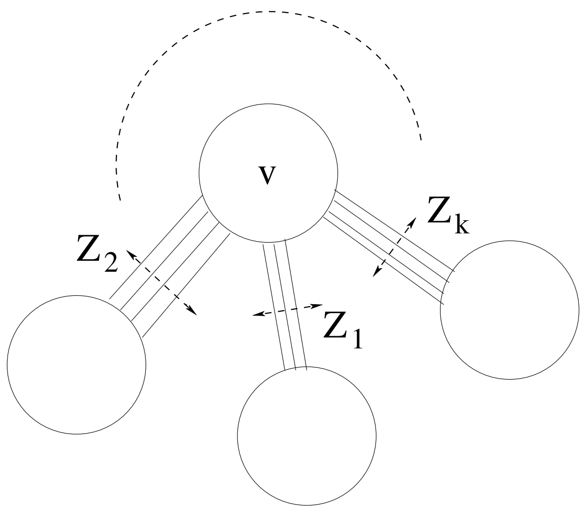

FIGURE 4. *Zones*

Recall that each zone Z_i is a class of parallel arcs. The number of arcs in Z_i is denoted by ω_i. If Z_i connects cliques u_i and v (possibly $u_i = v$), then Z_i determines cyclic subwords s_i, t_i of length $\omega_i - 1$ of the clique-labels of v, u_i respectively, such that $s_i \equiv t_i^{-1}$ in $G_1 * G_2$. The label of v has the form $s_1 x_1 \cdots s_k x_k$ (up to cyclic permutation) for some letters $x_i \in G_1 \cup G_2$.
We assume that label(u_i) and label(v) are of the form in (4.5), and let $s(i), t(i)$ denote the subscripts of the letters immediately preceding s_i, t_i in v, u_i respectively. (Thus for example label(v) has a cyclic subword of the form $z_{s(i)} s_i z_{s(i+1)}$.)

5. Theorem A

In this section we prove Theorem A. Throughout this section we assume that Hypothesis A holds. That is, (a, b) is an admissible pair in the free product $G_1 * G_2$, $n \geq 2$, R is cyclically reduced of free product length at least 2 in $A * B := \langle a \rangle * \langle UbU^{-1} \rangle$, $G = (G_1 * G_2)/N(R^n)$, and the generalised triangle group description Figure 1 is maximal. We also assume that v is an interior clique in a minimal clique-picture $\mathcal{P}$ over G. Our aim will be to show that v has degree at least 6 in $\mathcal{P}$.

We will use the following generalisation of the concept of periodic word, as applied to cyclic subwords of the clique-label label(v) in (4.5). We say that a cyclic subword $W = z_j \cdots z_k$ (subscripts modulo ml) of label(v) is *virtually periodic* with *virtual period* μ if, for each $i \in \{j, j+1, \dots, k-\mu\}$, one of the following happens:

(1) $z_i = z_{i+\mu}$;
(2) a special letter $z_d = a^\psi$ belongs to W, for some $d \equiv 0 \bmod l$, $i \equiv d \bmod \mu$, and each of z_i, $z_{i+\mu}$ is equal to a power of a;
(3) a special letter $z_d = b^\psi$ belongs to W, for some $d \equiv l/2 \bmod l$, $i \equiv d \bmod \mu$, and each of z_i, $z_{i+\mu}$ is equal to a power of b;
(4) a and b have a common root c in G_1 or G_2, a special letter $z_d = c^\psi$ belongs to W, for some $d \equiv 0 \bmod l/2$, $i \equiv d \bmod \mu$, and each of z_i, $z_{i+\mu}$ is equal to a power of c.

Recall that the pair (a, b) is assumed to be admissible. If a and b have a common power in G_1 and G_2, then they have a common root, and in that case the second and third possibilities in the above definition are subsumed in the fourth. Otherwise the fourth possibility cannot occur.
Note that in each case of the definition, z_i and $z_{i+\mu}$ belong to the same free factor G_1 or G_2. Since the z_i belong alternatingly to G_1 and G_2, it follows that any period or virtual period μ must be even.
By definition, the clique-label label(v) itself is virtually periodic of virtual period l. Other examples of virtually periodic words arise from zones incident at v.

Lemma 5.1. *Suppose that Z_i is a zone incident at v. Then there is a positive even integer $\mu \leq l/2$ and a cyclic subword s_i^+ of* label(v) *of length $\omega_i + \mu - 1$ and virtual period μ, such that s_i is either an initial or a terminal segment of s_i^+.*

Proof. Recall that s_i is a cyclic subword of

$$\text{label}(v) = \prod_{i=1}^{m} a^{\alpha_i} U b^{\beta_i} U^{-1} = z_0 z_1 \cdots z_{ml-1}.$$

Suppose that $s_i \equiv z_j z_{j+1} \cdots z_k$.
The zone Z_i joins v to an adjacent clique u and identifies s_i with t_i^{-1} for some cyclic subword t_i of label(u). Thus t_i^{-1} is a cyclic subword of label$(u)^{-1}$.

Write $t_i^{-1} = y_{j'}y_{j'+1}\cdots y_{k'}$ where

$$\mathrm{label}(u)^{-1} = y_0y_1\cdots y_{rl-1} = \prod_{k=1}^{r}[a^{\gamma(k)}Ub^{\delta(k)}U^{-1}].$$

Since $s_i \equiv t_i^{-1}$, then in particular $k'-j' \equiv \ell(t_i)-1 = \ell(s_i)-1 \equiv k-j$ mod l. If $j \equiv j'$ mod l, then we may amalgamate the cliques u and v, contrary to the hypothesis. Hence there are integers d and μ_i such that $0 \le d \le r$, $0 < \mu_i \le l/2$ such that $j' = j + dl \pm \mu_i$. Define

$$s_i^+ = \begin{cases} z_{j-\mu_i}\cdots z_k & \text{if} \quad j' = j+dl-\mu_i \\ z_j\cdots z_{k+\mu_i} & \text{if} \quad j' = j+dl+\mu_i \end{cases}$$

In the first case, s_i is a terminal segment of s_i^+, while the initial segment of the same length agrees with $t_i^{-1} \equiv s_i$, except possibly at special letters z_d ($d \equiv 0$ mod $l/2$) which may be a different power of a (or of b) than the corresponding letter of s_i. It follows that s_i^+ is virtually periodic of virtual period μ_i, as claimed.
The second case is entirely analogous, except that s_i is an initial rather than a terminal segment of s_i^+.
Finally, the fact that the virtual period $\mu = \mu_i$ is even follows from the fact that the letters in label(v) alternate between the factors G_1 and G_2. (See the remarks following the definition of virtual period at the start of this section.) □

We need to analyse the interaction of virtually periodic subwords of label(v) obtained by applying Lemma 5.1 to two adjacent zones at v. To do this we will use the following analogue of Theorem 2.3.

Lemma 5.2. *Suppose that the cyclic subword $W = z_j\cdots z_k$ (subscripts modulo ml) of* label(v) *is the union of a virtually periodic segment W_1 of virtual period μ and a virtually periodic segment W_2 of virtual period ν. Let $\gamma = \gcd(\mu,\nu)$. If the intersection of these segments has length at least $\mu+\nu-\gamma$, then W is virtually periodic of virtual period γ.*

Proof. Let i, r be such that z_i and $z_{i+r\gamma}$ are letters of W. Then we claim there is a finite chain of subscripts $i(0), i(1), \ldots, i(N)$ with $i(0) = i$ and $i(N) = i + r\gamma$ such that, for each t either $|i(t) - i(t+1)| = \mu$ and $z_{i(t)}$ and $z_{i(t+1)}$ are letters of W_1, or $|i(t)-i(t+1)| = \nu$ and $z_{i(t)}$ and $z_{i(t+1)}$ are letters of W_2. Certainly each letter in W_1 (resp. W_2) is linked to some letter in $W_0 := W_1 \cap W_2$ by such a chain, since $\ell(W_0) \ge \max(\mu,\nu)$, so it suffices to prove the claim when $z_i, z_{i+r\gamma}$ are letters of W_0. Write $r\gamma = \alpha\mu + \beta\nu$ where $\alpha, \beta \in \mathbb{Z}$, and argue by induction on $|\alpha| + |\beta|$. Without loss of generality, assume that $\alpha > 0$. If $z_{i+\mu}$ is a letter of W_0, then the result follows by applying the inductive hypothesis to $z_{i+\mu}, z_{i+r\gamma}$. Otherwise, $\beta < 0$ and $z_{i-\nu}$ is a letter of W_0, so we may apply the inductive hypothesis to $z_{i-\nu}, z_{i+r\gamma}$. This proves the claim.

Now we take $r = 1$ in the above, and prove that at least one of the alternative conditions for virtual periodicity holds.
If $z_{i(t)} = z_{i(t+1)}$ for all t then $z_i = z_{i+\gamma}$.
Suppose next that $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, and a, b have a common root c. Then there is a special letter z_d in W, $i \equiv i(t) \equiv d$ mod μ or mod ν (and hence in either case mod γ). Moreover for each t either $z_{i(t)} = z_{i(t+1)}$ or each of $z_{i(t)}, z_{i(t+1)}$ is a power of c. Since $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, it follows that each $z_{i(t)}$ is a power of c. In particular z_i and $z_{i+\gamma}$ are both powers of c.
Finally, suppose that $z_{i(t)} \neq z_{i(t+1)}$ for at least one value of t, and a, b have no common root. By admissibility, a, b also have no common non-trivial power. As above, W contains a special letter $z_d = a^{\psi}$ or $z_d = b^{\psi}$, and $i \equiv d$ mod γ. Consider the least t for which $z_{i(t)} \neq z_{i(t+1)}$. Then either $z_{i(t)}$ and $z_{i(t+1)}$ are both powers of a or both powers of b. Assume the former. Then $z_i = z_{i(0)} = z_{i(1)} = \cdots = z_{i(t)}$ are also powers of a. We claim that $z_{i(t+1)}$, $\ldots z_{i(N)} = z_{i+\gamma}$ are also powers of a, which will complete the proof. Suppose by way of contradiction that this is not true. Then there is an r for which $z_{i(r)}$ is a power of a and $z_{i(r+1)}$ is not a power of a. By the definition of virtual periodicity, it follows that both $z_{i(r)}$ and $z_{i(r+1)}$ must be powers of b. But then $z_{i(r)}$ is simultaneously a power of a and of b, contrary to the admissibility hypothesis. □

Corollary 5.3. *If a clique-label in a clique-picture over G satisfying Hypothesis A has virtual period $\mu < l$, then a refinement is possible.*

Proof. By Lemma 5.2 the clique-label label(v) in (4.5) has virtual period $\gcd(\mu, l)|l$, so without loss of generality $\mu|l$. Let $V = z_1 \cdots z_{\frac{\mu}{2}}$ and let $s = l/\mu$. Then by definition of virtual periodicity and by the admissibility hypothesis one of the following is true:

(1) s is odd and
$$\text{label}(v) = \prod_{j=1}^{sm} [a^{\gamma(j)} V b^{\delta(j)} V^{-1}]$$
for some $\gamma(j), \delta(j)$.

(2) s is even, a, b have a common root c, and
$$\text{label}(v) = \prod_{j=1}^{sm} [c^{\gamma(j)} V x V^{-1}]$$
for some letter x of order 2 and some $\gamma(j)$.

In either case, we have a refinement. □

Corollary 5.4. *Suppose that v is a clique in a minimal clique-picture over G satisfying Hypothesis A. Suppose also that the generalised triangle group description of G has no refinement. Then the length of any zone incident at v is strictly less than l.*

Proof. The i'th zone Z_i contains ω_i arcs. Assume that $\omega_i \geq l > l/2$. Then by Lemma 5.1, the word s_i^+ has virtual period μ_i. It also has length $\omega_i + \mu_i - 1$, which is strictly greater than $l + \mu_i - \gcd(l, \mu_i)$ since $\gcd(l, \mu_i)$ is even. But s_i^+ is a cyclic subword of the clique-label label(v) which is virtually periodic with virtual period l.
By Lemma 5.2 it follows that label(v) has virtual period $\gcd(l, \mu_i) \leq l/2$. But by Corollary 5.3 this leads to a refinement of our generalised triangle group description of G, contrary to the hypothesis.

□

Assuming Hypothesis A, we have a minimal clique-picture over a one-relator product $G = (G_1 * G_2)/N(R(a, UbU^{-1})^n)$ with $n \geq 2$ and $\ell(R) \geq 4$ as a word in $\langle a \rangle * \langle UbU^{-1} \rangle$. Recall (4.5) that any clique-label has the form

$$\text{label}(v) = \prod_{j=1}^{m} [a^{\alpha(j)} U b^{\beta(j)} U^{-1}]$$

By the Spelling Theorem 2.4 we must have $m \geq 2nl$ where $l = \ell(aUbU^{-1})$. But by Corollary 5.4 each zone has fewer than l arcs, so all cliques have degree at least $2n+1$. To prove the theorem, we assume that $n = 2$ and that v is a clique of degree 5, with zones $Z_1, \dots Z_5$ of sizes $\omega_1, \dots \omega_5$ respectively, in cyclic order around v, and aim to derive a contradiction.

The key tool in the proof of Theorem A is the following.

Lemma 5.5. *For each $i = 1, \dots, 5$, one of the following holds:*

(1) $\omega_i + \omega_{i-1} < 3l/2$;
(2) $\omega_i + \omega_{i+1} < 3l/2$.

Proof. By Lemma 5.1, s_i is either an initial segment or a terminal segment of a subword s_i^+ of label(v) of length $\omega_i + \mu_i - 1$ and virtual period μ_i, where $0 < \mu_i \leq l/2$. We will assume that s_i is an initial segment of s_i^+ and show that $\omega_i + \omega_{i+1} < 3l/2$. (An entirely analogous argument shows that, if s_i is a terminal segment of s_i^+, then $\omega_{i-1} + \omega_i < 3l/2$.)
The result follows immediately from Corollary 5.4 if either of ω_i, ω_{i+1} is less than or equal to $l/2$, so we may assume for the rest of the proof that $\omega_i, \omega_{i+1} > l/2$.
Now apply Lemma 5.1 to s_{i+1}: s_{i+1} is either an initial segment or a terminal segment of a cyclic subword s_{i+1}^+ of label(v) of length $\omega_{i+1} + \mu_{i+1}$ and virtual period μ_{i+1}, where $0 < \mu_{i+1} \leq l/2$. Our argument splits into two cases, depending on whether s_{i+1} is an initial or terminal segment of s_{i+1}^+.

Case 1. s_{i+1} is a terminal segment of s_{i+1}^+.
Consider the cyclic subword $W := s_i z_{s(i+1)} s_{i+1}$ of label(v). Since $\mu_i \leq l/2 < \omega_{i+1}$, the virtually periodic subword s_i^+ is an initial segment of W. Similarly, s_{i+1}^+ is a terminal segment of W. These segments intersect in a segment of length $\mu_i + \mu_{i+1} - 1$. Let $\lambda = \gcd(\mu_i, \mu_{i+1})$. Then $\lambda > 1$ is even,

so $\mu_i+\mu_{i+1}-1 > \mu_i+\mu_{i+1}-\lambda$. Hence by Lemma 5.2 W is virtually periodic with virtual period λ.
Now recall that W is a cyclic subword of label(v), which is virtually periodic with virtual period l. If W has length greater than $l+\lambda-2$, then label(v) has virtual period $\gcd(l,\lambda)<l$, by another application of Lemma 5.2.
But by Corollary 5.3 this leads to a refinement of our generalised triangle group description of G, contrary to the hypothesis. Thus

$$\omega_i+\omega_{i+1} = 1+\ell(W) \le l+\lambda-1 < 3l/2.$$

Case 2. s_{i+1} is an initial segment of s_{i+1}^+.
Let $\bar{s}_i$, $\bar{s}_i^+$ denote the cyclic subwords of label(v) of lengths $\ell(s_i)$, $\ell(s_i^+)$ respectively, that begin with the letter exactly l places after the first letter of s_i. By the virtual periodicity of label(v) it follows that $\bar{s}_i^+$ is also virtually periodic, of virtual period μ_i. Moreover, $\bar{s}_i^+$ has length $\omega_i+\mu_i-1$ and has $\bar{s}_i$ as an initial segment.
By construction, the union of the subwords s_{i+1} and $\bar{s}_i$ of label(v) has length $l-1$. Let W be the union of the subwords s_{i+1}^+ and $\bar{s}_i^+$ of label(v). Then W has length at least $l+\mu_i-1$. Arguing as in Case 1, we obtain a refinement of the generalised triangle group description of G, contrary to the hypothesis, if s_{i+1}^+ and $\bar{s}_i^+$ intersect in a segment of length $\mu_i+\mu_{i+1}-\gcd(\mu_i,\mu_{i+1})$ or greater. So we may assume that this does not happen.
In particular, $\mu_{i+1} < l-\omega_{i+1}+\mu_i$, for otherwise $\bar{s}_i^+$ is a subword of s_{i+1}^+, of length $\omega_i+\mu_i-1 > l/2+\mu_i-1 > \mu_{i+1}+\mu_i-\gcd(\mu_i,\mu_{i+1})$. Hence $\bar{s}_i^+$ is a terminal segment of W, and the intersection of s_{i+1}^+ and $\bar{s}_i^+$ has length precisely $\omega_{i+1}+\mu_{i+1}+\omega_i-l-1$.
Thus

$$\omega_i+\omega_{i+1} < \mu_i+\mu_{i+1}-\gcd(\mu_i,\mu_{i+1})+l+1-\mu_{i+1} < l+\mu_i \le 3l/2$$

as claimed.

□

Using Lemma 5.5, we complete the proof of Theorem A as follows. Recall that v is an interior clique in a clique-picture. Renumbering the zones if necessary, we may assume by Lemma 5.5 that $\omega_1+\omega_2 < 3l/2$. Applying Lemma 5.5 again, with $i=4$, either $\omega_3+\omega_4<3l/2$ or $\omega_4+\omega_5<3l/2$. In the first case $\omega_5>l$; in the second $\omega_3>l$. Either of these contradicts Lemma 5.4.

6. Applications

Here we give the proofs for Theorems 1.1, 1.2, 1.3 1.4 and 1.5. By Theorem A, a minimal clique-picture over G satisfies the $C(6)$ property.

Proof of Theorem 1.1. Suppose that there is a non-trivial word w in $\{a, UbU^{-1}\}$, G_1 or G_2 that is trivial in G. Then we obtain a minimal picture P over G on D^2 with boundary label w. We prove the theorem by

induction on the number of cliques in P; the case of 0 cliques corresponds to the empty picture P, for which there is nothing to prove.
Suppose first that some clique v in P is not simply connected, and let C be one of the boundary components of the surface carrying the clique v. By an innermost argument, we may assume that C bounds a disc $D \subset D^2$ such that every clique in $P \cap D$ is simply-connected.
Now the label on C is a word in $A * B$ which is the identity in G, and $P \cap D$ has at least one fewer clique than P, so by inductive hypothesis the label is trivial in H.
We then amend P by replacing $P \cap D$ by a picture over H, all of the arcs, vertices and regions of which will belong to the same clique as v in the amended picture P'. Since $P \cap D$ was not empty (for otherwise D is contained in v), the new picture P' also has fewer cliques than P, and the result follows from the inductive hypothesis.
Hence we are reduced to the situation where every clique in P is simply connected, and hence we may form from P a clique-picture Γ over G on D^2 with boundary label w. Without loss of generality, we may assume that Γ is minimal. It follows that Γ satisfies $C(6)$.

If Γ is empty, then w is already trivial in H, G_1 or G_2 and so we get a contradiction. On the other hand suppose that Γ is non-empty. If no arcs of Γ meet ∂D^2, then Γ is a spherical picture (i.e a picture on S^2) and the $C(6)$ property implies $\chi(S^2) \neq 2$. This contradiction implies that $G_1 \to G$ and $G_2 \to G$ are both injective.

Suppose then that some arcs of Γ meet ∂D^2. Then $w \notin G_1, G_2$. Moreover if w is a word in $\{a, UbU^{-1}\}$, then the $C(6)$ condition combined with [10, Chapter V, Corollary 3.3] implies that some boundary clique v_0 has at most degree 3.

By Corollary 5.4, v_0 is connected to ∂D^2 by a zone Z_i with $\omega_i > l$. Either a refinement is possible by Lemma 3.1 or we can amalgamate v_0 with ∂D^2 to form a new clique-picture with fewer cliques whose boundary label is also a word in $\{a, UbU^{-1}\}$ The former possibility contradicts the maximality of the triangle group description for G while in the latter the result follows by inductive hypothesis.

□

Our proof of Theorem 1.2 will use the following technical result, which is based on small-cancellation formulae from [10].

Lemma 6.1. *Let $\mathcal{P}$ be a minimal clique-picture on D^2 over G, whose boundary label is a word $W \in G_1 * G_2$. Then each clique in $\mathcal{P}$ has label of length at most $2\ell(W)$.*

Proof. Recall that $\mathcal{P}$ satisfies the small cancellation condition $C(6)$ by Theorem A. Hence, if we remove the boundary arcs from $\mathcal{P}$ and replace each parallel class of arcs by a single arc, we obtain a $[6,3]$-map Γ, in the sense of

[10, Chapter V]. Applying [10, Chapter 5, Equation (3.2)] to Γ, we obtain

$$\sum_v {}^{\circ}(deg_\Gamma(v) - 6) \leq \sum_v {}^{\bullet}(4 - deg_\Gamma(v)) - 6, \tag{6.2}$$

where $\sum^\circ, \sum^\bullet$ denote sums over interior and boundary cliques respectively, and $deg_\Gamma(v)$ denotes the degree of v in Γ. Recall (Corollary 5.4) that no zone in $\mathcal{P}$ has length l or greater. Recall also (Theorem 2.4) that every clique-label has length at least $4l$. It follows that, for any boundary clique v in $\mathcal{P}$ with $deg_\Gamma(v) < 4$, there are more than $(4 - deg_\Gamma(v))l$ boundary arcs of $\mathcal{P}$ incident at v. So the right-hand side of (6.2) is at most $\ell(W)/l - 6$.
By the C(6) property, each term on the left hand side of (6.2) is non-negative. Hence for any interior clique v in $\mathcal{P}$,

$$deg_\mathcal{P}(v) = deg_\Gamma(v) \leq 6 + \ell(W)/l - 6 = \ell(W)/l,$$

and hence

$$\ell(label(v)) < l \cdot deg_\mathcal{P}(v) \leq \ell(W).$$

Similarly, for any boundary clique v, we have

$$deg_\Gamma(v) \leq 4 + \ell(W)/l - 6 < \ell(W)/l,$$

so there are at most $l \cdot deg_\Gamma(v) < \ell(W)$ interior arcs of $\mathcal{P}$ incident at v. Clearly there are also at most $\ell(W)$ boundary arcs incident at v, so

$$\ell(label(v)) \leq 2\ell(W)$$

as claimed. □

Proof of Theorem 1.2. Any clique-picture Γ over G satisfies $C(6)$, and hence a quadratic isoperimetric inequality [10, the Area Theorem 3.1 of Chapter V]. In other words, there is a quadratic function f such that any word of length t representing the identity element of G is the boundary label of a clique-picture with at most $f(t)$ cliques. Also there is a bound (as a function of t) on the length of any clique-label of Γ given by Lemma 6.1.
Since both the number of cliques and the length of any clique-label are bounded, there are only a finite number of connected graphs that could arise as clique-pictures for words of length less than or equal to that of a given word w. Moreover, if a and b have finite order, any such graph can be labelled as a clique-picture only in a finite number of ways. For any such potential labelling, we may check whether or not the clique-labels are equal to the identity in H, and whether or not the region-labels are equal to the identity in G_1 or G_2, using the solution to the word problem in H, G_1 and G_2 respectively. Hence we may obtain an effective list of all words of length less than or equal to $\ell(w)$ that appear as boundary labels of connected clique-pictures over G. In particular, we may check, for all cyclic subwords w_1 of w, whether or not $w_1 g$ belongs to this list for some letter $g \in G_1 \cup G_2$. (Note that this check also uses the solution to the word problem in G_1 and G_2, and that the letter g, if it exists, is unique by the Freiheitssatz, Theorem 1.1). If so, then w is a cyclic conjugate of $w_1 w_2$ for some w_2 , so $w = 1$ in G

if and only if $g = w_2$ in G, which we may assume inductively is decidable. Hence the word problem is solvable for G.
If a, b both have infinite order, then H is a one-relator group with torsion, and hence hyperbolic. Each clique can then be replaced by a picture over H whose number of vertices is bounded above by a linear function of $2\ell(W)$. Hence G satisfies a quadratic isoperimetric inequality, and hence has solvable word problem.

□

Proof of Theorem 1.3. By Theorem 1.2 we are reduced to the case where precisely one of a, b has infinite order. Let us assume that a has infinite order and b has finite order.
Suppose that $Z \in G_1 * G_2$ is a cyclically reduced word of free-product length n. We wish to decide algorithmically whether or not $Z = 1$ in G. If so, then there is a minimal (and hence reduced) clique-picture P over G with boundary label a cyclically reduced conjugate Y of Z. Arguing as in the proof of Theorem 1.2, there are only finitely many finite graphs on which P might be modelled.
Since a has infinite order, it is not immediately clear that each clique has only finitely many possible labellings. However, there are only finitely many *labelling patterns*, each a cyclic conjugate of some

$$a^{\alpha(1)}Ub^{\beta(1)}U^{-1}\cdots a^{\alpha(k)}Ub^{\beta(k)}U^{-1},$$

where the $\beta(i)$ are fixed but the $\alpha(i) \in \mathbb{Z}$ are regarded as variables. In a similar way, there are only finitely many possible labelling patterns for the boundary, each a conjugate of $Z^{\pm 1}$).
We are thus reduced to the following problem. Choose a fixed finite planar graph Γ with one vertex v_0 of index equal to $\ell(Z)$ (representing the boundary) and every other vertex of index equal to a multiple of $l := \ell(aUbU^{-1})$. Choose a fixed labelling for v_0 as ∂P, and a fixed labelling pattern for every other vertex as a clique-label. We claim that there is at most one assignment of labels a^α to the variable labels which is compatible with this graph arising from a clique-picture over G; and that moreover we may determine algorithmically whether or not this assignment exists, and if so we can compute it. The result then follows from the argument in Theorem 1.2.
We prove this claim by induction on the number of vertices in Γ.
Let $\bar{\Gamma}$ be the graph obtained from Γ by identifying parallel edges. By the Freiheitssatz (Theorem 1.1) we may assume that $\bar{\Gamma}$ is connected, so every region is simply connected.
If $\bar{\Gamma}$ has a single vertex v_0 then there are no edges in Γ and hence nothing to prove. Otherwise it follows from the C(6) property (Theorem A) that v_0 has a neighbour v_1 such that either (i) v_1 has degree less than 4 in $\bar{\Gamma}$; or (ii) v_1 has degree 4 in $\bar{\Gamma}$ and at least 3 of the corners at v_1 belong to triangular regions.

Now no two interior corners which are adjacent in a region or around a clique can have variable labels a^n, a^m: in the first case because a clique-amalgamation along the adjoining edge would be possible, so the clique-picture would not be reduced; in the second because the intermediate zone would have size l, contradicting Corollary 5.4. In particular, any 2-gonal or 3-gonal region has at most one variable label, so its labels can all be identified (or Γ is recognisably incompatible with labelling of a reduced clique-picture) using the solution to the membership problem in $G_1 * G_2$.

Hence we may identify all but at most one of the variable labels of corners at v_1. This final label – if it exists – can then be uniquely identified using the solution to the membership problem in H (or again we can deduce that the labelling is incompatible with that of a clique-picture). The result now follows from the inductive hypothesis applied to the graph Δ obtained from Γ by shrinking v_1, v_0 and all edges between them to a single point (to be regarded as the "boundary vertex" u_0 of Δ.

□

Proof of Theorem 1.4. Suppose by contradiction that $W_1 = 1 = W_2$ in G. We can assume by Theorem 1.1 that $\ell(W_1) \neq 0 \neq \ell(W_2)$. We obtain a minimal clique-picture Γ over G with boundary label W_1 or W_2. Suppose without loss of generality that Γ has boundary label W_1, form a new clique picture $\tilde{\Gamma}$ with boundary label W_2^{-1} by adding a vertex labelled R^{-n}. $\tilde{\Gamma}$ has only one boundary vertex and is reduced since Γ is minimal. It follows from [10, Corollary 3.4 of Chapter V] that $\tilde{\Gamma}$ has a single vertex or clique. Hence up to conjugacy W_2 (and hence W_1) is a reduced word in $A * B$ which is trivial in H and has length strictly less than the length of R^n. This contradicts the Spelling Theorem 2.4, hence the result. □

Proof of Theorem 1.5. We construct a pushout square of aspherical CW-complexes and embeddings which realises Figure 1 on fundamental groups. The result follows from this construction.

To begin the construction, choose disjoint Eilenberg-MacLane complexes X_A, X_B of types $K(A,1), K(B,1)$ for the cyclic groups A, B respectively, and connect their base-points by a 1-cell e_0 to form a $K(A*B,1)$-complex $X_0 := X_A \cup e_0 \cup X_B$.

In a similar way, we choose disjoint Eilenberg-MacLane complexes $X_{G1} = K(G_1,1)$ and $X_{G2} = K(G_2,1)$ and connect their base-points by a 1-cell e_1 to form a $K(G_1 * G_2,1)$-complex $X_G := X_{G1} \cup e_1 \cup X_{G2}$.

The embedding $A * B \to G_1 * G_2$ can be realised by a continuous map $f : X_0 \to X_G$. Since each of A, B is contained in a conjugate of G_1 or of G_2, we may assume without loss of generality that each of X_A, X_B is mapped by f into $X_{G1} \sqcup X_{G2}$. Replacing X_{G1} and/or X_{G2} by appropriate mapping cylinders, we may assume that f maps $X_A \cup X_B$ injectively.

The 1-cell e_0 is mapped by f to a path in the homotopy class of the word $U \in G_1 * G_2$. Let X_1 be the mapping cylinder of $f|_{e_0} : e_0 \to X_G$. Then X_1

is a $K(G_1 * G_2, 1)$-complex, and $X_1 = X_{G1} \cup X_{G2} \cup e_0 \cup e_1 \cup e_2$, where e_2 is a 2-cell. Indeed, X_1 collapses to $X_G = X_1 \setminus \{e_0, e_2\}$ across e_2. Another important property of X_1 is that it contains X_0 as a subcomplex.

Now H is a one-relator quotient of $A*B$, so we may form a $K(H,1)$-complex X_2 from X_0 by adding a single 2-cell e_3, together with cells in dimensions 3 and above.

Finally, we form a complex X from X_1 and X_2 by identifying their isomorphic subcomplexes X_0. By the van-Kampen Theorem, the spaces $X_0 = X_1 \cap X_2$, X_1, X_2 and $X = X_1 \cup X_2$ realise the pushout diagram in Figure 1 on fundamental groups. It remains therefore only to show that X is a $K(G,1)$ space, namely that X is aspherical.

Now by Theorem 1.1, each of the maps $G_1 \to G$, $G_2 \to G$ and $H \to G$ is injective. It follows that the maps $A \to G$ and $B \to G$ are also injective. By the Kurosh Subgroup Theorem for free products, it follows that the kernels of the maps $A*B \to G$ and $G_1 * G_2 \to G$ are free, and hence have homological dimension 1. By [5, Theorem 4.2], it suffices to prove that $\pi_2(X) = 0$.

Suppose then that $h : S^2 \to X$ is a continuous map. We show that h is nullhomotopic in X in a series of stages. Up to homotopy, we may assume that h maps S^2 into the 2-skeleton of X, which consists of the 2-skeleta of X_{G1} and X_{G2} together with the 1-cells e_0, e_1 and the 2-cells e_2, e_3. We may also assume that h is transverse to the 2-cells of X, and that the restriction of h to $h^{-1}(X^{(1)})$ is transverse to the 1-cells of X. Thus the preimages of the 2-cells and 1-cells of X in S^2 under h form a picture on S^2.

Consider the sub-picture $\mathcal{P}$ formed by the preimages of the 2-cell e_3 and the 1-cell e_0. This is a picture over the generalised triangle group H. Suppose that some component $\mathcal{P}'$ of $\mathcal{P}$ is on a non-simply-connected surface $\Sigma \subset S^2$, and let γ be a boundary component of Σ that separates Σ from a disc D such that $\partial D = \gamma$ and $h(D)$ involves 2-cells from X_G. Then $h|_\gamma$ is null-homotopic in X, and hence in X_2 by Theorem 1.1. Let D' be a disc bounded by γ and extend $h|\gamma$ to a map $D' \to X_2$, which we also call h by abuse of notation. Then in $\pi_2(X)$ we can express the homotopy class of h as the sum of two classes $[h(D' \cup (S^2 \setminus D))]$ and $[h(D \cup -D')]$, where $-D'$ denotes D' with the opposite orientation. If we can show that each of these is nullhomotopic, then so is h, and the proof is complete.

This reduces the problem to the case where every component of $\mathcal{P}$ is a disc-picture over H. Collapsing each such component to a point gives a clique-picture over G. By Theorem A any reduced clique-picture satisfies C(6), so either the clique-picture is empty, or an amalgamation of cliques is possible. Amalgamation of cliques amounts to amending h by adding the class of a map $S^2 \to X_2$. Since X_2 is aspherical, this map is nullhomotopic in X_2 and hence also in X. Thus amalgamation of cliques does not change the homotopy-class of h in $\pi_2(X)$. After a finite number of clique-amalgamations, we are reduced to the case of an empty clique-picture.

In this case, $h(S^2) \subset X_1$. Since X_1 is aspherical, h is nullhomotopic in X_1, and hence also in X.
This completes the proof.

□

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