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SIMPLE BIFURCATION AND GLOBAL CURVES OF SOLUTIONS OF p-LAPLACIAN PROBLEMS WITH RADIAL SYMMETRY

BRYAN P. RYNNE

Abstract. We consider the boundary-value problem

$$-(r^{N-1}\phi_p(u'(r)))' = \lambda r^{N-1} f(r, u(r)), \quad r \in (0,1), \tag{1}$$

$$BC_N(u) = (0,0), \tag{2}$$

where $N \geqslant 1$ is an integer, $p \in \mathbb{R}$ satisfies $2 \neq p > 1$, $\phi_p(s) := |s|^{p-1} \operatorname{sign} s$, $s \in \mathbb{R}$, $\lambda \geqslant 0$, and

$$BC_N(u) = \begin{cases} (u(0), u(1)), & \text{if } N = 1, \\ (u'(0), u(1)), & \text{if } N > 1. \end{cases}$$

The case $N = 1$ is a standard, 1-dimensional, Dirichlet Sturm-Liouville problem, whereas the case $N > 1$ arises when searching for radially symmetric solutions of a PDE Dirichlet p-Laplacian problem on the unit ball in $\mathbb{R}^N$.

Under suitable additional assumptions on f we obtain a 'simple bifurcation' theorem giving C^1 curves of solutions bifurcating from trivial solutions at (weighted) eigenvalues of the p-Laplacian, and also obtain global C^1 curves of positive solutions. Similar results have been obtained before, but with the restriction that $p > 2$ for various differentiability reasons related to the p-Laplacian We extend these results to all $2 \neq p > 1$.

The crux of the proofs lies in extending various differentiability results for solution operators for the p-Laplacian to $2 \neq p > 1$ which had previously been obtained for $p > 2$.

1. Introduction

We consider the boundary-value problem

$$-(r^{N-1}\phi_p(u'(r)))' = \lambda r^{N-1} f(r, u(r)), \quad r \in (0,1), \tag{1.1}$$

$$BC_N(u) = (0,0), \tag{1.2}$$

where $N \geqslant 1$ is an integer, $p \in \mathbb{R}$ satisfies $2 \neq p > 1$, $\phi_p(s) := |s|^{p-1} \operatorname{sign} s$, $s \in \mathbb{R}$, $\lambda \geqslant 0$, and

$$BC_N(u) = \begin{cases} (u(0), u(1)), & \text{if } N = 1, \\ (u'(0), u(1)), & \text{if } N > 1. \end{cases}$$

We suppose that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ has the form

$$f(r, \xi) = g(r, \xi)\phi_p(\xi), \quad (r, \xi) \in [0,1] \times \mathbb{R}, \tag{1.3}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is C^1, and we suppose that

$$g_0(r) := g(r, 0) > 0, \quad r \in [0,1]. \tag{1.4}$$

This form of f means that f is C^1 if $p > 2$, but only C^0 if $1 < p < 2$. It also means that $u = 0$ is a (*trivial*) solution of (1.1)-(1.2) for any $\lambda \in \mathbb{R}$.

The case $N = 1$ is a standard, 1-dimensional, Dirichlet Sturm-Liouville problem, whereas the case $N > 1$ arises when searching for radially symmetric solutions of a PDE Dirichlet p-Laplacian problem (having a similar form to (1.1), with a PDE operator on the left hand side), on the unit ball in $\mathbb{R}^N$.

The problem (1.1)-(1.2) has been considered in a large number of papers, particularly in the case $N = 1$, $p = 2$. Among a variety of approaches to this problem, we mention that both local and global bifurcation theory, and degree theory, have been applied to yield continua of solutions (see [10] for a discussion of recent results using this type of approach, although the hypotheses in f are slightly different to those here). In addition, curves of solutions have been obtained using both quadrature and continuation methods (the latter relying on the implicit function theorem to 'continue' a curve). Our interest here is in the continuation approach. This approach is adopted in, for example, the papers [6, 7, 8, 12]; the case $N = 1$ is considered in [12], while the case $N > 1$ is considered in [6, 7, 8]. Each of these papers contains an extensive introductory survey of various aspects of the preceding literature on this problem. In particular, the paper [8] is, to some extent, a survey of various recent results. In view of this we will omit an extensive discussion here, and simply give a brief comparison of the results obtained here with preceding results.

Each of the papers [6, 7, 8] obtains a curve of solutions of the problem (1.1)-(1.2) (the curve is C^0 in [6] and C^1 in [7, 8]), essentially, by using a local bifurcation result to 'start off' a curve of solutions from a suitable trivial solution, and then using the implicit function theorem to continue this curve to 'infinity'; the paper [12] uses a similar approach, but under the main hypotheses on f in [12] the problem does not have 'trivial' solutions from which to bifurcate, so a different argument is used in [12] to 'start off' the curve. In each of these papers the problem is reformulated as a functional equation involving the inverse of the p-Laplacian Δ_p (expressed as an integral operator), and also involving ϕ_p and its inverse. The crux of the argument is to show that we can set up this reformulated problem in terms of a function which is sufficiently differentiable at solutions of the problem to apply the implicit function theorem at these solutions (or to derive a 'simple bifurcation' theorem).

When $p = 2$ the p-Laplacian is, of course, the usual linear Laplacian $\Delta = \Delta_2$, and $\phi_2(\xi) = \xi$. $\xi \in \mathbb{R}$. Hence, in this case, the inverse of Δ_2 is linear, so obtaining differentiability is, essentially, trivial, and so such continuation arguments have been used extensively to obtain results similar to those described here, and also a multitude of other results – for example, so called 'S-shaped' bifurcation diagrams. See [10] for a brief discussion of this case, with many references.

However, when $p \neq 2$, both Δ_p and ϕ_p are nonlinear, and finding a suitable formulation of the problem with the required differentiability is considerably more delicate (an obvious, but not the only, problem is that the function ϕ_p is not differentiable at $\xi = 0$ when $1 < p < 2$). In order to obtain the required differentiability it is assumed in each of the papers [6, 7, 8, 12] that $p > 2$ (the main results in [12] do not require this, but the differentiability assumptions in [12, Section 5], which considers a problem of the above form, do require this). In this paper we will remove this restriction and obtain a suitable formulation of the problem for all p satisfying $2 \neq p > 1$. We then obtain the results on simple bifurcation and existence of a curve of positive solutions alluded to above.

Of course, simple bifurcation and the implicit function theorem have many applications to a wide range of problems, so there are many other applications of these results (and the underlying differentiability results) which could be discussed, but for brevity we will not consider these. Similar methods could also be used to consider, for example, simple bifurcation from infinity (see [5] for the case $N = 1$, $p = 2$) or from half-eigenvalues of problems with jumping nonlinearities (Rabinowitz-type global bifurcation from half-eigenvalues for such problems, with $N = 1$ and $p > 1$, is discussed in [11]).

2. Preliminaries

2.1. **Notation.** For any integer $r \geqslant 0$, $C^r[0,1]$ will denote the standard Banach space of real valued, r-times continuously differentiable functions defined on $[0,1]$, with the norm $|u|_r = \sum_{i=0}^{r} |u^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[0,1]$ (throughout, all function spaces will be real). For any $q \geqslant 1$, $L^q(0,1)$ will denote the standard Banach space of real valued functions on $[0,1]$ whose qth power is integrable, with norm $\|\cdot\|_q$. We let $W^{1,q}(0,1)$, with norm $\|\cdot\|_{1,q}$, denote the usual Sobolev space of absolutely continuous functions u on $[0,1]$, with derivative $u' \in L^q(0,1)$.

If $F : X \to Z$ is a function between Banach spaces X and Z, then $DF(x) : X \to Z$ will denote the Fréchet derivative of F at $x \in X$; partial Fréchet derivatives will be indicated by subscripts, for example, $D_xG(x,y)$, $D_yG(x,y)$ will denote the partial derivatives of a function G depending on x and y.

If $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous then, for any $w \in C^0[0,1]$, we define $h(w) \in C^0[0,1]$ by

$$h(w)(r) := h(r, w(r)), \quad r \in [0,1].$$

Clearly, this 'Nemitskii' mapping $w \to h(w) : C^0[0,1] \to C^0[0,1]$ is continuous. When necessary, we will use the term 'Nemitskii mapping' to emphasize the distinction between these mappings and the underlying real-valued functions. In particular, we will repeatedly use the Nemitskii mapping $\phi_p : w \to \phi_p(w) : C^0[0,1] \to C^0[0,1]$. Also, it will be notationally convenient to write r^{N-1} for the Nemitskii mapping $r \to r^{N-1}$ (that is, r^{N-1} will denote a function, as well as a value).

2.2. **The p-Laplacian operator.** Letting

$$\mathcal{D}_p = \{u \in C^1[0,1] : u \text{ satisfies (1.2) and } r^{N-1}\phi_p(u') \in W^{1,1}(0,1)\} \ ,$$

we define the p-Laplacian operator $\Delta_p : \mathcal{D}_p \to L^1(0,1)$ by

$$\Delta_p(u) = (r^{N-1}\phi_p(u'))', \quad u \in \mathcal{D}_p.$$

This operator is $(p-1)$-*homogeneous*, that is, $\Delta_p(tu) = t^{p-1}\Delta_p(u)$, for any $t > 0$ and $u \in \mathcal{D}_p$. The problem (1.1)-(1.2), can now be rewritten as

$$-\Delta_p(u) = \lambda r^{N-1} g(u)\phi_p(u), \quad (\lambda, u) \in \mathbb{R} \times \mathcal{D}_p. \tag{2.1}$$

NB. Of course, most of the operators in this paper will depend on N (as well as on p), but we will regard this as fixed throughout and will not display it explicitly in the notation, except when it seems helpful to do so (e.g., in the boundary condition operator BC_N). We could, likewise, omit the dependence on p in the notation, but it is conventional to display this in the p-Laplacian operator Δ_p, so omitting it might cause confusion.

2.3. **A solution operator for Δ_p.** The following result is well known. When $N = 1$ it is proved in, for example, [2, Theorem 3.1] or [9, Theorem 20] (these references prove the result for periodic boundary conditions, but the proofs can readily be modified to deal with Dirichlet boundary conditions — in fact, the Dirichlet case is simpler, since in the periodic case the operator Δ_p has a 1-dimensional null-space consisting of the constant functions). When $N > 1$ see, for example, [7, Section 3].

Theorem 2.1. *For all $2 \neq p > 1$ the problem*

$$-\Delta_p(u) = r^{N-1}h, \quad h \in L^1(0,1), \tag{2.2}$$

has a unique solution $u = S_p(h) \in \mathcal{D}_p$. The operator $S_p : L^1(0,1) \to C^1[0,1]$ is continuous and p^*-homogeneous (where $p^* := (p-1)^{-1}$).

Remark 2.2. In the case $N > 1$, an explicit formula for the operator S_p is given in [7, Section 3] in terms of integral operators and the Nemitskii operator $\phi_{p^*+1} = \phi_p^{-1}$. This explicit formula is relatively easy to construct, but is notationally long-winded to describe so we will omit it here. The construction in [7] uses $C^0[0,1]$ as the domain of the operator S_p rather than $L^1(0,1)$, that is, $h \in C^0[0,1]$ in (2.2). However, it can be verified, using the explicit form of S_p in [7], that Theorem 2.1 holds with $h \in L^1(0,1)$ in (2.2). This extension of the domain of S_p to $L^1(0,1)$ will be crucial for the results below, in particular, for Theorem 3.1 (via Theorem 2.4). The explicit form for S_p also shows that the solution of (2.2) is C^1 at $r = 0$, which is perhaps not immediately obvious.

2.4. **Differentiability of the operators ϕ_p and S_p.** We now consider some differentiability properties of the above operators. We first consider the Nemitskii operator ϕ_p. When $1 < p < 2$, the function $\phi_p : \mathbb{R} \to \mathbb{R}$ is not C^1 at 0, which causes some difficulty in obtaining differentiability of the Nemitskii operator ϕ_p in this case.

Theorem 2.3 ([2, Lemma 3.8]). (*A*) *Suppose that $1 < p < 2$ and $h_0 \in C^1[0,1]$ has only simple zeros in $[0,1]$. Then there exists a neighbourhood V of h_0 in $C^1[0,1]$ such that if $h \in V$ then $|h|^{p-2} \in L^1(0,1)$ and $\phi_p : V \to L^1(0,1)$ is C^1, with derivative*

$$D\phi_p(h)\bar{h} = (p-1)|h|^{p-2}\bar{h}, \quad h \in V, \ \bar{h} \in C^1[0,1]. \tag{2.3}$$

(*B*) *Suppose that $p > 2$. Then $\phi_p : C^0[0,1] \to C^0[0,1]$ is C^1, with derivative given by (2.3) for any h, $\bar{h} \in C^0[0,1]$.*

Next, we consider the differentiability of the operator S_p. As mentioned in Remark 2.2, the explicit form of this operator (as described in [7]) contains the operator $\phi_{p^*+1} = \phi_p^{-1}$ and this causes some difficulty in obtaining differentiability of S_p when $1 < p^* + 1 < 2$, that is, when $p > 2$.

In fact, in this case we are unable to obtain differentiability of S_p as a mapping from $L^1(0,1)$ into $C^1[0,1]$ (the domain and range of S_p in Theorem 2.1), and we have to use alternative spaces.

Theorem 2.4. *For any $h \in L^1(0,1)$, let $u_h := S_p(h) \in C^1[0,1]$.*

(A) Suppose that $1 < p < 2$. Then $S_p : L^1(0,1) \to C^1[0,1]$ is C^1, and

$$h,\, \bar h \in L^1(0,1),\ v = DS_p(h)\bar h \in C^1[0,1] \implies \begin{cases} r^{N-1}|u_h'|^{p-2}v' \in W^{1,1}(0,1), \\ -(r^{N-1}|u_h'|^{p-2}v')' = p^* r^{N-1}\bar h, \\ BC_N(v) = (0,0). \end{cases} \tag{2.4}$$

(B) Suppose that $p > 2$ and $h_0 \in C^0[0,1]$ is such that if $u_{h_0}'(r) = 0$, for $r \in [0,1]$, then $h_0(r) \neq 0$. Then there exists a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that:

(a) $h \in V_0 \implies |u_h'|^{2-p} \in L^1(0,1)$;

(b) the mapping $h \to |u_h'|^{2-p} : V_0 \to L^1(0,1)$ is continuous;

(c) the mapping $S_p : V_0 \subset C^0[0,1] \to W^{1,1}(0,1)$ is C^1, and (2.4) holds for any $h \in V_0$, $\bar h \in C^0[0,1]$, with $v \in W^{1,1}(0,1)$.

NB. By (2.4), v' is continuous, except possibly at the zeros of u_h', and these are isolated (otherwise u_h would be trivial). When $N > 1$, the boundary condition (1.2) implies that $u_h'(0) = 0$; in this case, v satisfies the boundary condition in (2.4) at $r = 0$ in the sense that $\lim_{r\to 0} v'(r) = 0$.

Proof. When $N = 1$ the result is proved in [2, Theorem 3.4] for the periodic case; the proof in the Dirichlet case is similar. When $N > 1$ the result is proved in [7, Theorem 3.5], except that in Case (A) it is shown in [7] that $S_p : C^0[0,1] \to C^1[0,1]$ is C^1. However, the proof in [7] can be extended to give the result stated, and this will be crucial for the results below. A similar result is also described in Theorem 5 and Corollary 6 of [6].

We also note that when $N > 1$ and $p > 2$, the statement that $v = DS_p(h)\bar h$ satisfies the boundary condition $v'(0) = 0$ in (2.4) is delicate, since in this case the theorem only asserts that $v \in W^{1,1}(0,1)$ so, a priori, the value of $v'(0)$ does not seem to be well-defined. In fact, the arguments on p. 8 of [7] can be used to prove that, in this case, this boundary condition holds in the sense that $\lim_{r\to 0} v'(r) = 0$, as stated in the theorem. Unfortunately, the details given in [7] do not cover the case $p > 2$, but they can be adapted to do so as follows.

Since $u_{h_0} = S_p(h_0) \in C^1[0,1]$ satisfies (2.2), it follows that $u_{h_0}'(0) = 0$, and so $h_0(0) \neq 0$ (by the hypothesis in part (B)). Hence, we may suppose that V_0 is such that $h(0) \neq 0$ for all $h \in V_0$. Given this, for any fixed $h \in V_0$ the arguments in the proof of Theorem 3.5 on p. 8 of [7] can be used to show that there exists $\delta \in (0,1)$ and constants C_1, C_2 (which depend on h and $\bar h$) such that, for $r \in [0,\delta]$,

$$|u_{h_0}'(r)| \geqslant C_1 r^{p^*} \quad \text{(by the formula for } u_{h_0}'(r) \text{ four lines from the end of the proof in [7])},$$
$$|v'(r)| \leqslant C_2 r^{p^*} \quad \text{(by the estimate for } |v'(r)| \text{ five lines from the end of the proof in [7])},$$

which proves that $\lim_{r\to 0} v'(r) = 0$. We note that in [7] an inequality of the form $|u_{h_0}'(r)| \leqslant C_3 r^{p^*}$ is obtained, which is correct but is the wrong way round to derive the above inequality for $|v'(r)|$, when $p > 2$. □

2.5. **Eigenvalues of the p-Laplacian.** We briefly consider the weighted, nonlinear eigenvalue problem

$$-\Delta_p(v) = \lambda r^{N-1}\rho\phi_p(v), \quad v \in \mathcal{D}_p, \tag{2.5}$$

where $\lambda \in \mathbb{R}$ and the weight function $\rho \in L^1(0,1)$ satisfies $\rho \geqslant 0$ on $[0,1]$, and $\rho > 0$ on a set of positive Lebesgue measure in $[0,1]$. The following result is well-known, see, for example, [1, Section 3.1] (which deals with the case $N = 1$), and [1, Section 3.2] (which extends the results to the case $N > 1$). It is assumed in [1] that $\rho > 0$ a.e. in $[0,1]$, but the methods can be extended to deal with the above assumptions on ρ.

Lemma 2.5. *Under the above hypotheses on ρ, the eigenvalue problem* (2.5) *has a sequence of eigenvalues $0 < \lambda_0(\rho) < \lambda_1(\rho) < \dots$, accumulating at ∞. Each eigenvalue $\lambda_k(\rho)$, $k \geqslant 0$, has a corresponding normalised eigenfunction $v_k(\rho)$ with exactly k zeros in $(0,1)$, and all eigenfunctions of $\lambda_k(\rho)$ are multiples of $v_k(\rho)$. If ρ_1, ρ_2 satisfy the above hypotheses, with $\rho_1 \leqslant \rho_2$ on $[0,1]$ and $\rho_1 < \rho_2$ on a set of positive Lebesgue measure in $[0,1]$, then $\lambda_k(\rho_1) > \lambda_k(\rho_2)$, $k \geqslant 0$.*

2.6. **Solutions of** (1.1) **have simple zeros.** As a final 'preliminary', the following result will be useful below. It follows from [1, Lemma 3.1] and [1, Remark 3.2 (*iii*)].

Lemma 2.6. *Under the hypothesis* (1.3) *on the form of f, if $u \in C^1[0,1]$ is a non-trivial solution of the differential equation* (1.1) *then u has only simple zeros in $[0,1]$. In particular, if $N > 1$ and $u'(0) = 0$ then $u(0) \neq 0$. This result also holds for any eigenfunction v of* (2.5).

3. Differentiability of a solution operator for (2.1)

Of course, we are mainly interested in the differentiability of a suitable solution operator for the problem (2.1), and this will follow from Theorems 2.3 and 2.4. To describe this, in a single notational framework for the cases $1 < p < 2$ and $p > 2$, we introduce the following notation,

$$Y_p := \begin{cases} C^1[0,1], & 1 < p < 2, \\ W^{1,1}(0,1), & p > 2 \end{cases}$$

(for each $2 \neq p > 1$, Y_p is simply the codomain of S_p in the differentiability result Theorem 2.4), and we define a function $F : \mathbb{R} \times Y_p \to Y_p$ by

$$F(\lambda, u) := u - \lambda^{p^*} S_p(g(u)\phi_p(u)), \quad (\lambda, u) \in \mathbb{R} \times Y_p.$$

By Theorem 2.1, F is continuous, and equation (2.1) is equivalent to

$$F(\lambda, u) = 0, \quad (\lambda, u) \in \mathbb{R} \times Y_p. \tag{3.1}$$

The following theorem describes the differentiability of F at solutions of (3.1). The important point here is that this result is valid for all $2 \neq p > 1$, unlike the corresponding results in [6, 7, 8] which only hold for $p > 2$.

Theorem 3.1. *Suppose that $2 \neq p > 1$, and (λ_0, u_0) is a non-trivial solution of* (3.1)*. If $p > 2$, suppose also that if $u_0'(r) = 0$, for $r \in [0,1]$, then $g(r, u_0(r)) \neq 0$. Then the following results hold.*

- (*a*) *$F : \mathbb{R} \times Y_p \to Y_p$ is C^1 on a neighbourhood of (λ_0, u_0) in $\mathbb{R} \times Y_p$.*
- (*b*) *The derivative $D_uF(\lambda_0, u_0) : Y_p \to Y_p$ has the form $D_uF(\lambda_0, u_0) = I_p - \lambda^{p^*} K_p^0$, where $I_p : Y_p \to Y_p$ is the identity on Y_p and $K_p^0 : Y_p \to Y_p$ is given by*

$$K_p^0 \bar{w} := DS_p(g(u_0)\phi_p(u_0))\big[\big(g_\xi(u_0)\phi_p(u_0) + g(u_0)(p-1)|u_0|^{p-2}\big)\bar{w}\big], \quad \bar{w} \in Y_p. \tag{3.2}$$

- (*c*) *The operator K_p^0 is compact.*
- (*d*) *The derivative $D_uF(\lambda_0, u_0)$ is singular $\iff$ $\ker(D_uF(\lambda_0, u_0)) \neq \{0\}$.*
  Also, $\bar{w} \in \ker(D_uF(\lambda_0, u_0)) \iff \bar{w} \in W^{1,1}(0,1)$ satisfies the linear, Sturm-Liouville, boundary value problem

$$\begin{gathered} -(r^{N-1}|u_0'|^{p-2}\bar{w}')' = p^*\lambda r^{N-1}\big(g_\xi(u_0)\phi_p(u_0) + g(u_0)(p-1)|u_0|^{p-2}\big)\bar{w}, \\ BC_N(\bar{w}) = (0,0). \end{gathered} \tag{3.3}$$

Proof. Parts (*a*), (*b*) and (*d*) follow from Theorems 2.3 and 2.4, and the chain rule. We sketch some details. By Lemma 2.6, u_0 has only simple zeros in $[0,1]$. Hence, if $1 < p < 2$ then the required results follow immediately from parts (*A*) of Theorems 2.3 and 2.4. Now suppose that $p > 2$. In the notation of Theorem 2.4, we have $h_0 = g(u_0)\phi_p(u_0)$ and $u_{h_0} = u_0$, so the hypothesis in part (*B*) of Theorem 2.4 becomes

$$u_0'(r) = 0 \implies g(r, u_0(r))\phi_p(u_0(r)) \neq 0, \quad r \in [0,1],$$

and this is true by the hypothesis in the theorem and the simplicity of the zeros of u_0. So, the required results in this case follow from parts (*B*) of Theorems 2.3 and 2.4.

To prove part (c) we observe that, by the definition of K_p^0 in (3.2) (and the mapping properties of DS_p in Theorem 2.4), we can regard K_p^0 as a composition of continuous linear mappings as follows (where $\hookrightarrow$ denotes a compact embedding and $\to$ denotes a continuous mapping):

$$C^1[0,1] \hookrightarrow C^0[0,1] \to L^1(0,1) \to C^1[0,1], \qquad 1 < p < 2;$$
$$W^{1,1}(0,1) \hookrightarrow C^0[0,1] \to C^0[0,1] \to W^{1,1}(0,1), \qquad p > 2.$$

Hence, K_p^0 is compact. □

Remark 3.2. The coefficient functions in (3.3) satisfy the standard hypotheses in the theory of linear, second order, Sturm-Liouville problems with L^1 coefficients (see [4]). Specifically, $1/|u_0'|^{p-2} = |u_0'|^{2-p} \in L^1(0,1)$ (by Theorem 2.4) and $|u_0|^{p-2} \in L^1(0,1)$ (this is obvious when $p > 2$, and follows from the simplicity of the zeros of u_0 when $1 < p < 2$). Thus, when $N = 1$ the results of [4] hold for (3.3). When $N > 1$ this problem has an additional degeneracy at $r = 0$ due to the term r^{N-1}, but we will utilize the results of [4] on intervals of the form $[\delta, 1]$ for arbitrary $\delta \in (0,1)$.

In the following two sections we will use Theorem 3.1 (together with the implicit function theorem) to obtain a 'simple bifurcation' result, and curves of solutions of (2.1). As mentioned in the introduction, there are many other applications of this result.

4. Simple bifurcation for (2.1)

In this section we will obtain a 'simple bifurcation' result for the problem (2.1). For the rest of the section we consider some fixed $k \geqslant 0$, and let $\lambda_k = \lambda_k(g_0)$, $v_k = v_k(g_0)$, be the kth eigenvalue and (normalised) eigenfunction of the problem (2.5) with $\rho = g_0$ (recall Lemma 2.5), that is,

$$-\Delta_p(v_k) = \lambda_k r^{N-1} g_0 \phi_p(v_k), \tag{4.1}$$

and we search for a curve of non-trivial solutions (λ, u) of (2.1) bifurcating from $(\lambda_k, 0)$. Let

$$Z_p := \Big\{ z \in Y_p : \int_0^1 r^{N-1} g_0 \phi_p(v_k) z = 0 \Big\}.$$

It is clear that

$$Y_p = \operatorname{span}\{v_k\} \oplus Z_p. \tag{4.2}$$

We can now state the main result on simple bifurcation, in the spirit of the famous Crandall and Rabinowitz theorem in [3].

Theorem 4.1. *There exists $\epsilon > 0$, a neighbourhood U_p of $(\lambda_k, 0)$ in $\mathbb{R} \times Y_p$, and a C^1 mapping $s \to (\lambda(s), z(s)) : (-\epsilon, \epsilon) \to \mathbb{R} \times Z_p$, such that $(\lambda(0), z(0)) = (\lambda_k, 0)$ and*

$$\{(\lambda, u) \in U_p : u \neq 0 \text{ and (2.1) holds}\} = \{(\lambda(s), s(v_k + z(s))) : 0 \neq s \in (-\epsilon, \epsilon)\}. \tag{4.3}$$

Proof. Using the decomposition (4.2), we will search for non-trivial solutions of (3.1) (and hence of (2.1)) having the form

$$(\lambda, u) = (\lambda, s(v_k + z)), \quad 0 \neq s \in \mathbb{R},\ z \in Z_p \tag{4.4}$$

(it will be seen below, in the paragraph following the proof of Proposition 4.2, that this in fact yields all solutions of (3.1) in a neighbourhood of $(\lambda_k, 0)$ in $\mathbb{R} \times Y_p$). Substituting this form for (λ, u) into (3.1) and dividing by s yields

$$v_k + z - \lambda^{p^*} S_p\big(g(s(v_k + z))\phi_p(v_k + z)\big) = 0, \quad 0 \neq s \in \mathbb{R},\ z \in Z_p.$$

In view of this, we define $G : \mathbb{R}^2 \times Z_p \to Y_p$ by

$$G(s, \lambda, z) := \begin{cases} v_k + z - \lambda^{p^*} S_p\big(g(s(v_k + z))\phi_p(v_k + z)\big), & s \neq 0, \\ v_k + z - \lambda^{p^*} S_p\big(g_0 \phi_p(v_k + z)\big), & s = 0, \end{cases}$$

for $(\lambda, z) \in \mathbb{R} \times Z_p$. It follows from Theorem 2.1 that G is continuous, and by (4.1),

$$G(0, \lambda_k, 0) = 0. \tag{4.5}$$

Also, by construction, any solution (s, λ, z) of

$$G(s, \lambda, z) = 0, \quad 0 \neq s \in \mathbb{R},\ (\lambda, z) \in \mathbb{R} \times Z_p, \tag{4.6}$$

yields a non-trivial solution (λ, u) of (2.1), via (4.4).

In addition, a slight extension of the proof of Theorem 3.1 shows that G is C^1 in a neighbourhood of $(0, \lambda_k, 0)$ in $\mathbb{R}^2 \times Z_p$ (condition (1.4) ensures that the hypothesis on g in Theorem 3.1 is satisfied here), and the partial derivatives with respect to λ and z, at $(0, \lambda_k, 0)$, are given by

$$D_z G(0, \lambda_k, 0)\bar{z} = \bar{z} - \lambda_k^{p^*} DS_p\big(g_0 \phi_p(v_k)\big)\big((p-1) g_0 |v_k|^{p-2} \bar{z}\big), \quad \bar{z} \in Z_p,$$
$$D_\lambda G(0, \lambda_k, 0)\bar{\lambda} = -\frac{p^*}{\lambda_k}\bar{\lambda} v_k, \quad \bar{\lambda} \in \mathbb{R}.$$

The latter result follows immediately from the form of G, together with the definition of S_p and (4.1). In fact, these show that

$$S_p\big(t g_0 \phi_p(v_k)\big) = \left(\frac{t}{\lambda_k}\right)^{p^*} v_k, \quad t > 0,$$

and differentiating this identity with respect to t at $t = 1$ yields

$$DS_p\big(g_0 \phi_p(v_k)\big)(g_0 \phi_p(v_k)) = \frac{p^*}{\lambda_k^{p^*}} v_k, \tag{4.7}$$

which will be useful below.

The following proposition now describes the set of solutions of (4.6) in a neighbourhood of $(0, \lambda_k, 0)$.

Proposition 4.2. *There exists $\epsilon > 0$, a neighbourhood V_p of $(\lambda_k, 0)$ in $\mathbb{R} \times Z_p$, and a C^1 mapping $s \to (\lambda(s), z(s)) : (-\epsilon, \epsilon) \to V_p$ such that $(\lambda(0), z(0)) = (\lambda_k, 0)$ and*

$$\{(s, \lambda, z) \in (-\epsilon, \epsilon) \times V_p : (4.6)\ \textit{holds}\} = \{(s, \lambda(s), z(s)) : 0 \neq s \in (-\epsilon, \epsilon)\}. \tag{4.8}$$

Proof. In the case $p = 2$ the theorem, and the overall strategy of the proof, dates back to [3]. However, the details here, with $p \neq 2$, are more akin to the proofs of [6, Theorem 1] and [7, Lemma 5.1]. These papers consider the case $p > 2$, but the proof here is valid for $2 \neq p > 1$, so we provide the details necessary to demonstrate this — we will omit most of the details that are the same as in [6] and [7]. The idea of linearising about the eigenfunction, instead of at $u = 0$, was first used in [6].

We will apply the implicit function theorem to equation (4.6), in a neighbourhood of the point $(\lambda_k, 0)$, so we need to show that the linear operator $D_{(\lambda,z)}G(0, \lambda_k, 0) : \mathbb{R} \times Z_p \to Y_p$ is non-singular (that is, it is a bounded, linear isomorphism). To do this, we first define a linear operator $L : Z_p \to Y_p$ by

$$L\bar{z} := \lambda_k^{p^*} DS_p\big(g_0 \phi_p(v_k)\big)\big((p-1) g_0 |v_k|^{p-2} \bar{z}\big), \quad \bar{z} \in Z_p \tag{4.9}$$

(so that $D_z G(0, \lambda_k, 0) = I_p - L$). By the proof of Theorem 3.1, L is compact.

Lemma 4.3. *The range $\operatorname{ran}(L) \subset Z_p$, that is, L is invariant on Z_p.*

Proof. Let $\bar{z} \in Z_p$ be arbitrary, and define $z := L\bar{z}$. We will show that $z \in Z_p$. By (2.4), (4.1) and (4.9),

$$-(r^{N-1} |v_k'|^{p-2} z')' = \lambda_k r^{N-1} g_0 |v_k|^{p-2} \bar{z}, \quad \text{on } (0, 1), \tag{4.10}$$
$$BC_N(z) = (0, 0). \tag{4.11}$$

Also, by (2.4) and the fact that $v_k \in \mathcal{D}_p$, we have

$$v_k \in C^1[0, 1], \quad z,\ r^{N-1}\phi_p(v_k'),\ r^{N-1}|v_k'|^{p-2} z' \in W^{1,1}(0, 1), \tag{4.12}$$

so we can multiply both sides of (4.10) by v_k, use the fact that $\bar{z} \in Z_p$, and integrate by parts twice (it will be shown below that the boundary terms that arise in this process are zero — see (4.13)) to obtain

$$\begin{aligned} 0 &= \int_0^1 \lambda_k r^{N-1} g_0 \phi_p(v_k) \bar{z} = -\int_0^1 \big(r^{N-1}|v_k'|^{p-2}\, z'\big)' v_k \\ &= -\big[r^{N-1}|v_k'|^{p-2}\, z' v_k\big]_0^1 + \int_0^1 r^{N-1}\phi_p(v_k')\, z' \\ &= \big[r^{N-1}\phi_p(v_k')\, z\big]_0^1 - \int_0^1 \big(r^{N-1}\phi_p(v_k')\big)' z \qquad \text{(by (4.13))} \\ &= \int_0^1 \lambda_k r^{N-1} g_0 \phi_p(v_k) z \qquad \text{(by (4.1), (4.13))} \end{aligned}$$

which shows that $z \in Z_p$, and hence that $R(L) \subset Z_p$. So, to complete the proof of Lemma 4.3, we must prove that the boundary terms in the above calculation satisfy

$$\big[r^{N-1}|v_k'|^{p-2}\, z' v_k\big]_0^1 = 0, \quad \big[r^{N-1}\phi_p(v_k')\, z\big]_0^1 = 0, \quad N \geqslant 1, \quad 2 \neq p > 1. \tag{4.13}$$

We consider the values $N = 1$ and $N > 1$ separately. Some of the cases to consider are simple, but we write them all out for completeness. In each of these cases we use the properties in (4.12), without further comment.

<u>$N = 1$</u>

For $2 \neq p > 1$:

$$\begin{aligned} &v_k(0) = v_k(1) = 0 \implies \big[|v_k'|^{p-2}\, z' v_k\big]_0^1 = 0; \\ &z(0) = z(1) = 0 \implies \big[\phi_p(v_k')\, z\big]_0^1 = 0. \end{aligned}$$

<u>$N > 1$</u>

For $2 \neq p > 1$: $z(1) = 0 \implies \big[r^{N-1}\phi_p(v_k')\, z\big]_0^1 = 0.$

For $p > 2$: $z'(0) = 0$ (by Theorem 2.4 (B) (c)) and $v_k(1) = 0 \implies \big[r^{N-1}|v_k'|^{p-2}\, z' v_k\big]_0^1 = 0.$

When $1 < p < 2$, this latter boundary term is harder to deal with (which seems to be one of the reasons this case is not considered in [6]-[8], see [7, Remark 5.3]). We require an estimate on the behaviour of v_k' and z' near to $r = 0$, which can in fact be obtained from the results of [7]. The argument on p. 8 of [7] shows that in this case

$$v_{h,\bar{h}} := DS_p(h)\bar{h} \ \text{ for } h,\, \bar{h} \in C^0[0,1] \implies |v_{h,\bar{h}}'(r)| \leqslant C_{h,\bar{h}}\, r^{p^*}, \quad r \in [0,1], \tag{4.14}$$

for some constant $C_{h,\bar{h}}$ (which depends on h and $\bar{h}$). By (4.7) and the definition of L in (4.9), both v_k and $z = L\bar{z}$ have the form of $v_{h,\bar{h}}$ in (4.14), except that the function $(p-1)g_0|v_k|^{p-2}\bar{z}$ corresponding to $\bar{h}$ in (4.9) may be singular at zeros of v_k. However, v_k is continuous and $v_k(0) \neq 0$ (by Lemma 2.6), so repeating the calculations on p. 8 of [7], near to $r = 0$, shows that

$$|v_k'(r)| + |z'(r)| \leqslant C r^{p^*}, \quad r \in [0,\delta], \tag{4.15}$$

for some constant C and $\delta \in (0,1]$. Hence:

(4.15), $v_k(1) = 0$ and $v_k'(1) \neq 0 \implies \big[r^{N-1}|v_k'|^{p-2}\, z' v_k\big]_0^1 = 0.$

This completes the proof of (4.13), and hence of Lemma 4.3. □

Lemma 4.4. *The kernel* $\ker(I_p - L) = \{0\}$.

Proof. Suppose that $w \in \ker(I_p - L)$, that is, $w = Lw$, for some $w \in Z_p$. Then, by (4.10)-(4.11), w satisfies the linear, Sturm-Liouville boundary value problem

$$-(r^{N-1}|v_k'|^{p-2}w')' = \lambda_k r^{N-1} g_0 |v_k|^{p-2} w, \quad \text{on } (0,1), \tag{4.16}$$

$$BC_N(w) = (0,0). \tag{4.17}$$

However, the eigenfunction v_k also satisfies this problem (recall (4.1)), so we claim that $w = cv_k$, for some constant c. In fact, this follows from the standard uniqueness of solutions of the initial value problem consisting of equation (4.16) on $(\delta, 1]$, for arbitrary $\delta \in (0,1)$, together with the 'initial conditions' $w(1) = 0$, $w'(1) = \alpha$, for arbitrary $\alpha \in \mathbb{R}$ (see Problem 1 in Chapter 3 of [4] and recalling Remark 3.2). But $w \in Z_p$, so the decomposition (4.2) now implies that $c = 0$, and hence $w = 0$. □

It now follows from the compactness of L, together with Lemmas 4.3 and 4.4, that the operator $I_p - L : Z_p \to Z_p$ is non-singular, so the operator

$$(\bar\lambda, \bar z) \to D_{(\lambda,z)} G(0, \lambda_k, 0)(\bar\lambda, \bar z) = -\frac{p^*}{\lambda_k} \bar\lambda v_k + (I_p - L)\bar z : \mathbb{R} \times Z_p \to Y_p = \operatorname{span}\{v_k\} \oplus Z_p$$

is non-singular. Combining this with (4.5) and the implicit function theorem now yields the desired C^1 curve of solutions of equation (4.6), which proves (4.8), and so completes the proof of Proposition 4.2. □

Next, Proposition 4.2 together with the definition of G shows that the set on the left hand side of (4.3) contains the set on the right hand side. An argument similar to that on p. 39 of [6] proves the reverse inclusion. This completes the proof of Theorem 4.1. □

When $p > 2$, Theorem 4.1 obtains a C^1 solution curve in the space $Y_p = W^{1,1}(0,1)$. However, by Theorem 2.1, the solutions are actually in $C^1[0,1]$ – we state this in the following corollary.

Corollary 4.5. *When $p > 2$ the C^1 function $z(\cdot) : (-\epsilon, \epsilon) \to Y_p = W^{1,1}(0,1)$ given by Theorem 4.1 is continuous into $C^1[0,1]$.*

Proof. To show this we substitute the solutions $(\lambda(s), s(v_k + z(s)))$, $s \in (-\epsilon, \epsilon)$, into (2.1) and rearrange to yield

$$z(s) = -v_k + S_p\big(\lambda(s)\, g(s(v_k + z(s)))\, \phi_p(v_k + z(s))\big), \quad s \in (-\epsilon, \epsilon). \tag{4.18}$$

The result now follows from the continuity properties of ϕ_p and S_p (see Theorem 2.1). □

Given a solution u of (2.1), the number of (nodal) zeros of u in $(0,1)$ is often of interest. Lemma 2.5 and Corollary 4.5 immediately yield the following result.

Corollary 4.6. *For $2 \ne p > 1$, if $s \ne 0$ is sufficiently small then the solution $s(v_k + z(s)) \in C^1[0,1]$ of* (2.1) *given by Theorem 4.1 has only simple zeros in* $[0,1]$*, and exactly k such zeros in* $(0,1)$*. In particular, if $k = 0$ then this solution has no zeros in* $(0,1)$*, and so is of one sign.*

Remark 4.7. When $p > 2$, Theorem 4.1 is similar to [6, Theorem 1] and [7, Lemma 5.1] (although [6] considers a problem of the form $-\Delta_p(u) = r^{N-1}(\lambda\phi_p(u) + f(u))$, rather than (2.1)), but it holds for all $2 \ne p > 1$. However, Theorem 4.1 obtains a C^1 curve of solutions, whereas C^0 curves are obtained in [6] and [7]. The improved differentiability here follows from our assumption that the function f in (1.1) has the form (1.3), with a C^1 function g. Such an f is slightly smoother at $u = 0$ than the assumptions in [6] or [7] imply. We could use the same hypotheses on f as in [7], and we would then obtain a C^0 curve of solutions (using the form of the implicit function theorem in [3, Appendix A] to give a continuous curve of solutions of equation (4.6), as in [6] and [7]). We also observe that [6, Remark 4] notes that slightly stronger smoothness conditions on the nonlinearity considered in [6] yield improved smoothness of the solution curves found there.

5. Smooth curves of positive solutions

In this section we consider the set of 'positive' solutions of (2.1),

$$\mathcal{S}^+ := \{(\lambda, u) \in (0, \infty) \times \mathcal{D}_p : (2.1) \text{ holds, with } u \neq 0 \text{ and } u \geqslant 0\},$$

and we will show that, under suitable conditions on g, the set S^+ forms a C^1 curve in the space $(0, \infty) \times Y_p$.

We suppose that, in addition to the previous hypotheses, the function g satisfies

$$g(r, \xi) > 0, \quad g_\xi(r, \xi) \leqslant 0, \quad (r, \xi) \in [0, 1] \times (0, \infty), \tag{5.1}$$

$$g_\xi(r, \xi) < 0, \quad (r, \xi) \in E \subset [0, 1] \times (0, \infty), \tag{5.2}$$

where the set E has one of the following forms:

$$E = P \times [0, \infty) \text{ and } P \subset [0, 1] \text{ has positive measure;}$$
$$E = (1 - \delta, 1) \times (0, \delta), \text{ for some } \delta > 0.$$

When $N = 1$ the set E could also have the form $E = (0, \delta) \times (0, \delta)$, for some $\delta > 0$.

It follows from (5.1) that, for each $r \in [0, 1]$, the function $g(r, \cdot)$ is decreasing on $[0, \infty)$, and bounded below by 0, so the following limits exist

$$g_0(r) \geqslant g_\infty(r) := \lim_{\xi \to \infty} g(r, \xi) \geqslant 0, \quad r \in [0, 1], \tag{5.3}$$

with $g_\infty \in L^\infty(0, 1)$. In addition, (5.2) implies that $g_0 > g_\infty$ on a set of positive measure. So, by Lemma 2.5, we may define

$$\lambda_{\max} := \begin{cases} \lambda_0(g_\infty) < \infty, & \text{if } g_\infty \neq 0 \ \ (\text{in } L^\infty(0, 1)), \\ \infty, & \text{if } g_\infty = 0 \ \ (\text{in } L^\infty(0, 1)), \end{cases}$$

$$\lambda_{\min} := \lambda_0(g_0) < \lambda_{\max}.$$

We now have the following basic properties of positive solutions.

Lemma 5.1. *If $(\lambda, u) \in \mathcal{S}^+$ then:*
- (a) $u > 0$ *on* $(0, 1)$;
- (b) *if* $N = 1$*:* $u'(0) > 0$, $u'(1) < 0$; *if* $N > 1$*:* $u(0) > 0$, $u'(1) < 0$;
- (c) $\lambda \in (\lambda_{\min}, \lambda_{\max})$.

Proof. Properties (a) and (b) follow readily from (1.1) and (1.2) and the properties of g, see [7, Proposition 4.1] and [12, Lemma 5.1]. Next, by (5.1),

$$g_\infty \leqslant g(u) \leqslant g_0, \tag{5.4}$$

and, by (5.2) and properties (a) and (b) (together with the boundary condition $u(1) = 0$), these inequalities are strict on (possibly different) sets of positive measure, so (c) follows from Lemma 2.5. □

We now state the main result of this section.

Theorem 5.2. *Suppose that $2 \neq p > 1$ and g satisfies (5.1)–(5.2). Then there exists a C^1 function $u : (\lambda_{\min}, \lambda_{\max}) \to Y_p$ such that:*
- (a) $\lim_{\lambda \searrow \lambda_{\min}} |u(\lambda)|_0 = 0$ *and* $\lim_{\lambda \nearrow \lambda_{\max}} |u(\lambda)|_0 = \infty$;
- (b) *if $\lambda \in (\lambda_{\min}, \lambda_{\max})$ then $u(\lambda)$ is the unique positive solution of* (2.1);
- (c) *if $\lambda \notin (\lambda_{\min}, \lambda_{\max})$ then* (2.1) *has no positive solution.*

Hence, $\mathcal{S}^+ \subset (0, \infty) \times Y_p$ consists precisely of the C^1 curve of solutions $\{(\lambda, u(\lambda)) : \lambda \in (\lambda_{\min}, \lambda_{\max})\}$.

Remark 5.3. Theorem 5.2 is similar to [12, Theorem 5.2] (with $N = 1$, $p > 2$) and [7, Theorem 2.3] (with $N > 1$, $p > 2$), but it holds for $N \geqslant 1$ and all $2 \neq p > 1$. We also mention that [7] also obtains curves of negative solutions by imposing similar conditions to (5.1)–(5.2) when $\xi < 0$. For brevity we omit this here.

Proof. The proof is similar to the proofs of [7, Theorem 2.3] and [12, Theorem 5.2], so we will omit most of the details. The crux is to apply the implicit function theorem to a suitable formulation of the problem at any solution $(\lambda, u) \in \mathcal{S}^+$ to obtain a local curve of solutions passing through (λ, u). This is the point at which [7] and [12] required $p > 2$.

We first note that to find solutions $(\lambda, u) \in \mathcal{S}^+$ it suffices to consider equation (3.1) on $(0, \infty) \times Y_p$, and by Theorem 3.1 the function $F : (0, \infty) \times Y_p \to Y_p$ is C^1 in a neighbourhood of any $(\lambda, u) \in \mathcal{S}^+$, so we may attempt to apply the implicit function theorem at (λ, u). To do this we need the following result.

Proposition 5.4. *If $(\lambda, u) \in \mathcal{S}^+$ then $D_uF(\lambda, u) : Y_p \to Y_p$ is non-singular.*

Proof. By the compactness results of Theorem 3.1 it suffices to show that $\ker D_uF(\lambda, u) = \{0\}$. The argument to show this is as in the proofs of [12, Proposition 3.5] (when $N = 1$) and [7, Lemma 6.1] (when $N > 1$). □

The remainder of the proof is now, essentially, as in [7] and [12]. Briefly (see [7] and [12] for details):

- the above results show that, given a point $(\bar\lambda, \bar u) \in \mathcal{S}^+$, we can construct a C^1 solution curve (parametrised by λ) passing through $(\bar\lambda, \bar u)$ and extending to cover a maximal open interval $I \subset (\lambda_{\min}, \lambda_{\max})$; standard arguments (using the assumptions on g) then show that $I = (\lambda_{\min}, \lambda_{\max})$ and the solution curve must satisfy the properties described in part (a) of the theorem;
- by Theorem 4.1 there exists a unique curve of positive solutions in $\mathcal{S}^+$, in a neighbourhood of $(\lambda_0, 0)$, so this must be parametrised by λ and defined over some interval $(\lambda_0, \lambda_0 + \epsilon)$;
- combining these two results produces a unique solution curve having the desired properties.

This completes the (sketched) proof of Theorem 5.2. □

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