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Option prices under liquidity risk as weak solutions of semilinear diffusion equations

M. A. Fahrenwaldt and A. F. Roch

Abstract. Prices of financial options in a market with liquidity risk are shown to be weak solutions of a class of semilinear parabolic partial differential equations with nonnegative characteristic form. We prove the existence and uniqueness of such solutions, and then show the solutions correspond to option prices as defined in terms of replication in a probabilistic setup. We obtain an asymptotic representation of the price and the hedging strategy as a liquidity parameter converges to zero.

Mathematics Subject Classification. Primary 35K65, Secondary 91G20.

Keywords. Liquidity, Option pricing, Degenerate parabolic partial differential equations, Weak convergence.

1. Introduction

Nowadays, risk cannot be efficiently managed without taking into account liquidity risk. An important aspect of risk management for financial institutions is to understand the effect of liquidity on the pricing and hedging of derivative securities. The definition of liquidity depends on the market structure and the financial or economic questions being studied. For instance, in a limit order book market, one can measure liquidity by quantifying the number of shares being offered at each price, or the resilience of the order book after a large trade. On the other hand, to study liquidity in a dealers market, one must typically consider some notion of information flow, noise trading and utility functions for the market makers.

In this paper, we consider liquidity in terms of the depth of the market, namely the impact that trades have on prices. We give a framework for the problem of option pricing in a large trader model with a partial differential equation (PDE) perspective. We analyse the limit as the market becomes

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infinitely liquid (the impact of a trade becomes infinitely small) and provide first-order asymptotics of the option price and the hedging strategy in a liquidity parameter.

For the economic model we follow Jarrow and Roch [29] and consider a variation based on Brownian motion of the larger trader model of Bank and Baum [3] in which the dynamics of the price process depends specifically on the current holdings of the investor. The price process depends on economic variables which follow a degenerate diffusion process. Unlike [3] however we do not specifically assume the existence of a local martingale measure for all primal processes (Assumption 3 in [3]). The consequence is a nonlinear term in the wealth equation and a liquidity premium in option prices. Our analysis also has some common features with the liquidity cost model of [10], and in particular the Taylor expansion of the super-hedging cost of [39] who use the PDE characterisation of [11]. The main difference however is that in these models the price impact is momentary, whereas we regard it as a longer-lasting phenomenon.

In mathematical terms we derive a semilinear parabolic PDE on a bounded domain in $\mathbb{R}^n$ given as

$$\partial_\tau u(\tau, x) = \mathcal{L}(\tau, x)u(\tau, x) + F(\tau, x, \sigma^\top Du) \tag{1}$$

where the second-order partial differential operator is in divergence form

$$\mathcal{L}(\tau, x) = \sum_{i,j=1}^{n} \partial_i[\sigma\sigma^\top]_{ij}(\tau, x)\partial_j + \sum_{i=1}^{n} b_i(\tau, x)\partial_i$$

for a matrix σ and a vector b. We assume that the quadratic form defined by $\sigma\sigma^\top$, i.e. the characteristic form, is nonnegative. Moreover, the error term F, representing the liquidity cost, is quadratic in $\sigma^\top Du$. The characteristic form being nonnegative as opposed to bounded below by a positive number (i.e., uniform ellipticity of the operator $\mathcal{L}$) is a type of degeneracy.

Using variational methods we show existence and uniqueness of weak solutions of (1). We work with the concept of weak solutions for the simple reason that it allows us to obtain information about the growth of the gradient $\sigma^\top Du$ in an L^2-space. Many authors have approached the problem of existence and uniqueness of PDEs, and the relation to backward stochastic differential equations (BSDEs) with the theory of viscosity solutions [13, 23], but this type of solution is not differentiable in general.

A clear advantage of the weak solution approach is that we obtain the (economically crucial) hedging strategy as the gradient of the solution, which allows us to show that the hedging strategy converges to the hedging strategy in a frictionless setting as the market becomes more liquid. Finally, we investigate the regularity of the option price u and $\sigma^\top Du$ and show that these functions are Hölder continuous on certain subdomains.

We briefly summarise the related PDE literature distinguishing between analytic and stochastic perspectives. Due to the vastness of the literature we mainly concentrate on semilinear PDEs with quadratic growth in the gradient noting that other types of growth have also received considerable attention.

The reader is invited to consult the cited papers for further references. We remark that the study of the well-posedness of uniformly parabolic semilinear problems is standard, cf. [2,33,35], and for the case of nonnegative characteristic form we refer to [36]. To paraphrase [4], this is, however, not the kind of degeneracy that has received wide attention compared to the type found in quasilinear equations where the coefficients of the second order operator depend on u and Du.

To the best of our knowledge there is no treatment of parabolic PDEs of nonnegative characteristic form with error terms of quadratic growth. In [28], the author treats quasilinear PDEs of a special structure and growth conditions which do not cover (1). Closely related to our problem is [44] which which treats existence, uniqueness and regularity of weak solutions of PDEs of type (1), however under a uniform ellipticity assumption. More recent attention focused on the quasilinear case under a coerciveness assumption, cf. [7,8,21]. A quasilinear initial-boundary-value problem also in an economic context using the assumption of uniformly coercive operators was studied using weak solutions in [16].

From a stochastic perspective, the connection between BSDEs and semilinear partial differential equations goes back to [38] and was further exploited in terms of classical and viscosity solutions in [37]. The case of quadratic growth in the gradient was considered in [30] in two ways: the elliptic Dirichlet problem for a uniformly elliptic operator on a bounded domain was treated using weak solutions and the Cauchy problem on Euclidean space using viscosity solutions. Important for our paper is the seminal [5] where the link between BSDEs and weak solutions of the associated PDEs is explored. Here, the generator is assumed to be at most linear in the gradient Du and the authors consider the Cauchy problem on Euclidean space.

In the more recent literature, a generator $F(\tau, x, u, Du)$ with at most quadratic growth in u was treated in [40] using weak solutions on Euclidean space. Linear degenerate initial-boundary value problems on unbounded domains (also motivated by financial mathematics) are considered in [19,20] and further papers by the same authors.

This paper is organised as follows. The next section introduces the notation. Section 3 develops the probabilistic formulation for our liquidity models. The key results of the PDE analysis are stated and interpreted in Sect. 4 with detailed proofs contained in Sect. 5.

2. Notation

We introduce certain spaces and pieces of notation. The reader is referred to [1,18] for further details and proofs.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, denote by $\mathrm{vol}(\Omega)$ the Lebesgue measure of Ω and define $Q_T = [0, T] \times \Omega$ to be the parabolic cylinder. We denote by $C(\Omega)$ the linear space of bounded continuous functions on Ω and by $C_c(\Omega)$ the linear space of continuous functions that have compact support. Both are metric spaces under the usual sup norm $||f||_\infty = \sup_{x\in\Omega} |f(x)|$, analogously for $C(Q_T)$ under the norm $||f||_\infty = \sup_{(t,x)\in Q_T} |f(t,x)|$.

For $k \in \mathbb{N}$ we let $C^k(\Omega)$ be the space of k times continuously differentiable functions with norm $||f||_{C_b^k(\Omega)} = \sum_{|\alpha|\leq k} ||D^\alpha f||_\infty$. Here, $D^\alpha = (D_1^{\alpha_1}, \ldots, D_n^{\alpha_n})$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D_i = \partial_i = \partial/\partial_i$ denotes differentiation with respect to the ith coordinate. By $D = (D_1^1, \ldots, D_n^1)$ we denote the gradient operator. The analogous notation applies for functions of time and space where we have the function spaces $C^{k,l}([0,T]\times\Omega)$.

Let δ be a metric on Ω. The Hölder continuous functions with exponent $\alpha \in (0,1)$ are given by

$$C^\alpha(\Omega) = \left\{ f \in C_b(\Omega) \middle| [f]_\alpha = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{\delta(x,y)^\alpha} < \infty \right\}$$

with norm $||f||_{C^\alpha(\Omega)} = ||f||_\infty + [f]_\alpha$. Typically one uses the metric $\delta(x,y) = |x-y|$ with the Euclidean distance between x and y. The analogous definition holds on Q_T where we mention the special metric

$$\delta\left((\tau_1,x_1),(\tau_2,x_2)\right) = \max\left\{ |\tau_1-\tau_2|^{1/2}, |x_1-x_2| \right\}, \tag{2}$$

the so-called *parabolic metric*.

The space $C_c^\infty(\Omega)$ is the space of test functions, i.e. smooth functions of compact support. The Lebesgue spaces $L^p(\Omega)$ are defined as the set of equivalence classes of measurable real-valued functions f on Ω such that $||f||_p = \int_\Omega |f(x)|^p dx$ is finite; similarly for $L^p(Q_T)$. By $L^\infty(\Omega)$ we denote the space of measurable functions with the essential supremum norm.

Given $f, g \in L^1(\Omega)$ and α a multi-index, we say $g = D^\alpha f$ in a *weak sense* (αth weak partial derivative) if $\int_\Omega f D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega g\varphi dx$ for all $\varphi \in C_c^\infty(\Omega)$. The Sobolev spaces $W^{k,p}(\Omega)$ for $k \geq 1$ and $1 \leq p < \infty$ are then defined as the set of measurable functions f with weak derivatives $D^\alpha f$ up to order k such that the norm $||f||^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} ||D^\alpha f||_p^p$ is finite where α is a multi-index. By $W_0^{k,p}(\Omega)$ we denote the completion of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

For $\mathfrak{X}$ a Banach space of real-valued functions on Ω we define the space $C([0,T];\mathfrak{X})$ to be the set of functions $f : Q_T \to \mathbb{R}$ such that $t \mapsto f(t)$ is continuous from $[0,T]$ to $\mathfrak{X}$ under the norm $\sup_t ||f(t)||_\mathfrak{X}$, analogously for $L^p([0,T];\mathfrak{X})$.

We denote the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$. The inner product in $L^2(Q_T)$ (and by abuse of notation also in $L^2(Q_T)\otimes\mathbb{R}^d$) is denoted by $\langle\cdot,\cdot\rangle$.

Given a bounded domain Ω with C^1-boundary $\partial\Omega$ we have the integration by parts formula

$$\int_\Omega (\partial_i f) g dx = -\int_\Omega f(\partial_i g) dx + \int_{\partial\Omega} f g \nu^i dS$$

with $\nu = (\nu^1, \ldots, \nu^n)$ the outward pointing unit normal field and dS a surface element.

Given a second-order linear differential operator

$$\mathcal{L}(\tau) = \sum_{i,j=1}^{n} a_{ij}(\tau, x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(\tau, x)\partial_i \tag{3}$$

with suitable coefficient functions a_{ij}, b_i, we say that $\mathcal{L}(\tau)$ is *uniformly elliptic on* Ω if the quadratic form given on $Q_T \times \mathbb{R}^n$ defined by $L(\tau, x, \xi) = \sum a_{ij}(\tau, x)\xi_i\xi_j$ is uniformly bounded below, i.e. there is $\nu > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(\tau, x)\xi_i\xi_j \geq \nu|\xi|^2 \tag{4}$$

for all $\tau \in [0, T]$ and $(x, \xi) \in \Omega \times \mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

3. Probabilistic setup

We consider an investor who incurs liquidity costs due to the trading impact on prices of a risky asset. We present the price impacts model and define the replication problem for which the solution of the PDE will be shown to be associated to option prices.

3.1. The price impact model

Following [29], we consider a multi-asset variation of the large trader model of [3], which we base on Brownian motion. The interest rate is constant, and for simplicity we only consider discounted prices. We consider a market that consists of d traded risky assets.

Economic variables. We first define an n-dimensional process $(X_t)_{t\geq 0}$ that represents all economically relevant variables, i.e. fundamentals prices, volatilities, interest rates, market liquidity, etc. To this end let W be a d-dimensional Brownian motion, and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ its filtration. The process (X_t) is the unique strong solution of

$$dX_t = \beta(t, X_t)dt + \sigma(t, X_t)dW_t \quad (t \geq 0),$$

with β and σ Lipschitz continuous in the second argument uniformly t. Here, β is $\mathbb{R}^n$-valued and σ takes values in the $n \times d$-matrices over $\mathbb{R}$.

Price of risky assets. We assume there is a family of d-dimensional stochastic processes $S(\theta) = (S(t, X_t; \theta))_{t\geq 0, \theta\in\mathbb{R}^d}$ for which each component of $S(t, X_t; \theta)$ is interpreted as the price of a traded risky asset at time t when the investor holds a constant position of $\theta \in \mathbb{R}^d$ units of these assets. The i-th component of θ, denoted θ_i, gives the position in the i-th asset. In this sense, holding one of the d assets may have a price impact on any of the d assets. A negative value for θ_i represents a short sale in this asset. We often write $S_t(\theta)$ for $S(t, X_t; \theta)$ to simplify the notation.

To derive the dynamics of the prices, we suppose that $S(\cdot; \theta) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a deterministic function that is continuously differentiable with respect to t and twice continuously differentiable with respect to x. By Itô's

Formula, the family of processes can be expressed with the following stochastic differential equations

$$dS^i(t,X_t;\theta) = S^i(t,X_t;\theta)\left(\mu_i(t,X_t;\theta)dt + \bar{\sigma}_i(t,X_t;\theta)^\top dW_t\right),$$

for $t \geq 0$, $i \leq d$ and $\theta \in \mathbb{R}^d$ where $\mu_i : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ and $\bar{\sigma}_i : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d_+$ are deterministic functions.

We assume there is a local martingale measure $\mathbb{Q}$ for the *unaffected* price processes (the price processes observed if $\theta \equiv 0$):

$$dS^i_t(0) = S^i_t(0)\bar{\sigma}_i(t,X_t;0)^\top dB_t \quad (t \geq 0, i \leq d),$$

in which $B = W + \int \eta(s,X_s)ds$ defines a $\mathbb{Q}$-Brownian motion and

$$\eta^\top(t,x)\bar{\sigma}_i(t,x;0) = \mu_i(t,x;0),$$

for all $t \geq 0, x \in \mathbb{R}^n, i \leq d$. This assumption rules out arbitrage opportunities for small traders (traders who do not have an impact on prices), when the large trader does not trade. In terms of B, X takes the representation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \quad (t \geq 0),$$

in which $b = \beta - \sigma\eta$. Unlike [3], we do not assume that all $S(\theta)$ are local martingales under $\mathbb{Q}$. Indeed, for all θ, we assume there is a local martingale measure $\mathbb{Q}(\theta)$ for the price processes $S^i(\theta)$:

$$dS^i_t(\theta) = S^i_t(\theta)\bar{\sigma}_i(t,X_t;\theta)^\top dB_t(\theta) \quad (t \geq 0, i \leq d),$$

in which $B(\theta) = W + \int \eta(s,X_s,\theta)ds$ defines a $\mathbb{Q}(\theta)$-Brownian motion and

$$\eta^\top(t,x;\theta)\bar{\sigma}_i(t,x;\theta) = \mu_i(t,x;\theta),$$

for all $t \geq 0, x \in \mathbb{R}^n, i \leq d$. The quantity $\eta_i(t,X_t;\theta)$ is the risk premium associated to the i-th Brownian motion risk W^i when the investor holds a position θ. The above equation implies that the large trader has a direct impact on the risk premia of the traded assets.

The existence of $\eta(\cdot;\theta)$ is justified by the fact that when θ is kept constant by the large trader, small traders obtain the price $S^i_t(\theta)$, so the existence of the equivalent local martingale measure $\mathbb{Q}(\theta)$ rules out arbitrage opportunities for small traders, cf. Hypothesis 2 (NFLVR Infinitesimal Traders) in [29]. With these measure changes, $S^i(\theta)$ is represented as

$$dS^i_t(\theta) = -\psi_t(\theta)^\top\bar{\sigma}_i(t,X_t;\theta)S^i_t(\theta)dt + S^i_t(\theta)\bar{\sigma}_i(t,X_t;\theta)^\top dB_t \quad (t \geq 0, \theta \in \mathbb{R}^d)$$

in which $\psi_t(\theta) = \eta(t,X_t;0) - \eta(t,X_t;\theta)$.

Example 1. A simple example is the Bachelier model for $\theta = 0$ under $\mathbb{Q}(0)$:

$$S^i_t(0) = \sum_{j=1}^d \sigma_{ji}B^j_t,$$

with σ a $d \times d$ matrix, and

$$S^i_t(\theta) = \sum_{j=1}^d \sigma_{ji}B^j_t + 2\lambda(\sigma_i)^\top\sigma\theta(T-t), \theta \in \mathbb{R}^d,$$

with $\lambda > 0$. (σ_i is the i-th column of the matrix σ) We define $X^i = S^i(0)$, $1 \leq i \leq d$. For this model, $\bar{\sigma}_{ji}(t, X_t; \theta) S^i_t(\theta) = \sigma_{ji}$, and $\psi_t(\theta) = 2\lambda\sigma\theta$.

A well-known empirical feature of asset prices is that the risk premium depends on the volatility (cf. [24]). In this simple model, the large trader's impact on the risk premium is proportional to the volatility. The i-th component of the vector $\sigma\theta$ gives the large trader's exposure to the i-th Brownian risk W^i so that the change of drift associated to the i-th component of W is proportional to this exposure.

A first trade at time t makes the price process S^i jump by $2\lambda(\sigma_i)^\top \sigma\theta(T - t)$. Also, at time T, all processes $S^i(\theta)$ converge back to $S^i(0)$.

Liquidity costs. We follow [3] who define the asymptotic liquidation proceeds from a position θ at time t for the single asset setup by considering that the asset is liquidated in infinitesimal packets, infinitely fast. In other words, the liquidation value of an asset at time t is given by the integral

$$\int_0^\theta S_t(\theta - y)dy$$

in the case $d = 1$.

In [29], this definition is extended to d-dimensional trading strategies. The definition of the asymptotic liquidation proceeds $L_t(\theta)$ then involves a curvilinear integral from θ to the d-dimensional vector 0. To simplify the treatment, we adopt the convention that each asset is purchased (resp. liquidated) one by one, in the (resp. reverse) order given by their index i. Shares of the assets are liquidated in infinitesimally small packets, starting from asset d, down to asset 1. However, since prices are a function of the current holdings, the liquidation of the i-th asset is executed while still holding the first $i-1$ assets. Hence, the price obtained during this liquidation is $S^i_t(\Theta_i(y))$ and the liquidation value of asset i is

$$\int_0^{\theta_i} S^i_t(\Theta_i(\theta_i - y))dy,$$

with $\Theta_i(y) = (\theta_1, \theta_2, \ldots, \theta_{i-1}, y, 0, \ldots, 0)$, $(y \leq \theta_i)$. As such, for a position $\theta = (\theta_i)_{i \leq d}$ to liquidate, we define the asymptotic liquidation proceeds as

$$L_t(\theta) = \sum_{i=1}^d \int_0^{\theta_i} S^i_t(\Theta_i(\theta_i - y))dy.$$

Following the same logic, the cost of building a position θ (a negative quantity) is defined as

$$-\sum_{i=1}^d \int_0^{\theta_i} S^i_t(\Theta_i(y))dy,$$

by considering that assets are purchased in the order of their index. A simple change of variable shows that this is simply equal to $-L_t(\theta)$. More generally, the proceeds obtained from changing position from θ to ϑ is given by $L_t(\theta) - L_t(\vartheta)$ at time t.

Hypothesis 2. Assume the following conditions.

(i) $\sum_{i=1}^{d}\int_0^{\theta_i}\left(\int_0^t \left|\bar{\sigma}_{ji}\left(s,x;\Theta_i(y)\right)\right|^2 S_s^i\left(\Theta_i(y)\right)^2 ds\right)^{1/2} dy<\infty$, for $1\le j\le d$,

(ii) $\sum_{i=1}^{d}\int_0^{\theta_i}\int_0^t \left|\psi\left(s,x;\Theta_i(y)\right)^\top \bar{\sigma}_i\left(s,x;\Theta_i(y)\right) S_s^i\left(\Theta_i(y)\right)\right| dsdy<\infty$,

a.s. for all $\theta\in\mathbb{R}^d, t\ge 0$ and $x\in\mathbb{R}^n$.

By Theorem 2.2 of [45] (Fubini's Theorem for stochastic integrals) we can write the asymptotic liquidation proceeds as

$$L_t(\theta)=L_0(\theta)-\int_0^t \Psi(s,X_s;\theta)ds+\int_0^t \Sigma(s,X_s;\theta)^\top dB_s \quad (t\ge 0), \tag{5}$$

with

$$\Psi(t,x;\theta)=\sum_{i=1}^{d}\int_0^{\theta_i}\psi\left(t,x;\Theta_i(y)\right)^\top \bar{\sigma}_i\left(t,x;\Theta_i(y)\right) S_t^i\left(\Theta_i(y)\right) dy$$

and the components of the vector-valued process Σ are given by

$$\Sigma_j(t,x;\theta)=\sum_{i=1}^{d}\int_0^{\theta_i}\bar{\sigma}_{ji}\left(t,x;\Theta_i(y)\right) S_t^i\left(\Theta_i(y)\right) dy,$$

for $1\le j\le d$.

Remark 3. In Example 1, we have $\Sigma(t,x;\theta)=\sigma\theta$ so that $\psi_j(t,x;\theta)=2\lambda\Sigma_j(t,x;\theta)$. Consequently, Hypothesis 2 is clearly satisfied. Furthermore, $\Psi(t,x;\theta)=\lambda|\sigma\theta|^2$.

3.2. Trading strategies, and wealth processes

We now define the wealth processes associated to a self-financing trading strategy. As opposed to [3] who go through considerable details to define the wealth process, the representation for L in (5) allows us to take an easier route. This construction is taken from [29] and works for any value of $d\ge 1$.

We start by defining the notion of self-financing for simple trading strategies. Let $\theta_t=\sum_{i\ge 1}\xi_i \mathbf{1}_{\{\tau_i\le t\}}$ be a simple strategy in which $(\tau_i)_{i\ge 1}$ is an increasing sequence of stopping times and $\xi_i\in\mathcal{F}_{\tau_i}$. The vector θ_t denotes the number of shares owned by the investor in each risky asset at time t. We define the wealth process Π directly in terms of L. As shown in [3], this allows the investor to minimize transaction costs associated to the liquidity of assets. More precisely, at time t, the wealth is given by

$$\Pi_t=\Pi_0+L_t(\theta_t)-\sum_{i\ge 1}\left(L_{\tau_i}(\theta_{\tau_i})-L_{\tau_i}(\theta_{\tau_i-})\right)\mathbf{1}_{\{\tau_i\le t\}}.$$

By the fact that $\theta_{\tau_{i-1}} = \theta_{\tau_i -}$ for $i > 1$ and $\theta_0 = 0$, the sum becomes

$$\begin{aligned}\Pi_t &= \Pi_0 + \sum_{i\geq 1}(L_{\tau_i}(\theta_{\tau_{i-1}}) - L_{\tau_{i-1}}(\theta_{\tau_{i-1}}))\mathbf{1}_{\{\tau_i \leq t\}} \\ &\quad + \sum_{i\geq 1}(L_t(\theta_t) - L_{\tau_i}(\theta_{\tau_i}))\mathbf{1}_{\{\tau_i \leq t < \tau_{i+1}\}} \\ &= \Pi_0 + \int_0^t \Sigma(s, X_s; \theta_s)^\top dB_s - \int_0^t \Psi(s, X_s; \theta)ds,\end{aligned}$$

using (5) and the fact that $\theta_s = \theta_{\tau_{i-1}}$ for $\tau_{i-1} \leq s < \tau_i$. Because these two integrals are well-defined for more general processes θ, we can extend the definition of wealth processes to more general trading strategies:

Definition 4. A *trading strategy* is an adapted process θ such that $\Sigma(\cdot, X; \theta)$ is B-integrable, progressively-measurable, and

$$\int_0^T |\Psi(t, X_t; \theta_t)|dt < \infty \quad a.s.,$$

for $T > 0$. The *wealth process* associated to θ is then given by

$$\Pi_t = \Pi_0 + \int_0^t \Sigma(s, X_s; \theta_s)^\top dB_s - \int_0^t \Psi(s, X_s; \theta_s)ds \quad (t \geq 0). \tag{6}$$

3.3. Replication of contingent claims

The main problem in which we are interested is the pricing of contingent claims in the context of liquidity risk and price impacts. We proceed by calculating the replication cost of contingent claims payoffs. Let $T > 0$ denote the maturity of an option. If $\hat{h}$ denotes its payoff function at time T, then the replication problem consists in finding a trading strategy θ that sets the portfolio wealth at time T equal to $\hat{h}(S_T)$:

$$\hat{h}(S_T) = \Pi_T = \Pi_0 + \int_0^T \Sigma(t, X_t; \theta_t)^\top dB_t - \int_0^T \Psi(t, X_t; \theta_t)dt. \tag{7}$$

Remark 5. If an investor owns the option and wants to hedge away the risk, he needs to solve:

$$0 = \hat{h}(S_T) + \Pi_0 + \int_0^T \Sigma(t, X_t; \theta_t)^\top dB_t - \int_0^T \Psi(t, X_t; \theta_t)dt,$$

which corresponds to (7) with $\hat{h}$ replaced with $-\hat{h}$.

Of course, the expression $\hat{h}(S_T)$ is ambiguous here as it depends on θ_T. In order to avoid price manipulations, however, it is often assumed that the investor liquidates his position at time T, so that the observed price at maturity is $S(T, X_T; 0)$ and the associated payoff is $\hat{h}(S(T, X_T; 0))$, or that θ_T is replaced by an approximation $\Delta(T, X_T)$. In general, the option payoff can thus be represented as $h(X_T)$, for some deterministic function h.

We can make this a dynamic problem by making the following definition:

Definition 6. The *price process* of a replicable contingent claim is given by the wealth process Π satisfying

$$h(X_T) = \Pi_t + \int_t^T \Sigma(s, X_s; \theta_s)^\top dB_s - \int_t^T \Psi(s, X_s; \theta_s) ds, \quad (0 \le t \le T) \tag{8}$$

in which θ is called the *replication (or hedging) strategy.*

Equation (8) is a type of BSDE for which the pair (Π, θ) is a solution. We make the following standing assumption on Σ:

Hypothesis 7. The mapping $\theta \mapsto \Sigma(t, x; \theta)$ is surjective from $\mathbb{R}^d$ to $\mathbb{R}^d$ for all $t \ge 0$ and $x \in \mathbb{R}^n$.

Under this assumption, for each $t \le T$, $z \in \mathbb{R}^d$, $x \in \mathbb{R}^n$ we can find $\theta_0 \in \mathbb{R}^d$ such that $\Sigma(t, x; \theta_0) = z$. Accordingly, we can find a measurable function Φ (see Lemma 1 of [29]) such that

$$\Phi(t, x; z) = \Psi(t, x; \theta_0), \tag{9}$$

and Φ will play a key role in the PDE analysis.

In Example 1, Assumption 7 is satisfied if and only if σ is invertible. In this case, $\Phi(z) = \lambda |z|^2$.

The quadratic form for Φ can be obtained for much more general models than Example 1. In fact, for any choice of $\bar{\sigma}$ and μ it suffices to take

$$\psi_j(t, x; \theta) = \sum_{k=1}^d \tilde{f}_{jk}(t, x) \Sigma_k(t, x; \theta) \tag{10}$$

in which $\tilde{f}_{kj} = \tilde{f}_{jk}$ for all $1 \le k, j \le d$. In this case, consider the change of variable $z_j = \Sigma_j(t, x; \theta)$ so that

$$\Psi(t, x; \theta) = \sum_{j,k=1}^d \int_0^{\Sigma_j(t,x;\theta)} \tilde{f}_{jk}(t, x) z_k dz_j.$$

Since $\phi(z) := \left(\sum_{k=1}^d \tilde{f}_{jk}(t, x) z_k \right)_{1 \le j \le d}$ is a conservative vector field, it follows that

$$\Psi(t, x; \theta) = \tfrac{1}{2} \sum_{i,j=1}^d \tilde{f}_{ij}(t, x) \Sigma_i(t, x; \theta) \Sigma_j(t, x; \theta)$$

and

$$\Phi(t, x; z) = \tfrac{1}{2} \sum_{i,j=1}^d \tilde{f}_{ij}(t, x) z_i z_j$$

from the fact that $\nabla \Phi = \phi$. From an economic point of view, the representation of ψ as in Eq. 10 means that the required market premium for Brownian motion W^j is a linear combination of the volatility structure $\Sigma(t, x; \cdot)$, thus giving a multivariate generalisation of the well-known empirical feature of asset prices that risk premia depend on the volatility (cf. [24]). We make this a standing assumption.

Hypothesis 8. We assume that

$$\Phi(t,x;z)=\sum_{i,j=1}^{d} f_{ij}(t,x)z_iz_j$$

in which the $f_{ij}\in C^{0,1}([0,T]\times\mathbb{R}^d)$ and $f_{ij}=f_{ji}$ for all $1\le i,j\le d$.

In practically relevant situations, one could directly specify the parameters in the dynamics of X and the function Φ, (without the need to specify $\bar\sigma$ and μ) reflecting the properties of a given financial market (economic variables and liquidity costs associated to trading).

Hypothesis 7 implies that the existence of a solution (Π,Z) of the BSDE (8) is equivalent to the existence of a solution (Y,Z) of

$$h(X_T)=Y_t+\int_t^T Z_s dB_s-\int_t^T \Phi(s,X_s;Z_s)ds,\quad (0\le t\le T).$$

Since this is not a linear equation in Z, the replication cost of two units of h is not twice the replication cost of one unit. In order to emphasise the dependence on this nonlinear term and to study the asymptotic representation of Y and Z when liquidity costs are small, we introduce the parameter $\lambda>0$ in the previous equation:

$$h(X_T)=Y_t^{(\lambda)}+\int_t^T Z_s^{(\lambda)}dB_s-\lambda\int_t^T \Phi(s,X_s;Z_s)ds,\quad (0\le t\le T).\qquad (11)$$

By Theorem 2 of [9], the BSDE (11) has a solution when $z\mapsto\Phi(t,x;z)$ is continuous for all t,x and has at most quadratic growth in z, i.e. when there are constants $C_0,C_1\ge 0$ such that $|\Phi(t,x;z)|\le C_0+C_1|z|^2$ uniformly in t,x and $\mathbb{E}_{\mathbb{Q}}e^{C_1h(X_T)}<\infty$.

To this general replication problem we naturally associate a nonlinear PDE. We prove the existence and uniqueness of the solution of this PDE in a Sobolev space, and show in a second step that it also gives the solution of the BSDE. The clear advantage of this approach is that we explicitly obtain the replication strategy as the derivative of the option price with respect to the underlying. This also allows us to show that the replication strategy and the option price converges to the replication strategy and the option price in a frictionless setting when liquidity costs are small (λ is small).

In the setting of Example 1, Φ is given by $\Phi(z)=\lambda|z|^2$. Consider the variable $\chi=e^{2\lambda Y}$. By Itô's Formula and Eq. (11),

$$e^{2\lambda h(X_T)}=\chi_t+\int_t^T 2\lambda\chi_s Z_s dB_s,\quad (0\le t\le T).$$

In other words, χ is a martingale, and $\chi_t=\mathbb{E}_{\mathbb{Q}}\left(\exp(2\lambda h(X_T))\middle|\mathcal{F}_t\right)$. Consequently, Y can be represented as

$$Y_t^{(\lambda)}=\frac{1}{2\lambda}\log\left(\mathbb{E}_{\mathbb{Q}}\left(\exp(2\lambda h(X_T))\middle|\mathcal{F}_t\right)\right).$$

The solution Y can therefore be represented explicitly in terms of $h(X_T)$ in this special case.

3.4. Bounded domains and the PDE formulation

Our goal is to obtain the solution of the above replication problem in terms of an associated PDE, and study the analytical properties of the solution. The PDE results below are valid for a bounded domain Ω with C^2-boundary $\partial\Omega$. This is a common assumption in the PDE literature, and our results below cannot be extended to the case of domains of infinite volume.

In practice, the solution of the PDE gives a good approximation of the problem as one can take Ω as large as needed. Indeed, suppose that the initial condition of the process X at time t is given by $X_t = x$ a.s. Introducing the stopping time $\tau_{x,t} = \inf\left\{t \le s \le T | X_s \notin \overline{\Omega}\right\} \wedge T$, we modify BSDE (11) by

$$\Upsilon\left(\tau_{x,t}, X_{\tau_{x,t}}\right) = Y_s + \int_s^{\tau_{x,t}} Z_r dB_r - \lambda \int_s^{\tau_{x,t}} \Phi(r, X_r; Z_r) dr \tag{12}$$

for $t \le s \le T$ with $\Upsilon(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that $\Upsilon(T,x) = h(x)$ and $\Upsilon(t,x) = g(x)$ for $t < T$, $x \in \mathbb{R}^n$ for some $g(x) : \mathbb{R}^n \to \mathbb{R}$. If Ω_m is a domain that contains the ball of radius m centered at 0 and $\sup_{t\le T} \Upsilon(t, X_t) \in L^2$, then $\Upsilon(\tau^m_{x,t}, X_{\tau^m_{x,t}})$ converges to $h(X_T)$ in L^2 as $m \to \infty$ (where $\tau^m_{x,t} = \inf\left\{t \le s \le T | X_s \notin \overline{\Omega}_m\right\} \wedge T$) since $\lim_{m\to\infty} \mathbb{Q}(\tau^m_{x,t} < T) = 0$. The solution of (12) thus satisfies the approximation replication definition of [10]. In this sense, one can make the mean square hedging error arbitrarily close to zero, and that should be quite satisfactory for any practical implementation, in particular when using finite difference methods. A simple specification for the function H is of course obtained by setting $g = 0$. Note that [3] defined the related notion of approximately attainable contingent claim with a similar line of reasoning.

After time reversion $\tau = T - t$, the PDE associated to (12) is

$$\left.\begin{array}{ll} \partial_\tau u = \mathcal{L}(\tau)u + \lambda\Phi\left(T-\tau, \sigma^\top Du\right) & \text{on } (0,T] \times \Omega, \\ u(0) = h & \text{on } \Omega, \\ u = g & \text{on } [0,T] \times \partial\Omega, \end{array}\right\} \tag{13}$$

with

$$\mathcal{L}(\tau) = \tfrac{1}{2} \sum_{i,j=1}^n \left[\sigma\sigma^\top\right]_{ij} (\tau,x)\partial_i\partial_j + \sum_{i=1}^n b_i(\tau,x)\partial_i.$$

Here, Ω is the domain of X, and $b = (b_1, \ldots, b_n)$ and $\sigma = (\sigma_{ij})_{1\le i\le n, 1\le j\le d}$ are the drift vector and diffusion matrix of the process X.

When $\lambda = 0$, the market is perfectly liquid (the price impact of a trade is zero). In this case, option prices (solutions of (12) or (13)) are $\mathbb{Q}$-martingales and are related by

$$u(t, X_t) = Y_t = \mathbb{E}_{\mathbb{Q}}\left(H\left(\tau_{x,t}, X_{\tau_{x,t}}\right) \Big| \mathcal{F}_t\right), \tag{14}$$

which is the classical Feynman–Kac result that tells us that the solution of (13) can be written as a conditional expectation of the terminal condition.

4. Analysis of the PDE

We address existence and uniqueness of solutions to the PDE (13), first-order asymptotics in λ and the Hölder continuity of solution (and their gradients) on certain subdomains of Ω.

We consider the slightly more general initial-boundary value problem

$$\left.\begin{aligned} \partial_\tau u &= \mathcal{L}(\tau)u + \lambda F\left(\tau, \sigma^\top Du, \sigma^\top Du\right) + f(\tau) && \text{on } (0,T]\times\Omega, \\ u(0) &= h && \text{on } \Omega, \\ u &= g && \text{on } [0,T]\times\partial\Omega, \end{aligned}\right\} \tag{15}$$

where $\mathcal{L}(\tau)$ is written in divergence form

$$\mathcal{L}(\tau) = \tfrac{1}{2}\sum_{i,j=1}^{n} \partial_i[\sigma\sigma^\top]_{ij}(\tau,x)\partial_j + \sum_{i=1}^{n} \tilde{b}_i(\tau,x)\partial_i$$

and

$$\tilde{b}_i(\tau,x) = b_i(\tau,x) - \tfrac{1}{2}\sum_{k=1}^{n} \partial_k[\sigma\sigma^\top]_{ki}(\tau,x)$$

with detailed assumptions to be given below. We understand $u(\tau,x)$ as a Banach-space-valued function of τ so that we mostly drop any x-dependence.

The expression $\sigma^\top(\tau,x)Du(\tau,x)$ for a function u will always be treated as a single entity which we abbreviate as $\sigma^\top Du$. This is the $\sigma^\top$-gradient in the sense of Chapter 4.1 of [28].

Hypothesis 9. We make the following assumptions.

(i) *Domain:* the set Ω is open, bounded and has a boundary of class C^2.

(ii) *Coefficients:* The $n\times d$-matrix σ has components σ_{ij} which belong to $C^{0,2}(\overline{Q_T})$ and the components of the drift vector $b=(b_1,\ldots,b_n)$ belong to $C^{0,1}(\overline{Q_T})$. The quadratic form defined by the square matrix $\sigma\sigma^\top$ is not assumed to be positive definite.

(iii) *Generator:* The function F takes the form

$$F(\tau,x,z,z') = \sum_{i,j=1}^{d} f_{ij}(\tau,x)z_i z'_j$$

for $(\tau,x)\in Q_T$, $z,z'\in\mathbb{R}^d$ where $f_{ij}\in C^{0,1}(\overline{Q_T})$ and $f_{ij}=f_{ji}$.

(iv) *Inhomogeneous part:* The function f is in $L^\infty(Q_T)$.

(v) *Initial and boundary conditions:*

(a) The initial condition h belongs to $L^\infty(\Omega)$, and the boundary condition g is in $L^\infty(\partial\Omega)$.

(b) There is a function $H\in L^\infty([0,T];W^{2,\infty}(\Omega))$ such that $\partial_\tau H\in L^\infty(Q_T)$ and $H(0,x)=h(x)$ on Ω, $H(\tau,x)=g(x)$ on $[0,T]\times\partial\Omega$.

Remark 10. Since these assumptions may look restrictive at first sight we put them in perspective.

(i) The assumption on the coefficients σ and b is standard in the sense that it agrees with the usual Lipschitz hypothesis made in stochastic analysis, cf. Section 3.1 of this paper. Higher differentiability of σ_{ij} is required since we must write $\mathcal{L}$ in divergence form in order to apply the variational method.

The proofs of the existence and uniqueness result (more precisely of Lemmas 22, 23 and 26) depend on $||\operatorname{div}\tilde{b}||_{L^\infty(Q_T)}$ being finite so that we must require that $\tilde{b}$ (and hence b) belong to $L^\infty([0,T];W^{1,\infty}(\Omega))\otimes\mathbb{R}^n$ which means that $b(\tau,\cdot)$ is Lipschitz continuous.

(ii) The generator F is related to the function Φ of (9) by $F(\tau,x,z,z)=\Phi(T-\tau,x,z)$. The differentiability of the f_{ij} is a technical point needed for the proof of Proposition 28.

(iii) The function H is related to Υ of (12) by a time reversion $H(\tau,x)=\Upsilon(T-\tau,x)$. Note that by the Sobolev embedding theorem (Theorem 6 of Chapter 5.6 of [18]) we have that $H(\tau,\cdot)\in C^{1+\alpha}(\overline{\Omega})$ for any $\alpha\in(0,1)$ so that this assumption entails some regularity of H. The hypothesis on H is driven by the assumption on b as discussed in Remark 29.

Since the characteristic form of $\mathcal{L}$ may be zero, we seek solutions in a Sobolev space based on the $\sigma^\top$-gradient. This space is defined as

$$W^{1,2}_\sigma(\Omega)=\left\{w\in L^2(\Omega)\,\middle|\,\sigma^\top Dw\in L^2(\Omega)\otimes\mathbb{R}^d\right\},$$

with norm

$$||w||^2_{W^{1,2}_\sigma(\Omega)}=||w||^2_{L^2(\Omega)}+||\sigma^\top Dw||^2_{L^2(\Omega)\otimes\mathbb{R}^d}.$$

Standard arguments (cf., Theorem 3.3 of [1]) show that $W^{1,2}_\sigma(\Omega)$ is a Banach space. Also let $W^{1,2}_{\sigma,0}(\Omega)$ be the closure of $C^\infty_c(\Omega)$ in $W^{1,2}_\sigma(\Omega)$. The solution of the PDE (15) will then be in the Hilbert space $L^2([0,T];W^{1,2}_\sigma(\Omega))$ and the boundary condition will be interpreted in terms of $L^2([0,T];W^{1,2}_{\sigma,0}(\Omega))$.

Remark 11. For square matrices σ, the spaces $W^{1,2}_\sigma(\Omega)$ and $W^{1,2}_{\sigma,0}(\Omega)$ correspond to the energy spaces $H^{0,\Gamma}_{m,\mathbf{m}}(\Omega)$ defined in Chapter 4.2 of [28] for $\Gamma=\partial\Omega$ and $\Gamma=\emptyset$, respectively, where $m=2$ and $\mathbf{m}=(2,\ldots,2)$. We mention for completeness that the case of scalar weights applied to a function and its derivatives leads to the Sobolev spaces considered in [32].

Also recall the notion of a weak (or generalised) solution.

Definition 12. Let $v\in L^2([0,T];W^{1,2}_\sigma(\Omega))$. By a *weak solution* of the PDE

$$\left.\begin{array}{ll}\partial_\tau u=\mathcal{L}(\tau)u+\lambda F\left(\tau,\sigma^\top Dv,\sigma^\top Du\right)+f(\tau) & \text{on } (0,T]\times\Omega,\\ u(0)=h & \text{on } \Omega,\\ u=g & \text{on } [0,T]\times\partial\Omega,\end{array}\right\}$$

we mean a function $u\in L^2([0,T];W^{1,2}_\sigma(\Omega))$ such that for all test functions $\varphi\in C^\infty_c(Q_T)$ we have (leaving out any τ-dependence)

$$\begin{aligned}(u(T),\varphi(T))-(u(0),\varphi(0))-\int_0^T(u,\partial_\tau\varphi)d\tau+\int_0^T B[u,\varphi]d\tau\\ =\lambda\int_0^T(F(\sigma^\top Dv,\sigma^\top Du),\varphi)d\tau+\int_0^T(f,\varphi)d\tau.\end{aligned}\tag{16}$$

Here B is a function of τ given by the bilinear form

$$B[u,\varphi] = \tfrac{1}{2}\sum_{i,j}\int_\Omega [\sigma\sigma^\top]_{ij}\partial_i u\,\partial_j\varphi\,dx + \sum_i \int_\Omega u_i\partial_i(\tilde{b}_i\varphi)dx.$$

The initial-boundary condition is interpreted as $u(0)=g$ in $L^2(\Omega)$ and $u-H \in L^2([0,T];W^{1,2}_{\sigma,0}(\Omega))$.

Remark 13. The interpretation of the boundary condition looks unusual since one typically phrases this in terms of the trace operator. However, the space $W^{1,2}_\sigma(\Omega)$ does not possess a straightforward trace mapping into $L^2(\partial\Omega)$. For the classical trace $W^{1,2}(\Omega)\to L^2(\partial\Omega)$, the kernel of this map is given precisely by $W^{1,2}_0(\Omega)$ and it is this analogy that we exploit.

Note that even if the weak solution u were continuous, we cannot generally expect $u=g$ in a pointwise sense on the boundary. This is for two reasons. First, we do not assume that σ is a square matrix. Second, σ could be zero on the boundary so that near this part of $\partial\Omega$ we cannot control the gradient Du.

In the case of square σ, the existence of suitable trace mappings on the energy spaces with degenerate weighting is discussed in detail in Chapter 4.2 of [28]. We also refer to [22] and to Chapter I.1 of [36] for a detailed discussion on how to decompose the boundary into singular and regular parts based on the Fichera function.

4.1. A Feynman–Kac-type theorem

We first show the usefulness of studying this PDE: weak solutions of (15) can be used to obtain solutions of the BSDE (12) so that studying the PDE solutions yields valuable information about option prices and their gradients. This is expressed in a Feynman–Kac-type theorem the proof of which is nontrivial as we do not deal with classical PDE solutions, i.e. twice continuously differentiable functions. For ease of notation we suppress the initial condition $X_t=x$ a.s. from the statement of the theorem.

Theorem 14. *Assume Hypothesis 9 and let $u\in L^2([0,T];W^{1,2}_\sigma(\Omega))$ be a weak solution of the initial-boundary-value problem (15). Then the pair $(Y_s,Z_s) = \big(u(s,X_s),\sigma^\top(s,X_s)Du(s,X_s)\big)$ is a solution of the BSDE*

$$Y_s = \Upsilon(T\wedge\tau, X_{T\wedge\tau}) + \int_{s\wedge\tau}^{T\wedge\tau}\lambda\Phi(r,X_r;Z_r)dr - \int_{s\wedge\tau}^{T\wedge\tau} Z_r dB_r$$

where $t\le s\le T$ and $\tau=\inf\left\{s\ge t | X_s\notin\overline{\Omega}\right\}$ is a stopping time.

The proof of the theorem is strongly intertwined with the existence and uniqueness result for the PDE so that it is contained in Sect. 5.5.

4.2. Existence and uniqueness

We now state the existence and uniqueness of solutions to the aforementioned class of semilinear PDEs on bounded domains. In economic terms the result shows that there is a unique option price in an L^2-space whose $\sigma^\top$-gradient (on which the hedging of the contingent claim is based) also lives in an L^2-space. The latter assertion cannot be obtained through viscosity solutions.

Theorem 15. *Under Hypothesis 9, for $\lambda \geq 0$ sufficiently small the PDE (15) has a unique weak solution $u \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$. Moreover, we have higher regularity in the time variable in the sense that $u \in C([0,T]; L^2(\Omega))$.*

The proof relies on the Schauder fixed point theorem. In numerical implementations, a Galerkin scheme could be used to solve a regularised PDE and pass to the weak limit.

4.3. Liquidity asymptotics of the solution

In this section we describe the continuity of the PDE solutions with respect to the parameter λ leading to natural asymptotics. From an economic point of view, we study the marginal properties of prices by considering the limit as the assets become more liquid.

Theorem 16. *Assume Hypothesis 9 and let $u^{(\lambda)}$ be the unique weak solution of the PDE (15) for a suitably small $\lambda \geq 0$.*

(i) *Then as $\lambda \to 0$ we have $u^{(\lambda)} \to u^{(0)}$ in $L^2([0,T]; W^{1,2}_\sigma(\Omega))$ and the order of convergence is $O(\lambda)$.*

(ii) *For $\lambda > 0$ define $v^{(\lambda)} = \frac{1}{\lambda}\left(u^{(\lambda)} - u^{(0)}\right)$. Then there exists a $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$ such that $v^{(\lambda)} \to v$ in $L^2([0,T]; W^{1,2}_\sigma(\Omega))$ as $\lambda \to 0$. Moreover, v is the weak solution of the PDE*

$$\left.\begin{array}{ll} \partial_\tau v = \mathcal{L}(\tau)v + F(\tau, \sigma^\top Du^{(0)}, \sigma^\top Du^{(0)}) & \text{on } (0,T] \times \Omega, \\ v(0) = 0 & \text{on } \Omega, \\ v = 0 & \text{on } [0,T] \times \partial\Omega. \end{array}\right\} \tag{17}$$

The first assertion expresses the continuity from above at 0 of the derivative prices as a function of the liquidity parameter: as the market becomes more liquid, the derivative price continuously approaches the Black-Scholes price (14) of the derivative when price impact is absent.

The second assertion also makes precise a formal perturbation approach in powers of λ. The function $v^{(\lambda)}$ gives the additional liquidity cost in the option price $u^{(\lambda)}$ per unit of λ, i.e. the marginal liquidity cost of the option.

4.4. Regularity of the weak solution

In practical implementations one is naturally interested in the regularity of the weak solution of the PDE (15). The more regular the PDE solution, the better is the convergence of such a numerical scheme.

Hölder regularity of parabolic equations and systems is a topic well-covered in the literature. For quasilinear equations with at most quadratic growth in Du we refer to Chapter V.1 of [33] whose Theorem 1.1 establishes Hölder continuity given a smallness condition, cf. also [12] which shows Hölder continuity once Du is in an L^q-space. Other types of growth in Du are covered extensively in [15]. The corresponding results for semilinear parabolic system with quadratic growth in the gradient can be found in [26,27,43,44] to name just a few references. A broader overview is given in [31].

Theorem 17. *Under Hypothesis 9, let u be the unique weak solution of the PDE (15). Choose a domain $\Omega' \subseteq \Omega$ such that the operator $\mathcal{L}(\tau)$ is uniformly elliptic with constant of ellipticity ν in the sense of (4) on $[0,T] \times \Omega'$. Suppose*

$$2\left(\lambda \sup_{(\tau,x)\in Q'_T} ||(f_{ij})(\tau,x)||_{op} + \sup_{(\tau,x)\in Q'_T} |\tilde{b}(\tau,x)|\right) ||u_0||_\infty < \nu,$$

where $||A||_{op}$ denotes the operator norm of a real $d\times d$-matrix A acting on $\mathbb{R}^d$. Then the weak solution u and its gradient Du are Hölder continuous on Ω' with respect to the parabolic metric δ defined in (2) for some Hölder exponent $\alpha \in (0,1)$ which depends on the data of the PDE and on ν.

5. Proofs of the key results

This section is more technical as it contains the proofs of the key results. Each subsection contains one building block of the proofs, blocks 2.-5. depend on the estimates derived in 1.

1. Existence and uniqueness of weak solutions with zero boundary conditions (special case of Theorem 15)
2. Existence and uniqueness of weak solutions for nonzero boundary conditions (general case of Theorem 15)
3. Price asymptotics for small liquidity effects (Theorem 16)
4. Regularity of the weak solution (Theorem 17)
5. Feynman–Kac-type theorem (Theorem 14)

For ease of presentation we only consider time-independent generators of the diagonal form $F: \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is given as

$$F(x,z,z') = \sum_{i=1}^{d} f_i(x) z_i z'_i$$

for functions $f_i \in C^1(\overline{\Omega})$, where we set $\gamma = \max_{1\le i\le d} ||f_i||_\infty$. This is no restriction and the general case follows similarly in each instance.

5.1. Existence and uniqueness result with zero boundary conditions

This section contains the proof of Theorem 15 in the special case when the solution is required to vanish on the boundary. The strategy of the proof is the same as in [5] with two nontrivial complications: the quadratic nonlinearity and the boundary.

To solve the PDE we interpret it in a variational sense and use energy methods based on L^∞-a-priori estimates obtained via a classical maximum principle. Since the differential operator $\mathcal{L}$ is independent of the solution u, we do not need more abstract methods to treat quasilinear and fully nonlinear equations, cf. [41,47]. So our exposition is self-contained and easily accessible.

The diffusion degeneracy requires a regularisation of the equation by the method of vanishing viscosity (elliptic regularisation), cf. for example [6,34, 36,42] and the weak convergence of the corresponding solutions [17,18].

This approach is also suggested by the connection with stochastic analysis. We want to establish the link between a class of evolution equations and BSDEs. This is achieved by suitably approximating the PDE problem so that the classical Feynman–Kac theorem can be used.

Proposition 18. *Assume Hypothesis 9 with $g = 0$. Then for sufficiently small $\lambda \geq 0$ the PDE (15) has a unique weak solution $u \in L^2([0,T]; W^{1,2}_\sigma(\Omega)) \cap L^\infty(Q_T)$. Moreover, $u \in C\left([0,T]; L^2(\Omega)\right)$. If $f = 0$, the solution exists for any $\lambda \geq 0$.*

Proof. The idea (cf. Chapter 14 of [5]) is first to obtain a priori estimates of u in $L^\infty(Q_T)$ for the simplified problem

$$\left.\begin{aligned} \partial_\tau u &= \mathcal{L}(\tau)u + \lambda F(\sigma^\top Dv, \sigma^\top Du) + f(\tau) && \text{on } (0,T] \times \Omega, \\ u(0) &= h && \text{on } \Omega, \\ u &= 0 && \text{on } [0,T] \times \partial\Omega. \end{aligned}\right\} \tag{18}$$

for some fixed $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$. These estimates then allow the application of weak convergence methods to construct a solution of (18). Denoting the solution of (18) by $u = A[v]$ to highlight the dependence on v we define a nonlinear operator A. This turns out to be a compact operator preserving a suitable subset of a Hilbert space so that the fixed point of A guaranteed by the Schauder fixed point theorem is the desired solution to the original PDE. The precise argument is as follows:

(i) Let $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$ and approximate $\sigma^\top Dv$ in the L^2-norm by a sequence of smooth functions $\tilde{v}_\epsilon$ (Lemma 19) indexed by $\epsilon > 0$. Consider the corresponding uniformly elliptic PDE

$$\left.\begin{aligned} \partial_\tau u_\epsilon &= \left(\mathcal{L}(\tau) + \tfrac{1}{2}\epsilon^2\Delta\right) u_\epsilon \\ &\quad + \lambda F(\tilde{v}_\epsilon, \sigma^\top Du_\epsilon) + f(\tau) && \text{on } (0,T] \times \Omega, \\ u_\epsilon(0) &= h && \text{on } \Omega, \\ u_\epsilon &= 0 && \text{on } [0,T] \times \partial\Omega. \end{aligned}\right\} \tag{19}$$

Here, Δ stands for the Laplace operator on $\mathbb{R}^n$. By standard results (e.g., Theorem 4 in Chapter 9 of [25]), the PDE (19) has a unique classical solution u_ϵ.

First of all note that using mollifiers, any non-continuous h can be approximated arbitrarily closely in the $L^2(\Omega)$-norm by a sequence of smooth functions h_ϵ. By construction, these function satisfy $||h_\epsilon||_\infty \leq ||h||_\infty$. This allows us to extend all proofs to the case of $h \in L^\infty(\Omega)$. The same argument applies to f. Expressing u_ϵ using a Duhamel representation based on a fundamental solution shows that we can allow for h and f measurable.

(ii) Note that due to the linearity of F in z' we can "pull" the term in F into the first-order part of the operator $\mathcal{L}$ to obtain a linear Dirichlet problem. A classical maximum principle (Lemma 20) implies that $||u_\epsilon(\tau)||_{L^\infty(Q_T)} \leq m$ where $m = ||h||_{L^\infty(Q_T)} + ||f||_{L^\infty(Q_T)}$ for all τ independently of ϵ.

(iii) Using these a priori estimates and invoking a compactness argument we can extract a weakly converging sequence $u_\epsilon \in L^2([0,T]; W^{1,2}_\sigma(\Omega)) \cap L^\infty(Q_T)$ by Lemma 22 and Corollary 24. This sequence also converges strongly to a limit $u \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$ as shown in Lemma 23. The key point here is that $||\operatorname{div} \tilde{b}||_{L^\infty(Q_T)} < \infty$ which restricts the choice coefficients b_i and σ_{ij}.
Moreover, this function is regular in the time variable in the sense that $u \in C([0,T]; L^2(\Omega))$.

(iv) So far, the solution u depends on v. This allows us to define a continuous nonlinear map $A : v \mapsto u$ defined as the strong limit of the u_ϵ when $\tilde{v}_\epsilon \to \sigma^\top Dv$. We show that that A is well-defined (Lemma 25) and for $\lambda < 1/2\gamma m$ preserves the space $\mathcal{Z}$ (Lemma 26), where

$$\mathcal{Z} = \left\{ w \in L^2([0,T]; W^{1,2}_\sigma(\Omega)) \,\middle|\, w(0) = h, ||w||^2_{L^2([0,T]; W^{1,2}_\sigma(\Omega))} \le R^2 \right\},$$

with R sufficiently large depending on the data of the PDE. Since A is continuous and compact (Lemma 25), it has a fixed point in $\mathcal{Z}$ by the Schauder fixed point theorem, cf. for example Theorem 2.A of [46]. This fixed point is a weak solution of the PDE (Lemma 27).

(v) The PDE has at most one solution: the arguments of Appendix A.2 in [5] go through in our situation.

(vi) It remains to show that for $f = 0$ the existence claim holds for any $\lambda \ge 0$. Suppose that $\lambda < 1/2\gamma||h||_\infty$ and choose $\mu > 1$. Since $\lambda < 1/2\gamma||h||_\infty < 1/2\gamma||\frac{1}{\mu}h||_\infty$, by the above arguments the PDE

$$\left.\begin{array}{ll} \partial_\tau u = \mathcal{L}(\tau)u + \lambda F\left(\sigma^\top Du, \sigma^\top Du\right) & \text{on } (0,T] \times \Omega, \\ u(0) = \frac{1}{\mu} h & \text{on } \Omega, \\ u = 0 & \text{on } [0,T] \times \partial\Omega. \end{array}\right\}$$

also has a unique weak solution u. The homogeneity of F implies that the function $v = \mu u$ is a weak solution of

$$\left.\begin{array}{ll} \partial_\tau v = \mathcal{L}(\tau)v + \mu\lambda F(\sigma^\top Dv, \sigma^\top Dv) & \text{on } (0,T] \times \Omega, \\ v(0) = h & \text{on } \Omega, \\ v = 0 & \text{on } [0,T] \times \partial\Omega. \end{array}\right\}$$

Since $\mu > 1$ was arbitrary, the problem (15) has a solution for any λ.

This completes the proof. □

For an element in $L^2([0,T]; W^{1,2}_\sigma(\Omega))$ we need to approximate the weak derivative $\sigma^\top Dv$ in a controlled way.

Lemma 19. ([1], *Theorem 2.29)* *Let $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$. Then there is a sequence of smooth functions $\tilde{v}_\epsilon : Q_T \to \mathbb{R}^d$ indexed by $\epsilon > 0$ such that $\lim_{\epsilon\to 0} \tilde{v}_\epsilon = \sigma^\top Dv$ in $L^2(Q_T) \otimes \mathbb{R}^d$. Moreover, it holds that $||\tilde{v}_\epsilon||_{L^2(Q_T)\otimes\mathbb{R}^d} \le ||\sigma^\top Dv||_{L^2(Q_T)\otimes\mathbb{R}^d}$ for any ϵ.*

We shall also employ the following classical maximum principle.

Lemma 20. *(Maximum principle, [33]) Let Ω be a bounded domain in $\mathbb{R}^n$ and let u be a classical solution of the Dirichlet problem*

$$\left.\begin{aligned} \partial_\tau u &= \mathcal{T}(\tau)u + f(\tau) && \text{on } (0,T]\times\Omega,\\ u(0) &= h && \text{on } \Omega,\\ u &= 0 && \text{on } [0,T]\times\partial\Omega, \end{aligned}\right\}$$

where $\mathcal{T}(t)$ is a second-order uniformly elliptic operator of the form (3) with coefficients in $C^{0,1}([0,T]\times\overline{\Omega})$. Suppose that $h\in L^\infty(\Omega)$ and $f\in L^\infty(Q_T)$. Then

$$|u(\tau,x)| \le \operatorname*{ess\,sup}_{y\in\Omega}|h(y)| + \operatorname*{ess\,sup}_{(s,y)\in Q_T}|f(s,y)|$$

for every $(\tau,x)\in Q_T$.

As a final preliminary result we note a useful bound on the inner product of F with L^∞-functions. This follows from Cauchy's inequality.

Lemma 21. *Let $v,w\in L^2(\Omega)\otimes\mathbb{R}^d$ with $u\in L^\infty(\Omega)$. Then in $L^2(\Omega)$ we have*

$$|(F(v,w),u)| \le \tfrac{1}{2}\gamma||u||_\infty\left(||v||^2_{L^2(\Omega)\otimes\mathbb{R}^d} + ||w||^2_{L^2(\Omega)\otimes\mathbb{R}^d}\right),$$

where $\gamma = \max_i ||f_i||_\infty$.

Using Gronwall's lemma we obtain estimates for λ sufficiently small.

Lemma 22. *Let $v\in L^2([0,T];W^{1,2}_\sigma(\Omega))$ and let $\tilde{v}_\epsilon$ be as in Lemma 19. Denote by u_ϵ the classical solution of (19) corresponding to $\tilde{v}_\epsilon$. Set $m=||h||_\infty+||f||_\infty$ and suppose that $\lambda\le 1/\gamma m$. Then for all $\epsilon>0$ we have*

$$||u_\epsilon||^2_{L^2(Q_T)} \le \operatorname{vol}(\Omega)m^2T$$

$$||\sigma^\top Du_\epsilon||^2_{L^2(Q_T)\otimes\mathbb{R}^d} \le \frac{\operatorname{vol}(\Omega)m^2(1+\hat{b})T + \gamma\lambda m||\sigma^\top Dv||^2_{L^2(Q_T)\otimes\mathbb{R}^d}}{1-\gamma\lambda m},$$

where $\hat{b} = ||\operatorname{div}\tilde{b}||_{L^\infty(Q_T)}$. Thus both sequences u_ϵ and $\sigma^\top Du_\epsilon$ contain a weakly convergent subsequence in $L^2(Q_T)$ and $L^2(Q_T)\otimes\mathbb{R}^d$, respectively, with weak limits u and $\sigma^\top Du$.

The existence of the weak limits u and $\sigma^\top Du$ is related to the weak closure of the $\sigma^\top$-gradient operator, cf. the discussion preceding Lemma 1.1 of Chapter 4.1 in [28].

Proof. Unless otherwise indicated, all norms are $L^2(Q_T)$-norms.

1. Estimates pointwise in the time variable: let $\epsilon>0$ and consider the inner product of the time derivative of u_ϵ with u_ϵ in $L^2(\Omega)$. Let $\mathcal{L}_\epsilon(\tau) = \mathcal{L}(\tau)+\frac{1}{2}\epsilon^2\Delta$ be the regularised operator obtained by adding multiples of the Laplacian Δ. We have

$$\begin{aligned}&\left(\frac{d}{d\tau}u_\epsilon(\tau),u_\epsilon(\tau)\right)\\ &\quad= (\mathcal{L}_\epsilon(\tau)u_\epsilon(\tau),u_\epsilon(\tau)) + \lambda\left(F\left(\tilde{v}_\epsilon(\tau),\sigma^\top Du_\epsilon(\tau)\right),u_\epsilon(\tau)\right) + (f(\tau),u_\epsilon(\tau)).\end{aligned}$$

Integration by parts is allowed by the regularity of u'_ϵ and yields

$$\begin{aligned}\tfrac{1}{2}\frac{d}{d\tau}||u_\epsilon(\tau)||^2 = &- \tfrac{1}{2}||\sigma^\top Du_\epsilon(\tau)||^2 - \tfrac{1}{2}\epsilon^2||Du_\epsilon(\tau)||^2 \\ &- \tfrac{1}{2}(\operatorname{div}\tilde{b}(\tau), u_\epsilon^2(\tau)) + \lambda\left(F\left(\tilde{v}_\epsilon(\tau), \sigma^\top Du_\epsilon(\tau)\right), u_\epsilon(\tau)\right) \\ &+ (f(\tau), u_\epsilon(\tau)), \end{aligned} \tag{20}$$

where we used the zero boundary conditions. Using Lemma 21 we deduce

$$\begin{aligned}&\frac{d}{d\tau}||u_\epsilon(\tau)||^2 + ||\sigma^\top Du_\epsilon(\tau)||^2 + \epsilon^2||Du_\epsilon(\tau)||^2 \\ &\quad\le ||\operatorname{div}\tilde{b}(\tau)||_\infty ||u_\epsilon(\tau)||^2 + \lambda\gamma||u_\epsilon(\tau)||_\infty\left(||\tilde{v}_\epsilon(\tau)||^2 + ||\sigma^\top Du_\epsilon(\tau)||^2\right) \\ &\qquad + (f(\tau), u_\epsilon(\tau))\end{aligned}$$

With $||u_\epsilon(\tau)||_\infty \le m$ (maximum principle) and Cauchy's inequality we find

$$\begin{aligned}&\frac{d}{d\tau}||u_\epsilon(\tau)||^2 + (1-\gamma\lambda m)\,||\sigma^\top(\tau)Du_\epsilon(\tau)||^2 + \epsilon^2||Du_\epsilon(\tau)||^2 \\ &\quad\le ||\operatorname{div}\tilde{b}(\tau)||_\infty||u_\epsilon(\tau)||^2 + \gamma\lambda m||\tilde{v}_\epsilon(\tau)||^2 + \tfrac{1}{2}||f(\tau)||^2 + \tfrac{1}{2}||u_\epsilon(\tau)||^2. \end{aligned} \tag{21}$$

2. We draw two conclusions from this. First of all, by the maximum principle $||u_\epsilon(\tau)||^2 \le \operatorname{vol}(\Omega)m^2$ for every $\tau \in [0,T]$ and every $\epsilon > 0$. Also, the boundedness of $||u_\epsilon(\tau)||^2$ implies the boundedness of $||\sigma^\top Du_\epsilon(\tau)||^2$ by (21):

$$\begin{aligned}||\sigma^\top Du_\epsilon(\tau)||^2 &\le \frac{\left(\hat{b}+\frac{1}{2}\right)||u_\epsilon(\tau)||^2 + \gamma\lambda m||\tilde{v}_\epsilon(\tau)||^2 + \frac{1}{2}||f(\tau)||_2^2}{1-\gamma\lambda m} \\ &\le \frac{\operatorname{vol}(\Omega)m^2(1+\hat{b}) + \gamma\lambda m||\tilde{v}_\epsilon(\tau)||^2}{1-\gamma\lambda m},\end{aligned}$$

since $||f(\tau)||^2 \le \operatorname{vol}(\Omega)m^2$.

3. Integrating with respect to τ, the pointwise norms on $u_\epsilon(\tau)$ and $\sigma^\top Du_\epsilon(\tau)$ translate to $L^2(Q_T)$- and $L^2(Q_T)\otimes\mathbb{R}^d$-norms where we note that $||\tilde{v}_\epsilon||_{L^2(Q_T)\otimes\mathbb{R}^d} \le ||\sigma^\top Dv||_{L^2(Q_T)\otimes\mathbb{R}^d}$ by Lemma 19.

4. Both norms $||u_\epsilon||^2_{L^2(Q_T)}$ and $||\sigma^\top Du_\epsilon||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ are bounded independently of ϵ. Since we are working in a reflexive Banach space (indeed in a Hilbert space), the Eberlein-Smulian theorem guarantees the existence of a weakly convergent subsequence. If w is the weak limit of $\sigma^\top Du_\epsilon$, then we find that $w = \sigma^\top Du$ in a distributional sense so that u_ϵ converges to u and $\sigma^\top Du_\epsilon$ converges to $\sigma^\top Du$. □

We also note that weak convergence implies strong convergence. This hinges on the quantity $\hat{b}$ being finite.

Lemma 23. *The weak limits are also strong limits in $L^2(Q_T)$ and $L^2(Q_T)\otimes\mathbb{R}^d$, respectively. Moreover, u_ϵ converges in $C([0,T];L^2(\Omega))$.*

Proof. All norms are $L^2(Q_T)$-norms unless otherwise indicated.

1. Let $\epsilon, \epsilon' > 0$ and suppose that $\tilde{v}_\epsilon, \tilde{v}_{\epsilon'} \in L^2(Q_T) \otimes \mathbb{R}^d$ as in Lemma 19. As before, let $u_\epsilon, u_{\epsilon'}$ be the corresponding solutions of (19). We find

$$\begin{aligned}&\left(\frac{d}{d\tau}\left(u_\epsilon - u_{\epsilon'}\right)(\tau), \left(u_\epsilon - u_{\epsilon'}\right)(\tau)\right)\\ &\quad = \left(\left(\mathcal{L}_\epsilon u_\epsilon - \mathcal{L}_{\epsilon'} u_{\epsilon'}\right)(\tau), (u_\epsilon - u_{\epsilon'})(\tau)\right)\\ &\qquad + \lambda\left(F(\tilde{v}_\epsilon(\tau), \sigma^\top Du_\epsilon(\tau)) - F(\tilde{v}_{\epsilon'}(\tau), \sigma^\top Du_{\epsilon'}(\tau)), (u_\epsilon - u_{\epsilon'})(\tau)\right).\end{aligned}$$

The tricky term in this expression is

$$\begin{aligned}&\left(\epsilon^2 \Delta u_\epsilon(\tau) - \epsilon'^2 \Delta u_{\epsilon'}(\tau), (u_\epsilon - u_{\epsilon'})(\tau)\right)\\ &\quad = \epsilon^2\left(\Delta u_\epsilon(\tau), u_\epsilon(\tau)\right) + \epsilon'^2\left(\Delta u_{\epsilon'}(\tau), u_{\epsilon'}(\tau)\right)\\ &\qquad - \epsilon^2\left(\Delta u_\epsilon(\tau), u_{\epsilon'}(\tau)\right) - \epsilon'^2\left(\Delta u_{\epsilon'}(\tau), u_\epsilon(\tau)\right).\end{aligned}$$

Using $\epsilon^2(\Delta u_\epsilon(\tau), u_\epsilon(\tau)) = -\epsilon^2||Du_\epsilon(\tau)||^2$, we arrive at

$$\begin{aligned}&\frac{d}{d\tau}||(u_\epsilon - u_{\epsilon'})(\tau)||^2 + ||\sigma^\top Du_\epsilon - \sigma^\top Du_{\epsilon'}(\tau))||^2\\ &\qquad + \epsilon^2||Du_\epsilon(\tau)||^2 + \epsilon'^2||Du_{\epsilon'}(\tau)||^2\\ &\quad \leq ||\operatorname{div}\tilde{b}(\tau)||_\infty||(u_\epsilon - u_{\epsilon'})(\tau)||^2\\ &\qquad - \epsilon^2(\Delta u_\epsilon(\tau), u_{\epsilon'}(\tau)) - \epsilon'^2(\Delta u_{\epsilon'}(\tau), u_\epsilon(\tau))\\ &\qquad + 2\lambda(F(\tilde{v}_\epsilon(\tau), \sigma^\top Du_\epsilon(\tau)) - F(\tilde{v}_{\epsilon'}(\tau), \sigma^\top Du_{\epsilon'}(\tau)), (u_\epsilon - u_{\epsilon'})(\tau)). \quad (22)\end{aligned}$$

2. Via Gronwall's inequality we see that u_ϵ converges strongly to u. This is due to two reasons. First, after integrating (22) with respect to τ, the term $\epsilon^2\langle\Delta u_\epsilon, u_{\epsilon'}\rangle + \epsilon'^2\langle\Delta u_{\epsilon'}, u_\epsilon\rangle$ tends to zero as $\epsilon, \epsilon' \to 0$ since $\epsilon^2\Delta u_\epsilon$ converges weakly to zero in $L^2(Q_T)$ (the arguments of the proof of Lemma 14.8 of [5] go through, cf. also the proof of Theorem 5 of Chapter 7.1.3 in [18]). Second, the inner product in F tends to zero using the bilinearity of F, the $L^2(Q_T) \otimes \mathbb{R}^d$-convergence of $\tilde{v}_\epsilon$ and the weak convergence of $\sigma^\top Du_\epsilon$.

Moreover, the bound on $||(u_\epsilon - u_{\epsilon'})(\tau)||$ from Gronwall's inequality can be chosen independently of τ so we have convergence in $C([0,T]; L^2(\Omega))$.

3. As regards the convergence of $\sigma^\top Du_\epsilon - \sigma^\top Du_{\epsilon'}$ in $L^2(Q_T) \otimes \mathbb{R}^d$, we see from (22) that $||\sigma^\top Du_\epsilon - \sigma^\top Du_{\epsilon'}||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ also tends to 0 as $\epsilon, \epsilon' \to 0$.

The following corollary summarises important bounds on u.

Corollary 24. *In the above notation the following holds.*

(i) *The limit u belongs to $L^\infty(Q_T)$.*

(ii) *Let $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$. Then*

$$||u||^2_{L^2([0,T];W^{1,2}_\sigma(\Omega))} \leq \frac{\operatorname{vol}(\Omega)m^2(2 + \hat{b} - \gamma\lambda m)T + \gamma\lambda m||v||^2_{L^2([0,T];W^{1,2}_\sigma(\Omega))}}{1 - \gamma\lambda m} \quad (23)$$

in the notation of Lemma 22.

Proof. (i) As the sequence u_ϵ converges to u in $L^2(Q_T)$, there is a subsequence that converges a.e. to u. Since the subsequence is uniformly bounded in $L^\infty(Q_T)$ by $||h||_\infty + ||f||_\infty$, it follows that $u \in L^\infty(Q_T)$.

(ii) This follows from Lemma 22.

The function u depends on the given v and we express this correspondence by defining a nonlinear map A setting $u = A[v]$. We have seen that this map A acts on $L^2([0,T]; W^{1,2}_\sigma(\Omega))$ and we collect key properties of A.

Lemma 25. *Under the above assumptions the following assertions are true.*

(i) *The map A is well-defined, i.e. independent of the sequence $\tilde{v}_\epsilon$ chosen to approximate $\sigma^\top Dv$.*

(ii) *The map A continuous on $L^2([0,T]; W^{1,2}_\sigma(\Omega))$.*

(iii) *The map A is compact.*

Proof. (i) This follows from steps 2 and 3 of Lemma 23.

(ii) Let v_k be a convergent sequence of functions in $L^2([0,T]; W^{1,2}_\sigma(\Omega))$ with limit v. Define $u_k = A[v_k]$ and $u = A[v]$. We know from Corollary 24 that the sequence (u_k) is bounded in norm uniformly in k. So there is a weakly convergent subsequence (u_{k_j}) that converges to some w. By the definition of weak solutions in Definition 12 we see that $w = A[v]$. Using approximating sequences $u^\epsilon_{k_j} \to u_{k_j}$ and $u^\epsilon \to u$ the argument in the proof of Lemma 23 shows that $u_{k_j} \to u$ strongly. A contradiction argument shows that we must have $u_k \to u$ for the whole sequence.

(iii) The compactness claim follows by a similar argument also exploiting the uniform boundedness of $A[v_k]$ and the consequent existence of a strongly convergent subsequence. □

In order to apply the Schauder fixed point theorem we must first show that A preserves a smaller set within $L^2([0,T]; W^{1,2}_\sigma(\Omega))$.

Lemma 26. *Define $m = ||h||_\infty + ||f||_\infty$ and suppose that*

$$\lambda \le 1/2\gamma m, \qquad \text{and} \quad R^2 \ge \frac{\text{vol}(\Omega) m^2 (2 + \hat{b} - \gamma\lambda m)T}{1 - 2\gamma\lambda m}, \tag{24}$$

where $\hat{b} = ||\text{div}\, \tilde{b}||_{L^\infty(Q_T)}$. Then the nonlinear operator A preserves the set

$$\mathcal{Z} = \left\{ w \in L^2([0,T]; W^{1,2}_\sigma(\Omega)) \,\middle|\, w(0) = h, ||w||_{L^2([0,T];W^{1,2}_\sigma(\Omega))} \le R \right\},$$

which is a nonempty closed, bounded and convex subset of the Banach space $L^2([0,T]; W^{1,2}_\sigma(\Omega))$.

Proof. Fix $v \in L^2([0,T]; W^{1,2}_\sigma(\Omega))$. Suppose that $||v||^2_{L^2([0,T]:W^{1,2}_\sigma(\Omega))} \le R^2$ with R^2 satisfying (24). Then using (23) we find

$$\begin{aligned} ||u||^2_{L^2([0,T]:W^{1,2}_\sigma(\Omega))} \le& \frac{\text{vol}(\Omega) m^2 (2 + \hat{b} - \gamma\lambda m)T + \gamma\lambda m R^2}{1 - \gamma\lambda m} \\ \le& \frac{(1 - 2\gamma\lambda m)R^2 + \gamma\lambda m R^2}{1 - \gamma\lambda m} \\ =& R^2, \end{aligned}$$

so that A preserves $\mathcal{Z}$. The remaining assertions on $\mathcal{Z}$ are clear.

We link the space $\mathcal{Z}$ with the PDE.

Lemma 27. *Let $v \in \mathcal{Z}$. Then $u = A[v]$ is a weak solution of the PDE*

$$\left.\begin{array}{ll} \partial_\tau u = \mathcal{L}(\tau)u + \lambda F(\sigma^\top Dv, \sigma^\top Du) + f(\tau) & \text{on } (0,T]\times\Omega, \\ u(0) = h & \text{on } \Omega, \\ u = 0 & \text{on } [0,T]\times\partial\Omega, \end{array}\right\} \tag{25}$$

and u also belongs to $\mathcal{Z}$.

Proof. Each $u = A[v]$ is the limit of functions u_ϵ in reference to a sequence $\tilde{v}_\epsilon$ approximating $\sigma^\top Dv$ where aach of the u_ϵ solves the regularised PDE (19). By convergence of u_ϵ and $\sigma^\top Du_\epsilon$ we find that u is a weak solution of (25).

It is clear that the boundary condition is satisfied in the sense that $u \in L^2([0,T_1]; W^{1,2}_{\sigma,0}(\Omega))$: indeed, for every τ we have $u_\epsilon(\tau) \in W^{1,2}_0(\Omega)$. Since $W^{1,2}_0(\Omega) \subseteq W^{1,2}_{\sigma,0}(\Omega)$, we also have $u_\epsilon \in L^2([0,T]; W^{1,2}_{\sigma,0}(\Omega))$. □

5.2. Proof of the existence and uniqueness for nonzero boundary conditions

We extend the existence and uniqueness result to nonzero boundary conditions exploting the compatibility expressed by the function H.

Proposition 28. *The assertion of Proposition 18 also holds when the boundary condition g is nonzero and satisfies Hypothesis 9.*

Proof. The idea of the proof is to reduce the more general boundary value problem to a problem with zero boundary conditions, cf. Remark 6.4 of [6]. This exploits the fact that F is given by a quadratic form in Du.

Suppose that u is a weak solution of (15) with $f = 0$. Then formally $v = u - H$ is a weak solution of the PDE

$$\left.\begin{array}{ll} \partial_\tau v = \mathcal{L}'(\tau)v + \lambda F(\sigma^\top Dv, \sigma^\top Dv) + f(\tau) & \text{on } (0,T]\times\Omega, \\ v(0) = 0 & \text{on } \Omega, \\ v = 0 & \text{on } [0,T]\times\partial\Omega, \end{array}\right\} \tag{26}$$

where

$$\mathcal{L}'v = \mathcal{L}v + 2\lambda F(\sigma^\top DH, \sigma^\top Dv) \tag{27}$$

and

$$f = \mathcal{L}H - \partial_\tau H + \lambda F(\sigma^\top DH, \sigma^\top DH).$$

We must show that this PDE can be covered by our existence and uniqueness result of Proposition 18. We assumed in Hypothesis 9 that $\partial_\tau H$ and $\partial_i\partial_j H$ all belong to $C(Q_T)$. The operator $\mathcal{L}'$ can be brought into the form

$$\mathcal{L}'(\tau) = \tfrac{1}{2}\sum_{i,j=1}^n \partial_i[\sigma\sigma^\top]_{ij}(\tau,x)\partial_j + \sum_{i=1}^n b'_i(\tau,x)\partial_i$$

where the coefficients b'_i are such that $\partial_j b'_i \in C^{0,1}([0,T]\times\overline{\Omega})$ by the assumptions on H. The functions b'_i are obtained from the b_i and the second summand in (27).

The converse also holds, i.e. when v solves (26), then $u = v + H$ solves (15) with the correct boundary conditions. We have chosen all assumptions so that (26) has a unique weak solution.

Remark 29. We comment on the restrictions this approach places on H.

(i) We need both $\partial_\tau H$ and $\mathcal{L}H$ in $L^\infty(Q_T)$ since the inhomogeneous term must be in $L^\infty(Q_T)$ which suggests that $H(\tau,\cdot) \in W^{2,\infty}(\Omega)$. By the Sobolev embedding theorem this means that the function $H(\tau,\cdot)$ is at least once continuously differentiable and the derivative is Hölder continuous. All arguments on the existence of a classical solution to the approximating PDE (19) go through for Hölder continuous b'.

(ii) Moreover, we require $||\mathrm{div}\,\tilde{b}'||_{L^\infty(Q_T)}$ to be finite both for the strong convergence of u_ϵ and the contraction property of A. This requires $\sigma^\top DH(\tau,\cdot) \in W^{1,\infty}(\Omega)$ which triggers Lipschitz continuity of H.

5.3. Proof of the price asymptotics for small liquidity effects

Proof of Theorem 16. We first prove assertion (i) on the continuity of the solution $u^{(\lambda)}$ in the liquidity parameter λ. For ease of presentation we consider only zero boundary conditions.

1. From the the continuity of A in Lemma 25 there are sequences of differentiable functions $\tilde{v}_\epsilon^{(\lambda)}$ and $u_\epsilon^{(\lambda)}$ converging to $\sigma^\top Du^{(\lambda)}$ and $u^{(\lambda)}$, respectively in their L^2-spaces. These functions are related via

$$\left.\begin{aligned} \partial_\tau u_\epsilon^{(\lambda)} &= \mathcal{L}_\epsilon(\tau)u_\epsilon^{(\lambda)} + \lambda F(\tilde{v}_\epsilon^{(\lambda)}, \sigma^\top Du_\epsilon^{(\lambda)}) && \text{on } (0,T]\times\Omega,\\ u_\epsilon^{(\lambda)}(0) &= h && \text{on } \Omega,\\ u_\epsilon^{(\lambda)} &= 0 && \text{on } [0,T]\times\partial\Omega. \end{aligned}\right\}$$

We also pick another sequence of functions $u_{\epsilon'}^{(0)}$ corresponding to $\lambda = 0$ satisfying the PDEs

$$\left.\begin{aligned} \partial_\tau u_{\epsilon'}^{(0)} &= \mathcal{L}_{\epsilon'}(\tau)u_{\epsilon'}^{(0)} && \text{on } (0,T]\times\Omega,\\ u_{\epsilon'}^{(0)}(0) &= h && \text{on } \Omega,\\ u_{\epsilon'}^{(0)} &= 0 && \text{on } [0,T]\times\partial\Omega. \end{aligned}\right\}$$

Here, $\mathcal{L}_\epsilon(\tau) = \mathcal{L}(\tau) + \frac{1}{2}\epsilon^2\Delta$ and similarly for $\mathcal{L}_{\epsilon'}$

2. As in Step 1 of Lemma 23 we obtain

$$\begin{aligned} &\left(\frac{d}{d\tau}(u_\epsilon^{(\lambda)} - u_{\epsilon'}^{(0)})(\tau), (u_\epsilon^{(\lambda)} - u_{\epsilon'}^{(0)})(\tau)\right)\\ &\quad= \left((\mathcal{L}_\epsilon u_\epsilon^{(\lambda)} - \mathcal{L}_{\epsilon'} u_{\epsilon'}^{(0)})(\tau), (u_\epsilon^{(\lambda)} - u_{\epsilon'}^{(0)})(\tau)\right)\\ &\qquad+ \lambda\left(F(\tilde{v}_\epsilon^{(\lambda)}(\tau), \sigma^\top Du_\epsilon^{(\lambda)}(\tau)), (u_\epsilon^{(\lambda)} - u_{\epsilon'}^{(0)})(\tau)\right). \end{aligned}$$

3. Now proceed as in Lemma 23 to find bounds for $||u^{(\lambda)} - u^{(0)}||^2_{L^2(Q_T)}$ and $||\sigma^\top Du^{(\lambda)} - \sigma^\top Du^{(0)}||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ using Gronwall's inequality. Note that each $||\sigma^\top Du^{(\lambda)}||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ can be bounded by an $R(\lambda)$ and moreover by (24) we may assume that $R(\lambda) \le R$ for some R uniformly for all λ. The crucial bound is on the F-term and a direct calculation invoking Lemma 21 yields

$$\begin{aligned}&\left|\Big(F(\tilde v_\epsilon^{(\lambda)}(\tau),\sigma^\top Du_\epsilon^{(\lambda)}(\tau)),(u_\epsilon^{(\lambda)}-u_{\epsilon'}^{(0)})(\tau)\Big)\right|\\&\quad\le \tfrac12\|(u_\epsilon^{(\lambda)}-u_{\epsilon'}^{(0)})(\tau)\|_\infty\Big(\|\tilde v_\epsilon^{(\lambda)}(\tau)\|^2+\|\sigma^\top Du_\epsilon^{(\lambda)}(\tau)\|^2\Big)\\&\quad\le \tfrac12\cdot 2m\cdot 2R^2,\end{aligned}$$

using the fact that the uniform bounds of $u_\epsilon^{(\lambda)}$ are independent of both ϵ and λ. The bounds of $||u^{(\lambda)}-u^{(0)}||^2_{L^2(Q_T)}$ and $||\sigma^\top Du^{(\lambda)}-\sigma^\top Du^{(0)})||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ are thus linear in λ so that assertion (i) is proved.

To show assertion (ii) on the asymptotics proceed as follows. All of this can be made precise by the usual approximating arguments based on $u_\epsilon^{(\lambda)}\to u^{(\lambda)}$ etc.

1. Setting $v^{(\lambda)}=\frac{1}{\lambda}(u^{(\lambda)}-u^{(0)})$ we see that it is the weak solution of

$$\left.\begin{aligned}\partial_\tau v^{(\lambda)}&=\mathcal{L}(\tau)v^{(\lambda)}+F(\sigma^\top Du^{(\lambda)},\sigma^\top Du^{(\lambda)}) && \text{on } (0,T]\times\Omega,\\ v^{(\lambda)}(0)&=0 && \text{on } \Omega,\\ v^{(\lambda)}&=0 && \text{on } [0,T]\times\partial\Omega.\end{aligned}\right\}\tag{28}$$

Arguments used before show that $||v^{(\lambda)}||^2_{L^2(Q_T)}$ and $||\sigma^\top Dv^{(\lambda)}||^2_{L^2(Q_T)\otimes\mathbb{R}^d}$ are bounded independently of λ. Since we are working in reflexive Banach spaces (indeed Hilbert spaces), these sequences must have weakly converging subsequences with limits v and $\sigma^\top Dv$.

2. Considering the definition of weak solutions by duality, we see from (28) that upon letting $\lambda\to 0$, the function v is a weak solution of

$$\left.\begin{aligned}\partial_\tau v&=\mathcal{L}(\tau)v+F(\sigma^\top Du^{(0)},\sigma^\top Du^{(0)}) && \text{on } (0,T]\times\Omega,\\ v(0)&=0 && \text{on } \Omega,\\ v&=0 && \text{on } [0,T]\times\partial\Omega.\end{aligned}\right\}$$

as claimed. Note that this PDE has a unique weak solution by standard results, cf. [6].

3. To see that $v^{(\lambda)}\to v$ strongly we set $w=v^{(\lambda)}-v$. Then w is a weak solution of the PDE

$$\partial_\tau w=\mathcal{L}w+F(\sigma^\top Du^{(\lambda)},\sigma^\top Du^{(\lambda)})-F(\sigma^\top Du^{(0)},\sigma^\top Du^{(0)})$$

and zero initial-boundary conditions. The usual argument involving $\left(\frac{d}{d\tau}w,w\right)$ yields using Cauchy's inequality (see also (20))

$$\begin{aligned}&\tfrac12\frac{d}{d\tau}||w(\tau)||^2+\tfrac12||\sigma^\top Dw(\tau)||^2\\&\quad\le\tfrac12||\operatorname{div}\tilde b||_{L^\infty(Q_T)}||w(\tau)||^2\\&\qquad+\tfrac12||F(\sigma^\top Du^{(\lambda)},\sigma^\top Du^{(\lambda)})-F(\sigma^\top Du^{(0)},\sigma^\top Du^{(0)})||^2+\tfrac12||w(\tau)||^2\end{aligned}$$

with norms in $L^2(\Omega)$. Now exploit the bilinearity of F and write

$$\begin{aligned}&F(\sigma^\top Du^{(\lambda)},\sigma^\top Du^{(\lambda)})-F(\sigma^\top Du^{(0)},\sigma^\top Du^{(0)})\\&\quad=F(\sigma^\top Du^{(\lambda)},\sigma^\top Du^{(\lambda)}-\sigma^\top Du^{(0)})+F(\sigma^\top Du^{(\lambda)}-\sigma^\top Du^{(0)},\sigma^\top Du^{(0)}).\end{aligned}$$

The triangle inequality allows us to control the term in F in the PDE for w in terms of $\sigma^\top Du^{(\lambda)}-\sigma^\top Du^{(0)}$. Now apply Gronwall's inequality and use the strong convergence of $\sigma^\top Du^{(\lambda)}$ to $\sigma^\top Du^{(0)}$ to show that both $||w||$ and $\sigma^\top Dw$ tend to zero as $\lambda \to 0$ in their respective L^2-spaces. □

5.4. Proof of the regularity of the weak solution and its gradient

Proof of Theorem 17. This is a simple application of the Theorem of [26], cf. also Theorem 0.1 of [27]. These results cover systems of equations as opposed to equations which are treated by Theorem 1.1 of Chapter V.1 in [33] to the same effect. The smallness condition in [26] is, however, better suited to our purposes.

Define the open parabolic cylinder $Q'_T = (0,T)\times\Omega'$. First note that weak solutions in the sense of our Definition 12 are also weak solutions in the sense of [26]. This is since uniform ellipticity of $\mathcal{L}$ on $[0,T]\times\Omega'$ implies that u belongs to the Sobolev space $W^{1,2}(Q'_T)$ where we take only space derivatives.

We establish the precise connection with the PDE considered in [26,27] by writing our PDE as

$$\partial_t u - \sum_{i,j=1}^n \partial_i[\sigma\sigma^\top](\tau,x)\partial_j u = \psi(\tau,x,Du)$$

where

$$\psi(\tau,x,z) = \sum_{i=1}^n \tilde{b}_i(\tau,x)z_i + \lambda F(\tau,x,z,z) + f(\tau,x)$$

for $z \in \mathbb{R}^d$. To apply the cited results we must show that there are nonnegative constants $\alpha, \beta \in \mathbb{R}$ such that

$$|\psi(\tau,x,z)| \le \alpha|z|^2 + \beta$$

for $|z|$ the Euclidean norm of $z \in \mathbb{R}^d$. Moreover define two constants

$$c_1 = \sup_{(\tau,x)\in Q'_T} |\tilde{b}(\tau,x)|, \quad c_2 = \sup_{(\tau,x)\in Q'_T} ||(f_{ij})(\tau,x)||_{op},$$

where $||(f_{ij})(\tau,x)||_{op}$ denotes the operator norm of the matrix with components f_{ij} acting on $\mathbb{R}^d$.

A direct calculation yields the bound for $|z| \le 1$ given by

$$|\psi(\tau,x,z)| \le (||f||_\infty + c_1) + \lambda c_2|z|^2$$

and

$$|\psi(\tau,x,z)| \le ||f||_\infty + (c_1 + \lambda c_2)|z|^2$$

for $|z| > 1$. Overall we have

$$|\psi(\tau,x,z)| \le (||f||_\infty + c_1) + (c_1 + \lambda c_2)|z|^2$$

so that the condition in equation (0.4) in [27] is satisfied. Moreover, the coefficients σ are continuous and ψ is a Carathéodory function. Also, by Corollary 24 we find $\sup_{Q'_T} |u(\tau,x)| \le \sup_{Q_T} |u(\tau,x)| \le ||u_0||_\infty$. Let ν denote the uniform ellipticity constant of $\mathcal{L}$ on Q'_T as in (4) and let δ be the parabolic Hölder metric (2). By the Theorem of [26] the solution u is Hölder continuous with respect to δ with some exponent $\alpha \in (0,1)$ if

$$2(c_1 + \lambda c_2)||u_0||_\infty < \nu.$$

Hölder continuity of the gradient Du follows form Theorem 3.2 of [27] since by assumption on σ, $\tilde{b}$ these coefficients are Hölder continuous (indeed continuously differentiable). □

5.5. Proof of Feynman–Kac-type result

Proof of Theorem 14. We adjust the argument of the proof of Theorem 14.5 of [5]. The idea is to approximate the degenerate PDE by uniformly parabolic PDEs, use their solutions to construct stochastic processes satisfying a forward-backward system and show that the desired BSDE is obtained in the limit.

1. Approximation of the degenerate PDE by a non-degenerate semilinear problem with smooth data (cf. step (i) of Theorem 15).

We revert time back to $t = T - \tau$ and approximate Υ in $L^2(Q_T)$ by smooth functions Υ_ϵ such that $\epsilon\Upsilon_\epsilon(0)$ is bounded in $W^{1,2}(\Omega)$.

We can approximate u by a sequence u_ϵ corresponding to a sequence $\tilde{v}_\epsilon$ of smooth functions converging to $\sigma^\top Du$ in the space $L^2(Q_T) \otimes \mathbb{R}^d$. This means u_ϵ solves

$$\left.\begin{array}{rll} -\partial_t u_\epsilon &= \mathcal{L}_\epsilon(t)u_\epsilon + \lambda F(\tilde{v}_\epsilon, \sigma^\top Du_\epsilon) & \text{on } (0,T)\times\Omega, \\ u_\epsilon(T) &= \Upsilon_\epsilon(T) & \text{on } \Omega, \\ u_\epsilon &= \Upsilon_\epsilon & \text{on } [0,T]\times\partial\Omega, \end{array}\right\}$$

with $\mathcal{L}_\epsilon(t) = \mathcal{L}(t) + \frac{1}{2}\epsilon^2\Delta$.

2. Construct a BSDE for u_ϵ. Let $\tilde{B}$ be an n-dimensional Brownian motion, independent of B. Let $X_s^{t,x,\epsilon}$ be the unique strong solution of the SDE

$$\left.\begin{array}{rl} dX_s^{t,x,\epsilon} &= b(s, X_s^{t,x,\epsilon})ds + \sigma(s, X_s^{t,x,\epsilon})dB_s + \epsilon d\tilde{B}_s,\ t \le s \le T, \\ X_t^{x,t,\epsilon} &= x. \end{array}\right\}$$

Then set $\tau_{x,\epsilon} = \inf\{r \ge 0 | X_r^{t,x,\epsilon} \notin \overline{\Omega}\}$ and

$$\begin{aligned} Y_s^{t,x,\epsilon} &= u_\epsilon(s, X_s^{t,x,\epsilon}), \\ Z_s^{t,x,\epsilon} &= \sigma(s, X_s^{t,x,\epsilon})^\top Du_\epsilon(s, X_s^{t,x,\epsilon}), \\ \tilde{Z}_s^{t,x,\epsilon} &= \epsilon Du_\epsilon(s, X_s^{t,x,\epsilon}). \end{aligned}$$

The triplet $(Y_s^{t,x,\epsilon}, Z_s^{t,x,\epsilon}, \tilde{Z}_s^{t,x,\epsilon})$ solves the BSDE

$$\begin{aligned} Y_s^{t,x,\epsilon} = \Upsilon_\epsilon(T\wedge\tau_{x,\epsilon}, X_{T\wedge\tau_{x,\epsilon}}^{t,x,\epsilon}) &+ \int_{s\wedge\tau_{x,\epsilon}}^{T\wedge\tau_{x,\epsilon}} \lambda F(\tilde{v}_\epsilon(r, X_r^{t,x,\epsilon}), Z_r^{t,x,\epsilon})dr \\ &- \int_{s\wedge\tau_{x,\epsilon}}^{T\wedge\tau_{x,\epsilon}} Z_r^{t,x,\epsilon} dB_r - \int_{s\wedge\tau_{x,\epsilon}}^{T\wedge\tau_{x,\epsilon}} \tilde{Z}_r^{t,x,\epsilon} d\tilde{B}_r, \end{aligned}$$

cf. [14] for BSDEs with random terminal times.

3. Limit as $\epsilon \to 0$. We have that $\tau_{x,\epsilon} \to \tau_x$ and $X_s^{t,x,\epsilon} \to X_s^{t,x}$, in probability as $\epsilon \to 0$. Now by Lemma 23 we know that $u_\epsilon \to u$ in $C([0,T]; L^2(\Omega))$, $\sigma^\top Du_\epsilon \to \sigma^\top Du$ in $L^2(Q_T) \otimes \mathbb{R}^d$. Moreover, $\epsilon Du_\epsilon \to 0$ in $L^2(Q_T) \otimes \mathbb{R}^d$ from considering the limit as $\epsilon \to 0$ in Eq. (22). By the repeated application of

Theorem 14.4 of [5] we have the following convergence in $\mathcal{H}^2(t,T)$-spaces as $\epsilon \to 0$:

$$\begin{aligned} Y_s^{t,x,\epsilon} &\to Y_s^{t,x} := u(s, X_s^{t,x}), \\ Z_s^{t,x,\epsilon} &\to Z_s^{t,x} := \sigma(s, X_s^{t,x})^\top Du(s, X_s^{t,x}), \\ \tilde{Z}_s^{t,x,\epsilon} &\to 0, \\ F(\tilde{v}_\epsilon(s, X_s^{t,x}), Z_s^{t,x,\epsilon}) &\to \Phi(Z_s^{t,x}). \end{aligned}$$

Here, $\mathcal{H}^2(t,T)$ is the space of square-integrable predictable processes which are finite under the norm $\mathbb{E}\left(\int_t^T |X_t|^2 dt\right)$. Moreover, $\Upsilon_\epsilon(X_T^{t,x,\epsilon}) \to \Upsilon(X_T^{t,x})$ in probability. Thus, we find

$$Y_s^{t,x} = \Upsilon(T \wedge \tau_x, X_{T\wedge\tau_x}^{t,x}) + \int_{s\wedge\tau_x}^{T\wedge\tau_x} \lambda\Phi(Z_r^{t,x})dr - \int_{s\wedge\tau_x}^{T\wedge\tau_x} Z_r^{t,x} dB_r$$

in the limit $\epsilon \to 0$. □

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