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CLOSED INVERSE SUBSEMIGROUPS OF GRAPH INVERSE SEMIGROUPS

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ABSTRACT. As part of his study of representations of the polycylic monoids, M.V. Lawson described all the closed inverse submonoids of a polycyclic monoid P_n and classified them up to conjugacy. We show that Lawson's description can be extended to closed inverse subsemigroups of graph inverse semigroups. We then apply B. Schein's theory of cosets in inverse semigroups to the closed inverse subsemigroups of graph inverse semigroups: we give necessary and sufficient conditions for a closed inverse subsemigroup of a graph inverse semigroup to have finite index, and determine the value of the index when it is finite.

1. INTRODUCTION

Graph inverse semigroups were introduced by Ash and Hall [3]: the construction, recalled in detail in section 2.2, associates to any directed graph Γ an inverse semigroup $\mathcal{S}(\Gamma)$ whose non-zero elements are pairs of directed paths in Γ with the same initial vertex. If Γ has a single vertex and n edges with $n > 1$, then $\mathcal{S}(\Gamma)$ is the polycyclic monoid P_n as defined by Nivat and Perrot [12]: if $n = 1$ then $\mathcal{S}(\Gamma)$ is the bicyclic monoid B with an adjoined zero. Ash and Hall give necessary and sufficient conditions on the structure of Γ for $\mathcal{S}(\Gamma)$

to be congruence-free, and they use graph inverse semigroups to study the realisation of finite posets as the posets of $\mathcal{J}$–classes in finite semigroups. The structure of graph inverse semigroups as HNN extensions of inverse semigroups with zero was presented in [4, section 5]. For more recent work on the structure of graph inverse semigroups, we refer to [6, 10, 11]. Connections between graph inverse semigroups and graph C^*–algebras have been fruitfully studied in [13].

As part of his study of representations of the polycylic monoids, Lawson [8] described all the closed inverse submonoids of a polycyclic monoid P_n and classified them up to conjugacy. We show in section 3 that Lawson's description can be extended to closed inverse subsemigroups of graph inverse semigroups. As in Lawson's study, there are three types: finite chains of idempotents, infinite chains of idempotents, and closed inverse subsemigroups of *cycle type* that are generated (as closed inverse subsemigroups) by a single non-idempotent element. In section 4 we apply Schein's theory of cosets in inverse subgroups [14] to the closed inverse subsemigroups of graph inverse semigroups as classified in section 3: we give necessary and sufficient conditions for a closed inverse subsemigroup L of $\mathcal{S}(\Gamma)$ to have finite index, and determine the value of the index when it is finite.

2. Preliminaries

2.1. **Cosets.** Let S be an inverse semigroup with semilattice of idempotents $E(S)$. We recall that the *natural partial order* on S is defined by

$$s \leqslant t \iff \text{there exists } e \in E(S) \text{ such that } s = et\,.$$

A subset $A \subseteq S$ is *closed* if, whenever $a \in A$ and $a \leqslant s$, then $s \in A$. The closure $B^\uparrow$ of a subset $B \subseteq S$ is defined as

$$B^\uparrow = \{s \in S : s \geqslant b \text{ for some } b \in B\}\,.$$

A subset L of S is *full* if $E(S) \subseteq L$.

Let L be a closed inverse subsemigroup of S, and let $t \in S$ with $tt^{-1} \in L$. Then the subset

$$(Lt)^\uparrow = \{s \in S : \text{there exists } x \in L \text{ with } s \geqslant xt\}$$

is a (right) coset of L in S. For the basic theory of such cosets we refer to [14]: the essential facts that we require are contained in the following result.

Proposition 2.1. [14, Proposition 6.] *Let L be a closed inverse subsemigroup of S.*

   (a) *Suppose that C is a coset of L. Then $(CC^{-1})^\uparrow = L$.*
   (b) *If $t \in C$ then $tt^{-1} \in L$ and $C = (Lt)^\uparrow$. Hence two cosets of L are either disjoint or they coincide.*
   (c) *Two elements $a, b \in S$ belong to the same coset C of L if and only if $ab^{-1} \in L$.*

We note that the cosets of L partition S if and only if L is full in S. The cardinality of the set of cosets of L in S is the *index* of L in S, denoted by $[S : L]$.

The closed inverse submonoids of free inverse monoids were completely described by Margolis and Meakin in [9]. For other related work on inverse subsemigroups of finite index, see [2] and the first author's PhD thesis [1].

2.2. Graph inverse semigroups. Let Γ be a directed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *path category* $\mathscr{P}(\Gamma)$ of Γ is defined as follows. Its set of objects is $V(\Gamma)$, and an arrow in $\mathscr{P}(\Gamma)$ from x to y is a finite directed path p in Γ. The length $|p|$ of p is defined to be the number of edges that it contains. If a path p has initial vertex x and final vertex y, then we write $\mathbf{d}(p) = x$ and $\mathbf{r}(p) = y$. The composition pq of arrows p, q is then defined when $\mathbf{r}(p) = \mathbf{d}(q)$, and is given by the concatenation of the paths. Identity arrows in $\mathscr{P}(\Gamma)$ are *empty* (or *length zero*) paths that consist of a single vertex.
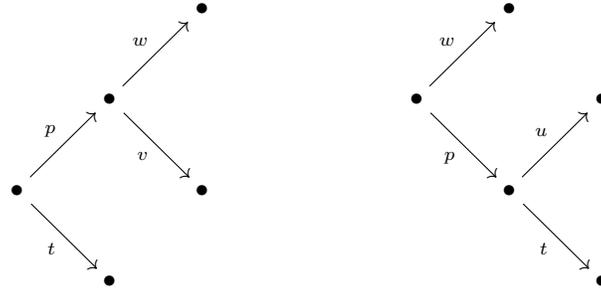
The *graph inverse semigroup* $\mathcal{S}(\Gamma)$ of Γ has underlying set

$$\{(v, w) : v, w \in \mathscr{P}(\Gamma), \mathbf{d}(v) = \mathbf{d}(w)\} \cup \{0\}$$

equipped with the binary operation

$$(t, u)(v, w) = \begin{cases} (t, pw) & \text{if } u = pv \text{ in } \mathscr{P}(\Gamma), \\ (pt, w) & \text{if } v = pu \text{ in } \mathscr{P}(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

This composition is illustrated in the following diagrams:

$$(t, pv)(v, w) = (t, pw) \qquad\qquad (t, u)(pu, w) = (pt, w)$$

The inverse of (v, w) is given by $(v, w)^{-1} = (w, v)$. The idempotents of $\mathcal{S}(\Gamma)$ are the pairs (u, u) and 0: if we identify $E(\mathcal{S}(\Gamma))$ with $\mathscr{P}(\Gamma) \cup \{0\}$, then $\mathscr{P}(\Gamma) \cup \{0\}$ becomes a semilattice with ordering given by

(2.1) $$u \leqslant v \quad \text{if and only if } v \text{ is a suffix of } u$$

and composition (meet)

(2.2) $$u \wedge v = \begin{cases} u & \text{if } v \text{ is a suffix of } u, \\ v & \text{if } u \text{ is a suffix of } v, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $u \wedge b$ is non-zero if and only if one of u, v is a suffix of the other: in this case we say that u, v are *suffix comparable*.

The natural partial order on non-zero elements of $\mathcal{S}(\Gamma)$ is then given by $(t, u) \leqslant (v, w)$ if and only if there exists a path $p \in \mathscr{P}(\Gamma)$ such that $t = pv$ and $u = pw$: that is, we descend in the natural partial order from (v, w) by prepending the same prefix to each of v and w, and ascend from (v, w) by deleting an identical prefix from each of v and w. Recall that an inverse semigroup S with zero $0 \in S$ is said to be E^*–*unitary*, if whenever $e \in E(S), e \neq 0$ and $s \in S$ with $s \geqslant e$ then $s \in E(S)$. It is clear that graph inverse

semigroups are E^*–unitary. For further structural results about graph inverse semigroups, we refer to [6, 10]. Graph inverse semigroups as topological inverse semigroups have been recently studied in [11].

3. Closed inverse subsemigroups of graph inverse semigroups

Our first result generalizes – and closely follows – Lawson's classification [8, Theorem 4.3] of closed inverse submonoids of the polycyclic monoids P_n to the closed inverse subsemigroups of graph inverse semigroups $\mathcal{S}(\Gamma)$. Given Lawson's insights, the generalization is largely routine, but it is perhaps slightly surprising that the classification extends from bouquets of circles (giving the polycyclic monoids as graph inverse semigroups) to arbitrary directed graphs, and so we have presented it in detail. Our notational conventions also differ slightly from those in [8], although we have retained the terminology used there.

Let u be a path in Γ, with u either finite, or semi-infinite with final vertex $\mathbf{r}(u)$. Then u determines a proper closed inverse subsemigroup K_u contained in $E(\mathcal{S}(\Gamma))$:

$$K_u = (u, u)^{\uparrow} = \{(w, w) : w \text{ is a suffix of } u\}.$$

If u is a finite path, then K_u consists of a finite chain of idempotents in $S(\Gamma)$, bounded above by $(\mathbf{r}(u), \mathbf{r}(u))$ and below by (u, u). In this case we say that K_u is of *finite chain type*. If u is a semi-infinite path with final vertex $\mathbf{r}(u)$, then K_u consists of an infinite descending chain of idempotents determined by the finite suffixes of u, bounded above by $(\mathbf{r}(u), \mathbf{r}(u))$. We say that K_u is of *infinite chain type*.

Theorem 3.1. *In a graph inverse semigroup $\mathcal{S}(\Gamma)$ there are three types of proper closed inverse subsemigroups L:*

(a) *Finite chain type,*
(b) *Infinite chain type,*
(c) *Cycle type: L has the form*

$$L_{p,d} = \{(vp^r d, vp^s d) : r, s \geqslant 0 \text{ with } v \text{ a suffix of } p\} \cup \{(q, q) : q \text{ a suffix of } d\},$$

where p is a directed circuit in Γ, d is a directed path in Γ starting at the initial point of p, and where p, d do not share a non-trivial prefix. In this case, L is the smallest closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ containing (d, pd).

Proof. It is clear, for any finite or semi-infinite path u in Γ, that K_u is a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$. We now verify that $L_{p,d}$ is a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$. Any two paths occurring in elements of $L_{p,d}$ are suffix comparable, and we have

$$(q, q)(q', q') = (q \wedge q', q \wedge q') \text{ for suffixes } q, q' \text{ of } d,$$
$$(q, q)(vp^r d, vp^s d) = (vp^r d, vp^s d) = (vp^r d, vp^s d)(q, q).$$

Now consider a product $(vp^r d, vp^s d)(wp^j d, wp^k d)$: write $p = v_0 v$ and suppose that $s < j$. Then $wp^j d = wp^{j-s-1} v_0 vp^s d$ and so

$$(vp^r d, vp^s d)(wp^j d, wp^k d) = (wp^{j-s-1} v_0 vp^r d, wp^k d) = (wp^{j-s+r} d, wp^k d) \in L_{p,d},$$

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and a similar calculation applies if $s > j$. If $s = j$ and v is a suffix of w, say $w = v_1 v$, then

$$(vp^r d, vp^s d)(wp^s d, wp^k d) = (vp^r d, vp^s d)(v_1 vp^s d, wp^k d)$$
$$= (v_1 vp^r d, wp^k d)$$
$$= (wp^r d, wp^k d) \in L_{p,d}$$

and a similar calculation applies if $s = j$ and w is a suffix of v. Hence $L_{p,d}$ is a sub-semigroup of $\mathcal{S}(\Gamma)$, and since the inverse of an element of $L_{p,d}$ is clearly also in $L_{p,d}$, we deduce that $L_{p,d}$ is an inverse subsemigroup of $\mathcal{S}(\Gamma)$. Since we ascend in the natural partial order in $L_{p,d}$ by deleting identical prefixes from the paths $vp^r d$ and $vp^s d$, or from a given suffix of d, it is also clear that $L_{p,d}$ is closed.

If F is a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ and contains (d, pd), then for any $m, n \geqslant 0$ we have $(pd, d)^m (d, pd)^n = (p^m d, d)(d, p^n d) = (p^m d, p^n d) \in F$. Ascending in the natural partial order, we may obtain any element of $L_{p,d}$, and so $L_{p,d} \subseteq F$.

Let L be a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$. If w and w' are paths occurring in elements of L and are not suffix comparable, then the product of the idempotents (w, w) and (w', w') in L is equal to 0, and so $0 \in L$ and by closure $L = \mathcal{S}(\Gamma)$. Hence if L is proper, any two paths occurring in elements of L are suffix comparable and hence have the same terminal vertex. By definition, if $(u, v) \in \mathcal{S}(\Gamma)$ then u, v have the same initial vertex: hence if $(u, v) \in L$ then u, v have the same initial and the same terminal vertex in Γ. Suffix comparability then ensures that any proper closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ consisting entirely of idempotents is either a finite or an infinite chain.

We shall now describe those closed inverse subsemigroups of $\mathcal{S}(\Gamma)$ which contain non-idempotent elements. Suppose that $L \neq E(L)$ is a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$. Then there exists $(u, v) \in L$, with $u \neq v$, and we may assume that the path u is shorter than the path v. Hence u is a suffix of v and so $v = pu$ for some path p. Since u and v have the same initial and terminal vertices, p must be a directed circuit in Γ. If p and u share a common prefix, with $p = ap_1$ and $u = au_1$ then

$$(u_1, p_1 u) \geqslant (au_1, ap_1 u) = (u, pu)$$

and so by closure, $(u_1, p_1 u) \in L$.

Amongst the non-idempotent elements $(d, pd) \in L$, choose d and p so that $|pd| = |p| + |d|$ is minimal. Then d, p do not share a non-trivial prefix, and as above, $L_{p,d} \subseteq L$. Now for any $m \geqslant 0$ we have $(p^m d, p^m d) \in E(L)$ and so, if $(w_1, w_2) \in L$, each w_i is a suffix of some directed path $p^m d$ for sufficiently large m. If $|w_i| \leqslant |d|$, then by suffix comparability, w_i is a suffix of d. If each of w_1 and w_2 is a suffix if d, then we may suppose that $|w_1| \leqslant |w_2| \leqslant |d|$, so that $w_2 = hw_1$ for some path h. Since w_1 and w_2 have the same inital and terminal vertices, h is a circuit, and

$$|w_1| + |h| = |w_1 h| = |w_2| \leqslant |d| < |p| + |d|.$$

By minimality of $|p| + |d|$, (w_1, w_2) must be an idempotent, and so $(w_1, w_2) = (q, q)$ for some suffix q of d.

Now suppose that $|w_1| \leqslant d$ but $|w_2| > d$. Then w_1 is a suffix q of d, and $w_2 = vp^s d$ for some $s \geqslant 0$ and (non-empty) suffix v of p. Since $(vp^s d, vd) \in L_{p,d} \subseteq L$ we have $(q, vd) = (q, vp^s d)(vp^s d, vd) \in L$. Then q and vd have the same initial and final vertices. Set $d = hq$: then $\mathbf{d}(v) = \mathbf{d}(q) = \mathbf{r}(h)$ and $\mathbf{r}(v) = \mathbf{d}(d) = \mathbf{d}(h)$. Hence vh is a circuit,

and $(q, vd) = (q, vhq) \in L$. But $|vhq| = |v| + |hq| = |v| + |d|$: by minimality of $|p| + |d|$ we have $|v| = |p|$ and so $v = p$. Hence $(q, pd) \in L$, and since $(pd, d) \in L_{p,d}$ we have $(q, d) = (q, pd)(pd, d) \in L$, with $|q| \leqslant |d|$. We have now reduced to the previous case, and it follows that $q = d$. Hence

$$(w_1, w_2) = (q, vp^s d) = (d, p^{s+1} d) \in L_{p,d}\,.$$

We may now assume that for $i = 1, 2$ we have $|w_i| \geqslant |d|$ and so $w_1 = up^r d, w_2 = vp^s d$ for some $r, s \geqslant 0$ and suffixes u, v of p. Since $(ud, up^r d)$ and $(vp^s d, vd)$ are in $L_{p,d} \subseteq L$, we have

$$(ud, vd) = (ud, up^r d)(up^r d, vp^s d)(vp^s d, vd) \in L\,.$$

Again by suffix comparability, we may assume that ud is a suffix of vd, and so u is a suffix of v. We write $v = hu$ and, as above, h must be a circuit. If $|h| > 0$, w have $(ud, hud) \in L$ with

$$|h| + |ud| = |hud| = |v| + |d| \leqslant |p| + |d|\,.$$

By minimality of $|p| + |d|$ we deduce that $|v| = |p|$ and so $v = p$. Therefore $(ud, pd) \in L$, and since $(pd, d) \in L_{p,d} \subseteq L$, we now have $(ud, d) = (ud, pd)(pd, d) \in L$ with $|ud| = |u| + |d| < |p| + |d|$, which contradicts the minimality of $|p| + |d|$. Hence h is empty, and $u = v$. In this last case then, we have $(w_1, w_2) = (up^r d, up^s d) \in L_{p,d}$, and so $L \subseteq L_{p,d}$. □

From the result of the previous Theorem, we may immediately conclude the following:

Corollary 3.2. *If the graph Γ contains no directed circuit, then every proper closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ is a chain of idempotents.*

Our next result, based on [8, Theorem 4.4] which treats the polycyclic monoids, classifies the closed inverse subsemigroups of a graph inverse semigroup up to conjugacy. We recall that two inverse subsemigroups H and K of an inverse semigroup S are *conjugate* if there exists $s \in S$ with $s^{-1} H s \subseteq K$ and $sKs^{-1} \subseteq H$. We begin with two further definitions:

Definition 3.3. Let $L = (u, u)^{\uparrow}$ be a closed inverse subsemigroup of finite chain type in a graph inverse semigroup $\mathcal{S}(\Gamma)$. We call the initial vertex of the directed path u the *root* of L.

Adapting ideas of [7, Section 1.3] from words to paths in Γ :

Definition 3.4. Two paths p, q in Γ are *conjugate* if there are paths u, v in Γ such that $p = uv$ and $q = vu$. Equivalently, (see [7, Proposition 1.3.4]) there exists a path w in Γ such that $wp = qw$. Conjugate paths must be directed circuits in Γ, and conjugacy amounts to the selection of an alternative initial edge.

The following Lemma is due to Lawson and is extracted from the proof of [8, Theorem 4.4].

Lemma 3.5. *Let S be an E^*–unitary inverse semigroup. If H and K are conjugate closed inverse subsemigroups of S with $H \neq S \neq K$ and $H \subseteq E(S)$ then $K \subseteq E(S)$. Moreover, if H has a minimum idempotent, then so does K.*

Proof. There exists $s \in S$ with $s^{-1} H s \subseteq K$ and $sKs^{-1} \subseteq H$. Let $k \in K$: then $k \neq 0$ and $sks^{-1} \in H$ and so $sks^{-1} \in E(S)$. It follows that $s^{-1}(sks^{-1})s = (s^{-1}s)k(s^{-1}s) \in E(S)$ and $(s^{-1}s)k(s^{-1}s) \leqslant k$. Since S is E^*–unitary, we deduce that $k \in E(S)$.

Now suppose that $m \in H \subseteq E(S)$ is the minimum idempotent in H and that $e \in K$. Then $m \leqslant ses^{-1}$ and so

$$s^{-1}ms \leqslant s^{-1}ses^{-1}s = es^{-1}s \leqslant e\,.$$

Therefore $s^{-1}ms$ is the minimum idempotent in K. □

Theorem 3.6.

(a) *Let L be a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ of finite chain type. Then all closed inverse subsemigroups conjugate to L are of finite chain type. Two closed inverse subsemigroups K_u and K_v of finite chain type are conjugate in $\mathcal{S}(\Gamma)$ if and only if they have the same root.*

(b) *Let L be a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ of infinite chain type. Then all closed inverse subsemigroups conjugate to L are also of infinite chain type. Two closed inverse subsemigroups K_u and K_v of infinite chain type are conjugate if and only if there are finite suffixes s and t of u and v respectively, with $\mathbf{d}(s) = \mathbf{d}(t)$, and a semi-infinite path p such that $u = ps$ and $v = pt$*

(c) *Let L be a closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ of cycle type. Then all closed inverse subsemigroups conjugate to L are also of cycle type. Moreover, $L_{p,d}$ is conjugate to $L_{q,k}$ if and only if p and q are conjugate directed circuits in Γ.*

Proof. (a) It follows from Lemma 3.5 that if L has finite chain type then so does every closed inverse subsemigroup conjugate to L.

Suppose that K_u and K_v have the same root in $V(\Gamma)$. Then $(u, v) \in \mathcal{S}(\Gamma)$, and for any suffix w of u we have

$$(v, u)(w, w)(u, v) = (v, v) \in K_v\,.$$

Similarly, for any suffix t of v, $(u, v)(t, t)(v, u) = (u, u) \in K_u$. Hence K_u and K_v are conjugate.

Conversely, suppose that K_u and K_v are conjugate, with conjugating element $(p, q) \in \mathcal{S}(\Gamma)$, so that for any suffixes s of u and t of v we have

$$(q, p)(s, s)(p, q) \in K_v \text{ and } (p, q)(t, t)(q, p) \in K_u\,.$$

Then $(q, p)(\mathbf{r}(u), \mathbf{r}(u))(p, q) \in K_v$, so that p and $\mathbf{r}(u)$ are suffix-comparable: hence p also ends at $\mathbf{r}(u)$, and $(q, p)(\mathbf{r}(u), \mathbf{r}(u))(p, q) = (q, q) \in K_v$. Therefore q is a suffix of v. Similarly, p is a suffix of u.

Let $v = v_1 q$: then

$$(p, q)(v, v)(q, p) = (p, q)(v_1 q, v_1 q)(q, p) = (v_1 p, v_1 p) \in K_u$$

and so $v_1 p$ is a suffix of u. Let $u = u_0 v_1 p$: then

$$(q, p)(u, u)(p, q) = (q, p)(u_0 v_1 p, u_0 v_1 p)(p, q) = (u_0 v_1 q, u_0 v_1 q) \in K_v$$

and so $u_0 v_1 q$ is a suffix of v. But $v = v_1 q$ and so u_0 is a vertex: hence $u_0 = \mathbf{d}(u)$ and $u = v_1 p$. Hence u and v have the same initial vertex.

(b) By Lemma 3.5 any closed inverse subsemigroup K_v conjugate to K_u must be of chain type, and by part (a) K_v must be infinite. Suppose that $(s, t) \in \mathcal{S}(\Gamma)$ with $(t, s)K_u(s, t) \subseteq K_v$ and $(s, t)K_v(t, s) \subseteq K_u$. Since $0 \notin K_v$ we have, for all $(q, q) \in K_u$, that s is suffix comparable with q and similarly for all $(w, w) \in K_v$, that t is suffix comparable with w. If we consider q with $|q| \geqslant |s|$ then s must be a suffix of q, and hence a suffix of u, and similarly t is a suffix of v. Now for any suffix hs of u containing s we have $(hs, hs) \in K_u$

and $(t, s)(hs, hs)(s, t) = (ht, ht) \in K_v$. Hence ht is a suffix of v. Similarly, for any suffix kt of v, the path ks is a suffix of u. It follows that, for some semi-infinite path p, we have $u = ps$ and $v = pt$.

Conversely, if s and t exist as in the Theorem and $(w, w) \in K_u$ then s is suffix comparable with w. If w is a suffix of s, with $s = hw$, then

$$(t, s)(w, w)(s, t) = (t, hw)(w, w)(hw, t) = (t, t) \in K_v$$

and if s is a suffix of w with $w = hs$ then

$$(t, s)(w, w)(s, t) = (t, s)(hs, hs)(s, t) = (ht, ht) \in K_v.$$

Similarly $(s, t)K_v(t, s) \subseteq K_u$, and K_u and K_v are conjugate.

(c) By parts (a) and (b), any closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ that is conjugate to $L_{p,d}$ must be of cycle type. Suppose that the closed inverse subsemigroups $L_{p,d}$ and $L_{q,k}$ are conjugate in $\mathcal{S}(\Gamma)$, and so there exists $(s, t) \in \mathcal{S}(\Gamma)$ such that

(3.1) $$(t, s)L_{p,d}(s, t) \subseteq L_{q,k}$$

(3.2) $$(s, t)L_{q,k}(t, s) \subseteq L_{p,d}.$$

Since $L_{q,k}$ is closed and $L_{p,d}$ is the smallest closed inverse subsemigroup of $\mathcal{S}(\Gamma)$ containing (d, pd), then (3.1) is equivalent to $(t, s)(d, pd)(s, t) \in L_{q,k}$. Also, since $0 \notin L_{q,k}$ we must have s suffix-comparable with u and v whenever (u, v) is an element of $L_{p,d}$. Hence $(s, s) \in L_{p,d}$, and similarly $(t, t) \in L_{q,k}$.

First suppose that $s = up^a d$ and $t = vq^b k$ for some $a, b \geqslant 0$, where u is a suffix of p and v is a suffix of q. Write $p = hu$: then

$$\begin{aligned}(t, s)(d, pd)(s, t) &= (vq^b k, up^a d)(d, pd)(up^a d, vq^b k) \\ &= (vq^b k, up)(u, vq^b k) \\ &= (vq^b k, uhvq^b k) \in L_{q,k}.\end{aligned}$$

It follows that $uhvq^b k = vq^m k$ for some $m \geqslant 0$. Comparing lengths of these directed paths, we see that $m > b$, and then after cancellation we obtain $uhv = vq^{m-b}$. Hence uh is conjugate to some power of q, and since uh is a conjugate of p, we conclude that p is conjugate to some power of q.

Now suppose that s is a suffix of d and write $d = cs$. With t as before, we now obtain

$$(t, s)(d, pd)(s, t) = (vq^b k, s)(cs, pcs)(s, vq^b k) = (cvq^b k, pcvq^b k) \in L_{q,k}.$$

It follows that $pcvq^b k = cvq^m k$ for some $m \geqslant 0$. Again $m > b$ and after cancellation we obtain $pcv = cvq^{m-b}$. Here we see directly that p is conjugate to a power of q.

Now suppose that s is a suffix of d and write $d = cs$, and that t is a suffix of k and write $k = jt$. We now obtain

$$(t, s)(d, pd)(s, t) = (t, s)(cs, pcs)(s, t) = (ct, pct) \in L_{q,k}.$$

Since by assumption p is not the empty path, we have $ct = wq^a k$ and $pct = wq^b k$ for some suffix w of q and some $a, b \geqslant 0$. Again comparing lengths, we see that $b > a$, and then $pct = pwq^a k = wq^b k$. After cancellation we obtain $pw = wq^{b-a}$ and again p is conjugate to a power of q.

Hence for each possibility of s, we deduce from (3.1) that p is conjugate to some power of q. Using equation (3.2) we deduce similarly that q is conjugate to a power of p. Again comparing lengths, we conclude that p and q are conjugate.

Conversely, if p, q are conjugate, suppose that $p = uv$ and $q = vu$. Then it is easy to check that setting $s = k$ and $t = vd$ furnishes a pair (s, t) satisfying (3.1) and (3.2). $\qquad\square$

Remark 3.7. For the polycyclic monoids P_n $(n \geqslant 2)$, we obtain the classification of closed inverse submonoids up to conjugacy given in [8, Theorem 4.4] by applying Theorem 3.6 to the graph Γ with one vertex and n loops labelled $a_1, \ldots, a_n$. For the case $n = 1$, with a single loop labelled a, we obtain the graph inverse semigroup $\mathcal{S}(\Gamma) = B \cup \{0\}$, where B is the bicyclic monoid. A proper closed inverse subsemigroup L of $\mathcal{S}(\Gamma)$ cannot contain 0 and so is a closed inverse subsemigroup of B. If $L \subseteq E(B)$ then by Theorem 3.1, L is either $E(B)$ itself or is of finite chain type, and part (a) of Theorem 3.6, then shows that all closed inverse subsemigroups of finite chain type in B are then conjugate. By Theorem 3.1, a closed inverse subsemigroup L of B of cycle type consists of elements of the form (qp^r, qp^s) with $r, s \geqslant 0$, and where $p = a^m$ for some $m \geqslant 1$ and $q = a^k$ for some k with $0 \leqslant k \leqslant m - 1$: that is, elements of the form (a^{rm+k}, a^{sm+k}). The subsemigroup L is therefore isomorphic to the fundamental simple inverse ω–semigroup B_m, discussed in [5, section 5.7].

4. THE INDEX OF CLOSED INVERSE SUBSEMIGROUPS

We first discuss the index of closed inverse subsemigroups of finite and infinite chain type in $\mathcal{S}(\Gamma)$. For a fixed path w in Γ and a vertex v of w, we define $N^{\Gamma}_{v,w}$ to be the number of distinct directed paths in Γ whose initial vertex is v but whose first edge is not in w. The empty path v is one such path.

Theorem 4.1.

(a) *Let $L = K_w$ be a closed inverse subsemigroup of finite chain type in $\mathcal{S}(\Gamma)$. Then L has infinite index in $\mathcal{S}(\Gamma)$ if and only if there exist infinitely many distinct finite paths beginning at some vertex of w.*

(b) *If $K_w = (w, w)^{\uparrow}$ has finite index in $\mathcal{S}(\Gamma)$ then*

$$[\mathcal{S}(\Gamma) : K_w] = \sum_{v \in V(w)} N^{\Gamma}_{v,w}.$$

(c) *Let L be a closed inverse subsemigroup of infinite chain type in $\mathcal{S}(\Gamma)$. Then L has infinite index in $\mathcal{S}(\Gamma)$.*

Proof. (a) Let $L = K_w$ have finite chain type. A coset representative of K_w has the form (s, u) where s is some suffix of w, and s, u have the same initial vertex. If L has infinite index, then there are infinitely many distinct choices for (s, u) and since w is a finite path, there exist infinitely many distinct finite paths that share a common initial vertex on w. (We note that these paths may arise as prefixes of some semi-infinite path whose initial vertex is on w.)

Conversely, let v be a vertex of w at which infinitely many distinct finite paths $p_i, i \geqslant 1$ begin. Let s be the suffix of w beginning at v. Then the coset $T_i = L(s, p_i)^{\uparrow}$ exists. If p_i and p_j are not suffix comparable then T_i and T_j are distinct. If p_i is a proper suffix of p_j,

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with $p_j = hp_i$, then $(s, p_i)(p_j, s) = (s, p_i)(hp_i, s) = (hs, s) \notin K_w$, and again T_i and T_j are distinct.

(b) By part (a) there are only finitely many distinct finite paths beginning at each vertex of w, and so $N_{v,w}^{\Gamma}$ is finite for each vertex v of w. A coset representative of L has the form (s, t) where s is a suffix of w. Suppose that two such elements, (s_1, t_1) and (s_2, t_2), represent the same coset. Then $(s_1, t_1)(t_2, s_2) \in L$: in particular the product is non-zero and so t_1, t_2 are suffix comparable. We may assume that $t_2 = ht_1$: then $(s_1, t_1)(ht_1, s_2) = (hs_1, s_2)$ and this is in L if and only if $s_2 = hs_1$. Therefore $L(s_1, t_1)^{\uparrow} = L(s_2, t_2)^{\uparrow}$ if and only if $(s_2, t_2) = (hs_1, ht_1)$, and so the distinct coset representatives are the pairs (s, t) where s is a suffix of w, s and t have the same initial vertex, but do not share the same initial edge. It follows that the number of distinct cosets is $\sum_{v \in V(w)} N_{v,w}^{\Gamma}$, and L itself is represented by $(\mathbf{r}(w), \mathbf{r}(w))$.

(c) If $L = K_w$ is of infinite chain type, then the elements of L comprise the idempotents determined by the finite suffixes of w. For $i \geqslant 1$ let w_i be the suffix of length i and suppose that $w_i = e_i w_{i-1}$. Then the coset $U_i = L(w_i, e_i)^{\uparrow}$ exists, and for $i \neq j$ we have

$$(w_i, e_i)(e_j, w_j) = \begin{cases} 0 & \text{if } e_i \neq e_j \\ (w_i, w_j) & \text{if } e_i = e_j. \end{cases}$$

In each case, the result is not in L, and so the cosets U_i and U_j are distinct, and L has infinite index. $\qquad\square$

Example 4.2. We illustrate the index computation in part (b) of Theorem 4.1 with Γ equal to the finite chain with n edges $e_1, \ldots, e_n$ and $n + 1$ vertices $v_0, v_1, \ldots, v_n$:

$$v_n \xrightarrow{\;e_n\;} v_{n-1} \xrightarrow{\;e_{n-1}\;} \ldots \xrightarrow{\quad} v_1 \xrightarrow{\;e_1\;} v_0$$

Here $\mathcal{S}(\Gamma)$ is finite, and every closed inverse subsemigroup is of finite cycle type and has finite index. The number of paths in Γ with initial vertex v_j is $j + 1$, and so

$$|\mathcal{S}(\Gamma)| = \sum_{j=0}^{n}(j+1)^2 = \sum_{j=1}^{n+1} j^2 = \frac{1}{6}(n+1)(n+2)(2n+3) \,.$$

We let w be the path $e_n \cdots e_1$ and $L = (w, w)^{\uparrow}$. Since w has $n + 1$ suffixes, we have $|L| = n + 1$. An element (s, t) lies in a coset of L if and only if s is a suffix of w and $\mathbf{d}(s) = \mathbf{d}(t)$: hence the total number of elements in all the cosets of L is $\sum_{j=0}^{n}(j+1) = \sum_{j=1}^{n+1} j = \frac{1}{2}(n+1)(n+2)$.

Now $N_{v_i,w}^{\Gamma} = 1$ since only the length zero path at v_i is counted, and so $[\mathcal{S}(\Gamma) : L] = n+1$.

Let q_i be the path $e_i \ldots e_1$, so that $q_n = w$, and set $q_0 = v_0$. The $n + 1$ cosets are then represented by the elements (q_i, v_i), $0 \leqslant i \leqslant n$, and

$$\begin{aligned} L(q_i, v_i)^{\uparrow} &= \{(q_k, q_k)(q_i, v_i) : 0 \leqslant k \leqslant n\}^{\uparrow} \\ &= (\{(q_k, e_k \cdots e_{i+1} : i < k \leqslant n\} \cup \{(q_i, v_i)\})^{\uparrow} \\ &= \{(q_n, e_n \ldots e_{i+1}), \ldots, (q_{i+1}, e_{i+1}), (q_i, v_i)\} \end{aligned}$$

and so $|L(q_i, v_i)^{\uparrow}| = n - i + 1$. Counting the total number of elements in all the cosets of L we obtain

$$\sum_{i=0}^{n}(n - i + 1) = \frac{1}{2}(n+1)(n+2)$$

as before.

We now discuss the closed inverse subsemigroups of cycle type.

Theorem 4.3. *A closed inverse subsemigroup $L_{p,d}$ of cycle type in $\mathcal{S}(\Gamma)$, such that p is a circuit with at least two distinct edges, has infinite index in $\mathcal{S}(\Gamma)$.*

Proof. Write $p = uv$ where each of u, v is non-empty and one contains an edge not in the other. Let c be the conjugate circuit vu. Then for $k \geqslant 1$, the element $(vd, c^k) \in \mathcal{S}(\Gamma)$ and determines a coset $X_k = L_{p,d}(vd, c^k)^\uparrow$. Then for $k > l$,

$$(vd, c^k)\,(vd, c^l)^{-1} = (vd, c^k)\,(c^l, vd) = (vd, c^{k-l}vd) = (vd, up^{k-l}d) \notin L_{p,d}$$

and the cosets X_k and X_l are distinct. $\qquad\square$

We now consider a graph Γ containing a loop a (a directed circuit of length one) and let L be a closed inverse subsemigroup $L_{a^m, d}$ of cycle type.

Theorem 4.4.

(a) *A closed inverse subsemigroup $L = L_{a^m, d}$ of $\mathcal{S}(\Gamma)$ is of infinite index if and only if there exist infinitely many distinct finite paths beginning at some vertex of d, and not equal to a power of a.*

(b) *If L has finite index in $\mathcal{S}(\Gamma)$ then*

$$[\mathcal{S}(\Gamma) : L_{a^m, d}] = (m - 1)N^\Gamma_{\mathbf{d}(a), a} + \sum_{v \in V(d)} N^{\Gamma \setminus \{a\}}_{v, d}\,.$$

Proof. (a) As in the proof of part (a) of Theorem 4.1, in order to construct infinitely many distinct cosets, there must exist infinitely many distinct finite paths beginning at some vertex of d. Powers of a can only produce coset representatives for finitely many cosets, since for two such possible representatives $(a^{mr}d, a^k)$ and $(a^{ms}d, a^l)$ with $k > l$ we have

$$(a^{mr}d, a^k)(a^l, a^{ms}d) = (a^{mr}d, a^{k-l+ms}d)$$

and this is in L if and only if $k \equiv l \pmod m$.

Conversely, suppose that infinitely many finite paths p_i, none equal to a power of a, begin at the vertex v of d, and let s be the suffix of d beginning at v. Then $(s, p_i)(p_j, s) = 0 \notin L$, unless p_i and p_j are suffix comparable. It follows that L has infinite index if the p_i fall into infinitely many suffix comparability classes. If there are only finitely many such classes, at least one is infinite and so there exists a semi-infinite path q starting at a vertex of d, and containing an edge $b \neq a$.

We let s be the suffix of d that has initial vertex $\mathbf{d}(q)$. Let p be a finite prefix of q containing b: then the coset $T_p = L(s, p)^\uparrow$ exists and for distinct such prefixes p_1 and p_2 with $p_2 = p_1 t$, either

$$(s, p_1)(p_2, s) = (s, p_1)(p_1 t, s) = 0 \notin L$$

and so by part (c) of Proposition 2.1, the cosets T_{p_1} and T_{p_2} are distinct, or p_1 is a suffix of $p_1 t$. In the latter case, we write $p_1 t = h p_1$: then

$$(s, p_1)(p_2, s) = (s, p_1)(p_1 t, s) = (s, p_1)(h p_1, s) = (hs, s)\,.$$

If we choose p_2 such that $|p_2| \geqslant 2|p_1|$ then p_1 is a subpath of h and so h contains b. It follows that $(hs, s) \notin L$. We can now take an infinite sequence $(p_i)_{i \geqslant 1}$ of prefixes of q, with b an edge of p_1 and $|p_{i+1}| > |p_i|$ for all $i \geqslant 1$, and the cosets T_{p_i} will be distinct.

(b) By part (a), only finitely many distinct finite paths begin at each vertex of d, with the exception of its initial vertex, which is also the initial vertex of the paths a^k for $k \geqslant 0$. A coset representative of $L = L_{a^m, d}$ has the form $(a^r d, w)$ with $r \geqslant 0$, or (s, w) where s is a proper suffix of d. Hence w has the same initial vertex v as d or of some proper suffix of d.

We can only construct finitely many representatives of the form (s, w). We note that w cannot contain the vertex v, or else we can construct infinitely many distinct finite paths starting at $\mathbf{d}(w)$ by following w to v and then repeating the loop a. The analysis in the proof of part (b) of Theorem 4.1 can then be repeated to show that the number of cosets with representatives of the form (q, w) is $\sum_{v \in V(d)} N_{v,d}^{\Gamma \setminus \{a\}}$.

We now consider representatives of the form $(a^r d, w)$ with $r \geqslant 0$. Here w must have the form $w = a^s t$ for some $s \geqslant 0$ and some (possibly empty) directed path t not containing the edge a. If $r \equiv s \pmod{m}$ then $(a^r d, a^s t)(t, d) = (a^r d, a^s d) \in L$ and so $L(a^r d, a^s t)^\uparrow = L(d, t)^\uparrow = L(s, t_1)^\uparrow$ for some suffix s of d and path t_1 with the same initial vertex as s but not sharing the same first edge. Hence $L(a^r d, a^s t)^\uparrow$ will be counted within the sum $\sum_{v \in V(d)} N_{v,d}^{\Gamma \setminus \{a\}}$. Now fix t and consider the cosets $L(a^r d, a^s t)^\uparrow$ with $r \not\equiv s \pmod{m}$. Now given $L(a^{r_1} d, a^{s_1} t)^\uparrow$ and $L(a^{r_2} d, a^{s_2} t)^\uparrow$ with $s_1 \geqslant s_2$, we have

$$(a^{r_1} d, a^{s_1} t)(a^{r_2} d, a^{s_2} t)^{-1} = (a^{r_1} d, a^{s_1} t)(a^{s_2} t, a^{r_2} d) = (a^{r_1} d, a^{s_1 - s_2 + r_2} d)$$

and $(a^{r_1} d, a^{s_1 - s_2 + r_2} d) \in L$ if and only if $r_2 - s_2 \equiv r_1 - s_1 \pmod{m}$. Hence for a fixed t we can produce exactly $m - 1$ distinct cosets of the form $L(a^r d, a^s t)^\uparrow$.

But for distinct paths t_1 and t_2, $a^{s_1} t_1$ cannot be suffix comparable with $a^{s_2} t_2$ and so

$$(a^{r_1} d, a^{s_1} t_1)(a^{r_2} d, a^{s_2} t_2)^{-1} = 0 \neq L$$

and the cosets determined by distinct paths t_1 and t_2 are distinct. Hence each of the $N_{\mathbf{d}(a), a}^\Gamma$ paths t starting at $\mathbf{d}(a)$, but not having a as its initial edge, contributes $m - 1$ cosets. $\quad\square$

Example 4.5. As in Remark 3.7, we suppose that Γ consists only of the vertex x and a loop a at x so that the graph inverse semigroup $\mathcal{S}(\Gamma)$ is the bicyclic monoid B with a zero adjoined. From Theorem 4.1, the closed inverse submonoids of B contained in $E(B)$ have infinite index. Part (b) of Theorem 4.4 tells us that that the closed inverse submonoid $B_m = L_{a^m, x}$ of B has index m.

Example 4.6. Let Γ be the following graph:
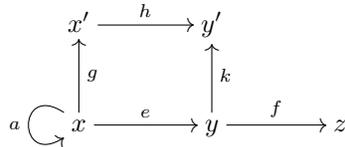
and let $L = L_{a^2, ef}$. Then we have

$$N_{z, ef}^{\Gamma \setminus a} = 1 \,, N_{y, ef}^{\Gamma \setminus a} = 2 \,, N_{x, ef}^{\Gamma \setminus a} = 3 \,,$$

counting the paths in the sets $\{z\}$, $\{y, k\}$ and $\{x, g, gh\}$ respectively, and $N_{x, a}^\Gamma = 6$, counting the paths in the set $\{x, e, g, ef, ek, gh\}$. From part (b) of Theorem 4.4 we find

that $[\mathcal{S}(\Gamma), L] = 12$ and a complete set of coset representatives is

$$\{(z,z),(f,y),(f,k),(ef,x),(ef,g),(ef,gh),$$
$$(ef,a),(ef,ag),(ef,agh),(ef,ae),(ef,aek),(ef,aef)\} \ .$$

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