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Labelled oriented graph groups and crossed modules

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Abstract. A labelled oriented graph (LOG) group is a group given by a presentation constructed in a certain way from a labelled oriented graph: examples include Wirtinger presentations of knot groups. We show how to obtain generators for the Schur Multiplier $H_2(G)$ of a LOG group from the underlying LOG, and by exhibiting the n–string braid group B_n as a LOG group, we compute $H_2(B_n)$.

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1. Labelled oriented graph groups

A *labelled oriented graph* (LOG) is a graph Γ with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$, and initial and terminal vertex maps $\iota, \tau : E \to V$; together with a third map $\lambda : E \to V$ called the *labelling*. In the case where the underlying graph is a tree we speak of a *labelled oriented tree* (LOT).

To any LOG Γ we associate a presentation

$$\mathcal{P} = \mathcal{P}(\Gamma) = \langle V(\Gamma) \mid \iota(e)\lambda(e) = \lambda(e)\tau(e) \ (e \in E(\Gamma)\rangle$$

of a group $G = G(\Gamma)$ (see [5]). Any group arising in this way is called a LOG group (a LOT group if Γ is a tree). Similarly we speak of LOG and LOT presentations, and connected presentations arising from connected LOGs. The best known examples of LOG presentations are the Wirtinger presentations of knot or link groups coming from planar projections of the knots or links. In this case Γ has one component for each component of the link, and each component is a cycle. In the case of a knot we can remove any single edge from Γ, since any relation is a consequence of the others, and the resulting

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presentation is a LOT presentation. Hence knot groups are LOT groups, and link groups in general are LOG groups.

Conversely, the presentation 2–complex of a finite LOT presentation is homotopy equivalent to the complement of a ribbon n-disk in D^{n+2}, for each $n \geq 2$ [5], and hence groups given by finite LOT presentations are ribbon 2-knot groups (since we may obtain a ribbon 2–knot by doubling the ribbon disk along its boundary).

We also remark that the presentation 2–complex $K(\Gamma)$ of any LOG-presentation $\mathcal{P}(\Gamma)$ has the same integral homology as the suspension of $(\Gamma \cup \{\text{point}\})$. This follows from the form of the relators of $\mathcal{P}(\Gamma)$. Specifically, we can replace each relation $\iota(e)\lambda(e) = \lambda(e)\tau(e)$ of $P(\Gamma)$ by $\iota(e) = \tau(e)$ without altering the homology type of $K(\Gamma)$. The resulting 2-complex is the suspension of $(\Gamma \cup \{\text{point}\})$. For recent results on the asphericity of $K(\Gamma)$ see [3].

The *weight* of a finitely generated group is the minimum number of elements needed to generate it as a normal subgroup. If G is a group of weight k then a *weight set* in G is a subset $X \subseteq G$ with $|X| = k$ whose normal closure is G.

It is easy to observe certain necessary conditions for an abstract group G to be given by a finite (connected) LOG presentation.

Lemma 1.1. *Let Γ be a finite, connected LOG and let $G = G(\Gamma)$. Then*

1. *G is finitely presented,*
2. *$H_1(G) = G/[G, G]$ is infinite cyclic,*
3. *G has weight 1.*

These conditions occur in Kervaire's classification of higher knot groups:

Theorem 1.2 (Kervaire [7]). *Let G be a group and let $n \geq 3$. Then there is a knotted n-sphere $k \subset S^{n+2}$ such that $G \cong \pi_1(S^{n+2} \backslash k)$ if and only if:*

1. *G is finitely presented,*
2. *$H_1(G) = G/[G, G]$ is infinite cyclic,*
3. *G has weight 1,*
4. *$H_2(G) = 0$.*

Thus the Schur multiplier is the obstruction to the group of a finite connected LOG being a higher knot group. However, there is no constraint on the structure of the Schur multiplier of a LOG group. A group G is the group of a finite connected LOG if and only if it is the fundamental group of the complement of a closed, orientable surface smoothly embedded in S^4 (see the introduction to [10] and the references cited there) and by a theorem of Litherland [8], any finitely generated abelian group can be realised as H_2 of such a fundamental group.

The present paper was motivated by work of Huebschmann [6] on the braid groups B_n. The n–string braid group B_n is a LOG group. It is easy to see that Artin's presentation of B_n can be rewritten as a LOG presentation in which the underlying LOG consists of a directed chain with loops attached at certain vertices. Huebschmann [6] considers a free crossed B_n–module $\partial : C_n \to B_n$ and shows that the kernel of ∂ is isomorphic to $H_2(B_n)$ (which is zero for $n = 3$ and is cyclic of order 2 for $n \geqslant 4$). We observe in section 2 that this approach can be followed in a more general setting, and further we show how to obtain generators for the Schur multiplier of a LOG group directly from the structure of the underlying LOG. The application to braid groups in section 3 then not only gives generators for the Schur multiplier of such a group, but also gives a new proof of the structure of its Schur multiplier.

2. LOG groups and crossed modules

We shall use J.H.C. Whitehead's concept of a *crossed module* [12] and the construction of *free* crossed modules: for a survey of these ideas and their relationships to group presentations, see [1]. Let X be a set, let G be a group, and let $f : X \to G$ be a function. Let $\partial : C \to G$ denote the free crossed module on f, which is constructed as follows. Using the notation of [9], let E be the free group with basis $X \times G$, and let C be the quotient of E by the normal closure of

$$W = \{(x, g)^{-1}(y, h)(x, g)(y, hg^{-1}(xf)g) : x, y \in X, g, h \in G\}.$$

The action of G on C is induced by the action of G on E given by $(x, g) \lhd h = (x, gh)$, and the map $\partial : C \to G$ is induced by the homomorphism $\widehat{\partial} : E \to G$ given by $(x, g) \mapsto g^{-1}(xf)g$. Theorem 2.2 of [9] then gives an algebraic characterisation of free crossed modules.

Theorem 2.1 (Ratcliffe, [9]**).** *Let $\partial : C \to G$ be a crossed module, with $N = \operatorname{im}\partial$ and $Q = G/N$. Then C is a free crossed module on the subset $X \subseteq C$ if and only if*

- *the abelianisation C^{ab} is a free Q–module, with the image of X in C^{ab} as a basis,*
- *the image of X in G is a weight set for N,*
- *the map $\partial_* : H_2(C) \to H_2(N)$ induced by ∂ is trivial.*

We shall also need the following general observation about H_2. Let G be an arbitrary group given by a free presentation $G = F/R$. For $g \in G$, let $C_G(g)$ denote the centralizer of g in G. Then we obtain a group homomorphism $\zeta_g : C_G(g) \to H_2(G)$ defined by $\zeta_g(a) = [\widehat{a}, \widehat{g}][R, F]$, where $\widehat{a}, \widehat{g}$ are preimages in F of the elements a and g of G, and we are using Hopf's formula (see [4, section VI.9]) to identify $H_2(G)$ as $(R \cap [F, F])/[R, F]$. For any subset $Y \subseteq G$ we obtain a homomorphism

$$\zeta_Y : \bigoplus_{y \in Y} C_\Gamma(y) \to H_2(G),\ (a_y)_{y\in Y} \mapsto \sum_{y \in Y} \zeta_y(a_y).$$

Theorem 2.2. (a) *Let G be a group of weight k whose abelianisation G^{ab} is free abelian of rank k and let X be a weight set in G. Let $\partial : C \to G$ be* *the free crossed G–module on the inclusion $X \hookrightarrow G$. Then C is a LOG group and the kernel of ∂ is naturally isomorphic to the homology group $H_2(G)$.*

(b) *Suppose Γ is a finite LOG with k components and that $G = G(\Gamma)$. Let $Y = \{v_1, \ldots, v_k\}$ be a choice of vertices, one from each component of Γ. Then the homomorphism ζ_Y is surjective: in particular, $H_2(G)$ is generated by the images under ζ_Y of elements $\lambda(g)$, where g runs through a basis of the free group $*_{i=1}^k \pi_1(\Gamma, v_i)$.*

Proof. (a) The group C that underlies the free crossed module $\partial : C \to G$ acquires a LOG group structure over the graph Γ with vertex set $X \times G$ and where, for all $x, y \in X$ and $g, h \in G$, there is an edge in Γ labelled (x, g) from (y, h) to $(y, hg^{-1}xg)$.

Since the normal closure of X exhausts G, the group N coincides with G: that is, the crossed module map $\partial : C \to G$ is surjective. Now by Ratcliffe's Theorem 2.1, $\partial : C \to G$ induces an isomorphism on abelianisations and the zero map on H_2. Since $\ker \partial$ is central, the five-term exact sequence for the group extension

$$1 \to \ker \partial \to C \stackrel{\partial}{\to} G \to 1$$

(see [11] or [4, section VI.8]) shows that $\ker \partial \cong H_2(G)$.

(b) Choose a maximal tree in each connected component of Γ. The collection of chosen maximal trees is then a maximal forest Ξ in Γ, and we set $\widehat{G} = G(\Xi)$. Then $G = \widehat{G}/N$, where N is the normal closure in $\widehat{G}$ of the set $R_\Xi = \{\iota(e)\lambda(e)\tau(e)^{-1}\lambda(e)^{-1}; e \in E(\Gamma)\backslash E(\Xi)\}$.

For each $v_i \in Y$, the labelling function $\lambda : E(\Gamma) \to V(\Gamma)$ induces a group homomorphism $\lambda_i : \pi_1(\Gamma, v_i) \to \widehat{G}$, and it follows from the relations of the presentation $\mathcal{P}(\Gamma)$ that $[\lambda(g), v_i] \in N$ for every $g \in \pi_1(\Gamma, v_i)$: hence $\lambda(g)N \in C_G(v_i)$. Indeed, modulo the relations of $\mathcal{P}(\Xi)$, we may replace the set R_Ξ above by a set

$$U_\Xi = \bigcup_{i=1}^{k} \{[\lambda(g_j), v_i], 1 \leq j \leq m_i\}$$

for appropriate bases $Z_i = \{g_{i,1}, \ldots, g_{i,m_i}\}$ of the free groups $\pi_1(\Gamma, v_i)$. In particular the homomorphism $\theta : \bigoplus_{i=1}^k \pi_1(\Gamma, w) \to N/[N, \widehat{G}]$ induced by the λ_i is surjective, and factors through ζ_Y.

To complete the proof, we show that $N/[N, \widehat{G}] \cong H_2(G)$, using the five-term exact sequence

$$H_2(\widehat{G}) \to H_2(G) \to N/[N, \widehat{G}] \to H_1(\widehat{G}) \to H_1(G) \to 0$$

of the group extension $N \to \widehat{G} \to G$. The inclusion $\Xi \subseteq \Gamma$ induces an isomorphism $H_0(\Xi) \to H_0(\Gamma)$, so the map $H_1(\widehat{G}) \to H_1(G)$ is an isomorphism, and $H_2(\widehat{G})$ is a homomorphic image of $H_2(K(\Xi)) \cong H_1(\Xi) = 0$. $\square$

2.1. Right-angled Artin groups as LOG groups

We illustrate the approach to the computation of $H_2(G)$ for a LOG group G given in Theorem 2.2. A *right-angled Artin group* is a group G given by a presentation

$$\mathscr{R}_{n,I} = \langle x_1, \ldots, x_n : x_j^{-1} x_i x_j = x_i \ (i,j) \in I \rangle$$

where I is some subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$. We may assume that if $(i,j) \in I$ then $i < j$. Results on right-angled Artin groups are comprehensively surveyed in [2].

Given a presentation $\mathscr{R}_{n,I}$ of a right-angled Artin group G, consider the graph $\Gamma_{N,I}$ having vertex set $X = \{x_1, \ldots, x_n\}$ and, at the vertex x_i $(1 \leqslant i \leqslant n)$, a loop labelled by x_j whenever $(i,j) \in I$. Then $\mathcal{P}(\Gamma_{n,I})$ is a LOG presentation, and inspection shows that $\mathcal{P}(\Gamma_{n,I})$ coincides with $\mathscr{R}_{n,I}$. The maximal forest in $\Gamma_{n,I}$ is just X; so in the notation given in the proof of Theorem 2.2, $\widehat{G}$ is the free group $F = F(X)$, and $H_2(G)$ is generated by the images $[x_i, x_j]$ of the loop labels under ζ_X, (so $(i,j) \in I$). Let N be the kernel of the quotient map $F \to G$. Then the inclusion-induced mapping

$$\frac{N}{[N,F]} \to \frac{[F,F]}{[[F,F],F]}$$

is injective, and $[F,F]/[[F,F],F]$ is the second homology group $H_2(F^{ab})$ which is free abelian of rank $n(n-1)/2$ with a basis given the image of the commutators $[x_i, x_j]$. It follows that $H_2(G)$ is free abelian of rank $|I|$, with a basis given by the images under ζ_X of the elements $[x_i, x_j]$ where $(i,j) \in I$.

3. Braid groups as LOG groups

The n-string braid group B_n is presented by Artin's presentation

$$\mathscr{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| \geqslant 2), \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ (1 \leqslant i \leqslant n-1) \rangle.$$

If we rewrite the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ as

$$\sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1}$$

and define $\tau_i = \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}$ we obtain the LOG presentation

$$\mathscr{L}_n = \langle \sigma_1, \ldots, \sigma_{n-1}, \tau_1, \ldots, \tau_{n-2} : \tau_i = \sigma_{i+1}^{-1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| \geqslant 2), \sigma_i^{-1} \tau_i \sigma_i = \sigma_{i+1} \ (1 \leqslant i \leqslant n-1) \rangle$$

of B_n. Here the underlying LOG Γ_n consists of a directed chain Ξ_n with vertices $\sigma_1, \tau_1, \sigma_2, \ldots, \tau_{n-2}, \sigma_{n-1}$ with edge labels $\sigma_2, \sigma_1, \sigma_3, \sigma_2, \ldots \sigma_{n-1}, \sigma_{n-2}$, together with loops labelled σ_j attached at each vertex $\sigma_1, \ldots, \sigma_{j-2}$ $(3 \leqslant j \leqslant n-1)$.

We let $\widehat{B_n}$ be the Artin group $G(\Xi_n)$: $\widehat{B_n}$ is presented by the subpresentation

$$\mathscr{A}_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \, (1 \leqslant i \leqslant n-1) \rangle$$

of $\mathscr{B}_n$. We let N be the kernel of the quotient map $\widehat{B_n} \to B_n$. For $k \geqslant 2$ we define $\lambda_k = \sigma_2\sigma_1\sigma_3\sigma_2 \cdots \sigma_k\sigma_{k-1} \in \widehat{B_n}$. Then λ_k is the label of the path in Ξ_n from σ_1 to σ_k.

Applying Theorem 2.2 to the presentation $\mathscr{L}_n$ we obtain:

Proposition 3.1. *The group $H_2(B_3)$ is zero, and for $n \geqslant 4$ the group $H_2(B_n)$ is cyclic of order two and is generated by the image of the commutator $[\sigma_1, \sigma_3]$ in $N/[N, \widehat{B_n}]$.*

Proof. We have $\Gamma_3 = \Xi_3$ and so $H_2(B_3) = 0$ (as is well-known: B_3 is isomorphic to the trefoil knot group). For $n \geqslant 4$, Theorem 2.2 tells us that $H_2(G)$, identified with the quotient $N/[N, \widehat{B_n}]$, is generated by the images of the commutators $[\sigma_1, \sigma_j]\,, (j \geqslant 3)$ and $[\sigma_1, \lambda_k\sigma_m\lambda_k^{-1}]\,, (k \geqslant 2, m \geqslant k+2)$. But in $\widehat{B_n}$ we have $\lambda_k^{-1}\sigma_1\lambda_k = \sigma_k$ and so in $N/[N, \widehat{B_n}]$ we have $[\sigma_1, \lambda_k\sigma_m\lambda_k^{-1}] = [\sigma_k, \sigma_m]$. Hence $H_2(B_n)$ is generated by the images of the commutators that occur as relations in $\mathscr{B}_n$ (but not in $\mathscr{A}_n$).

Now free cancellation and just two applications of relations in $\mathscr{A}_n$ shows that the product of conjugates of commutators

$$[\sigma_j, \sigma_i]^{\sigma_j^{-1}\sigma_{i+1}^{-1}} \cdot [\sigma_{i+1}, \sigma_j]^{\sigma_j^{-1}\sigma_{i+1}^{-1}} \cdot [\sigma_j, \sigma_{i+1}]^{\sigma_i\sigma_{i+1}^{-1}\sigma_j^{-1}} \cdot [\sigma_i, \sigma_j]^{\sigma_j^{-1}} \cdot [\sigma_{i+1}, \sigma_j]^{\sigma_j^{-1}\sigma_i} \cdot [\sigma_j, \sigma_i]^{\sigma_i^{-1}\sigma_j^{-1}\sigma_{i+1}\sigma_i}$$

is trivial in $N \subseteq \widehat{B_n}$, whereas modulo $[N, \widehat{B_n}]$ the product is clearly equal to

$$[\sigma_j, \sigma_i][\sigma_{i+1}, \sigma_j]$$

and so $[\sigma_j, \sigma_i] = [\sigma_j, \sigma_{i+1}]$ in $N/[N, \widehat{B_n}]$. Hence all the commutators $[\sigma_k, \sigma_l]$ with $k+2 \leqslant l$ are equal in $N/[N, \widehat{B_n}]$ to $[\sigma_1, \sigma_3]$.

Again working with the presentation $\mathscr{A}_3$ as a subpresentation of $\mathscr{A}_n$, we find that the following product of conjugates of the commutator $[\sigma_1, \sigma_3]$ and its inverse $[\sigma_3, \sigma_1]$

$$[\sigma_1, \sigma_3]^{\sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_3} \cdot [\sigma_1, \sigma_3]^{\sigma_1^{-1}\sigma_2^{-1}} \cdot [\sigma_3, \sigma_1]^{\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}} \cdot [\sigma_1, \sigma_3]^{\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}} \cdot [\sigma_1, \sigma_3]^{\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}} \cdot [\sigma_3, \sigma_1]^{\sigma_1^{-1}}$$

is trivial in N, whereas modulo $[N, \widehat{B_n}]$ the product is clearly equal to $[\sigma_1, \sigma_3]^2$, and so $[\sigma_1, \sigma_3]$ has order 2 in $N/[N, \widehat{B_n}]$. □

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