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Asymptotic Spatial Behaviour in Linearised Thermoelasticity for Non-Compact Regions.

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Dedicated with esteem to Giuseppe Saccomandi on the 60th Anniversary of his Birthday

Abstract

A quasi-static approximation is studied of linearised nonhomogeneous anisotropic compressible thermoelasticity on a non-compact region. Differential inequalities are constructed which under appropriate conditions lead to algebraic spatial growth and decay estimates for various cross-sectional and volume measures of the temperature, displacement and their spatial gradients.

Keywords: Linearised anisotropic thermoelasticity, non-compact regions, spatial behaviour, volume and spherical cross-sectional measures.

1 Introduction

Spatial behaviour is examined for a linearised nonhomogeneous anisotropic compressible thermoelastic material that occupies a non-compact region of Euclidean three-space. Examples of the non-compact region are a semi-infinite cone, half-space and exterior region. The cylinder has been treated elsewhere in the literature; see, for example, [13, 14]. In the present study, the relevant partial differential equations, simplified by appeal to a quasi-static approximation that neglects both inertia and velocity, consist of a parabolic equation for the
temperature and an elliptic system coupling temperature with displacement. By a linearised theory is meant the theory of small deformations superposed upon large in contrast to a linear theory where the reference state is assumed stress free and for which the elastic moduli accordingly possess both major and minor symmetry. We refer only briefly to the linear theory since arguments developed for the anisotropic case, relevant to the study of advanced materials, are only broadly applicable. A slightly different approach, to be reported separately, leads to improved estimates for isotropy. See Remark 2.1 and Section 8.

Subject to conditions prescribed on the data, algebraically increasing lower bounds and algebraically decreasing upper bounds are established for the respective spatial gradients of temperature and displacement in mean-square measures, including those defined over spherical cross-sections. Pointwise behaviour is not examined. Spherical cross-sections allow the half-space and exterior regions to be included without the need to specify asymptotic spatial behaviour which here is to be determined. The present treatment constructs and integrates differential inequalities for certain cross-sectional fluxes and in this respect is motivated by well-known arguments developed for uncoupled elliptic systems. (See, for example, [3, 12] and the references there cited). As far as the authors are aware, these methods have not previously been applied to the thermoelastic coupled system of present concern. Moreover, extension of the discussion to include the growth behaviour of temperature and displacement requires new techniques and has not been previously treated.

Decay of both temperature and displacement in cross-sectional and other measures is implied when the corresponding spatial gradients have algebraically decreasing upper bounds. On the other hand, it does not immediately follow that the temperature and displacement in suitable measure grow unboundedly when their spatial gradients possess algebraically increasing lower bounds. For the purely thermal problem this conclusion is proved in [11]. Here, we establish for the first time unbounded growth of the mean-square volume measure of the displacement but only when the temperature and its spatial gradient decay. The proof requires a Caccioppoli (or reverse Poincaré) inequality combined with further geometric structures. Other procedures in thermoelasticity discuss mainly stability and uniqueness and are not based on differential inequalities. They include those surveyed in [6–8]; see also [9].

Both growth and decay measures are averaged over space rather than space-time. Consequently, the respective estimates hold at a particular instant of time and not necessarily throughout the interval in which smooth solutions exist. This presents no difficulty regarding temperature which has infinite speed of propagation, but for the displacement the implications are less clear. They are overcome by tacitly assuming that the same data is prescribed during the whole interval of existence.

These issues may be attributed to the introduction of the quasi-static approximation which itself requires rigorous justification. Plausibility should not to be mistaken for proof (cp., [1, 2, 10]). Furthermore, application of a quasi-static approximation presumes that the initial and other data for which it is valid are compatible with that prescribed for the problem under consideration.

Section 2 disposes of notation, the statement of the initial boundary value problem, and certain geometric preliminaries including specification of the Cartesian coordinate system. Radial distance from the origin, located on a finite part of the boundary, is denoted by $r$, while $t$ denotes the time variable. Section 3 for ease of reference, provides a simplified version of the comprehensive discussion in [11] for the thermal component of the problem. Conditions
are formulated under which the relevant measures of the temperature and its spatial gradient either algebraically grow or decay. The Poincaré-Wirtinger inequality, pivotal to these and later calculations, is stated without proof. The coupled heat and displacement problem is studied in Section 4 in terms of the sum $H(r,t)$ of cross-sectional mechanical and heat fluxes whose radial derivative is proved positive for all sufficiently large radial values. Various inequalities are employed in this Section to also derive the basic differential inequality for $H$. Section 5 assumes that $H(r_0,t)$ is positive for some $r_0 > 0$ which enables the basic differential inequality to be integrated to yield algebraically increasing lower bounds in appropriate mean-square measures for the spatial gradients of temperature and displacement. A corresponding lower growth bound for the displacement, derived with the help of Caccioppoli’s inequality stated and proved in Section 6.1, is obtained in Section 6.2. The condition $H(r_0,t) \leq 0$ is shown in Section 7 to be sufficient for decay in various measures with respect to radial distance, while $H(r_0,t) = 0$ is sufficient for uniqueness. In particular, the appropriate form of the basic differential inequality derived in Section 4 is integrated to show with the help of the Poincaré-Wirtinger inequality that the temperature and displacement together with the respective spatial gradients decay at least algebraically in certain mean-square measures. Nevertheless, a supplementary calculation is required to establish decay of the displacement in mean-square cross-sectional measure under the same conditions. Section 8 briefly mentions some outstanding problems.

The growth and decay estimates obtained in this investigation are analogous to a Phragmén-Lindelöf principle with decay behaviour corresponding to a Saint-Venant principle. They are not only of intrinsic mathematical interest, but apply to general conduction and diffusion processes arising, for example, in biomedicine. In particular, asymptotic spatial behaviour of temperature and displacement, apart from that of the respective energies, appear to be of special practical importance in the design of structures for the nuclear and aerospace industries.

Standard conventions are adopted of the subscript comma to denote partial spatial differentiation and of summation over repeated Latin suffixes in the range 1, 2, 3. Greek suffixes range over 1, 2. An indicial notation generally indicates Cartesian components of a vector, but there is no typographical distinction between scalar and tensor quantities.

2 Geometry and other preliminaries

Let $\Omega \subseteq \mathbb{R}^3$ denote an open region with non-compact boundary; for example, the cone, half-space or exterior region. The non-compact boundary $\partial \Omega$ of $\Omega$ is supposed to be Lipschitz smooth and the union of disjoint parts $\partial \Omega_1$ and $\partial \Omega_2$ where $\partial \Omega_1$ satisfies

$$\partial \Omega_1 \subset \partial \Omega \cap B(0,r_0)$$

for given $r_0 > 0$, and $B(x,r)$ is the ball of radius $r > 0$ centred at $x \in \mathbb{R}^3$. The time-independent spatial region $\Omega(r_1,r_2)$ is denoted by

$$\Omega(r_1,r_2) := \Omega \cap [B(0,r_2) \setminus B(0,r_1)], \quad 0 < r_1 < r_2,$$

while the spherical cross-section $\Sigma(r)$ is defined by

$$\Sigma(r) := \Omega \cap \partial B(0,r), \quad r = |x| \equiv (x_ix_i)^{1/2},$$

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and in particular

\[ \Sigma(\infty) := \lim_{r \to \infty} \Sigma(r). \] (2.4)

The region \( \Omega \) is occupied by a linearised nonhomogeneous anisotropic compressible thermelastic material for which \( \sigma(x, t) \) is the stress tensor, \( q(x, t) \) the heat flux vector, and \( t \) denotes the time variable. Select a rectangular Cartesian coordinate system to have its origin at a point of \( \partial \Omega_1 \) and suppose the positive \( x_3 \)-axis lies entirely within \( \Omega \). With respect to this coordinate system, body forces and heat sources are restricted to \( \Omega(r_0, r) \), \( r_0 > 0 \). The quasi-static approximation of present concern neglects velocity and inertia and in consequence the thermoelastic stress and heat flux in \( \Omega(r_0, \infty) \) satisfy

\[ \sigma_{ij,j} = 0, \quad (x, t) \in \Omega(r_0, \infty) \times (0, T), \] (2.5)

\[ q_i,i = c \dot{\theta}, \quad (x, t) \in \Omega(r_0, \infty) \times (0, T), \] (2.6)

where \( \theta(x, t) \) is the positive temperature, \( (0, T) \) is the maximal (possibly finite) interval of existence of smooth solutions to the coupled initial boundary value problem stated below, and \( \sigma_{ij}(x, t) \), the Cartesian components of the stress tensor, and \( q_i(x, t) \), the components of the vector heat flux, are respectively related to the displacement vector \( u(x, t) \) and temperature \( \theta(x, t) \) by the constitutive assumptions

\[ \sigma_{ij} = c_{ijkl} u_{k,l} + \beta_{ij} \theta, \quad (x, t) \in \Omega \times [0, T), \] (2.7)

\[ q_i = \kappa_{ij} \theta_j, \quad (x, t) \in \Omega \times [0, T), \] (2.8)

where \( c_{ijkl}(x) \), \( \kappa_{ij}(x) \) and \( \beta_{ij}(x) \) are the non-homogeneous Cartesian components of the elastic moduli, heat conduction and thermal coupling tensors respectively. The elastic moduli and heat conduction tensors are assumed to satisfy the symmetry conditions

\[ c_{ijkl} = c_{klij}, \quad (2.9) \]

\[ \kappa_{ij} = \kappa_{ji}, \quad (2.10) \]

and to be positive definite in the sense that there exists uniformly bounded positive constants \( c_0 \) and \( \kappa_0 \) such that

\[ c_0 \psi_{ij} \psi_{ij} \leq c_{ijkl} \psi_{ij} \psi_{kl}, \quad (2.11) \]

\[ \kappa_0 \xi_i \xi_i \leq \kappa_{ij} \xi_i \xi_j, \quad (2.12) \]

for all second order tensors \( \psi \) and vectors \( \xi \). The elastic moduli, heat conduction and thermal coupling tensors are supposed bounded in \( \Omega \) and so we may put

\[ c_1^2 = \max_{\Omega} c_{ijkl}, \quad (2.13) \]

\[ \kappa_1^2 = \max_{\Omega} \kappa_{ij}, \quad (2.14) \]

\[ \beta^2 = \max_{\Omega} \beta_{ij}, \quad (2.15) \]

for uniformly bounded positive constants \( c_1 \), \( \kappa_1 \) and \( \beta \).
Remark 2.1 (Isotropy). In the classical linear theory both the heat conduction and elastic modulus tensors are constant and isotropic. Consequently, in the above notation and with δ_{ij} denoting the usual Kronecker delta, we have \( \kappa_{ij} = \kappa_0 \delta_{ij}, \beta_{ij} = \beta \delta_{ij}, \) while components of the elastic modulus satisfy
\[
c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]
where \( \lambda, \mu \) are the constant Lamé parameters. The elastic moduli now possess both major and minor symmetries given by
\[
c_{ijkl} = c_{jikl} = c_{klij},
\]
and the positive-definite condition (2.11) is defined on the set of symmetric second order tensors necessitating a slightly different treatment to that described in the following sections.

Remark 2.2 (Simple inequalities). In what follows, frequent use is made of the Schwarz inequality for fourth order tensors in the form
\[
|a_{ijkl}b_{ijkl}| \leq (a_{ijkl}a_{ijkl})^{1/2} (b_{ijkl}b_{ijkl})^{1/2},
\]
whose proof relies upon the inequality
\[
(a_{ijkl} + \mu b_{ijkl}) (a_{ijkl} + \mu b_{ijkl}) \geq 0,
\]
and on the selection and substitution of the optimal value of the constant \( \mu \) given by
\[
\mu = -\frac{(a_{ijkl}b_{ijkl})}{(b_{ijkl}b_{ijkl})}.
\]
An integral form of this inequality holds while corresponding inequalities hold for second order tensors.

Substitution of (2.7) and (2.8) respectively in (2.5) and (2.6) yields
\[
(c_{ijkl}u_{k,l})_{,j} + (\beta_{ij} \theta)_{,j} = 0, \quad (x,t) \in \Omega(r_0, \infty) \times (0,T),
\]
\[
(\kappa_{ij} \theta)_{,i} = c \dot{\theta}, \quad (x,t) \in \Omega(r_0, \infty) \times (0,T),
\]
to which are adjoined the initial and boundary conditions
\[
\theta(x,0) = 0, \quad x \in \Omega(r_0, \infty),
\]
\[
\theta(x,t) = 0, \quad (x,t) \in \partial \Omega_2 \times (0,T),
\]
\[
u_i(x,t) = 0, \quad (x,t) \in \partial \Omega_2 \times (0,T).
\]
Asymptotic spatial behaviour of neither the temperature nor the displacement is prescribed but is to be determined.

Observe that the temperature \( \theta(x,t) \) is calculated from the parabolic equation (2.19) subject to (2.20) and (2.21) and is therefore independent of the displacement. On the other hand, the elliptic system (2.18) shows that the displacement is coupled to the temperature.
Indeed, we could regard (2.18) as the displacement equilibrium equations subject to a body force
\[(\beta_{ij}\theta)_j.\]

In consequence, spatial growth or decay of the temperature becomes a factor in determining how the displacement behaves.

We first derive the relevant behaviour of the temperature which is then used to determine that of the displacement.

3 Heat equation

For ease of reference, certain growth and decay properties are established for the solution to the heat equation (2.19) subject to the homogeneous initial and boundary conditions (2.20) and (2.21). The method of proof, details of which are presented in [11], follows the general argument for the temperature spatial gradient used in Section 4 for exploring similar properties of the displacement gradient which is based upon that developed in [3]. Accordingly, consider the heat flow cross-sectional measure that for each \( t \in (0,T) \) is given by
\[
Q(r,t) := \int_0^t \int_{\Sigma(r)} q_i n_i dSd\eta, \quad r_0 \leq r \leq \infty,
\]
where \( n_i \) are the cartesian components of the unit normal on \( \Sigma(r) \) in the increasing radial direction, and \( dS \) is the element of area on \( \Sigma(r) \). The divergence theorem applied to (3.1) after appeal to (2.19), (2.20) and (2.21) leads to
\[
Q(r,t) = \left( \frac{c}{2} \right) \int_{\Omega(r_0,r)} \theta^2 dx + \int_0^t \int_{\Omega(r_0,r)} \kappa_{ij} \theta, i \theta, j dxd\eta + Q(r_0,t), \quad t \in (0,T),
\]
which may be written as
\[
Q(r,t) = V(r_0,t) + Q(r_0,t),
\]
where the thermal energy for each \( t \in (0,T) \) is defined to be
\[
V(r_1,r_2; t) := \left( \frac{c}{2} \right) \int_{\Omega(r_1,r_2)} \theta^2 dx + \int_0^t \int_{\Omega(r_1,r_2)} \kappa_{ij} \theta, i \theta, j dxd\eta, \quad 0 < r_1 < r_2 \leq \infty.
\]

It is immediate from (3.2) that the radial derivative of \( Q \), given by
\[
Q_r(r,t) = \left( \frac{c}{2} \right) \int_{\Sigma(r)} \theta^2 dS + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta, i \theta, j dSd\eta,
\]
is non-negative. In consequence for each \( t \in (0,T) \) we have
\[
Q(r,t) \geq Q(r_0,t), \quad 0 < r_0 < r \leq \infty.
\]

Crucial to our argument is the following version of the Poincaré-Wirtinger inequality [12]:

\[
\int_0^t \int_{\Sigma(r)} \theta^2 dS + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta, i \theta, j dSd\eta,
\]
Proposition 3.1 (Poincaré-Wirtinger inequality). Let \( w : \Omega \to \mathbb{R} \) be a smooth function that vanishes on \( \Sigma(r) \cap \partial \Omega_2 \). Then

\[
\int_{\Sigma(r)} w^2 \, dS \leq kr^2 \int_{\Sigma(r)} |\nabla_S w|^2 \, dS,
\]

(3.7)

where \( \nabla_S \) denotes the tangential gradient of \( w \) on \( \Sigma(r) \), and \( k \) is a constant whose value depends on \( \Omega \). For example, when \( \Omega \) is the half-space, \( k = 1/2 \).

Next consider the expression (3.1) for \( Q(r,t) \) and use Schwarz’s inequality to obtain

\[
|Q(r,t)| = \left| \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_j \theta_i dS d\eta \right| \leq \kappa_1^{1/2} \left( \int_0^t \int_{\Sigma(r)} \theta^2 dS d\eta \int_0^t \int_{\Sigma(r)} \theta_j \theta_j dS d\eta \right)^{1/2}
\]

\[
\leq a^{-1} r \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_j \theta_i dS d\eta,
\]

(3.9)

where

\[
a := \left( \frac{\kappa_0}{k \kappa_1} \right)^{1/2},
\]

(3.10)

and \( k \) is the constant appearing in the Poincaré-Wirtinger inequality (3.7). In conjunction with (3.5), we therefore obtain the basic differential inequality at each \( t \in (0,T) \):

\[
|Q(r,t)| \leq a^{-1} r Q_r(r,t), \quad r_0 < r \leq \infty, \quad t \in (0,T).
\]

(3.11)

Remark 3.1. Heat sources and initial data in \( \Omega(0,r_0) \) together with specified thermal boundary conditions on \( \partial \Omega_1 \) are assumed sufficient to determine the sign of \( H(r_0,t) \) which of course may vary with \( t \in (0,T) \). Consequently, conclusions of this and later Sections hold only for that value of \( t \) for which the corresponding sign of \( H(r_0,t) \) holds. Although infinite speed of heat propagation implies instantaneous change in the growth and decay of thermal measures caused by the sign change in \( H(r_0,t) \), the same may not hold for measures involving the displacement.

Growth in suitable measures is established on assuming that the data prescribed in \( \Omega(0,r_0) \) is such that

\[
Q(r_0,t) > 0,
\]

(3.12)

for then by (3.6) we have \( Q(r,t) \geq Q(r_0,t) > 0 \) and (3.11) may be written

\[
rQ_r - aQ \geq 0,
\]

(3.13)

which integrates to give for each \( t \in (0,T) \) the growth estimate

\[
Q(r,t) \geq Q(r_0,t) \left( \frac{r}{r_0} \right)^a.
\]

(3.14)
By virtue of (3.5) and (3.13) we conclude that
\[ a \frac{r^{a-1}}{r_0^a} Q(r_0, t) \leq \left( \frac{c}{2} \right) \int_{\Sigma(r)} \theta^2 dS + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_i \theta_j dS d\eta, \tag{3.15} \]
but when \( a > 1 \) we cannot deduce which term on the right grows with \( r \). Nevertheless, it is proved in [11] that (3.12) is sufficient for the temperature to become pointwise unbounded in \( \Omega(r_0, r) \) as \( r \to \infty \). Observe also, again provided \( a > 1 \), that we obtain from (3.9) a growth estimate for the mean-square cross-sectional measure of the temperature gradient:
\[ \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_i \theta_j dS d\eta \geq a \frac{r^{a-1}}{r_0^a} Q(r_0, t). \tag{3.16} \]
Of course, growth estimates with unrestricted \( a \) may be expressed in terms of the thermal energy \( V(r_0, r; t) \) defined in (3.4) since (3.3) and (3.14) show that
\[ V(r_0, r; t) \geq \left( \frac{r}{r_0} \right)^a - 1 \right) Q(r_0, t), \tag{3.17} \]
and the thermal energy ceases to exist as \( r \to \infty \).

Now consider temperatures in the class for which the thermal energy remains bounded as \( r \to \infty \) for all \( t \in (0, T) \). Then
\[ V(r_0, \infty; t) < \infty, \tag{3.18} \]
and we must have
\[ Q(r_0, t) \leq 0, \quad t \in (0, T). \tag{3.19} \]
Assume that \( Q(r_0, t) = 0 \). Since \( Q(r, t) \) is non-decreasing with \( r \), it follows that \( Q(r, t) \) is identically zero for all \( r_0 \leq r \leq \infty \) (otherwise \( Q(r_1, t) > 0 \) at \( r_1 > r_0 \) contradicting the assumption of bounded thermal energy). In consequence, by (3.2) we conclude that \( \theta(r, t) \) globally vanishes.

Therefore, we suppose that \( Q(r_0, t) < 0 \) and that
\[ Q(r, t) < 0, \quad r_0 \leq r \leq \infty, \quad t \in (0, T). \tag{3.20} \]
We can then write (3.11) as
\[ r Q_{,r}(r, t) + aQ(r, t) \geq 0, \quad r_0 < r \leq \infty, \tag{3.21} \]
for each \( t \in (0, T) \). Integration leads to the decay estimate
\[ -Q(r, t) \leq - \left( \frac{r_0}{r} \right)^a Q(r_0, t), \tag{3.22} \]
and consequently \( Q(r, t) \to 0 \) as \( r \to \infty \) which implies that
\[ V(r, \infty; t) = -Q(r, t) \leq \left( \frac{r_0}{r} \right)^a V(r_0, \infty; t) \tag{3.23} \]
for each \( t \in (0, T) \). The last inequality together with definition (3.4) leads to the conclusion
\[ \lim_{r \to \infty} \int_{\Omega(r, t)} \theta^2 dx = 0. \]
4 The coupled problem

We now seek to derive a differential inequality similar to (3.11) for the cross-sectional measure $H(r, t)$ defined for each $t \in [0, T)$ by

$$H(r, t) := W(r, t) + 2\lambda Q(r, t), \quad r_0 < r \leq \infty,$$

where $\lambda$ is a positive constant to be determined, and $W(r, t)$ is the cross-sectional flux specified by

$$W(r, t) := \int_{\Sigma(r)} \sigma_{ij} u_i n_j \, dS.$$  

(4.1)

The argument is the counterpart of that developed for $Q(r, t)$ in Section 3. The divergence theorem combined with (2.5)-(2.6) and (2.20)-(2.22) leads to the alternative expression

$$H(r, t) = \int_{\Omega(r_0, r)} (\sigma_{ij} u_i + \lambda c \theta^2) \, dx + 2\lambda \int_0^t \int_{\Sigma(r_0, r)} \kappa_{ij} \theta_i \theta_j \, dxd\eta + H(r_0, t),$$

(4.3)

which together with (2.7) implies that the radial derivative of $H(r, t)$ satisfies

$$H_r(r, t) = \int_{\Sigma(r)} (c_{ijkl} u_i u_k u_l + \beta_{ij} u_i \theta \theta_j + c\lambda \theta^2) \, dS + 2\lambda \int_0^t \int_{\Sigma(r_0, r)} \kappa_{ij} \theta_i \theta_j \, dSd\eta.$$  

(4.4)

On using the bounds (2.11) and (2.15) and Young’s inequality, we have the pointwise inequality

$$c_{ijkl} u_i u_k u_l + \beta_{ij} u_i \theta \theta_j + c\lambda \theta^2 \geq \left( c_0 - \frac{1}{2\epsilon_1} \right) u_i u_j + \left( c_\lambda - \frac{\epsilon_1 \beta^2}{2} \right) \theta^2,$$  

(4.5)

where the positive constant $\epsilon_1$ is chosen to lie in the interval

$$\frac{1}{2c_0} < \epsilon_1 < \frac{2c\lambda}{\beta^2},$$

from which we conclude that $\lambda$ must possess the lower bound

$$\lambda > \frac{\beta^2}{4c c_0}. $$

(4.6)

Indeed, we set

$$4cc_0 \lambda > \max \left( 4c_0^2, \beta^2 \right),$$

(4.7)

and take $\epsilon_1$ to be

$$\epsilon_1 := \frac{1}{2} \left( \frac{1}{2c_0} + \frac{2c\lambda}{\beta^2} \right).$$  

(4.8)

Substitution then enables the lower bound (4.5) to be written

$$c_{ijkl} u_i u_k u_l + \beta_{ij} u_i \theta \theta_j + c\lambda \theta^2 > c_2 \left( u_i u_j + \theta^2 \right),$$

(4.9)
where
\[ c_2 := \frac{(4c_0\lambda - \beta^2)}{8c_0}, \]  
(4.10)
and \( \lambda \) satisfies the bound (4.7).

Insertion into (4.4) immediately yields the lower bound
\[ H_{r}(r, t) \geq c_2 \int_{\Sigma(r)} (u_{i,j} u_{i,j} + \theta^2) \, dS + 2\lambda \int_{0}^{t} \int_{\Sigma(r)} \kappa_{ij} \theta_{i} \theta_{j} \, dS \, d\eta, \]  
(4.11)
while insertion into (4.3) leads to
\[ H(r, t) - H(r_0, t) \geq c_2 \int_{\Omega(r_0, r)} (u_{i,j} u_{i,j} + \theta^2) \, dx + 2\lambda \int_{0}^{t} \int_{\Omega(r_0, r)} \kappa_{ij} \theta_{i} \theta_{j} \, dx \, d\eta. \]  
(4.12)
Consequently, we have that the radial derivative of \( H(r, t) \) is non-negative for each \( t \in (0, T) \) and for \( r_0 \leq r \leq \infty \); that is \( H_{r}(r, t) \geq 0 \), or from (4.12)
\[ H(r, t) \geq H(r_0, t), \quad r_0 \leq r \leq \infty, \quad t \in (0, T). \]  
(4.13)
Derivation of corresponding upper bounds employs (2.13)-(2.15) with standard inequalities (cf., Remark 2.2) to obtain the point-wise estimate
\[ c_{ijkl} u_{i,j} u_{k,l} + \beta_{ij} u_{i,j} \theta + c\lambda \theta^2 \leq \left( c_1 + \frac{1}{2\epsilon_2} \right) u_{i,j} u_{i,j} + \left( c\lambda + \frac{\epsilon_2 \beta^2}{2} \right) \theta^2, \]  
(4.14)
where the positive constant \( \epsilon_2 \), when chosen to be
\[ \beta^2 \epsilon_2 = (c_1 - c\lambda) + \sqrt{(\beta^2 + (c\lambda - c_1)^2)}, \]  
(4.15)
leads to the upper bound
\[ c_{ijkl} u_{i,j} u_{k,l} + \beta_{ij} u_{i,j} \theta + c\lambda \theta^2 \leq c_3 (u_{i,j} u_{i,j} + \theta^2), \]  
(4.16)
where
\[ c_3 := \frac{1}{2} \left( (c\lambda + c_1) + \sqrt{(\beta^2 + (c\lambda - c_1)^2)} \right). \]  
(4.17)
Note that \( c_3 > c_2 \).

Therefore, we have the upper bounds
\[ H_{r}(r, t) \leq c_3 \int_{\Sigma(r)} (u_{i,j} u_{i,j} + \theta^2) \, dS + 2\lambda \int_{0}^{t} \int_{\Sigma(r)} \kappa_{ij} \theta_{i} \theta_{j} \, dS \, d\eta, \]  
(4.18)
\[ H(r, t) - H(r_0, t) \leq c_3 \int_{\Omega(r_0, r)} (u_{i,j} u_{i,j} + \theta^2) \, dx + 2\lambda \int_{0}^{t} \int_{\Omega(r_0, r)} \kappa_{ij} \theta_{i} \theta_{j} \, dx \, d\eta. \]  
(4.19)
In accordance with the procedure of Section 3, we next derive an upper bound for $|H(r,t)|$ in terms of cross-sectional measures. We have

$$\begin{align*}
|H(r,t)| &= |W(r,t) + 2\lambda Q(r,t)| \\
&\leq \left| \int_{\Sigma(r)} (c_{ijkl} u_{k,i} u_{n,j} + \beta_{ij} u_i \theta_n) \, dS \right| + 2\lambda \left| \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_{j,n} \, dS \, d\eta \right| \\
&\leq c_1 \int_{\Sigma(r)} (u_{k,i} u_{k,j} u_{i,j})^{1/2} \, dS + \beta \int_{\Sigma(r)} |\theta| (u_i u_i)^{1/2} \, dS \\
&\quad + 2\lambda \kappa_1 \int_0^t \int_{\Sigma(r)} |\theta| (\theta,\theta)^{1/2} \, dS \, d\eta \\
&\leq c_1 \left[ \int_{\Sigma(r)} u_{i,j} u_{j,i} \, dS \int_{\Sigma(r)} u_i u_i \, dS \right]^{1/2} + \beta \left[ \int_{\Sigma(r)} \theta^2 \, dS \int_{\Sigma(r)} u_i u_i \, dS \right]^{1/2} \\
&\quad + 2\lambda \kappa_1 \left[ \int_0^t \int_{\Sigma(r)} \theta^2 \, dS \, d\eta \int_0^t \int_{\Sigma(r)} \theta_{,i} \theta_{,j} \, dS \, d\eta \right]^{1/2} \\
&\leq r k^{1/2} \left( c_1 + \frac{\beta}{2\epsilon_3} \right) \int_{\Sigma(r)} u_{i,j} u_{j,i} \, dS + \frac{\epsilon_3 \beta^{1/2} r}{2} \int_{\Sigma(r)} \theta^2 \, dS \\
&\quad + 2\lambda \kappa_1 k^{1/2} r \int_0^t \int_{\Sigma(r)} \theta_{,i} \theta_{,j} \, dS \, d\eta,
\end{align*}$$

(4.20)

where $k$ is the constant occurring in the Poincaré-Wirtinger inequality (3.7), and $\epsilon_3$ is a positive constant taken to be $\beta \epsilon_3 = c_1 + \sqrt{c_1^2 + \beta^2}$.

Insertion into (4.20) yields the bound

$$|H(r,t)| \leq c_4 k^{1/2} r \left[ c_2 \int_{\Sigma(r)} (u_{i,j} u_{j,i} + \theta^2) \, dS + 2\lambda \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_{,i} \theta_{,j} \, dS \, d\eta \right],$$

(4.21)

where we set

$$c_4 := \max \left( \kappa_1, \frac{\sqrt{\beta^2 + c_1^2} + c_1}{2c_2} \right).$$

(4.22)

The required differential inequality is obtained on combining inequalities (4.21) and (4.11) to give at each $t \in (0,T)$

$$|H(r,t)| \leq \gamma^{-1} r^H(r,t), \quad r_0 < r \leq \infty,$$

(4.23)

where

$$\gamma = \frac{1}{k^{1/2} c_4}.$$

(4.24)

The sign of $H(r_0,t)$ may change with $t$ (see Remark 3.1). Separate consideration is now given when $H(r_0,t)$ is positive, zero or negative for given $t \in (0,T)$. 

11
5 Elastic growth

We suppose within $\Omega(0, r_0)$ that body forces, heat sources, kinematical and thermal initial and boundary conditions are such that for some $t \in (0, T)$ we have

$$H(r_0, t) > 0,$$

(5.1)

which by (4.13) implies

$$H(r, t) > 0, \quad r_0 \leq r \leq \infty, \quad t \in (0, T).$$

(5.2)

Consequently, (4.23) may be written

$$rH_r(r, t) - \gamma H(r, t) > 0,$$

(5.3)

which on integration gives

$$H(r_0, t) \left( \frac{r}{r_0} \right)^{\gamma} \leq H(r, t) \leq \gamma^{-1} r H_r(r, t),$$

(5.4)

where (5.3) is used. Hence, from (4.18), we obtain

$$\gamma H(r_0, t) \frac{r^{\gamma - 1}}{r_0^{\gamma}} \leq c_3 \int_{\Sigma(r)} (u_{i,j} u_{i,j} + \theta^2) \, dS + 2\lambda \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_i \theta_j \, dS \, d\eta,$$

(5.5)

and conclude that the right side of (5.5) grows at least algebraically provided $\gamma > 1$. Alternatively, we conclude from (4.19) that

$$\left( \left( \frac{r}{r_0} \right)^{\gamma} - 1 \right) H(r_0, t) \leq c_3 \int_{\Omega(r_0, r)} (u_{i,j} u_{i,j} + \theta^2) \, dx + 2\lambda \int_0^t \int_{\Omega(r_0, r)} \kappa_{ij} \theta_i \theta_j \, dx \, d\eta.$$  

(5.6)

Observe that not all terms on the right of (5.5) or (5.6) may grow, although at least one must become unbounded as $r \to \infty$. For example, it is established in Section 3 that when at time instant $t \in (0, T)$ we have

$$Q(r_0, t) < 0,$$

then there is at most algebraic decay of $V(r, \infty; t)$; or by (3.4),(3.18) and (3.23) that there is corresponding separate decay of

$$\int_{\Omega(r_0, \infty)} \theta^2 \, dS,$$

$$\int_0^t \int_{\Omega(r_0, r)} \kappa_{ij} \theta_i \theta_j \, dS \, d\eta, \quad r_0 \leq r \leq \infty,$$

It follows from (5.6) that subject to these conditions there is at least algebraic growth as $r \to \infty$ of the term

$$\int_{\Omega(r_0, r)} u_{i,j} u_{i,j} \, dS.$$

(5.7)

It becomes of interest to establish whether growth of the spatial gradient measured by (5.7) implies that of the displacement in appropriate measure. For this purpose, we state and prove a version of Caccioppoli’s inequality suitable to our needs based upon notes by Giaquinta [4]. We also assume that both the displacement $u_i$ and temperature $\theta$ are sufficiently smooth although relaxation of this condition appears possible.

The problem of establishing displacement behaviour when the temperature gradient and therefore temperature grow is not considered in the present study.
6 Caccioppoli’s inequality

We consider the system (2.18) subject to boundary conditions (2.22) and the growth condition (5.1). The temperature satisfies (2.19), the initial and boundary conditions (2.20) and (2.21), and the decay condition (3.20). The temperature and its spatial gradient therefore decay in appropriate measures.

6.1 Caccioppoli’s inequality and its proof

Additional notation required for the statement and proof of Caccioppoli’s inequality is now introduced. Let \( z \in \Sigma(r) \) and let \( \rho_1 < \rho_2 \leq \infty \) be scalar quantities where \( \rho_2 > \text{dist}(z, \partial \Omega_2) \).

Moreover, define the regions \( \tilde{\Omega} \) and \( \tilde{\Omega}_0 \) by

\[
\tilde{\Omega} := [B(z, \rho_2) \setminus B(z, \rho_1)] \cap \Omega(r_0, r) \quad (6.1)
\]

\[
\tilde{\Omega}_0 := B(z, \rho_2) \cap \Omega(r_0, r) \quad (6.2)
\]

and let \( \partial \tilde{\Omega} = \partial \tilde{\Omega}_1 \cup \partial \tilde{\Omega}_2 \) where

\[
\partial \tilde{\Omega}_2 := \partial \Omega_2 \cap B(z, \rho_2). \quad (6.3)
\]

Observe that \( \partial \tilde{\Omega} \setminus \partial B(z, \rho_1) = \partial \tilde{\Omega}_0 \).

Remark 6.1. When \( \rho_2 \) is chosen to satisfy

\[
\rho_2 \leq \text{dist}(z, \partial \Omega_2),
\]

then

\[
\tilde{\Omega} = B(z, \rho_2) \setminus B(z, \rho_1),
\]

and \( \partial \tilde{\Omega}_2 = 0 \).

The following statement and proof of Caccioppoli’s inequality are motivated by the account in [4] by Giaquinta.

**Theorem 6.1** (Caccioppoli’s inequality). Let \( u_i(x) \) be components of a smooth vector field that satisfies (2.18) subject to boundary conditions (2.22). The smooth temperature \( \theta(x, t) \) satisfies (2.19), (2.20), (2.21) and (3.20) and therefore decays in suitable measure as \( r \to \infty \) for each \( t \in (0, T) \). Let \( z \in \Sigma(r) \). Then

\[
\int_{B(z, \rho_1)} u_{i,j} u_{i,j} \, dx \leq \frac{2c_5}{c_0^2} \left\{ \frac{1}{(\rho_2 - \rho_1)^2} \int_{\tilde{\Omega}} u_i u_i \, dz + \int_{\tilde{\Omega}_0} \theta^2 \, dx \right\}, \quad (6.4)
\]

where \( c_5 \) is a positive constant given by

\[
c_5 := \max_{\tilde{\Omega}} \left( 4c_0^2 + c_0, \beta^2(1 + c_0) \right). \quad (6.5)
\]
**Proof:** Define the vector test function \( \phi(x) \) to have components specified by
\[
\phi_i(x) := u_i(x) \zeta^2(x), \quad x \in \Omega,
\]
where the cut-off function \( \zeta(x) \) satisfies
\[
0 \leq \zeta \leq 1, \quad x \in \Omega, \tag{6.6}
\]
together with
\[
\zeta(x) = 1, \quad x \in B(z, \rho_1), \tag{6.7}
\]
\[
\zeta(x) = 0, \quad x \in \partial \tilde{\Omega}_0 \setminus (\partial \Omega_2 \cup B(z, \rho_2)), \tag{6.8}
\]
\[
\zeta_i \zeta, i \leq \frac{1}{(\rho_2 - \rho_1)^2}, \quad x \in \tilde{\Omega}, \tag{6.9}
\]
which allows the gradient of \( \zeta \) to become infinite as \( \rho_1 \to \rho_2 \). Then we have
\[
c_0 \int_{\tilde{\Omega}_0} \zeta^2 u_{i,j} u_{i,j} \, dx \leq \int_{\tilde{\Omega}_0} \zeta^2 c_{ijkl} u_{i,j} u_{k,l} \, dx
\]
\[
= \int_{\partial \tilde{\Omega}_0} \zeta^2 u_i c_{ijkl} u_{k,l} n_j \, dS - 2 \int_{\Omega_0} \zeta \zeta,j u_i c_{ijkl} u_{k,l} \, dx
\]
\[
- \int_{\Omega_0} \zeta^2 (c_{ijkl} u_{k,l})_j \, dx
\]
\[
= -2 \int_{\Omega_0} \zeta \zeta,j u_i c_{ijkl} u_{k,l} \, dx + \int_{\Omega_0} \zeta^2 u_i (\beta_{ij})_j \, dx
\]
\[
= -2 \int_{\Omega_0} \zeta \zeta,j u_i c_{ijkl} u_{k,l} \, dx + \int_{\partial \tilde{\Omega}_0} \zeta^2 u_i \beta_{ij} \theta n_j \, dS - 2 \int_{\Omega_0} \zeta \zeta,j u_i \beta_{ij} \theta \, dx
\]
\[
- \int_{\Omega_0} \zeta^2 u_i \beta_{ij} \theta \, dx
\]
\[
= -2 \int_{\Omega_0} \zeta \zeta,j u_i c_{ijkl} u_{k,l} \, dx - 2 \int_{\Omega_0} \zeta \zeta,j u_i \beta_{ij} \theta \, dx
\]
\[
- \int_{\Omega_0} \zeta^2 u_i \beta_{ij} \theta \, dx, \tag{6.10}
\]
where \( \tilde{\Omega}_0 \) is defined by (6.2) over whose surface the respective integrals vanish.

Young’s inequality, Remark 2.2 and the properties (6.6)-(6.9) of the cut-off function \( \zeta(x) \)
are used to bound each integral on the right of the last equation. We have

\[-2 \int_{\tilde{\Omega}_0} \zeta \zeta, j u_i, c_{ijkl} u_{k,l} \, dx \leq 2 \left| \int_{\Omega_0} \zeta \zeta, j u_i, c_{ijkl} u_{k,l} \, dx \right| \]

\[\leq 2c_1 \int_{\Omega_0} (\zeta^2 \zeta, j u_i, u_{k,l} u_{k,l})^{1/2} \, dx \]

\[\leq c_1 \varepsilon_4 \int_{\Omega_0} \zeta^2 u_{i,j} u_{i,j} \, dx + \frac{c_1}{\varepsilon_4} \int_{\Omega_0} \zeta, j u_i u_i \, dx \]

\[\leq c_1 \varepsilon_4 \int_{\Omega_0} \zeta^2 u_{i,j} u_{i,j} \, dx + \frac{c_1}{\varepsilon_4 (\rho_2 - \rho_1)^2} \int_{\Omega} u_i u_i \, dx, \quad (6.11)\]

\[-2 \int_{\Omega_0} \zeta, j \beta_{ij} \theta \, dx \leq \int_{\Omega_0} \zeta, j \zeta, j u_i u_i \, dx + \int_{\Omega_0} \zeta^2 \beta_{ij} \beta_{ij} \theta^2 \, dx \]

\[\leq \frac{1}{(\rho_2 - \rho_1)^2} \int_{\Omega} u_i u_i \, dx + \beta^2 \int_{\Omega} \theta^2 \, dx \quad (6.12)\]

\[-\int_{\Omega_0} \zeta^2 u_{i,j} \beta_{ij} \theta \, dx \leq \frac{\varepsilon_5}{2} \int_{\Omega_0} \zeta^2 u_{i,j} u_{i,j} \, dx + \frac{1}{2 \varepsilon_5} \int_{\Omega_0} \zeta^2 \beta_{ij} \beta_{ij} \theta^2 \, dx \]

\[\leq \frac{\varepsilon_5}{2} \int_{\Omega_0} \zeta^2 u_{i,j} u_{i,j} \, dx + \frac{\beta^2}{2 \varepsilon_5} \int_{\Omega_0} \zeta^2 \theta^2 \, dx, \quad (6.13)\]

where \(\varepsilon_4, \varepsilon_5\), are positive constants chosen to be

\[\varepsilon_4 = \frac{c_0}{4 c_1}, \quad \varepsilon_5 = \frac{c_0}{2} .\]

On insertion of (6.11)-(6.13) into (6.10), on noting that

\[\int_{B(z, \rho_1)} u_{i,j} u_{i,j} \, dx \leq \int_{\Omega_0} \zeta^2 u_{i,j} u_{i,j} \, dx,\]

we are led after some rearrangement to (6.4) and Theorem 6.1 is proved.

**Remark 6.2 (Extended Inequality).** Examination of the proof of Theorem 6.1 indicates that it remains valid when

\[\rho_1 > \text{dist} (z, \partial \Omega_2), \quad (6.14)\]

so that

\[\tilde{B}(z, \rho_1) := B(z, \rho_1) \cap \Omega \subset B(z, \rho_1). \quad (6.15)\]

In which case, in the statement of the Theorem, the integral over \(B(z, \rho_1)\) on the left is replaced by that over \(\tilde{B}(z, \rho_1)\).

### 6.2 Growth of displacement

Subject to the asymptotic decay of temperature and its spatial gradient together with certain other conditions, including those of this Section, we use the modification of Theorem 6.1 as
described in Remark 6.2 to establish the unbounded growth of the displacement $u(x,t)$ in suitable mean-square measure as $r \to \infty$ for each $t \in (0,T)$. Further geometric structures are required.

Let $z \in \Sigma(r)$ be the position vector of the point of intersection of the $x_3$-axis with the cross-section surface $\Sigma(r)$. Consider the great circle $\Delta(r, \phi)$ on $\Sigma(r)$ through $z$ in the plane that intercepts the $x_3$-axis at angle $\phi$ to the $x_1$-axis. For fixed $r, \phi$, let the $(r, \phi, \psi_\alpha)$, $\alpha = 1, 2$, be the spherical polar coordinates of vector points $z_\alpha(r, \phi) \in \Delta(r, \phi)$, where $z_1$ and $z_2$ are on adjacent sides of $z$ and let the radii $\rho(r, \phi)$ be such that

$$B(z_\alpha, \rho_{\alpha+2}) \cap \Omega \subset B(z_\alpha, \rho_{\alpha+2}),$$

satisfy the conditions of Remark 6.2. Let $\Gamma(r_1, r; \phi)$ lie in the plane that intercepts the $x_3$-axis and is inclined to the $x_1$-axis at angle $\phi$. Therefore the great circle $\Delta(r, \phi)$ belongs to part of the boundary of $\Gamma(r_1, r; \phi)$. Indeed, the boundary of $\Gamma(r_1, r; \phi)$ is specified by

$$\partial \Gamma(r_1, r; \phi) := \Delta(r, \phi) \cup \partial \Omega_2^{(1)} \cup \partial \Omega_2^{(2)} \cup \Delta(r_1, \phi),$$

where

$$\partial \Omega_2^{(\alpha)} := \partial \Omega_2 \cap B(z_\alpha, \rho_{\alpha+2}), \quad \alpha = 1, 2,$$

and $\Delta(r, \phi)$ is the great circle in the same plane as $\Delta(r, \phi)$ but on the cross-sectional surface $\Sigma(r_1)$. For position vectors $w_\alpha, \alpha = 1, 2$, the radius $r_1 < r$ is defined by

$$r_1 := \max (w_1, w_2), \quad w_\alpha := \partial B(z, \rho_1) \cap \partial B(z_\alpha, \rho_{\alpha+2}).$$

Consequently, we have

$$\Omega(r_1, r) = \bigcup_{0 \leq \phi < \pi} \Gamma(r_1, r; \phi), \quad r_1 < r,$$

which implies

$$\int_{\Omega(r_1, r)} u_{i,j} u_{i,j} \, dx \leq \int_{B(z, \rho_1)} u_{i,j} u_{i,j} \, dx + \sum_{\alpha=1}^{2} \int_{B(z_\alpha, \rho_{\alpha+2})} u_{i,j} u_{i,j} \, dx$$

$$\leq c_6 \int_{\Omega(r_1, r)} u_i u_i \, dx + c_7 \int_{\Omega(r_1, r)} \theta^2 \, dx,$$

where $c_6, c_7$ are computable constants and we have employed Theorem 6.1 and the modification of Remark 6.2. Let $r_1 > r_0$. From (4.19) and (5.4) we obtain

$$c_3 \int_{\Omega(r_1, r)} \left( u_{i,j} u_{i,j} + \theta^2 \right) \, dx \geq 2\lambda \int_{\Omega(r_1, r)} \kappa_{i,j} \theta_i \theta_j \, dx \eta$$

$$\geq H(r, t) - H(r_1, t)$$

$$\geq H(r_1, t) \left[ \left( \frac{r}{r_1} \right)^\gamma - 1 \right]$$

$$\geq H(r_0, t) \left[ \left( \frac{r_1}{r_0} \right)^\gamma - 1 \right] \left[ \left( \frac{r}{r_1} \right)^\gamma - 1 \right].$$
Combine (6.21) and (6.22) to write
\[
H(r_0, t) \left[ \left( \frac{r_1}{r_0} \right)^\gamma - 1 \right] \left[ \left( \frac{r}{r_1} \right)^\gamma - 1 \right] \leq c_8 \int_{\Omega(r_1, r)} u_i u_i \, dx + c_9 \int_{\Omega(r_1, r)} \theta^2 \, dx \\
+ 2\lambda \int_0^t \int_{\Omega(r_1, r)} \kappa_{ij} \theta_i \theta_j \, dx \, d\eta,
\] (6.23)

where \( c_8, c_9 \) are computable constants. But the thermal terms in the last expression decay as \( r \to \infty \) and we conclude that the mean-square volume measure of the displacement becomes unbounded.

It remains open whether the mean-square measure over the cross-section \( \Sigma(r) \) of the displacement likewise becomes unbounded with \( r \) and for \( \gamma > 1 \).

7 Coupled Problem: Decay

We now examine displacements and temperature in the class for which
\[
c_3 \int_{\Sigma(r)} (u_{i,j} u_{i,j} + \theta^2) \, dS + 2\lambda \int_0^t \int_{\Sigma(r)} \kappa_{ij} \theta_i \theta_j \, dS \, d\eta < \infty, \quad r \to \infty,
\] (7.1)
which contradicts the growth estimate (5.5). Displacements and temperature in the class for which (5.6) is likewise contradicted can similarly be treated. Both assumptions therefore imply that within these classes we must have for each \( t \in (0, T) \):
\[
H(r_0, t) \leq 0.
\]
The case \( H(r_0, t) = 0 \) may be treated similarly to the case when \( Q(r_0, t) = 0 \) in Section 3 and again provides a condition for uniqueness. Let us therefore suppose that
\[
H(r, t) < 0, \quad \forall t \in (0, T), \quad r_0 \leq r \leq \infty,
\] (7.2)
since otherwise \( H(r, t) \) becomes positive at some \( r_1 > r_0 \) and (7.1) is contravened. We conclude that the basic differential inequality (4.23) assumes the form
\[
rH_{,r}(r, t) + \gamma H(r, t) \geq 0,
\] (7.3)
which on integration yields
\[
-H(r, t) \leq - \left( \frac{r_0}{r} \right)^\gamma H(r_0, t),
\]
and we have \( H(r, t) \to 0 \) as \( r \to \infty \). Accordingly, we obtain from (4.12) the decay estimate
\[
c_2 \int_{\Omega(r, \infty)} (u_{i,j} u_{i,j} + \theta^2) \, dx + 2\lambda \int_0^t \int_{\Omega(r, \infty)} \kappa_{ij} \theta_i \theta_j \, dx \, d\eta \leq -H(r, t) \\
\leq - \left( \frac{r_0}{r} \right)^\gamma H(r_0, t),
\] (7.4)
which implies that each term on the left of (7.4) decays at most algebraically.

We proceed to prove that the mean-square cross-sectional measure of the displacement likewise decays at most algebraically provided that \( \gamma > 1 \) where \( \gamma \) is defined in (4.24) (cp., [12]). Indeed, from (7.4) we deduce that

\[
\int_{\Sigma(r)} u_{i,j} u_{i,j} \, dS = o(r^{-(\gamma+1)}), \quad \text{as } r \to \infty,
\] (7.5)

which on using the Poincaré-Wirtinger inequality (3.7) implies

\[
r^{-1} \int_{\Sigma(r)} u_i u_t \, dS = o(r^{-\gamma}), \quad \text{as } r \to \infty.
\] (7.6)

Young’s inequality together with integration by parts then gives

\[
r^{-2} \int_{\Sigma(r)} u_i u_j x_j \, dS = - \int_{\Omega(r,\infty)} \left( 3u_i u_t d^2 - 2u_i u_t x_j x_j d^4 \right) \, dx - 2 \int_{\Omega(r,\infty)} u_i u_j x_j d^2 \, dx
\]
\[
\leq - \int_{\Omega(r,\infty)} u_i u_t d^2 \, dx + \int_{\Omega(r,\infty)} u_i u_j d^2 \, dx + \int_{\Omega(r,\infty)} u_i u_j u_i u_j \, dx,
\]

where \( d^2 = x_i x_i \), so that on recalling (7.4) we have the decay estimate

\[
r^{-1} \int_{\Sigma(r)} u_i u_t \, dS \leq \int_{\Omega(r,\infty)} u_i u_j u_i u_j \, dx
\]
\[
= c_2^{-1} \left[ c_2 \int_{\Omega(r,\infty)} \left( u_{i,j} u_{i,j} + \theta^2 \right) \, dx + 2\lambda \int_0^t \int_{\Omega(r,\infty)} \kappa_{ij} \theta_i \theta_j \, dx \, d\eta \right]
\]
\[
\leq - c_2^{-1} \left( \frac{r_0}{r} \right)^\gamma H(r_0, t),
\]

and the argument is complete subject to \( \gamma > 1 \) and the other conditions of this subsection.

**Remark 7.1.** Suppose that

\[
Q(r_0, t) > 0,
\] (7.7)

which from Section 3 implies algebraic growth of

\[
(c/2) \int_{\Omega(r_0, r)} \theta^2 \, dx + \int_0^t \int_{\Omega(r_0, r)} \kappa_{ij} \theta_i \theta_j \, dx \, d\eta
\]

as \( r \to \infty \). Consequently, when the flux \( W(r, t) \) satisfies

\[
W(r_0, t) < -2\lambda Q(r_0, t),
\] (7.8)

we have a contradiction and conclude that condition (7.2) is incompatible with (7.7).
8 Concluding Remarks

The present investigation concerns only the quasi-static approximation of linearised thermoelasticity on non-compact regions and leaves open at least two other problems. The first is to derive corresponding results for the complete linearised coupled system of equations for thermoelasticity in which the quasi-static approximation is relaxed and inertia retained. The second is to obtain companion results for classical linear thermoelasticity for which the elastic moduli have both major and minor symmetries. Consequently, elastic strain and not displacement gradient occurs in the theory and the positive-definite assumption (2.11) becomes restricted to symmetric second order tensors. The Poincaré-Wirtinger inequality probably should be replaced by an appropriate Korn’s inequality valid over spherical cross-sections that is derived, for example, in [5]. In the particular case of the classical linear theory outlined in Remark 2.1 and where the moduli are homogeneous and isotropic, the previous arguments may be broadly applied but with the function \( W(r,t) \) defined in (4.2) replaced by

\[
W_1(r,t) = \int_{\Sigma(r)} (u_{i,j} + \nu u_{p,p} \delta_{ij} + \beta \delta_{ij}) n_j u_i dS,
\]

where \( \nu = \lambda/2\mu \). A basic differential inequality, derived along similar lines as before, upon integration leads to improved estimates. Details are to be reported elsewhere.

Introduction of space-time measures, other coupled linearised and linear systems with or without quasi-static approximation, and ultimately analysis of mathematical models of climate change await investigation by these or related methods.

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References


Castellar del Vallés February, 13, 2024

Dear Professor Destrade,

I am submitting the manuscript entitled “Asymptotic Spatial Behaviour in Linearised Thermoelasticity for Non-Compact Regions” written by professor Knops and myself. We want to submit it for the special issue for the 60th birthday of Giuseppe Saccomandi.

We inform you that this manuscript has no any conflict of interest as far as we know.

Sincerely yours,

Ramón Quintanilla