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PARAMETRIC SET-THEORETIC YANG-BAXTER EQUATION: p -RACKS, SOLUTIONS & QUANTUM ALGEBRAS

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ABSTRACT. The theory of the parametric set-theoretic Yang-Baxter equation is established from a purely algebraic point of view. The first step towards this objective is the introduction of certain generalizations of the familiar shelves and racks called parametric (p)-shelves and racks. These objects satisfy a *parametric self-distributivity* condition and lead to solutions of the Yang-Baxter equation. Novel, non-reversible solutions are obtained from p -shelve/rack solutions by a suitable parametric twist, whereas all reversible set-theoretic solutions are reduced to the identity map via a parametric twist. The universal algebras associated to both p -rack and generic parametric, set-theoretic solutions are next presented and the corresponding universal \mathcal{R} -matrices are derived. The admissible universal Drinfel'd twist is constructed allowing the derivation of the general set-theoretic universal \mathcal{R} -matrix. By introducing the concept of a parametric coproduct we prove the existence of a parametric co-associativity. We show that the parametric coproduct is an algebra homomorphism and the universal \mathcal{R} -matrices satisfy intertwining relations with the algebra coproducts.

1. INTRODUCTION

The Yang-Baxter equation (YBE) is a central object in contemporary mathematics and mathematical physics. The breadth of its applications extends from one dimensional statistical systems and integrable quantum field theories to quantum group theory [25, 40, 41] and low-dimensional topology [42, 43] and a plethora of other areas in mathematics and physics. The equation was first introduced in a purely physical context in [57] as the main mathematical tool for the investigation of quantum systems with many particle interactions, and in [4] for the study of statistical model known as the anisotropic Heisenberg magnet. The idea of set-theoretic solutions to the Yang-Baxter equation was suggested later by Drinfel'd [24] and since then, set-theoretic solutions have been extensively investigated primarily by means of representations of the braid group, but almost exclusively for the parameter free Yang-Baxter equation (see for instance [27, 35, 49, 50]). The investigation of set-theoretic solutions of the non-parametric Yang-Baxter equation and the associated algebraic structures is a highly active research field that has been particularly prolific, given that a significant number of related studies has been produced over the past few years (see for instance [5]–[16] [17]–[22], [38, 52, 53], [31]–[33], [37, 38]). The theory of the set-theoretic Yang-Baxter equation has numerous significant connections to distinct mathematical areas, such as group theory, algebraic number theory, Hopf-Galois extensions, non-commutative rings, knot theory, Hopf algebras and quantum groups, universal algebras, groupoids, trusses and heaps [8], pointed Hopf algebras, Yetter-Drinfeld modules and Nichols algebras (see for instance among [2, 3, 5, 8, 27], [37]–[39],[42]–[45], [50, 52]). Moreover, interesting links between the set-theoretic Yang-Baxter equation

and geometric crystals [6, 28], or soliton cellular automata [36, 55] have been shown. Concrete connections with quantum spin-chain like systems were also made in [17, 18].

The main objectives of the present investigation are the derivation of set-theoretic solutions of the parametric Yang-Baxter equation as well as the rigorous formulation of the associated Yang-Baxter algebraic structures underpinning the parametric set-theoretic Yang-Baxter equation, and the identification of the universal \mathcal{R} -matrices. Set-theoretic solutions for the parametric Yang-Baxter equation (Yang-Baxter maps) have been studied up to date only in the context of classical discrete integrable systems connected also to the notion of Darboux-Bäcklund transformation in the Lax pair formulation [1, 47, 48, 56] and the refactorization frame. In this investigation for the first time to our knowledge an entirely algebraic analysis for the parametric set-theoretic Yang-Baxter equation is undertaken and purely algebraic solutions are produced. Earlier works on the algebraic structures of the non-parametric set-theoretic Yang-Baxter equation provide a basic algebraic blueprint [19, 20, 23], however the parametric case turns out to be considerably more involved in comparison as dealing with the various parameters involved is a combinatorial problem on its own. This statement concerns all the main findings of the present investigation, such as the proof of the main Theorem 2.12 as well as the formulation of the underlying quantum algebras associated to the set-theoretic parametric version of the Yang-Baxter equation. For instance the usual notion of (co)-associativity does not apply any more, however a version of parametric (co)-associativity together with consistent recursion relations are provided making the notion of a parametric co-product a well defined mathematical object as an algebra homomorphism.

We introduce at this point the parametric set-theoretic Yang-Baxter equation. Let $X, Y \subseteq X$ be non-empty sets, $z_{i,j} \in Y$, $i, j \in \mathbb{Z}^+$ and $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $x, y \in X$, $R^{z_{ij}}(y, x) = (\sigma_x^{z_{ij}}(y), \tau_y^{z_{ij}}(x))$. The notation z_{ij} denotes dependence on (z_i, z_j) . We say that $(X, R^{z_{ij}})$ is a solution of the parametric, set-theoretic Yang-Baxter equation (or simply a solution) if

$$R_{12}^{z_{12}} R_{13}^{z_{13}} R_{23}^{z_{23}} = R_{23}^{z_{23}} R_{13}^{z_{13}} R_{12}^{z_{12}}, \quad (1.1)$$

where $R_{12}^{z_{ij}}(c, b, a) = (\sigma_b^{z_{ij}}(c), \tau_c^{z_{ij}}(b), a)$, $R_{13}^{z_{ij}}(c, b, a) = (\sigma_a^{z_{ij}}(c), b, \tau_c^{z_{ij}}(a))$ and $R_{23}^{z_{ij}}(c, b, a) = (c, \sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$. We say that $R^{z_{ij}}$ is a left non-degenerate if for all $z_{i,j} \in Y$, $\sigma_x^{z_{ij}}$ is a bijective function and non-degenerate if both $\sigma_x^{z_{ij}}$, $\tau_y^{z_{ij}}$ are bijective functions. Also, the solution $(X, R^{z_{ij}})$ is called “reversible” [1, 47, 48] if $R_{21}^{z_{21}} R_{12}^{z_{12}} = \text{id}$. Interestingly, all the solutions from the point of view of discrete integrable systems [1, 47, 48, 56] or the re-factorization approach are reversible and it is shown in Proposition 2.9, Remark 2.10 that they are reduced to the identity map. Here, for the first time we present non-reversible solutions of the parametric set-theoretic Yang-Baxter equation.

We summarize below the main outcomes of each of the following sections. In Section 2, we introduce the notions of parametric (p) -shelves and racks to describe solution of the parametric set-theoretic Yang-Baxter equation. These algebraic objects may be thought of as parametric generalizations of the familiar shelves and racks, they satisfy a generalized *parametric self-distributivity* and naturally yield solutions of the Yang-Baxter equation. Parametric shelf or rack solutions of the Yang-Baxter equation are derived here for the first time. We then show in the second subsection that every left non-degenerate solution is Drinfel’d equivalent to a solution given by a shelf, (see

Proposition 2.9. In fact, by Theorem 2.12). Every left non-degenerate solution can be obtained from a shelf solution by finding a special map of the shelf itself, which we call *twist*, Definition 2.11. Bijective solutions naturally correspond to p -racks. In the third subsection we introduce the notions of parametric Yang-Baxter operators and structures by generalizing the idea of braiding to the parametric case. This section serves mainly as a warm up to Section 3, given that the generalized Yang-Baxter structures encode part of the associated underlying universal algebras, which are introduced next.

In Section 3, we focus on the linearized version of the set-theoretic Yang-Baxter equation. In the first subsection we introduce the p -rack algebra and we then construct the associated universal \mathcal{R} -matrix. By introducing the concept of a parametric coproduct we prove the existence of a parametric co-associativity. These notions allow us to show that the parametric coproduct is an algebra homomorphism and the universal \mathcal{R} -matrices intertwine with the algebra generators coproducts. Moreover, due to the parametric coassociativity the n -coproducts are consistently expressed in terms of 2^{n-2} suitable parametric binary trees. This algebraic structure is a rather new paradigm of quantum algebra, as the parameters are part of the universal structure. We also note that a quantum integrability statement is presented, as a family of mutually commuting non-local objects is identified. In the second subsection, we suitably extend the p -rack algebra and present the decorated p -rack and the p -set Yang-Baxter algebra. By means of a suitable admissible universal Drinfel'd twist we construct the universal-set theoretic \mathcal{R} -matrix. Fundamental representations of the aforementioned algebras are also considered leading naturally to the p -rack and general set-theoretic solutions of the parametric Yang-Baxter equation.

2. SOLVING THE PARAMETRIC SET-THEORETIC YBE

In this section we formulate the basic problem, which is the derivation of solutions of the parametric set-theoretic Yang-Baxter equation. Specifically, we identify the sets of conditions that such solutions satisfy and then we move one to the identification of concrete solutions. The key step into achieving these goals in the presentation of some new algebraic objects called p -shelves and p -racks that satisfy parametric self-distributivity conditions. Use of the newly introduced p -shelves/racks together with the identification of a suitable parametric twist allows the derivation of generic parametric set-theoretic solutions.

2.1. Parametric shelves and racks. We are first exploring a certain class of solutions of the parametric set-theoretic equation that are generalisations of the shelf/rack [10] type solutions of the non-parametric YBE. Such solutions are derived here for the first time. We start our analysis in this section with a definition that generalises the notion of shelves and racks by introducing the *parametric shelves and racks* (p -shelves and p -racks). Note that henceforth we write $z_{i,j,k,\dots} \in Y$ as a shorthand notation for $z_i, z_j, z_k, \dots \in Y, i, j, k, \dots \in \mathbb{Z}^+$.

Definition 2.1. (*p -shelves and p -racks*) Let $X, Y \subseteq X$ be non-empty sets. We define for all $z_{i,j} \in Y$, the binary operation $\triangleright_{z_{ij}} : X \times X \rightarrow X, (a, b) \mapsto a \triangleright_{z_{ij}} b$. The pair $(X, \triangleright_{z_{ij}})$ is said to be a

left parametric (p)-shelf if $\triangleright_{z_{ij}}$ satisfies the generalized left p -self-distributivity:

$$a \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} c) = (a \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (a \triangleright_{z_{ik}} c) \quad (2.1)$$

for all $a, b, c \in X$, $z_{i,j,k} \in Y$. Moreover, a left p -shelf $(X, \triangleright_{z_{ij}})$ is called a left p -rack if the maps $L_a^{z_{ij}} : X \rightarrow X$ defined by $L_a^{z_{ij}}(b) := a \triangleright_{z_{ij}} b$, for all $a, b \in X$, $z_{i,j} \in Y$, are bijective.

From now on whenever we say p -shelf or p -rack we mean left p -shelf or left p -rack.

Proposition 2.2. *Let $X, Y \subseteq X$ be non-empty sets. We define for $z_{i,j} \in Y$ the binary operation $\triangleright_{z_{ij}} : X \times X \rightarrow X$, $(a, b) \mapsto a \triangleright_{z_{ij}} b$. Then $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$, $R^{z_{ij}}(b, a) = (b, b \triangleright_{z_{ij}} a)$ is a solution of the parametric set-theoretic Yang-Baxter equation if and only if $(X, \triangleright_{z_{ij}})$ is a p -shelf.*

Proof. Let $R^{z_{ij}}$ be a solution of the parametric Yang-Baxter equation, then for all $a, b, c \in X$ and $z_{1,2,3} \in Y$ the LHS of the Yang-Baxter equation (1.1) gives

$$R_{12}^{z_{12}} R_{13}^{z_{13}} R_{23}^{z_{23}}(a, b, c) = (a, a \triangleright_{z_{12}} b, a \triangleright_{z_{13}} (b \triangleright_{z_{23}} c)) \quad (2.2)$$

whereas the RHS gives

$$R_{23}^{z_{23}} R_{13}^{z_{13}} R_{12}^{z_{12}}(a, b, c) = (a, a \triangleright_{z_{12}} b, (a \triangleright_{z_{12}} b) \triangleright_{z_{23}} (a \triangleright_{z_{13}} c)) \quad (2.3)$$

Equating (2.2) and (2.3) we conclude:

$$a \triangleright_{z_{13}} (b \triangleright_{z_{23}} c) = (a \triangleright_{z_{12}} b) \triangleright_{z_{23}} (a \triangleright_{z_{13}} c),$$

i.e. $(X, \triangleright_{z_{ij}})$ is a p -shelf. Conversely, if $(X, \triangleright_{z_{ij}})$ is a p -shelf then automatically the map $R^{z_{ij}}$ is a solution of the parametric set-theoretic Yang-Baxter equation. \square

Lemma 2.3. *Let $(X, \triangleright_{z_{ij}})$ be a p -rack and $R^{z_{ij}} : X \times X \rightarrow X \times X$, $R^{z_{ij}}(a, b) = (a, a \triangleright_{z_{ij}} b)$ be a non-degenerate solution of the parametric set-theoretic YBE, i.e. $a \triangleright_{z_{ij}}$ is a bijection for all $a \in X$, $z_{i,j} \in Y$. Then, $R^{z_{ij}} : X \times X \rightarrow X \times X$ is invertible with $(R^{z_{ij}})^{-1}(a, a \triangleright_{z_{ij}} b) = (a, b)$.*

Proof. The proof is straightforward, using the fact that $a \triangleright_{z_{ij}} : X \rightarrow X$ is a bijection. \square

Before we further proceed it is useful to recall the notion of skew braces as this will allow us to derive concrete solutions of the parametric set-theoretic Yang-Baxter equation.

Definition 2.4. [49]-[51], [35]. *A left skew brace is a set B together with two group operations $+, \circ : B \times B \rightarrow B$, the first is called addition and the second is called multiplication, such that for all $a, b, c \in B$,*

$$a \circ (b + c) = a \circ b - a + a \circ c. \quad (2.4)$$

If $+$ is an abelian group operation B is called a left brace. Moreover, if B is a left skew brace and for all $a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a two sided skew brace. Analogously if $+$ is abelian and B is a skew brace, then B is called a two sided brace.

The additive identity of a skew brace B will be denoted by 0 and the multiplicative identity by 1 . In every skew brace $0 = 1$.

From now on when we say skew brace we mean left skew brace. We may now state the following proposition regarding concrete solutions of the parametric set-theoretic Yang-Baxter equation coming from skew braces.

Proposition 2.5. *Let $(X, +, \circ)$ be a skew brace and $Y \subseteq X$, such that*

- for all $a, b \in X$, $z \in Y$, $(a + b) \circ z = a \circ z - z + b \circ z$,
- $z \in Y$ are central in $(X, +)$.

Define also for all $z_{i,j} \in Y$ the binary operation $\triangleright_{z_{ij}} : X \times X \rightarrow X$, such that for all $a, b \in X$, $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$. Then the map $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$,

$$R^{z_{ij}}(a, b) = (a, a \triangleright_{z_{ij}} b)$$

is a solution of the parametric Yang-Baxter equation. Moreover, the map $R^{z_{ij}}$ is invertible.

Proof. It suffices to show that the binary operation satisfies the p -self-distributivity condition, i.e. $(X, \triangleright_{z_{ij}})$ is a p -shelf. Let $z_{i,j,k} \in Y$, then the LHS of condition (2.1)

$$\begin{aligned} a \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} c) &= -a \circ z_i \circ z_k^{-1} + b \triangleright_{z_{jk}} c + a \circ z_i \circ z_k^{-1} \\ &= -a \circ z_i \circ z_k^{-1} - b \circ z_j \circ z_k^{-1} + c + b \circ z_j \circ z_k^{-1} + a \circ z_i \circ z_k^{-1}. \end{aligned}$$

Similarly the RHS of condition (2.1):

$$\begin{aligned} (a \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (a \triangleright_{z_{ik}} c) &= -(a \triangleright_{z_{ij}} b) \circ z_j \circ z_k^{-1} + a \triangleright_{z_{ik}} c + (a \triangleright_{z_{ij}} b) \circ z_j \circ z_k^{-1} \\ &= -a \circ z_i \circ z_k^{-1} - b \circ z_j \circ z_k^{-1} + c + b \circ z_j \circ z_k^{-1} + a \circ z_i \circ z_k^{-1}. \end{aligned}$$

Indeed, LHS = RHS which concludes our proof.

Moreover, it is easy to see that there exists $a \triangleright_{z_{ij}}^{-1} : X \rightarrow X$, such that $a \triangleright_{z_{ij}}^{-1} (a \triangleright_{z_{ij}} b) = a \triangleright_{z_{ij}} (a \triangleright_{z_{ij}}^{-1} b) = b$ and we immediately extract from $a \triangleright_{z_{ij}} (a \triangleright_{z_{ij}}^{-1} b) = b$ that $a \triangleright_{z_{ij}}^{-1} b = a \circ z_i \circ z_j^{-1} + b - a \circ z_i \circ z_j^{-1}$, i.e. $a \triangleright_{z_{ij}}$ is a bijection for all $a \in X$, $z_{i,j} \in Y$. Hence, $(X, \triangleright_{z_{ij}})$ is a p -rack, and $R^{z_{ij}}$ is invertible. \square

Remark 2.6. *In the special case where $(X, +, \circ)$ is a brace, i.e. $(X, +)$ is an abelian group, then for all $a, b \in X$, $z_{i,j} \in Y$, $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1} = b$, and hence $R^{z_{ij}} = id$.*

We now focus on the more general solution of the set-theoretic Yang-Baxter equation of the type $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$,

$$R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a)).$$

We next derive the conditions satisfied by the general set-theoretic solution of the parametric Yang-Baxter equation.

Proposition 2.7. *Let $X, Y \subseteq X$, be non-empty sets, and define for all $a, b \in X$, $z_{i,j} \in Y$, the maps $\sigma_a^{z_{ij}}, \tau_b^{z_{ij}} : X \rightarrow X$, $b \mapsto \sigma_a^{z_{ij}}(b)$ and $a \mapsto \tau_b^{z_{ij}}(a)$. Then $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$, $R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$ is a solution of the parametric set-theoretic Yang-Baxter equation if and only if, for all $z_{1,2,3} \in Y$,*

$$\sigma_a^{z_{13}}(\sigma_b^{z_{12}}(c)) = \sigma_{\sigma_a^{z_{23}}(b)}^{z_{12}}(\sigma_{\tau_b^{z_{23}}(a)}^{z_{13}}(c)) \quad (2.5)$$

$$\tau_c^{z_{13}}(\tau_b^{z_{23}}(a)) = \tau_{\tau_c^{z_{12}}(b)}^{z_{23}}(\tau_{\sigma_b^{z_{12}}(c)}^{z_{13}}(a)) \quad (2.6)$$

$$\sigma_{\tau_{\sigma_b^{z_{12}}(c)}^{z_{13}}(a)}^{z_{23}}(\tau_c^{z_{12}}(b)) = \tau_{\sigma_{\tau_b^{z_{23}}(a)}^{z_{13}}(c)}^{z_{12}}(\sigma_a^{z_{23}}(b)). \quad (2.7)$$

Proof. Let $R^{z_{ij}}$ be a solution. We compute explicitly the LHS and RHS of the parametric Yang-Baxter equation (1.1). The LHS of the Yang-Baxter equation gives, $a, b, c \in X$, $z_{1,2,3} \in Y$,

$$R_{12}^{z_{12}} R_{13}^{z_{13}} R_{23}^{z_{23}}(c, b, a) = (\sigma_{\sigma_a^{z_{23}}(b)}^{z_{12}}(\sigma_{\tau_b^{z_{23}}(a)}^{z_{13}}(c)), \tau_{\sigma_{\tau_b^{z_{23}}(a)}^{z_{13}}(c)}^{z_{12}}(\sigma_a^{z_{23}}(b)), \tau_c^{z_{13}}(\tau_b^{z_{23}}(a))), \quad (2.8)$$

whereas the RHS gives

$$R_{23}^{z_{23}} R_{13}^{z_{13}} R_{12}^{z_{12}}(c, b, a) = (\sigma_a^{z_{13}}(\sigma_b^{z_{12}}(c)), \sigma_{\tau_{\sigma_b^{z_{12}}(c)}^{z_{13}}(a)}^{z_{23}}(\tau_c^{z_{12}}(b)), \tau_{\tau_c^{z_{12}}(b)}^{z_{23}}(\tau_{\sigma_b^{z_{12}}(c)}^{z_{13}}(a))). \quad (2.9)$$

By equating (2.8) and (2.9) we arrive at (2.5)-(2.7). And conversely, if conditions (2.5)-(2.7) are satisfied then $R^{z_{ij}}$ automatically satisfies the parametric Yang-Baxter equation. \square

2.2. Generalized solutions from Drinfel'd twists. In this subsection we construct generic solutions of the parametric set-theoretic Yang-Baxter equation by suitably twisting p -shelve solutions. We first introduce the notion of a parametric *Drinfel'd twist* and extend some earlier results shown for the non-parametric Yang-Baxter equation [19, 20, 21, 23] to the parametric case. We note that a non-local twist type transformation for the non-parametric case was first introduced in [54] and then further studied and exploited in [44, 45].

Definition 2.8. *Let $(X, R^{z_{ij}})$ and $(X, S^{z_{ij}})$ be solutions of the parametric set-theoretic Yang-Baxter equation. We say that a map $\varphi^{z_{ij}} : X \times X \rightarrow X \times X$ is a Drinfel'd twist (D-twist) if for all $z_{i,j} \in Y$,*

$$\varphi^{z_{ij}} R^{z_{ij}} = S^{z_{ij}} (\varphi^{z_{ji}})^{(op)},$$

where $(\varphi^{z_{ji}})^{(op)} = \pi \circ \varphi^{z_{ji}}$, and $\pi : X \times X \rightarrow X \times X$ is the “flip” map, such that for all $x, y \in X$, $\pi(x, y) = (y, x)$. If $\varphi^{z_{ij}}$ is a bijection we say that $(X, R^{z_{ij}})$ and $(X, S^{z_{ij}})$ are D -equivalent (via $\varphi^{z_{ij}}$), and we denote it by $R^{z_{ij}} \cong_D S^{z_{ij}}$.

Proposition 2.9. *Let $(X, R^{z_{ij}})$ be a left non-degenerate solution, such that for all $a, b \in X$, $z_{i,j} \in Y$, $R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$ and let $(X, S^{z_{ij}})$ be a solution, such that for all $a, b \in X$, $z_{i,j} \in Y$, $S^{z_{ij}}(b, a) = (b, b \triangleright_{z_{ij}} a)$ and $\tau_b^{z_{ij}}(a) := (\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}})^{-1}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$. Then $R^{z_{ij}}$ is D -equivalent to $S^{z_{ij}}$.*

Proof. Let $\varphi^{z_{ij}} : X \times X \rightarrow X \times X$ be the map defined by $\varphi^{z_{ij}}(a, b) := (a, \sigma_a^{z_{ji}}(b))$, for all $a, b \in X$, $z_{i,j} \in Y$. $R^{z_{ij}}$ is left non-degenerate, hence $\varphi^{z_{ij}}$ is bijective and $(\varphi^{z_{ij}})^{-1}(a, b) = (a, (\sigma_a^{z_{ji}})^{-1}(b))$,

also $(\varphi^{z_{ji}})^{(op)}(b, a) = (\sigma_a^{z_{ij}}(b), a)$ for all $a, b \in X$. Then

$$\begin{aligned} & (\varphi^{z_{ij}})^{-1} S^{z_{ij}} (\varphi^{z_{ji}})^{(op)}(b, a) = (\varphi^{z_{ji}})^{-1} S^{z_{ij}} (\sigma_a^{z_{ij}}(b), a) = \\ & (\varphi^{z_{ji}})^{-1} (\sigma_a^{z_{ij}}(b), \sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a) = (\sigma_a^{z_{ij}}(b), (\sigma_a^{z_{ji}})_{\sigma_a^{z_{ij}}(b)}^{-1} (\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)) \\ & (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(b)) = R^{z_{ij}}(b, a), \end{aligned}$$

where we have defined $\tau_b^{z_{ij}}(a) := (\sigma_a^{z_{ji}})_{\sigma_a^{z_{ij}}(b)}^{-1} (\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$. That is $R^{z_{ij}} \cong_D S^{z_{ij}}$. \square

Remark 2.10. *In the special case of reversible R -matrices we observe from the fundamental relation $R_{21}^{z_{ji}} R_{12}^{z_{ij}} = id$, that $\sigma_a^{z_{ji}}(\tau_b^{z_{ij}}(a)) = a$, which leads to $b \triangleright_{z_{ij}} a = a$, and hence $S^{z_{ij}}(b, a) = (b, a)$ for all $a, b \in X$, $z_{i,j} \in Y$, i.e. $S^{z_{ij}} = id$.*

Definition 2.11. *Let $(X, \triangleright_{z_{ij}})$ be a p -shelf. We say that the twist $\varphi^{z_{ij}} : X \times X \rightarrow X \times X$, such that for all $a, b \in X$, $z_{i,j} \in Y$, $\varphi^{z_{ij}}(a, b) := (a, \sigma_a^{z_{ji}}(b))$ and $\sigma_a^{z_{ij}}$ is a bijection, is an admissible twist, if for all $a, b, c \in X$, $z_{i,j,k} \in Y$: (1) $\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) = \sigma_a^{z_{jk}}(\sigma_b^{z_{ik}}(c))$ and (2) $\sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a)$.*

In the following theorem we show that any left non-degenerate solution $(X, R^{z_{ij}})$ can be expressed in terms of the p -shelf $(X, \triangleright_{z_{ij}})$ and its admissible twist.

Theorem 2.12. *Let $(X, \triangleright_{z_{ij}})$ be a p -shelf and $\varphi^{z_{ij}} : X \times X \rightarrow X \times X$, such that $\varphi^{z_{ij}}(a, b) := (a, \sigma_a^{z_{ji}}(b))$ for all $a, b \in X$, $z_{i,j} \in Y$. Then, the map $R^{z_{ij}} : X \times X \rightarrow X \times X$ defined by*

$$R^{z_{ij}}(b, a) = \left(\sigma_a^{z_{ij}}(b), (\sigma_a^{z_{ji}})_{\sigma_a^{z_{ij}}(b)}^{-1} (\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a) \right), \quad (2.10)$$

for all $a, b \in X$, $z_{i,j} \in Y$ is a solution if and only if $\varphi^{z_{ij}}$ is an admissible twist.

Proof. We first assume that $\varphi^{z_{ij}}$ is an admissible twist. Then (2.5) immediately follows from (1). We set $\tau_b^{z_{ij}}(a) := (\sigma_a^{z_{ji}})_{\sigma_a^{z_{ij}}(b)}^{-1} (\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$, for all $a, b \in X$, $z_{i,j} \in Y$. For $a, b, c \in X$, $z_{i,j,k} \in Y$ we have:

$$\begin{aligned} \tau_{\sigma_b^{z_{jk}}(a)}^{z_{ij}}(\sigma_a^{z_{jk}}(b)) &= (\sigma_a^{z_{ji}})_{\sigma_a^{z_{jk}}(b)}^{-1} (\sigma_a^{z_{ij}})_{\sigma_a^{z_{jk}}(b)} (\sigma_b^{z_{ik}})_{\tau_b^{z_{jk}}(a)} (\sigma_a^{z_{ik}})_{\sigma_b^{z_{jk}}(a)} (c) \triangleright_{z_{ij}} \sigma_a^{z_{jk}}(b) \\ &= (\sigma_a^{z_{ji}})_{\sigma_b^{z_{ik}}(\sigma_b^{z_{ij}}(c))}^{-1} (\sigma_a^{z_{ik}})_{\sigma_b^{z_{ij}}(c)} (\sigma_b^{z_{ij}}(c)) \triangleright_{z_{ij}} \sigma_a^{z_{jk}}(b) && \text{by (1) Definiiton 2.11} \\ &= (\sigma_a^{z_{ji}})_{\sigma_b^{z_{ik}}(\sigma_b^{z_{ij}}(c))}^{-1} (\sigma_a^{z_{jk}})_{\sigma_b^{z_{ij}}(c)} (\sigma_b^{z_{ij}}(c) \triangleright_{z_{ij}} b) && \text{by (2) Definiiton 2.11} \\ &= (\sigma_a^{z_{ji}})_{\sigma_b^{z_{ik}}(\sigma_b^{z_{ij}}(c))}^{-1} (\sigma_a^{z_{jk}})_{\sigma_b^{z_{ij}}(c)} (\sigma_b^{z_{ij}}(c) (\tau_c^{z_{ij}}(b)))) \\ &= (\sigma_a^{z_{ji}})_{\sigma_b^{z_{ik}}(\sigma_b^{z_{ij}}(c))}^{-1} (\sigma_a^{z_{ji}})_{\sigma_b^{z_{ik}}(\sigma_b^{z_{ij}}(c))} (\sigma_b^{z_{jk}})_{\tau_{\sigma_b^{z_{ij}}(c)}^{z_{ij}}(a)} (\tau_c^{z_{ij}}(b)))) && \text{by (1) Definiiton 2.11} \\ &= \sigma_{\tau_{\sigma_b^{z_{ij}}(c)}^{z_{jk}}(a)}^{z_{jk}} (\tau_c^{z_{ij}}(b)), \end{aligned}$$

hence (2.7) holds.

We now show (2.6) using repeatedly the definition $\tau_b^{z_{ij}}(a) := (\sigma_a^{z_{ji}})^{-1}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$, for all $a, b \in X, z_{i,j} \in Y$:

$$\begin{aligned} \tau_{\tau_c^{z_{ij}}(b)}^{z_{jk}}(\tau_{\sigma_b^{z_{ij}}(c)}^{z_{ik}}(a)) &= (\sigma_{\sigma_b^{z_{ij}}(c)}^{z_{kj}})^{-1}(\sigma_{\tau_c^{z_{ij}}(b)}^{z_{jk}})^{-1}(\sigma_{\tau_c^{z_{ij}}(b)}^{z_{ik}})^{-1}(\tau_c^{z_{ij}}(b) \triangleright_{z_{jk}} \tau_{\sigma_b^{z_{ij}}(c)}^{z_{ik}}(a)) \\ &= (\sigma_{\tau_b^{z_{ij}}(a)}^{z_{kj}})^{-1}(\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{jk}})^{-1}((\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}})^{-1}(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ki}})^{-1}(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) \triangleright_{z_{ik}} a)) \end{aligned}$$

by (2.5), (2.7)

$$= (\sigma_{\tau_b^{z_{ij}}(a)}^{z_{kj}})^{-1}(\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{jk}})^{-1}((\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}})^{-1}(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) \triangleright_{z_{ik}} (\sigma_a^{z_{jk}}(b) \triangleright_{z_{jk}} a)))$$

by the p -self-distributivity and (2) of Definition 2.11,

$$= (\sigma_{\sigma_b^{z_{ij}}(a)}^{z_{ki}})^{-1}((\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{jk}})^{-1}(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) \triangleright_{z_{ik}} \tau_b^{z_{jk}}(a)))$$

by the p -self-distributivity, the definition of $\tau_b^{z_{ij}}(a)$ and (2) from Definition 2.11,

$$= (\sigma_{\sigma_b^{z_{ij}}(a)}^{z_{ki}})^{-1}((\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{jk}})^{-1}(\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{jk}})^{-1}(\sigma_{\tau_b^{z_{ij}}(a)}^{z_{ik}})^{-1}(\sigma_{\tau_b^{z_{ij}}(a)}^{z_{jk}}(a)))$$

by (2) from Definition 2.11,

$$= (\sigma_{\sigma_b^{z_{ij}}(a)}^{z_{ki}})^{-1}(\sigma_{\tau_b^{z_{ij}}(a)}^{z_{ik}})^{-1}(\sigma_{\tau_b^{z_{ij}}(a)}^{z_{jk}}(a)) = \tau_c^{z_{ik}}(\tau_b^{z_{jk}}(a)),$$

and we conclude that (2.6) is satisfied.

Conversely, let the map $R^{z_{ij}}$ be a solution on the set X . Then, condition (2.5) coincides with the identity (1) of Definition 2.11. We now show that, for all $a, b, c \in X, z_{i,j,k} \in Y$ identity (2) of Definition 2.11 holds. Indeed,

$$\begin{aligned} \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a) &= \sigma_c^{z_{ji}}(\sigma_{\tau_b^{z_{ij}}(a)}^{z_{jk}})^{-1}(\tau_{(\sigma_a^{z_{ij}})^{-1}(b)}^{z_{ij}}(a)) && \text{by (2.5)} \\ &= \sigma_c^{z_{ji}}(\sigma_{\tau_{\sigma_a^{z_{ij}}((\sigma_a^{z_{ij})^{-1}(b))}}^{z_{jk}}(c)}^{z_{jk}})^{-1}(\tau_{(\sigma_a^{z_{ij}})^{-1}(b)}^{z_{ij}}(a)) \\ &= \sigma_c^{z_{ji}}(\tau_{\sigma_a^{z_{ij}}((\sigma_a^{z_{ij})^{-1}(b))}}^{z_{ij}})^{-1}(\sigma_c^{z_{jk}}(a)) && \text{by (2.7)} \\ &= \sigma_c^{z_{ji}}(\tau_{(\sigma_{\sigma_c^{z_{jk}}(a)}^{z_{ij}})^{-1}(\sigma_c^{z_{ik}}(b))}^{z_{ij}})^{-1}(\sigma_c^{z_{jk}}(a)), \\ &= \sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) && \text{by (2.5)} \end{aligned}$$

and this concludes our proof. \square

Corollary 2.13. *Any left non-degenerate solution $R^{z_{ij}} : X \times X \rightarrow X \times X$, $R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$, for all $a, b \in X, z_{i,j} \in Y$, can be obtained from a p -shelve solution, where $a \triangleright_{z_{ij}} b = \sigma_a^{z_{ji}}(\tau_{(\sigma_b^{z_{ij}})^{-1}(a)}^{z_{ij}}(b))$, via an admissible twist.*

Proof. To show that any left non-degenerate solution can be obtained from a p -shelf solution, it is enough to show that $\sigma_a^{z_{ij}}$ satisfies the properties of Definition 2.11, where $a \triangleright_{z_{ij}} b := \sigma_a^{z_{ji}} (\tau_{(\sigma_b^{z_{ij}})^{-1}(a)}^{z_{ij}}(b))$. These follow from the proof of Theorem 2.12 (specifically the second part of the proof). \square

We conclude, given the findings of this subsection, that the problem of finding generic solutions of the parametric set-theoretic Yang-Baxter equation is reduced to the classification of p -shelve/rack solutions and the identification of admissible twists.

Corollary 2.14. *A left non-degenerate solution $(X, R^{z_{ij}})$ is bijective if and only if $(X, \triangleright_{z_{ij}})$ is a p -rack for all $z_{i,j} \in Y$.*

Proof. This follows from the fact that $R^{z_{ij}}$ is invertible if and only if the p -shelve solution is invertible, i.e. it is a p -rack solution. \square

Recall Remark 2.10, which together with Theorem 2.12 show that all reversible solutions are obtained from the identity map via an admissible twist.

Proposition 2.15. *Let $(X, +, \circ)$ be a skew brace and let $Y \subseteq X$, be such that*

- for all $w, z \in Y$, $z \circ w = w \circ z$,
- for all $a, b \in X$, $z \in Y$, $(a + b) \circ z = a \circ z - z + b \circ z$,
- $z \in Y$ are central in $(X, +)$.

Let also $\varphi^{z_{ij}} : X \times X \rightarrow X \times X$, be such that for all $a, b \in X$, and $z_{i,j} \in Y$, $\varphi^{z_{ij}}(a, b) = (a, \sigma_a^{z_{ji}}(b))$, where $\sigma_a^{z_{ij}}(b) = z_i^{-1} - a \circ z_i^{-1} \circ z_j + a \circ b \circ z_j$. We also define for all $z_{i,j} \in Y$, $\triangleright_{z_{ij}} : X \rightarrow X$, such that $a \triangleright_{z_{ij}} b := -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$, $a, b \in X$. Then:

- (1) For all $a, b \in X$ and $z_{i,j} \in Y$, $a \circ b = \sigma_a^{z_{ij}} \circ \tau_b^{z_{ij}}(a)$.
- (2) $\varphi^{z_{ij}}$ is an admissible twist, $z_{i,j} \in Y$.

Proof. Let us first observe that we can identify the inverse map $(\sigma_b^{z_{ij}})^{-1}$, i.e. $\sigma_b^{z_{ij}}$ is a bijection. It follows from $\sigma_a^{z_{ij}}((\sigma_a^{z_{ij}})^{-1}(b)) = b$ that $(\sigma_a^{z_{ij}})^{-1}(b) = z_i^{-1} - a^{-1} \circ z_i^{-1} \circ z_j^{-1} + a^{-1} \circ b \circ z_j^{-1}$, $a, b \in X$, $z_{i,j} \in Y$.

- (1) We may now extract the map $\tau_b^{z_{ij}} :$

$$\begin{aligned} \tau_b^{z_{ij}}(a) &:= (\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}})^{-1}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a) = \dots \\ &= (\sigma_a^{z_{ij}}(b))^{-1} \circ a \circ z_i^{-1} - (\sigma_a^{z_{ij}}(b))^{-1} \circ z_i^{-1} \circ z_j^{-1} + z_j^{-1} \end{aligned}$$

We now directly compute for all $a, b \in X$ and $z_{i,j} \in Y$,

$$\begin{aligned} \sigma_a^{z_{ij}}(b) \circ \tau_b^{z_{ij}}(a) &= \sigma_a^{z_{ij}}(b) \circ ((\sigma_a^{z_{ij}}(b))^{-1} \circ a \circ z_i^{-1} - (\sigma_a^{z_{ij}}(b))^{-1} \circ z_i^{-1} \circ z_j^{-1} + z_j^{-1}) \\ &= a \circ z_i^{-1} - z_i^{-1} \circ z_j^{-1} + \sigma_a^{z_{ij}}(b) \circ z_j^{-1} \\ &= a \circ z_i^{-1} - z_i^{-1} \circ z_j^{-1} + (z_i^{-1} - a \circ z_i^{-1} \circ z_j + a \circ b \circ z_j) \circ z_j^{-1} \end{aligned} \quad (2.11)$$

$$= a \circ b. \quad (2.12)$$

- (2) To prove that $\varphi^{z_{ij}}$ is an admissible twist we need to show properties (1) and (2) of Definition 2.11. Indeed,

$$\begin{aligned}
\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) &= z_i^{-1} - a \circ z_i^{-1} \circ z_k + a \circ \sigma_b^{z_{ij}}(c) \circ z_k \\
&= z_i^{-1} - a \circ b \circ z_i^{-1} \circ z_j \circ z_k + a \circ b \circ c \circ z_j \circ z_k \\
&= z_i^{-1} - \sigma_a^{z_{jk}} \circ \tau_b^{z_{jk}}(a) \circ z_i^{-1} \circ z_j \circ z_k + \sigma_a^{z_{jk}} \circ \tau_b^{z_{jk}}(a) \circ c \circ z_j \circ z_k, \\
&= \sigma_{\sigma_a^{z_{jk}}(b)}^{z_{ij}}(\sigma_{\tau_b^{z_{jk}}(a)}^{z_{ik}}(c)),
\end{aligned} \tag{2.13}$$

i.e. we conclude that conditions (1) of Definition 2.11; we also observe that $\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) = \sigma_a^{z_{ij}}(\sigma_b^{z_{ik}}(c))$.

Now we show condition (2) of Definition 2.11. We first directly compute

$$\begin{aligned}
\sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a) &= z_j^{-1} - c \circ z_j^{-1} \circ z_k + c \circ (z_j^{-1} - b \circ z_i \circ z_j^{-1} + a - z_j^{-1} + b \circ z_i \circ z_j^{-1}) \circ z_k \\
&= z_j^{-1} - c \circ b \circ z_i \circ z_j^{-1} \circ z_k + c \circ a \circ z_k - c \circ z_j^{-1} \circ z_k + c \circ b \circ z_i \circ z_j^{-1} \circ z_k.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
\sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) &= -\sigma_c^{z_{ik}}(b) \circ z_i \circ z_j^{-1} + \sigma_c^{z_{jk}}(a) + \sigma_c^{z_{ik}}(b) \circ z_i \circ z_j^{-1} \\
&= z_j^{-1} - c \circ b \circ z_i \circ z_j^{-1} \circ z_k + c \circ a \circ z_k - c \circ z_j^{-1} \circ z_k + c \circ b \circ z_i \circ z_j^{-1} \circ z_k.
\end{aligned} \tag{2.14}$$

Comparing the two expressions above we conclude that condition (2) of Definition 2.11 also holds, i.e. $\varphi^{z_{ij}}$ is indeed an admissible twist. \square

Remark 2.16. In the special case where $(X, +, \circ)$ is a brace, i.e. $(X, +)$ is an abelian group, then for all $a, b \in X$, $z_{i,j} \in Y$, $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1} = b$, hence $S^{z_{ij}}(a, b) = (a, a \triangleright_{z_{ij}} b) = (a, b)$, i.e. $S^{z_{ij}}(a, b) = id$. Also, $\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}}(\tau_b^{z_{ij}}(a)) = a$, hence the map $R^{z_{ij}} : X \times X \rightarrow X \times X$, $R^{z_{ij}}(b, a) = (\sigma_a^{z_{ij}}(b), \tau_b^{z_{ij}}(a))$ obtained from $S^{z_{ij}}$ via the admissible twist, is reversible, i.e. $R_{12}^{z_{ij}} R_{21}^{z_{ij}} = id$.

2.3. Parametric Yang-Baxter operators. This subsection serves as a predecessor of the subsequent section given that the generalized algebraic structures studied here encapsulate part of the underlying set-theoretic Yang-Baxter algebras, which are introduced in the next section. Bearing in mind the definition of braided groups and braidings in [46], [31] and their deformations [21, 22, 23] we further generalize the definitions to introduce the parametric Yang-Baxter structures.

It is useful for the purposes of this subsection to introduce the following. Let X and $Y \subseteq X$ be non-empty sets and introduce for all $z_{i,j,k} \in Y$ the maps, $M_y^{z_{ijk}} \in \{f_y^{z_{ijk}}, g_y^{z_{ijk}}, \hat{f}_y^{z_{ijk}}, \hat{g}_y^{z_{ijk}}\}$, $M_y^{z_{ijk}} : X \rightarrow X$, $x \mapsto M_y^{z_{ijk}}(x)$, and the maps $S_y^{z_{ij}} \in \{\sigma_y^{z_{ij}}, \tau_y^{z_{ij}}\}$, $S_y^{z_{ij}} : X \rightarrow X$, $x \mapsto S_y^{z_{ij}}(x)$.

Definition 2.17. Let (X, \circ) be a group, $Y \subseteq X$ and consider the following maps for $z_{i,j,k} \in Y$, $m : X \times X \rightarrow X$, $(x, y) \mapsto x \circ y$, $\pi : X \times X \rightarrow X \times X$, $(x, y) \mapsto (y, x)$, and $R^{z_{ij}}, \xi^{z_{ijk}}, \zeta^{z_{ijk}} : X \times X \rightarrow X \times X$:

$$R^{z_{ij}}(y, x) = (\sigma_x^{z_{ij}}(y), \tau_y^{z_{ij}}(x)), \quad \xi^{z_{ijk}}(y, x) = (f_x^{z_{ijk}}(y), g_y^{z_{ijk}}(x)), \quad \zeta^{z_{ijk}}(y, x) = (\hat{f}_x^{z_{ijk}}(y), \hat{g}_y^{z_{ijk}}(x)),$$

such that for all $x, y \in X$, $z_{i,j,k} \in Y$, $f_x^{z_{ijk}}(y) = f_x^{z_{ikj}}(y)$, $g_y^{z_{ijk}}(x) = g_y^{z_{ikj}}(x)$, $\hat{f}_x^{z_{ijk}}(y) = \hat{f}_x^{z_{ikj}}(y)$, $\hat{g}_y^{z_{ijk}}(x) = \hat{g}_y^{z_{ikj}}(x)$ and $M_y^{z_{ijk}}, S_y^{z_{ij}}$ are bijections. Let also $\hat{m} := m\pi$. The map $R^{z_{ij}}$ is called a

parametric (p)-set Yang-Baxter operator, and the group is called a parametric (p)-set Yang-Baxter group, if for all $x, y, w \in X$, $z_{i,j,k} \in Y$:

- (1) $\hat{m}(y, x) = m(R^{z_{ij}}(y, x))$.
- (2) $\xi^{z_{ijk}}(id \times \hat{m})(w, y, x) = (id_X \times \hat{m})R_{13}^{z_{ik}}R_{12}^{z_{ij}}(w, y, x)$.
- (3) $\zeta^{z_{ijk}}(\hat{m} \times id_X)(w, y, x) = (\hat{m} \times id_X)R_{13}^{z_{ik}}R_{23}^{z_{jk}}(w, y, x)$.

Proposition 2.18. *Let (X, \circ) be a p -set Yang-Baxter group and the map $R^{z_{ij}} : X \times X \rightarrow X \times X$ be a p -Yang-Baxter operator, then:*

- (1) $R^{z_{ij}}$ satisfies the parametric Yang-Baxter equation.
- (2) $x \circ y = f_x^{z_{ijk}}(y) \circ g_y^{z_{ijk}}(x) = \hat{f}_x^{z_{ijk}}(y) \circ \hat{g}_y^{z_{ijk}}(x)$.

Proof. For the first part it suffices to show that conditions (2.5)-(2.7) hold.

- (1) Indeed, from part (2) of Definition 2.17 we obtain for all $x, y, w \in X$, $z_{i,j,k} \in Y$:

$$\sigma_x^{z_{ik}}(\sigma_y^{z_{ij}}(w)) = f_{x \circ y}^{z_{ijk}}(w) = f_{\sigma_x^{z_{jk}}(y) \circ \tau_y^{z_{jk}}(x)}^{z_{ijk}}(w) = \sigma_{\sigma_x^{z_{jk}}(y)}^{z_{ij}}(\sigma_{\tau_y^{z_{jk}}(x)}^{z_{ik}}(w)), \quad (2.15)$$

hence condition (2.5) of the Yang-Baxter equation is satisfied.

From part (3) of Definition 2.17 we obtain:

$$\tau_w^{z_{ik}}(\tau_y^{z_{jk}}(x)) = \hat{g}_{y \circ w}^{z_{ijk}}(x) = \hat{g}_{\sigma_y^{z_{ij}}(w) \circ \tau_w^{z_{ij}}(y)}^{z_{ijk}}(x) = \tau_{\tau_w^{z_{ij}}(y)}^{z_{jk}}(\tau_{\sigma_y^{z_{ij}}(w)}^{z_{ik}}(x)), \quad (2.16)$$

hence condition (2.6) of the Yang-Baxter equation is also satisfied. It remains now to show (2.7). From part (3) of definition 2.17 we also have,

$$\sigma_x^{z_{jk}}(y) \circ \sigma_{\tau_y^{z_{jk}}(x)}^{z_{ik}}(w) = \hat{f}_x^{z_{ijk}}(y \circ w) = \hat{f}_x^{z_{ijk}}(\sigma_y^{z_{ij}}(w) \circ \tau_w^{z_{ij}}(y)) \quad (2.17)$$

$$= \sigma_x^{z_{ik}}(\sigma_y^{z_{ij}}(w)) \circ \sigma_{\tau_{\sigma_y^{z_{ij}}(w)}^{z_{ij}}(x)}^{z_{jk}}(\tau_w^{z_{ij}}(y)). \quad (2.18)$$

Consider the RHS of (2.7), which via part (1) of Definition 2.17 becomes

$$\begin{aligned} \tau_{\sigma_{\tau_y^{z_{jk}}(x)}^{z_{ik}}(w)}^{z_{ij}}(\sigma_x^{z_{jk}}(y)) &= (\sigma_x^{z_{ik}}(\sigma_y^{z_{ij}}(w)))^{-1} \circ \sigma_x^{z_{jk}}(y) \circ \sigma_{\tau_y^{z_{jk}}(x)}^{z_{ik}}(w), \quad \text{by (2.18)} \\ &= \sigma_{\tau_{\sigma_y^{z_{ij}}(w)}^{z_{ij}}(x)}^{z_{jk}}(\tau_w^{z_{ij}}(y)) \end{aligned} \quad (2.19)$$

and this is the proof of condition (2.7). That is, we conclude that the p -set Yang-Baxter operator is a solution of the parametric Yang-Baxter equation.

- (2) From (2.16), (2.17) and (1) of Definition 2.17 we conclude that $x \circ y = \hat{f}_x^{z_{ijk}}(y) \circ \hat{g}_y^{z_{ijk}}(x)$. From (2) of Definition 2.17 we also have that

$$\tau_{\sigma_y^{z_{ij}}(w)}^{z_{ik}} \circ \tau_w^{z_{ij}}(y) = g_w^{z_{ijk}}(x \circ y).$$

The latter relation together with (2.15) and (1) of Definition 2.17 lead also to $x \circ y = f_x^{z_{ijk}}(y) \circ g_y^{z_{ijk}}(x)$. \square

Example 2.19. Let $(X, +, \circ)$ be a skew brace and recall the map of Proposition 2.15 for all $z_{i,j} \in Y$, $\sigma_a^{z_{ij}} : X \rightarrow X$, $\sigma_a^z(b) = z_i^{-1} - a \circ z_i^{-1} \circ z_j + a \circ b \circ z_j$, $a, b \in X$, which satisfies:

$$a \circ b = \sigma_a^{z_{ij}}(b) \circ \tau_b^{z_{ij}}(a) \quad \text{and} \quad \sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c)) = \sigma_{a \circ b}^{z_{ij \circ k}}(c), \quad (2.20)$$

where the shorthand notation $z_{ij \circ k}$ denotes dependence on $(z_i, z_j \circ z_k)$, i.e. $f_a^{z_{ij \circ k}}(b) = \sigma_a^{z_{ij \circ k}}(b)$. Moreover, from (2.17) we obtain that $\hat{f}_a^{z_{ij \circ k}}(b) = \sigma_a^{z_{i \circ j \circ k}}(b)$, and $z_{i \circ j \circ k}$ denotes dependence on $(z_i \circ z_j, z_k)$.

Definition 2.20. Let X and $Y \subseteq X$ be non-empty sets and let for all $z_{i,j} \in Y$ the binary operation $\bullet_{z_{ij}} : X \times X \rightarrow X$, $(x, y) \mapsto x \bullet_{z_{ij}} y$. Consider also for all $z_{i,j} \in Y$ the following maps, $m_{z_{ij}} : X \times X \rightarrow X$, $(x, y) \mapsto x \bullet_{z_{ij}} y$, $\pi : X \times X \rightarrow X \times X$, $(x, y) \mapsto (y, x)$:

$$R^{z_{ij}}(y, x) = (y, \tau_y^{z_{ij}}(x)), \quad \zeta^{z_{ijk}}(y, x) = (y, \hat{g}_y^{z_{ijk}}(x))$$

such that $\tau_y^{z_{ij}}$, $\hat{g}_y^{z_{ijk}}$ are bijections for all $x, y \in X$, $z_{i,j,k} \in Y$ and $\hat{g}_y^{z_{ijk}}(x) = \hat{g}_y^{z_{jik}}(x)$. Let also $\hat{m}_{z_{ij}} := m_{z_{ij}} \pi$. The map $R^{z_{ij}}$ is called a p -rack operator, and $(X, \bullet_{z_{ij}})$ is called a p -rack magma, if for all $x, y, w \in X$, $z_{i,j,k} \in Y$:

- (1) $\hat{m}_{z_{ji}}(y, x) = m_{z_{ij}}(R^{z_{ij}}(y, x))$.
- (2) $\zeta^{z_{ijk}}(\hat{m}_{z_{ji}} \times id_X)(w, y, x) = (\hat{m}_{z_{ji}} \times id_X)R_{13}^{z_{ik}}R_{23}^{z_{jk}}(w, y, x)$.

We note that Definition 2.20 could be seen as a special case of Definition 2.17, however the underlying algebraic structure in 2.20 is a parametric one and is not necessarily a group as in 2.17.

Proposition 2.21. Let $(X, \bullet_{z_{ij}})$ be a p -rack magma and the map $R^{z_{ij}} : X \times X \rightarrow X \times X$, such that $R^{z_{ij}}(x, y) = (y, \tau_y^{z_{ij}}(x))$ be a p -rack braiding operator for all $z_{i,j} \in Y$. Then $R^{z_{ij}}$ satisfies the parametric Yang-Baxter equation.

Proof. Let $\tau_b^{z_{ij}}(a) := b \triangleright_{z_{ij}}(a)$. Then, for all $x, y, w \in X$, $z_{i,j,k} \in Y$, we obtain by condition (1) of Definition 2.20: $x \bullet_{z_{ji}} y = y \bullet_{z_{ij}}(y \triangleright_{z_{ij}} x)$ and from condition (3):

$$w \triangleright_{z_{ik}}(y \triangleright_{z_{jk}} x) = \hat{g}_{y \bullet_{z_{ji}} w}^{z_{ijk}}(x) = \hat{g}_{w \bullet_{z_{ij}}(y \triangleright_{z_{ij}} y)}^{z_{jik}}(x) = (w \triangleright_{z_{ij}} y) \triangleright_{z_{jk}}(w \triangleright_{z_{ik}} x).$$

The latter condition is the p self-distributivity, i.e. $(X, \triangleright_{z_{ij}})$ is a p -rack (recall from Definition 2.20 $w \triangleright_{z_{ij}}$ is a bijection for all $w \in X$, $z_{i,j} \in Y$) and hence according to Proposition 2.2 $R^{z_{ij}}$ is a solution of the Yang-Baxter equation. Due to the fact that $(X, \triangleright_{z_{ij}})$ is a p -rack the solution is invertible. \square

Example 2.22. Let $(X, +, \circ)$ be a skew brace and recall the binary operations from Proposition 2.15, $\bullet_{z_{ij}}, \triangleright_{z_{ij}} : X \times X \rightarrow X$, such that $a \bullet_{z_{ij}} b = a \circ z_i + b \circ z_j$ and $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$, for $a, b \in X$, $z_{i,j} \in Y$. Then the following relations are satisfied:

$$a \bullet_{z_{ji}} b = b \bullet_{z_{ij}}(b \triangleright_{z_{ij}} a), \quad \text{and} \quad b \triangleright_{z_{ik}}(a \triangleright_{z_{jk}} c) = (a \bullet_{z_{ji}} b) \triangleright_{z_{ok}} c,$$

where $z_o = 1$, and hence the function $\hat{g}_y^{z_{ijk}}(x)$ of Definition 2.20 is $\hat{g}_y^{z_{ijk}}(x) =: \hat{g}_y^{z_{ok}}(x) = y \triangleright_{z_{ok}} x$.

Note. The linearized version of the set-theoretic Yang-Baxter equation:

Consider a free vector space $V = \mathbb{C}X$ of dimension equal to the cardinality of X . Let $\mathbb{B} = \{e_a\}_{a \in X}$ be the basis of V and $\mathbb{B}^* = \{e_a^*\}_{a \in X}$ be the dual basis: $e_a^* e_b = \delta_{a,b}$, also $e_{a,b} := e_a e_b^*$. Let also $R^{z_{ij}} \in \text{End}(V \otimes V)$, $z_{i,j} \in Y$, be a solution of the tensor (quantum) parametric Yang-Baxter equation

$$R_{12}^{z_{12}} R_{13}^{z_{13}} R_{23}^{z_{23}} = R_{23}^{z_{23}} R_{13}^{z_{13}} R_{12}^{z_{12}}, \quad (2.21)$$

where in general $R^{z_{ij}} = \sum_{a,c,b,d} R^{z_{ij}}(b, c|a, d) e_{b,c} \otimes e_{a,d}$ and $R_{12}^{z_{ij}} = \sum_{a,c,b,d} R^{z_{ij}}(b, c|a, d) e_{b,c} \otimes e_{a,d} \otimes 1_V$, $R_{13}^{z_{ij}}(c, b, a) = \sum_{a,c,b,d} R^{z_{ij}}(b, c|a, d) e_{b,c} \otimes 1_V \otimes e_{a,d}$, $R_{23}^{z_{ij}}(c, b, a) = 1_V \otimes \sum_{a,c,b,d} R^{z_{ij}}(b, c|a, d) e_{b,c} \otimes e_{a,d}$, $z_{i,j} \in Y$.

In the case of set-theoretic solutions specifically, $R^{z_{ij}}(b, c|a, d) = \delta(c, \sigma_a^{z_{ij}}(b)) \delta(d, \tau_b^{z_{ij}}(a))$, whereas for p -shelves solutions, $R^{z_{ij}}(b, c|a, d) = \delta(c, b) \delta(d, b \triangleright_{z_{ij}} a)$. This description can be formally generalized to infinite countable sets, but for compact sets the description requires the use of functional analysis and the study of kernels of integral operators that correspond to the solution $R^{z_{ij}}$. \square

In the subsequent section we establish the algebraic framework in the tensor product formulation. This naturally provides solutions to the parametric set-theoretic Yang-Baxter equation, thus the linearized version presented above is certainly relevant in what follows.

3. PARAMETRIC SET-THEORETIC ALGEBRAS

In this section we are exploring the underlying algebraic structures that provide the universal \mathcal{R} -matrices associated to p -rack and set-theoretic solutions of the parametric Yang-Baxter equation. We show that universal set-theoretic solutions are obtained from the p -shelf ones via suitable admissible universal twists.

We first introduce the parametric rack (p -rack) and restricted p -rack algebras. Using these algebras we are able to extract the associated universal \mathcal{R} -matrices and coproduct structures. Certain fundamental representations of these algebras lead to set-theoretic solutions of the parametric Yang-Baxter equation.

3.1. The parametric rack algebras. We start our analysis by defining the algebra associated to p -rack solutions of the parametric Yang-Baxter equation called *parametric rack (p -rack) algebra* (see also relevant findings for the non-parametric case [23]).

Definition 3.1. (*The p -rack algebra.*) Let $X \subseteq Y$ be non-empty sets. We define for all $z_{i,j,k} \in Y$, $(i, j, k \in \mathbb{Z}^+)$ the binary operation, $\triangleright_{z_{ij}} : X \times X \rightarrow X$, $(a, b) \mapsto a \triangleright_{z_{ij}} b$. Let also $(X, \triangleright_{z_{ij}})$ be a finite magma, or such that $a \triangleright_{z_{ij}}$ is surjective, for every $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra \mathcal{Q} , over a field k generated by indeterminates, $1_{\mathcal{Q}}$ (unit element) $q_a^{z_{ij}}$, $(q_a^{z_{ij}})^{-1}$, $h_a \in \mathcal{Q}$ ($h_a = h_b \Leftrightarrow a = b$) and relations for all $a, b \in X$, $z_{i,j,k} \in Y$:

$$q_a^{z_{ij}} (q_a^{z_{ij}})^{-1} = (q_a^{z_{ij}})^{-1} q_a^{z_{ij}} = 1_{\mathcal{Q}}, \quad q_a^{z_{jk}} q_b^{z_{ik}} = q_b^{z_{ik}} q_{b \triangleright_{z_{ij}} a}^{z_{jk}}, \quad h_a h_b = \delta_{a,b} h_a^2, \quad q_b^{z_{ij}} h_{b \triangleright_{z_{ij}} a} = h_a q_b^{z_{ij}} \quad (3.1)$$

is a p -rack algebra.

The choice of the name p -rack algebra is justified by the following result.

Proposition 3.2. *Let \mathcal{Q} be the p -rack algebra, then for all $a, b, c \in X$, $z_{i,j,k} \in Y$, $c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a) = (c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a)$, i.e. $(X, \triangleright_{z_{ij}})$ is a p -rack.*

Proof. We compute $h_a q_b^{z_{jk}} q_c^{z_{ik}}$ using the associativity of the algebra:

$$h_a q_b^{z_{jk}} q_c^{z_{ik}} = q_b^{z_{jk}} h_{b \triangleright_{z_{jk}} a} q_c^{z_{ik}} = q_b^{z_{jk}} q_c^{z_{ik}} h_{c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a)} = q_c^{z_{ik}} q_{c \triangleright_{z_{ij}} b}^{z_{jk}} h_{c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a)}, \quad (3.2)$$

$$h_a q_b^{z_{jk}} q_c^{z_{ik}} = h_a q_c^{z_{ik}} q_{c \triangleright_{z_{ij}} b}^{z_{jk}} = q_c^{z_{ik}} h_{c \triangleright_{z_{ik}} a} q_{c \triangleright_{z_{ij}} b}^{z_{jk}} = q_c^{z_{ik}} q_{c \triangleright_{z_{ij}} b}^{z_{jk}} h_{(c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a)}. \quad (3.3)$$

Due to invertibility of $q_a^{z_{ij}}$ for all $a \in X$ we conclude from (3.2), (3.3) that

$$h_{c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a)} = h_{(c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a)} \Rightarrow c \triangleright_{z_{ik}} (b \triangleright_{z_{jk}} a) = (c \triangleright_{z_{ij}} b) \triangleright_{z_{jk}} (c \triangleright_{z_{ik}} a).$$

We assume $c \triangleright_{z_{ij}} a = c \triangleright_{z_{ij}} b$, then $q_c^{z_{ij}} h_{c \triangleright_{z_{ij}} a} = q_c^{z_{ij}} h_{c \triangleright_{z_{ij}} b}$, by the fourth relation in 3.1, we get $h_a q_c^{z_{ij}} = h_b q_c^{z_{ij}}$ and by the invertibility of $q_c^{z_{ij}}$, $h_a = h_b$, hence $a = b$, i.e. $a \triangleright_{z_{ij}}$ is bijective and thus $(X, \triangleright_{z_{ij}})$ is a p -rack. \square

Lemma 3.3. *Let $c = \sum_{a \in X} h_a$, then c is a central element of the p -rack algebra \mathcal{Q} . Also, $h_a^2 = h_a$, for all $a \in X$.*

Proof. The proof is straightforward by means of the definition of the algebra \mathcal{Q} and the fact that $a \triangleright_{z_{ij}}$ is a bijection. Without loss of generality we set $c = 1_{\mathcal{Q}}$, then it immediately follows that $h_a^2 = h_a$, for all $a \in X$. \square

Lemma 3.4. *(Commuting quantities.) Let $t^{z_{ik}} := \sum_{a \in X} q_a^{z_{ik}}$, for $z_{i,k} \in Y$. Then, $t^{z_{jk}} t^{z_{ik}} = t^{z_{ik}} t^{z_{jk}}$, for all $z_{i,j,k} \in Y$.*

Proof. The proof is straightforward by direct computation by using of the relation $q_a^{z_{jk}} q_b^{z_{ik}} = q_b^{z_{ik}} q_{b \triangleright_{z_{ij}} a}^{z_{jk}}$ and the fact that $a \triangleright_{z_{ij}}$ is a bijection. \square

Having defined the p -rack algebra we may now show that there exists an associated universal \mathcal{R} -matrix, solution of the parametric Yang-Baxter equation.

Proposition 3.5. *Let \mathcal{Q} be the p -rack algebra and $\mathcal{R}^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be an invertible element, such that $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}}$, $z_{i,j} \in Y$. Then $\mathcal{R}^{z_{ij}}$ satisfies the parametric Yang-Baxter equation,*

$$\mathcal{R}_{12}^{z_{12}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{23}^{z_{23}} = \mathcal{R}_{23}^{z_{23}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{12}^{z_{12}},$$

where $\mathcal{R}_{12}^{z_{12}} = \sum_{a \in X} h_a \otimes q_a^{z_{12}} \otimes 1_{\mathcal{Q}}$, $\mathcal{R}_{13}^{z_{13}} = \sum_{a \in X} h_a \otimes 1_{\mathcal{Q}} \otimes q_a^{z_{13}}$, and $\mathcal{R}_{23}^{z_{23}} = \sum_{a \in X} 1_{\mathcal{Q}} \otimes h_a \otimes q_a^{z_{23}}$. The inverse \mathcal{R} -matrix is $(\mathcal{R}^{z_{ij}})^{-1} = \sum_{a \in X} h_a \otimes (q_a^{z_{ij}})^{-1}$.

Proof. The proof is a direct computation of the two sides of the Yang-Baxter equation (and use of the fundamental relations (3.1)): The LHS of the Yang-Baxter equation becomes

$$\sum_{a,b,c \in X} h_a h_b \otimes q_a^{z_{12}} h_c \otimes q_b^{z_{13}} q_c^{z_{23}} = \sum_{a,b,c \in X} h_a \otimes q_a^{z_{12}} h_c \otimes q_a^{z_{13}} q_c^{z_{23}}$$

whereas the RHS gives

$$\sum_{a,b,c \in X} h_b h_a \otimes h_c q_a^{z_{12}} \otimes q_c^{z_{23}} q_b^{z_{13}} = \sum_{a,b,c \in X} h_a \otimes q_a^{z_{12}} h_{a \triangleright_{z_{12}} c} \otimes q_a^{z_{13}} q_{a \triangleright_{z_{12}} c}^{z_{23}},$$

by setting $a \triangleright_{z_{12}} c = \hat{c}$ in the final expression for the RHS (using that $a \triangleright_{z_{12}}$ is bijective) we show that LHS=RHS.

We recall from Lemma 3.3 that $\sum_{a \in X} h_a = 1_{\mathcal{Q}}$, then $(\mathcal{R}^{z_{ij}})^{-1} = \sum_{a \in X} h_a \otimes (q_a^{z_{ij}})^{-1}$. \square

Remark 3.6. Fundamental representation: We first recall the $n \times n$ matrices $e_{i,j}$, with elements $(e_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}$. Let \mathcal{Q} be the p -rack algebra and $\rho : \mathcal{Q} \rightarrow \text{End}(V)$ (V is an n -dimensional vector space) be the map defined by

$$q_a^{z_{ij}} \mapsto \sum_{x \in X} e_{x, a \triangleright_{z_{ij}} x}, \quad h_a \mapsto e_{a,a}. \quad (3.4)$$

Then $\mathcal{R}^{z_{ij}} \mapsto R^{z_{ij}} = \sum_{a,b \in X} e_{b,b} \otimes e_{a, b \triangleright_{z_{ij}} a}$, which is the linearized version of the p -rack solution.

We recall that $a \triangleright_{z_{ij}} : X \rightarrow X$ is a bijection, and $(R^{z_{12}})^{-1} = \sum_{a,b \in X} e_{b,b} \otimes e_{b \triangleright_{z_{12}} a, a}$.

We briefly formulate the Faddeev-Reshetikhin-Takhtajan (FRT) construction [30] in order to check the consistency of our algebraic construction. From the parametric Yang-Baxter equation (3.4, and after recalling the representations of Remark 3.6 and setting: $(\rho \otimes \text{id})\mathcal{R}^{z_{ij}} := L^{z_{ij}} = \sum_{a \in X} e_{a,a} \otimes q_a^{z_{ij}}$, $(\text{id} \otimes \rho)\mathcal{R}^{z_{ij}} = \hat{L}^{z_{ij}} = \sum_{a,b \in X} h_b \otimes e_{a, b \triangleright_{z_{ij}} a}$, and $(\rho \otimes \rho)\mathcal{R}^{z_{ij}} := R^{z_{ij}} = \sum_{a,b \in X} e_{b,b} \otimes e_{a, b \triangleright_{z_{ij}} a}$, we derive consistent algebraic relations:

$$R_{12}^{z_{12}} L_{13}^{z_{13}} L_{23}^{z_{23}} = L_{23}^{z_{23}} L_{13}^{z_{13}} R_{12}^{z_{12}}, \quad \hat{L}_{12}^{z_{12}} \hat{L}_{13}^{z_{13}} R_{23}^{z_{23}} = R_{23}^{z_{23}} \hat{L}_{13}^{z_{13}} \hat{L}_{12}^{z_{12}}, \quad L_{12}^{z_{12}} R_{13}^{z_{13}} \hat{L}_{23}^{z_{23}} = \hat{L}_{23}^{z_{23}} R_{13}^{z_{13}} L_{12}^{z_{12}}, \quad (3.5)$$

which lead to the p -rack algebra given in Definition (3.1) and provide a strong consistency check on the algebraic relations (3.1).

Proposition 3.7. (Algebra homomorphism) Let \mathcal{Q} be the p -rack algebra and $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be a solution of the Yang-Baxter equation. We also define for $z_{i,j,k} \in Y$, $\Delta'_{z_{ij}} : \mathcal{Q} \rightarrow \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,

$$\Delta'_{z_{jk}}((q_a^{z_{ik}})^{\pm 1}) := (q_a^{z_{ij}})^{\pm 1} \otimes (q_a^{z_{ik}})^{\pm 1}, \quad \Delta'_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \triangleright_{z_{ij}} c = a}.$$

Then the following statements hold:

- (1) $\Delta'_{z_{ij}}$ is a \mathcal{Q} algebra homomorphism for all $z_{i,j} \in Y$.
- (2) $\mathcal{R}^{z_{jk}} \Delta'_{z_{jk}}(q_a^{z_{ik}}) = \Delta'_{z_{kj}}^{(op)}(q_a^{z_{ik}}) \mathcal{R}^{z_{jk}}$, for all $a \in X$, $z_{i,j,k} \in Y$. Recall, $\Delta'_{z_{ij}}^{(op)} := \pi \circ \Delta'_{z_{ij}}$, where π is the flip map.

Proof. For our proof we use the definition of the p -rack algebra.

- (1) In order to show that $\Delta'_{z_{ij}}$ is an algebra homomorphism it suffices to show the following statements for all $z_{i,j,k,l} \in Y$. We first check that indeed:

$$\begin{aligned} \Delta'_{z_{kl}}((q_a^{z_{jl}})^{-1})\Delta'_{z_{kl}}(q_a^{z_{jl}}) &= \Delta'_{z_{kl}}(q_a^{z_{jl}})\Delta'_{z_{kl}}((q_a^{z_{jl}})^{-1}) = 1_{\mathcal{Q}} \otimes 1_{\mathcal{Q}}. \text{ Moreover,} \\ \Delta'_{z_{kl}}(q_a^{z_{jl}})\Delta'_{z_{kl}}(q_b^{z_{il}}) &= (q_a^{z_{jk}} \otimes q_b^{z_{jl}})(q_b^{z_{ik}} \otimes q_b^{z_{il}}) = \\ (q_b^{z_{ik}} \otimes q_b^{z_{il}})(q_{b \triangleright_{z_{ij}} a}^{z_{jk}} \otimes q_{b \triangleright_{z_{ij}} a}^{z_{jl}}) &= \Delta'_{z_{kl}}(q_b^{z_{il}})\Delta'_{z_{kl}}(q_{b \triangleright_{z_{ij}} a}^{z_{jl}}). \end{aligned}$$

Via the algebraic relations (3.1) we obtain

$$\begin{aligned} \Delta'_{z_{jk}}(h_a)\Delta'_{z_{jk}}(q_b^{z_{ik}}) &= (h_{a_1} \otimes h_{a_2} \Big|_{a_1 \triangleright_{z_{jk}} a_2 = a}) (q_b^{z_{ij}} \otimes q_b^{z_{ik}}) = \\ (q_b^{z_{ij}} \otimes q_b^{z_{ik}})(h_{b \triangleright_{z_{ij}} a_1} \otimes h_{b \triangleright_{z_{ik}} a_2} \Big|_{b \triangleright_{z_{ik}} (a_1 \triangleright_{z_{jk}} a_2) = b \triangleright_{z_{ik}} a}) &= \Delta'_{z_{jk}}(q_b^{z_{ik}})\Delta'_{z_{jk}}(h_{b \triangleright_{z_{ik}} a}), \end{aligned}$$

where we have used the self-distributivity for racks, $(b \triangleright_{z_{ij}} a_1) \triangleright_{z_{jk}} (b \triangleright_{z_{ik}} a_2) = b \triangleright_{z_{ik}} (a_1 \triangleright_{z_{jk}} a_2)$. Similarly, it is straightforward to show via (3.1) that $\Delta'_{z_{jk}}(h_a)\Delta'_{z_{jk}}(h_b) = \delta_{a,b}\Delta'_{z_{jk}}(h_a)$. And this concludes the proof of the first part.

- (2) For the proof of the second part we first notice that $\Delta'^{(op)}_{z_{kj}}(q_b^{z_{ij}}) = \Delta'_{z_{jk}}(q_b^{z_{ij}})$. We then compute for all $b \in X$, $z_{i,j,k} \in Y$:

$$\begin{aligned} \mathcal{R}^{z_{jk}}\Delta'_{z_{jk}}(q_b^{z_{ik}}) &= \left(\sum_{a \in X} h_a \otimes q_a^{z_{jk}}\right)(q_b^{z_{ij}} \otimes q_b^{z_{ik}}) = \\ (q_b^{z_{ij}} \otimes q_b^{z_{ik}})\left(\sum_{a \in X} h_{b \triangleright_{z_{ij}} a} \otimes q_{b \triangleright_{z_{ij}} a}^{z_{jk}}\right) &= \Delta'_{z_{jk}}(q_b^{z_{ik}})\mathcal{R}^{z_{jk}}. \end{aligned}$$

□

Lemma 3.8. (*Parametric coassociativity.*) Let \mathcal{Q} be the p -rack algebra. We also define for $z_{i,1,2,\dots,n} \in Y$, $\Delta'^{(n)}_{z_{12\dots n}} : \mathcal{Q} \rightarrow \mathcal{Q}^{\otimes n}$, such that

$$\Delta'^{(n)}_{z_{12\dots n}}((q_a^{z_{in}})^{\pm 1}) := (q_a^{z_{i1}})^{\pm 1} \otimes (q_a^{z_{i2}})^{\pm 1} \otimes \dots \otimes (q_a^{z_{in}})^{\pm 1}, \quad (3.6)$$

$$\Delta'^{(n)}_{z_{12\dots n}}(h_a) := \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes h_{a_2} \otimes \dots \otimes h_{a_n} \Big|_{a_1 \triangleright_{z_{1n}} (a_2 \triangleright_{z_{2n}} (\dots (a_{n-1} \triangleright_{z_{n-1n}} a_n) \dots)) = a}. \quad (3.7)$$

Then,

- (1) For all $a \in X$, $z_{i,1,2,\dots,n} \in Y$,

$$\Delta'^{(n)}_{z_{12\dots n}}((q_a^{z_{in}})^{\pm 1}) := (\Delta'^{(n-1)}_{z_{12\dots n-1}} \otimes id)\Delta'_{z_{n-1n}}((q_a^{z_{in}})^{\pm 1}) = (id \otimes \Delta'^{(n-1)}_{z_{23\dots n}})\Delta'_{z_{1n}}((q_a^{z_{in}})^{\pm 1}), \quad (3.8)$$

$$\Delta'^{(n)}_{z_{12\dots n}}(h_a) := (id \otimes \Delta'^{(n-1)}_{z_{23\dots n}})\Delta'_{z_{1n}}(h_a). \quad (3.9)$$

- (2) For all $a, b \in X$, $z_{i,1,2,\dots,n} \in Y$, $\Delta'^{(n)}_{z_{12\dots n}}$ is an algebra homomorphism.

Proof. These are shown by direct computation and use of the p self-distributivity. □

- Graphical representations of $\Delta'_{z_{12}}$, and the parametric coassociativity via binary trees:

- (1) We graphically depict below the parametric co-product $\Delta'_{z_{12}}$:

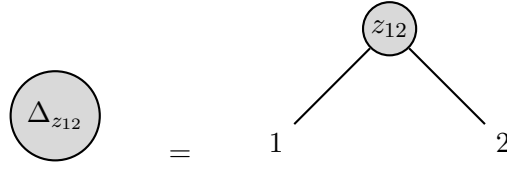


Figure 1.

- (2) We also see below the typical binary tree graphical representation of the parametric co-associativity condition for $n = 3$ and $\Delta'_{z_{123}}{}^{(3)}$, i.e. $\Delta'_{z_{123}}{}^{(3)} := (\Delta'_{z_{12}} \otimes \text{id})\Delta'_{z_{13}} = (\text{id} \otimes \Delta'_{z_{23}})\Delta'_{z_{12}}$:

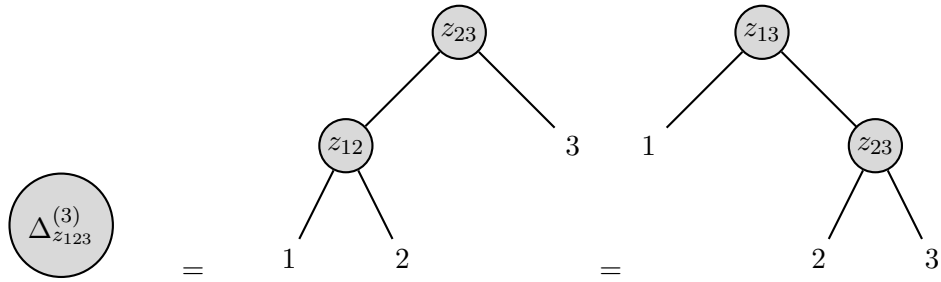


Figure 2.

Notice that $\Delta'_{z_{12}}(h_a)$ is still graphically represented by Figure 1, however the parametric coassociativity does not hold in this case and the three coproduct $\Delta'_{z_{123}}{}^{(3)}$ is then represented by the right part of the graphical equation Figure 2.

- (3) In general, in the case where coassociativity holds the n^{th} coproduct $\Delta'_{z_{12\dots n}}{}^{(n)}(q_a^{z_{kn}})$, $a \in X$, $z_{k,1,2,\dots,n} \in Y$ is depicted by 2^{n-2} equivalent diagrams:

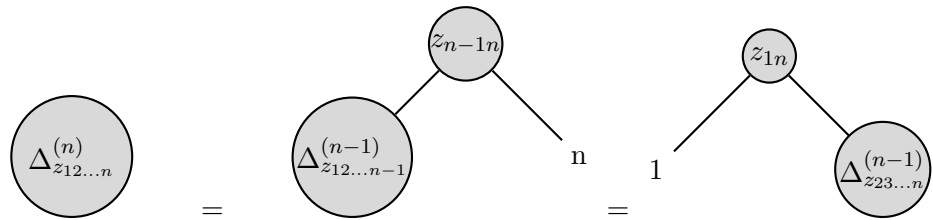


Figure 3.

Unfolding $\Delta'{}^{(n-1)}$ in the LHS and RHS of Figure 3 yields 2^{n-2} binary tree diagrams. $\Delta'_{z_{12\dots n}}{}^{(n)}(h_a)$ on the other hand (no co associativity applies) is depicted by only one diagram, shown in the right part of Figures 2 & 3.

- **Example.** The four equivalent binary trees that depict $\Delta'_{z_{1234}}{}^{(4)}(q_a^{z_{k4}})$, emerging from Figures 2 and 3:

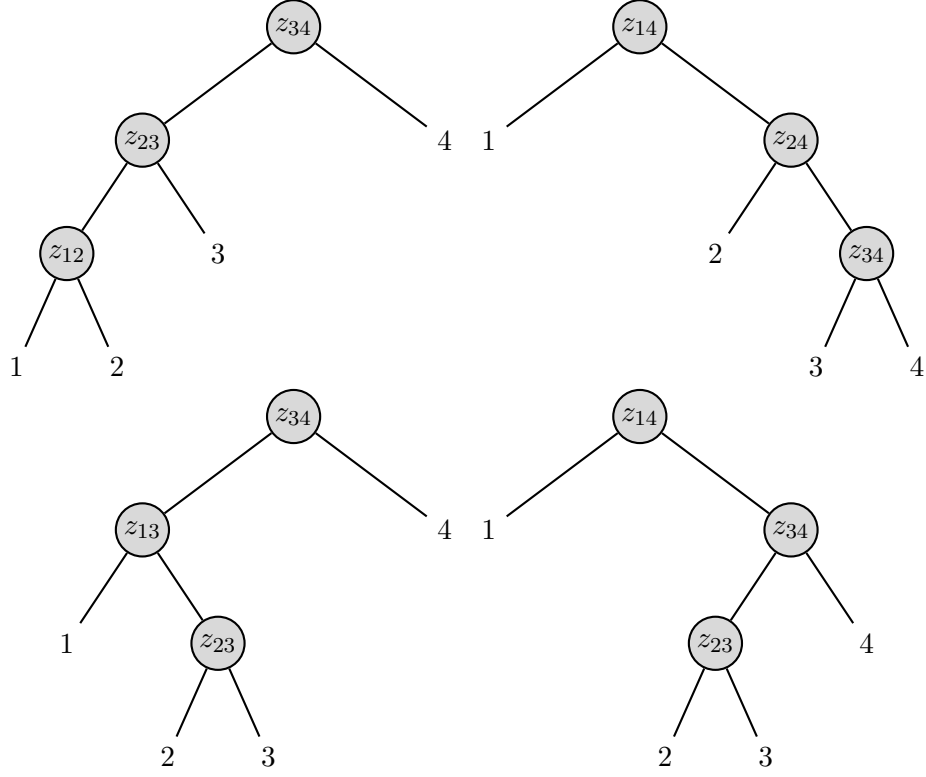


Figure 4.

$\Delta'_{z_{1234}}{}^{(4)}(h_a)$ is depicted by the top right binary tree above.

Definition 3.9. (The restricted p -rack algebra.) A p -rack algebra \mathcal{Q} is called a restricted p -rack algebra if for all $z_{i,j} \in Y$ there exists a binary operation $\bullet_{z_{i,j}} : X \times X \rightarrow X$, $(a, b) \mapsto a \bullet_{z_{i,j}} b$, such that, $a \bullet_{z_{i,j}}$, is a bijection and $a \bullet_{z_{j,i}} b = b \triangleright_{z_{i,j}} a$, for all $a, b \in X$, $z_{i,j} \in Y$.

Recall Example 2.22, where the condition of Definition 3.9 is satisfied.

Theorem 3.10. Let \mathcal{Q} be the restricted p -rack algebra and $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be a solution of the Yang-Baxter equation. Moreover, assume that for all $z_{i,j,k} \in Y$, $a, b \in X$, $(b \triangleright_{z_{ij}} a_1) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} a_2) = b \triangleright_{z_{ik}} (a_1 \bullet_{z_{jk}} a_2)$. We also define for $z_{i,j,k} \in Y$, $\Delta_{z_{ij}} : \mathcal{Q} \rightarrow \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,

$$\Delta_{z_{jk}}((q_a^{z_{ik}})^{\pm 1}) := (q_a^{z_{ij}})^{\pm 1} \otimes (q_a^{z_{ik}})^{\pm 1}, \quad \Delta_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet_{z_{ij}} c = a}.$$

Then the following statements hold:

- (1) $\Delta_{z_{ij}}$ is a \mathcal{Q} algebra homomorphism for all $z_{i,j} \in Y$.
- (2) $\mathcal{R}^{z_{jk}} \Delta_{z_{jk}}(y) = \Delta_{z_{kj}}^{(op)}(y) \mathcal{R}^{z_{jk}}$, for all $z_{j,k} \in Y$, $y \in \mathcal{Q}$. Recall $\Delta_{z_{ij}}^{(op)} := \pi \circ \Delta_{z_{ij}}$, where π is the flip map.

Proof. For the proof we use the definition of the restricted the p -rack algebra.

- (1) In order to show that $\Delta_{z_{ij}}$ is an algebra homomorphism it suffices to show the following statements for all $z_{i,j,k,l} \in Y$. We note that $\Delta_{z_{kl}}(q_a^{z_{jl}})\Delta_{z_{kl}}((q_a^{z_{jl}})^{-1}) = 1_{\mathcal{Q}} \otimes 1_{\mathcal{Q}}$ and $\Delta_{z_{kl}}(q_a^{z_{jl}})\Delta_{z_{kl}}(q_b^{z_{il}}) = \Delta_{z_{kl}}(q_b^{z_{il}})\Delta_{z_{kl}}(q_{b \triangleright_{z_{ij}} a}^{z_{jl}})$ are shown as in the first part of the proof of Proposition 3.7. Via the algebraic relations (3.1) we obtain

$$\begin{aligned} \Delta_{z_{jk}}(h_a)\Delta_{z_{jk}}(q_b^{z_{ik}}) &= (h_{a_1} \otimes h_{a_2} \Big|_{a_1 \bullet_{z_{jk}} a_2 = a}) (q_b^{z_{ij}} \otimes q_b^{z_{ik}}) = \\ &= (q_b^{z_{ij}} \otimes q_b^{z_{ik}}) (h_{b \triangleright_{z_{ij}} a_1} \otimes h_{b \triangleright_{z_{ik}} a_2} \Big|_{a_1 \bullet_{z_{jk}} a_2 = a}) = \Delta_{z_{jk}}(q_b^{z_{ik}})\Delta_{z_{jk}}(h_{b \triangleright_{z_{ik}} a}), \end{aligned}$$

where we have used $(b \triangleright_{z_{ij}} a_1) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} a_2) = b \triangleright_{z_{ik}} (a_1 \bullet_{z_{jk}} a_2)$. Similarly, it is straightforward to show via (3.1) that $\Delta_{z_{jk}}(h_a)\Delta_{z_{jk}}(h_b) = \delta_{a,b}\Delta_{z_{jk}}(h_a)$. And this concludes the proof of the first part.

- (2) We recall the proof of the second part of the proof of Proposition 3.7. Also,

$$\begin{aligned} \Delta_{z_{ji}}^{(op)}(h_a)\mathcal{R}^{z_{ij}} &= \left(\sum_{a_1, a_2 \in X} h_{a_2} \otimes h_{a_1} \Big|_{a_1 \bullet_{z_{ji}} a_2 = a} \right) \left(\sum_{b \in X} h_b \otimes q_b^{z_{ij}} \right) = \\ &= \left(\sum_{b \in X} h_b \otimes q_b^{z_{ij}} \right) \left(\sum_{a_1, a_2 \in X} h_{a_2} \otimes h_{a_2 \triangleright_{z_{ij}} a_1} \Big|_{a_2 \bullet_{z_{ij}} (a_2 \triangleright_{z_{ij}} a_1) = a} \right) = \mathcal{R}^{z_{ij}} \Delta_{z_{ij}}(h_a), \end{aligned} \quad (3.10)$$

where we have used $a_1 \bullet_{z_{ji}} a_2 = a_2 \bullet_{z_{ij}} (a_2 \triangleright_{z_{ij}} a_1)$. \square

Proposition 3.11. (*Parametric (co)-associativity.*) *Let \mathcal{Q} be the restricted p -rack algebra, assume also that for all $a, b, c \in X$ and $z_{i,j,k} \in Y$, $(b \triangleright_{z_{ij}} a) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = b \triangleright_{z_{ik}} (a \bullet_{z_{jk}} c)$ and $(a \bullet_{z_{ij}} b) \bullet_{z_{jk}} c = a \bullet_{z_{ik}} (b \bullet_{z_{jk}} c)$.*

We also define for $z_{i,1,2,\dots,n} \in Y$, $\Delta_{z_{12\dots n}}^{(n)} : \mathcal{Q} \rightarrow \mathcal{Q}^{\otimes n}$, such that

$$\Delta_{z_{12\dots n}}^{(n)}((q_a^{z_{in}})^{\pm 1}) = (q_a^{z_{i1}})^{\pm 1} \otimes (q_a^{z_{i2}})^{\pm 1} \otimes \dots \otimes (q_a^{z_{in}})^{\pm 1}, \quad (3.11)$$

$$\Delta_{z_{12\dots n}}^{(n)}(h_a) := \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes h_{a_2} \otimes \dots \otimes h_{a_n} \Big|_{\Pi_{z_{1\dots n}}(a_1, a_2, \dots, a_n) = a}, \quad (3.12)$$

where for all $a_1, a_2, \dots, a_n \in X$, $z_1, \dots, z_n \in Y$:

$$\Pi_{z_{12}}(a_1, a_2) := a_1 \bullet_{z_{12}} a_2 \quad (3.13)$$

$$\begin{aligned} \Pi_{z_{12\dots n}}(a_1, a_2, \dots, a_n) &:= a_1 \bullet_{z_{1n}} (a_2 \bullet_{z_{2n}} (a_3 \dots \bullet_{z_{n-2n}} (a_{n-1} \bullet_{z_{n-1n}} a_n) \dots)) \\ &= ((\dots ((a_1 \bullet_{z_{12}} a_2) \bullet_{z_{23}} a_3) \dots \bullet_{z_{n-2n-1}} a_{n-1}) \bullet_{z_{n-1n}} a_n), \quad n > 2. \end{aligned}$$

Then:

- (1) For all $a \in X$, $z_{i,1,2,\dots,n} \in Y$, the “parametric” coassociativity holds, $y \in \{q_a^{z_{in}}, h_a\}$:

$$\Delta_{z_{12\dots n}}^{(n)}(y) := (\Delta_{z_{12\dots n-1}}^{(n-1)} \otimes id)\Delta_{z_{n-1n}}(y) = (id \otimes \Delta_{z_{23\dots n}}^{(n-1)})\Delta_{z_{1n}}(y). \quad (3.14)$$

- (2) For all $a, b \in X$, $z_{i,1,2,\dots,n} \in Y$, $\Delta_{z_{12\dots n}}^{(n)}$ is an algebra homomorphism.

Proof. The proof of both parts is straightforward.

- (1) The n -coassociativity (3.14) for the coproducts of $q_a^{z_{ij}}$ for all $a \in X$, $z_{i,j} \in Y$ is immediately shown by direct computation. Also, using (3.13) we conclude for $n > 2$:

$$\Pi_{z_{12}\dots z_n}(a_1, a_2, \dots, a_n) = a_1 \bullet_{z_{1n}} \Pi_{z_{2\dots n}}(a_2, \dots, a_n) = \Pi_{z_{12}\dots z_{n-1}}(a_1, a_2, \dots, a_{n-1}) \bullet_{z_{n-1n}} a_n.$$

Then (3.14) immediately follows for the n -coproduct of h_a , $a \in X$.

- (2) Indeed, the p -rack algebra relations hold for the n -coproducts of the generators (see also Theorem 3.10). The algebra relations are shown as a generalization of the proof of part 1 of Theorem 3.10 bearing also in mind expressions (3.6), (3.12). \square

The n -coproducts as defined in Proposition 3.11 are naturally depicted by binary trees (see Figures 1-4, pages 17-18). We provide in the following corollary a concrete statement of quantum integrability as we identify an explicit set of mutually commuting non-local quantities.

Corollary 3.12. *(Commuting non-local quantities.) We define for all $z_{i,k_1,\dots,k_n} \in Y$,*

$$\mathfrak{t}^{z_{ik_1\dots k_n}} := \sum_{a \in X} \Delta_{z_{k_1 k_2 \dots k_n}}^{(n)}(q_a^{z_{ik_n}}),$$

then $\mathfrak{t}^{z_{jk_1\dots k_n}} \mathfrak{t}^{z_{ik_1\dots k_n}} = \mathfrak{t}^{z_{ik_1\dots k_n}} \mathfrak{t}^{z_{jk_1\dots k_n}}$, for all $z_{i,j,k_1,\dots,k_n} \in Y$.

Proof. This is a consequence of Lemma 3.4 and the form of the coproduct $\Delta_{z_{k_1 k_2 \dots k_n}}^{(n)}(q_a^{z_{ik_n}})$. \square

Lemma 3.13. *Let $\mathcal{R}^{z_{ij}}$ be a solution of the parametric Yang-Baxter equation and define:*

$$T_{1,23\dots n+1}^{z_{12}\dots z_{n+1}} := \mathcal{R}_{1n+1}^{z_{1n+1}} \mathcal{R}_{1n}^{z_{1n}} \dots \mathcal{R}_{12}^{z_{12}}, \quad T_{12\dots n,n+1}^{z_{12}\dots z_{n+1}} := \mathcal{R}_{1n+1}^{z_{1n+1}} \mathcal{R}_{2n+1}^{z_{2n+1}} \dots \mathcal{R}_{nn+1}^{z_{nn+1}}. \quad (3.15)$$

Let also \mathcal{Q} be the restricted p -rack algebra, $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ and for all $z_{i,j,k} \in Y$, $a, b, c \in X$, $(b \triangleright_{z_{ij}} a) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = b \triangleright_{z_{ik}} (a \bullet_{z_{jk}} c)$, $q_a^{z_{jk}} q_b^{z_{ik}} = q_{a \bullet_{z_{ji}} b}^{z_{ik}}$ and $(a \bullet_{z_{ij}} b) \bullet_{z_{jk}} c = a \bullet_{z_{ik}} (b \bullet_{z_{jk}} c)$, then

$$T_{1,23\dots n+1}^{z_{12}\dots z_{n+1}} = \sum_{a \in X} h_a \otimes \Delta_{z_{23\dots n+1}}^{(n)}(q_a^{z_{1n+1}}) = (id \otimes \Delta_{z_{23\dots n+1}}^{(n)}) \mathcal{R}^{z_{1n+1}} \quad (3.16)$$

$$T_{12\dots n,n+1}^{z_{12}\dots z_{n+1}} = \sum_{a \in X} \Delta_{z_{12\dots n}}^{(n)}(h_a) \otimes q_a^{z_{nn+1}} = (\Delta_{z_{12\dots n}}^{(n)} \otimes id) \mathcal{R}^{z_{nn+1}}. \quad (3.17)$$

Proof. The proof is immediate:

$$\begin{aligned} T_{1,23\dots n+1}^{z_{12}\dots z_{n+1}} &= \sum_{a \in X} h_a \otimes q_a^{z_{12}} \otimes q_a^{z_{13}} \otimes \dots \otimes q_a^{z_{1n+1}}, \\ T_{12\dots n,n+1}^{z_{12}\dots z_{n+1}} &= \sum_{a \in X} \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes \dots \otimes h_{a_n} \Big|_{a := \Pi_{z_{12}\dots z_n}(a_1, a_2, \dots, a_n)} \otimes q_a^{z_{nn+1}}. \end{aligned}$$

Recalling the definitions of the n -coproducts in Proposition 3.11, we arrive at (3.16), (3.17) \square

Lemma 3.13 with Theorem 3.10 and Proposition 3.11 provide a structure that generalizes in a sense the notion of a quasi-triangular (quasi)-bialgebra [25, 26] to the parametric frame. In the parameter free case the structure formulated in Lemma 3.13, Theorem 3.10 and Proposition 3.11 corresponds indeed to a quasi-triangular Hopf algebra if (X, \bullet) is group [23]. Recall that quantities (3.15) are tensor realizations of the algebras defined by (3.5) via the FRT construction [30].

Remark 3.14. *Theorem 3.10 and Proposition 3.11 can be generalised as follows. Let \mathcal{Q} be the restricted p -rack algebra, assume also that for all $a, b, c \in X$ and $z_{i,j,k} \in Y$, there exist $z_{\hat{o}}, z_o \in Y$, such that $(b \triangleright_{z_{ij}} a) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = b \triangleright_{z_{i\hat{o}}} (a \bullet_{z_{jk}} c)$ and $(a \bullet_{z_{ij}} b) \bullet_{z_{ok}} c = a \bullet_{z_{io}} (b \bullet_{z_{jk}} c)$.*

We also define for $z_{i,1,2,\dots,n} \in Y$, $\Delta_{z_{12\dots n}}^{(n)} : \mathcal{Q} \rightarrow \mathcal{Q}^{\otimes n}$, such that

$$\Delta_{z_{12\dots n}}^{(n)}((q_a^{z_{in}})^{\pm 1}) = (q_a^{z_{i1}})^{\pm 1} \otimes (q_a^{z_{i2}})^{\pm 1} \otimes \dots \otimes (q_a^{z_{in}})^{\pm 1}, \quad (3.18)$$

$$\Delta_{z_{12\dots n}}^{(n)}(h_a) := \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes h_{a_2} \otimes \dots \otimes h_{a_n} \Big|_{\Pi_{z_{1\dots n}}(a_1, a_2, \dots, a_n) = a}, \quad (3.19)$$

where for all $a_1, a_2, \dots, a_n \in X$, $z_1, \dots, z_n \in Y$:

$$\begin{aligned} \Pi_{z_{12}}(a_1, a_2) &:= a_1 \bullet_{z_{12}} a_2 & (3.20) \\ \Pi_{z_{12\dots n}}(a_1, a_2, \dots, a_n) &:= a_1 \bullet_{z_{1o}} (a_2 \bullet_{z_{2o}} (a_3 \dots \bullet_{z_{n-2o}} (a_{n-1} \bullet_{z_{n-1n}} a_n) \dots)) \\ &= ((\dots ((a_1 \bullet_{z_{12}} a_2) \bullet_{z_{o3}} a_3) \dots \bullet_{z_{on-1}} a_{n-1}) \bullet_{z_{on}} a_n, \quad n > 2. \end{aligned}$$

Then, it is shown by direct computation:

(1) For all $a \in X$, $z_{i,1,2,\dots,n} \in Y$, a parametric coassociativity holds as

$$\begin{aligned} \Delta_{z_{12\dots n}}^{(n)}((q_a^{z_{in}})^{\pm 1}) &:= (\Delta_{z_{12\dots n-1}}^{(n-1)} \otimes id) \Delta_{z_{n-1n}}((q_a^{z_{in}})^{\pm 1}) = (id \otimes \Delta_{z_{23\dots n}}^{(n-1)}) \Delta_{z_{1n}}((q_a^{z_{in}})^{\pm 1}). \\ \Delta_{z_{12\dots n}}^{(n)}(h_a) &:= (\Delta_{z_{12\dots n-1}}^{(n-1)} \otimes id) \Delta_{z_{on}}(h_a) = (id \otimes \Delta_{z_{23\dots n}}^{(n-1)}) \Delta_{z_{1o}}(h_a) \end{aligned} \quad (3.21)$$

(2) For all $a, b \in X$, $z_{i,1,2,\dots,n} \in Y$, $\Delta_{z_{12\dots n}}^{(n)}$ is a “weak” algebra homomorphism i.e. almost all the p -rack algebra relations hold, but

$$\Delta_{z_{12\dots n}}^{(n)}(h_a) \Delta_{z_{12\dots n}}^{(n)}(q_b^{z_{in}}) = \Delta_{z_{12\dots n}}^{(n)}(q_b^{z_{in}}) \Delta_{z_{12\dots n}}^{(n)}(h_{b \triangleright_{z_{i\hat{o}}} a}). \quad (3.22)$$

Example 3.15. Consider the binary operations $\bullet_{z_{ij}}, \triangleright_{z_{ij}} : X \times X \rightarrow X$, such that $a \bullet_{z_{ij}} b = a \circ z_i + b \circ z_j$ and $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$, where $(X, +, \circ)$ is a skew brace (see also Example 2.22), then one shows by direct computation that for all $a, b, c \in X$, $z_{i,j,k} \in Y$,

$$(a \triangleright_{z_{ij}} b) \bullet_{z_{jk}} (a \triangleright_{z_{ik}} c) = a \triangleright_{z_{i\hat{o}}} (b \bullet_{z_{jk}} c), \quad (a \bullet_{z_{ij}} b) \bullet_{z_{ok}} c = a \bullet_{z_{io}} (b \bullet_{z_{jk}} c),$$

where $z_o = z_{\hat{o}} = 1$ and

$$\Pi_{z_{1\dots n}}(a_1, a_2, \dots, a_n) = a_1 \circ z_1 + a_2 \circ z_2 + \dots + a_{n-1} \circ z_{n-1} + a_n \circ z_n.$$

Example 3.16. Recall the fundamental representation of the restricted p -rack algebra \mathcal{Q} , $\rho : \mathcal{Q} \rightarrow \text{End}(V)$, $q_a^{z_{ij}} \mapsto q_a^{z_{ij}} := \sum_{b \in X} e_{b, a \triangleright_{z_{ij}} b}$, $h_a \mapsto e_{a,a}$, $a \in X$, $z_{i,j} \in Y$, then:

$$q_a^{z_{jk}} q_b^{z_{ik}} = \sum_{c \in X} e_{c, b \triangleright_{z_{ik}} (a \triangleright_{z_{jk}} c)}. \quad (3.23)$$

Let also $(X, +, \circ)$ be a skew brace and recall the binary operations for $z_{i,j} \in Y$, $\bullet_{z_{ij}}, \triangleright_{z_{ij}} : X \times X \rightarrow X$, $a \bullet_{z_{ij}} b = a \circ z_i \circ z_j^{-1} + b$ and $a \triangleright_{z_{ij}} b = -a \circ z_i \circ z_j^{-1} + b + a \circ z_i \circ z_j^{-1}$, also

$$a \bullet_{z_{ji}} b = b \bullet_{z_{ij}} (b \triangleright_{z_{ij}} a) \quad \text{and} \quad b \triangleright_{z_{ik}} (a \triangleright_{z_{jk}} c) = (a \bullet_{z_{ji}} b) \triangleright_{z_{ok}} c, \quad (3.24)$$

where $z_o = 1$, then via (3.23) we conclude that $q_a^{z_{jk}} q_b^{z_{ik}} = q_{a \bullet_{z_{ji}} b}^{z_{ok}} = q_b^{z_{ik}} q_{b \triangleright_{z_{ij}} a}^{z_{jk}}$.

We conclude the subsection by noting that the rack (co)-homology has been studied in [2, 29, 16]. One of the natural future questions is the generalization of the (co)-homological analysis in the case of parametric racks, based on the parametric co-structure constructed here. This issue however will be addressed separately in a future work.

3.2. The p -set Yang-Baxter algebra. In this subsection we suitably extend the p -rack algebra \mathcal{Q} in order to construct the algebras associated to general set-theoretic solutions of the parametric Yang-Baxter equation.

We start our analysis by defining the *decorated p -rack algebra*.

Definition 3.17. (The decorated p -rack algebra.) Let \mathcal{Q} be the p -rack algebra. Let also $\sigma_a^{z_{ij}}, \tau_b^{z_{ij}} : X \rightarrow X$, and $\sigma_a^{z_{ij}}$ be a bijection for all $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\hat{\mathcal{Q}}$ over k , generated by indeterminates $q_a^{z_{ij}}, (q_a^{z_{ij}})^{-1}, h_a, \in \mathcal{Q}$ and $w_a^{z_{ij}}, (w_a^{z_{ij}})^{-1} \in \hat{\mathcal{Q}}$, $a \in X$, $1_{\hat{\mathcal{Q}}} = 1_{\mathcal{Q}}$ (the unit element) and relations, for $a, b \in X$, $z_{i,j,k} \in Y$:

$$\begin{aligned} q_a^{z_{ij}} (q_a^{z_{ij}})^{-1} &= (q_a^{z_{ij}})^{-1} q_a^{z_{ij}} = 1_{\hat{\mathcal{Q}}}, & q_a^{z_{jk}} q_b^{z_{ik}} &= q_b^{z_{ik}} q_{b \triangleright_{z_{ij}} a}^{z_{jk}}, & h_a h_b &= \delta_{a,b} h_a, \\ q_b^{z_{ij}} h_{b \triangleright_{z_{ij}} a} &= h_a q_b^{z_{ij}}, & w_a^{z_{ij}} (w_a^{z_{ij}})^{-1} &= 1_{\hat{\mathcal{Q}}}, & w_a^{z_{ki}} w_b^{z_{ji}} &= w_{\sigma_a^{z_{jk}}(b)}^{z_{ji}} w_{\tau_b^{z_{kj}}(a)}^{z_{ki}}, \\ w_a^{z_{ji}} h_b &= h_{\sigma_a^{z_{ij}}(b)} w_a^{z_{ji}}, & w_a^{z_{kj}} q_b^{z_{ij}} &= q_{\sigma_a^{z_{ik}}(b)}^{z_{ij}} w_a^{z_{kj}} \end{aligned} \quad (3.25)$$

is a decorated p -rack algebra.

Lemma 3.18. Let $c = \sum_{a \in X} h_a$, then c is a central element of the decorated p -rack algebra $\hat{\mathcal{Q}}$.

Proof. The proof is straightforward by means of the definition of the algebra $\hat{\mathcal{Q}}$ and the fact that $a \triangleright_{z_{ij}}$ and $\sigma_a^{z_{ij}}$ are bijections. \square

From now on we consider, without loss of generality, $c = 1_{\hat{\mathcal{Q}}}$ (see also Lemma 3.3).

Proposition 3.19. Let $\hat{\mathcal{Q}}$ be the decorated p -rack algebra, then for all $a, b, c \in X$, $z_{i,j,k} \in Y$:

$$\sigma_a^{z_{ik}} (\sigma_b^{z_{ij}}(c)) = \sigma_{\sigma_a^{z_{jk}}(b)}^{z_{ij}} (\sigma_{\tau_b^{z_{jk}}(a)}^{z_{ik}}(c)) \quad \& \quad \sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a).$$

Proof. We compute $w_a^{z_{ik}} w_b^{z_{ij}} h_c$ using the associativity of the algebra,

$$\begin{aligned} w_a^{z_{ki}} w_b^{z_{ji}} h_c &= w_{\sigma_a^{z_{jk}}(b)}^{z_{ji}} w_{\tau_b^{z_{kj}}(a)}^{z_{ki}} h_c = h_{\sigma_a^{z_{ij}}(\sigma_{\sigma_a^{z_{jk}}(b)}^{z_{ik}}(c))} w_{\sigma_b^{z_{jk}}(c)}^{z_{ji}} w_{\tau_c^{z_{kj}}(b)}^{z_{ki}}, \\ w_a^{z_{ki}} w_b^{z_{ji}} h_c &= h_{\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c))} w_a^{z_{ki}} w_b^{z_{ji}} = h_{\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c))} w_{\sigma_b^{z_{jk}}(c)}^{z_{ji}} w_{\tau_c^{z_{kj}}(b)}^{z_{ki}}. \end{aligned}$$

From the equations above and the invertibility of $w_a^{z_{ij}}$, for all $a \in X$, $z_{i,j} \in Y$, we conclude for all $a, b, c \in X$, $z_{i,j,k} \in Y$,

$$h_{\sigma_a^{z_{jk}}(b)}^{\sigma_a^{z_{ij}}(\sigma_a^{z_{jk}}(b))}(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c))) = h_{\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c))} = \sigma_a^{z_{ij}}(\sigma_b^{z_{jk}}(a))(\sigma_a^{z_{ik}}(\sigma_b^{z_{ij}}(c))),$$

for all $a, b, c \in X$, $z_{i,j,k} \in Y$.

We also compute $h_a q_b^{z_{ij}} w_c^{z_{kj}}$:

$$\begin{aligned} h_a q_b^{z_{ij}} w_c^{z_{kj}} &= h_a w_c^{z_{kj}} q_{(\sigma_c^{z_{ik}})^{-1}(b)}^{z_{ij}} = w_c^{z_{kj}} q_{(\sigma_c^{z_{ik}})^{-1}(b)}^{z_{ij}} h_{(\sigma_c^{z_{ik}})^{-1}(b) \triangleright_{z_{ij}} (\sigma_c^{z_{jk}})^{-1}(a)} \\ h_a q_b^{z_{ij}} w_c^{z_{kj}} &= q_b^{z_{ij}} h_{b \triangleright_{z_{ij}} a} w_c^{z_{kj}} = q_b^{z_{ij}} w_c^{z_{kj}} h_{(\sigma_c^{z_{ik}})^{-1}(b \triangleright_{z_{ij}} a)} = w_c^{z_{kj}} q_{(\sigma_c^{z_{ik}})^{-1}(b)}^{z_{ij}} h_{(\sigma_c^{z_{jk}})^{-1}(b \triangleright_{z_{ij}} a)}. \end{aligned}$$

From the equations above and the invertibility of $q_a^{z_{ij}}$, $w_a^{z_{ij}}$, for all $a \in X$, $z_{i,j} \in Y$, we conclude for all $a, b, c \in X$, $z_{i,j,k} \in Y$:

$$h_{(\sigma_c^{z_{ik}})^{-1}(b) \triangleright_{z_{ij}} (\sigma_c^{z_{jk}})^{-1}(a)} = h_{(\sigma_c^{z_{jk}})^{-1}(b \triangleright_{z_{ij}} a)} \Rightarrow (\sigma_c^{z_{ik}})^{-1}(b) \triangleright_{z_{ij}} (\sigma_c^{z_{jk}})^{-1}(a) = (\sigma_c^{z_{jk}})^{-1}(b \triangleright_{z_{ij}} a).$$

From the latter it immediately follows, $\sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a)$, for all $a, b, c \in X$, $z_{i,j,k} \in Y$. \square

It is interesting to note that the two key relations shown in Proposition 3.19 are precisely the conditions that appear in the definition of an admissible twist (see Definition 2.11) and are intrinsic properties of the underlying associative algebra. That is, the decorated p -rack algebra guarantees the existence of an admissible twist and hence the existence of a generic invertible set-theoretic solution.

Proposition 3.20. (Algebra homomorphism) *Let $\hat{\mathcal{Q}}$ be the decorated p -rack algebra and $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be a solution of the Yang-Baxter equation. We also define for $z_{i,j,k} \in Y$, $\Delta'_{z_{ij}} : \mathcal{Q} \rightarrow \mathcal{Q} \otimes \mathcal{Q}$, such that for all $a \in X$,*

$$\Delta'_{z_{jk}}((y_a^{z_{ik}})^{\pm 1}) := (y_a^{z_{ij}})^{\pm 1} \otimes (y_a^{z_{ik}})^{\pm 1}, \quad \Delta'_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \triangleright_{z_{ij}} c = a}, \quad y_a^{z_{ik}} \in \{q_a^{z_{ik}}, w_a^{z_{ik}}\}.$$

Then the following statements hold:

- (1) $\Delta'_{z_{ij}}$ is a $\hat{\mathcal{Q}}$ algebra homomorphism for all $z_{i,j} \in Y$.
- (2) $\mathcal{R}^{z_{jk}} \Delta'_{z_{jk}}(y_a^{z_{ik}}) = \Delta'_{z_{kj}}^{(op)}(y_a^{z_{ik}}) \mathcal{R}^{z_{jk}}$, for $y_a^{z_{ik}} \in \{q_a^{z_{ik}}, w_a^{z_{ik}}\}$, $a \in X$, $z_{i,j,k} \in Y$. Recall, $\Delta'_{z_{ij}}^{(op)} := \pi \circ \Delta'_{z_{ij}}$, where π is the flip map.

Proof. Bearing in mind Proposition 3.7 it suffices to show:

- (1) (i) $\Delta'_{z_{ij}}(w_a^{z_{kj}}) \Delta'_{z_{ij}}(w_b^{z_{lj}}) = \Delta'_{z_{ij}}(w_{\sigma_a^{z_{lk}}(b)}^{z_{lj}}) \Delta'_{z_{ij}}(w_{\tau_b^{z_{lk}}(a)}^{z_{kj}})$,
- (ii) $\Delta'_{z_{ij}}(w_a^{z_{kj}}) \Delta'_{z_{ij}}(q_a^{z_{lj}}) = \Delta'_{z_{ij}}(q_{\sigma_a^{z_{lk}}(b)}^{z_{lj}}) \Delta'_{z_{ij}}(w_a^{z_{kj}})$,
- (iii) $\Delta'_{z_{ij}}(w_a^{z_{kj}}) \Delta'_{z_{ij}}(h_b) = \Delta'_{z_{ij}}(h_{\sigma_a^{z_{jk}}(b)}) \Delta'_{z_{ij}}(w_a^{z_{kj}})$ and (2) $\mathcal{R}^{z_{jk}} \Delta'_{z_{jk}}(w_a^{z_{ik}}) = \Delta'_{z_{kj}}^{(op)}(w_a^{z_{ik}}) \mathcal{R}^{z_{jk}}$.

- (1) First it is straightforward, via the algebraic relations of the decorated p -rack algebra and the form of the coproducts to show (i), (ii). To show (iii) we use in addition the condition, $\sigma_c^{z_{ik}}(b) \triangleright_{z_{ij}} \sigma_c^{z_{jk}}(a) = \sigma_c^{z_{jk}}(b \triangleright_{z_{ij}} a)$ (see Proposition 3.19).
- (2) This part is also shown directly by means of the relations of the p -decorated algebra. \square

Definition 3.21. (The p -set Yang-Baxter algebra.) Let \mathcal{Q} be the restricted p -rack algebra. Let also $\sigma_a^{z_{ij}}, \tau_b^{z_{ij}} : X \rightarrow X$, and $\sigma_a^{z_{ij}}$ be bijective for all $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\hat{\mathcal{Q}}$ over k , generated by indeterminates $q_a^{z_{ij}}, (q_a^{z_{ij}})^{-1}, h_a \in \mathcal{Q}$ ($h_a = h_b \Leftrightarrow a = b$), $1_{\hat{\mathcal{Q}}}$ (unit element), $w_a^{z_{ij}}, (w_a^{z_{ij}})^{-1} \in \hat{\mathcal{Q}}$, for $a \in X$, $z_{i,j} \in Y$ and relations, (3.25) is a p -set Yang-Baxter algebra.

Proposition 3.22. Let $\hat{\mathcal{Q}}$ be the p -set Yang-Baxter algebra and $\mathcal{R}^{z_{ij}} = \sum_{b \in X} h_b \otimes q_b^{z_{ij}} \in \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$ is a solution of the Yang-Baxter equation. Let also for all $a, b, c \in X$, $z_{i,j,k} \in Y : (b \triangleright_{z_{ij}} a) \bullet_{z_{jk}} (b \triangleright_{z_{ik}} c) = b \triangleright_{z_{ik}} (a \bullet_{z_{jk}} c)$ and

$$\sigma_c^{z_{ik}}(a) \bullet_{z_{ij}} \sigma_c^{z_{jk}}(b) = \sigma_c^{z_{jk}}(a \bullet_{z_{ij}} b) \quad (3.26)$$

We also define for $z_{i,j,k} \in Y$, $\Delta_{z_{ij}} : \hat{\mathcal{Q}} \rightarrow \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$, such that for all $a \in X :$

$$\Delta_{z_{ij}}(q_a^{z_{kj}}) := q_a^{z_{ki}} \otimes q_a^{z_{kj}}, \quad \Delta_{z_{ij}}(h_a) := \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet_{z_{ij}} c = a}, \quad \Delta_{z_{ij}}(w_a^{z_{kj}}) = w_a^{z_{ki}} \otimes w_a^{z_{kj}}.$$

Then the following statements hold:

- (1) $\Delta_{z_{ij}}$ is an algebra homomorphism for all $z_{i,j} \in Y$.
- (2) $\Delta_{z_{ji}}^{(op)}(y) \mathcal{R}^{z_{ij}} = \mathcal{R}^{z_{ij}} \Delta_{z_{ij}}(y)$, $y \in \{h_a, q_a^{kj}, w_a^{kj}\}$, for all $a \in X$, $z_{i,j,k} \in Y$, recall also that $\Delta_{z_{ij}}^{(op)} := \pi \circ \Delta_{z_{ij}}$, where π is the flip map.

Proof. In our proof below we are using the definition of the p -set Yang-Baxter algebra and (3.26).

- (1) Recalling part 1 of Theorem 3.10 it is sufficient to check the consistency of the following algebraic relations using (3.26), for all $a, b \in X$, $z_{i,j,k,l} \in Y :$

$$\begin{aligned} \Delta_{z_{ij}}(w_a^{z_{lj}}) \Delta_{z_{ij}}(w_b^{z_{kj}}) &= \Delta_{z_{ij}}(w_{\sigma_a^{z_{kl}}(b)}^{z_{kj}}) \Delta_{z_{ij}}(w_{\tau_b^{z_{kl}}(a)}^{z_{lj}}), \\ \Delta_{z_{ij}}(w_a^{z_{kj}}) \Delta_{z_{ij}}(h_b) &= \Delta_{z_{ij}}(h_{\sigma_a^{z_{jk}}(b)}) \Delta_{z_{ij}}(w_a^{z_{kj}}). \\ \Delta_{z_{ij}}(w_a^{z_{lj}}) \Delta_{z_{ij}}(q_b^{z_{kj}}) &= \Delta_{z_{ij}}(q_{\sigma_a^{z_{kl}}(b)}^{z_{kj}}) \Delta_{z_{ij}}(w_a^{z_{lj}}). \end{aligned}$$

By using the algebra $\hat{\mathcal{Q}}$ relations and the form of the coproducts we show by direct computation that the above equations hold. In order to show the second of the three equalities above we also use identity (3.26).

- (2) Given Theorem 3.10 it suffices to show that for all $a \in X$, $z_{i,j,k} \in Y$, $\Delta_{z_{ij}}(w_a^{z_{kj}}) \mathcal{R}^{z_{ij}} = \mathcal{R}^{z_{ij}} \Delta_{z_{ij}}(w_a^{z_{kj}})$. Indeed, this is shown by a direct computation using the algebraic relations of Definition 3.17. \square

Lemma 3.23. (*Parametric (co)-associativity.*) Let $\hat{\mathcal{Q}}$ be the decorated p -rack algebra and consider the subalgebra $\hat{\mathcal{Q}}^-$ consisting of the elements $1_{\hat{\mathcal{Q}}}$, $(q_a^{z_{ij}})^{\pm 1}$, $(w_a^{z_{ij}})^{\pm 1}$. We also define for all $z_{i,1,2,\dots,n} \in Y$, $a \in X$, and $y_a^{z_{ij}} = \{q_a^{z_{ij}}, w_a^{z_{ij}}\}$, the map $\Delta_{z_{12\dots n}}^{(n)} : \hat{\mathcal{Q}} \rightarrow \hat{\mathcal{Q}}^{\otimes n}$, such that

$$\Delta_{z_{12\dots n}}^{(n)}((y_a^{z_{in}})^{\pm 1}) := (y_a^{z_{i1}})^{\pm 1} \otimes (y_a^{z_{i2}})^{\pm 1} \otimes \dots \otimes (y_a^{z_{in}})^{\pm 1}. \quad (3.27)$$

Then, for all $z_{i,1,2,\dots,n} \in Y$,

- (1) $\Delta_{z_{12\dots n}}^{(n)}((y_a^{z_{in}})^{\pm 1}) := (\Delta_{z_{12\dots n-1}}^{(n-1)} \otimes id)\Delta_{z_{n-1n}}((y_a^{z_{in}})^{\pm 1}) = (id \otimes \Delta_{z_{23\dots n}}^{(n-1)})\Delta_{z_{1n}}((y_a^{z_{in}})^{\pm 1})$, $a \in X$.
- (2) $\Delta_{z_{12\dots n}}^{(n)}$ is a $\hat{\mathcal{Q}}^-$ algebra homomorphism.

Proof. The proof of both (1) and (2) follows directly from the form of the n -coproduct and the algebraic relations 3.1. \square

We come now to the derivation of the admissible Drinfel'd twist that will be used to provide the universal \mathcal{R} -matrix associated to the general set-theoretic solution. The n -fold twist will be also explicitly derived. It is useful to introduce some practical notation that can be applied in the following propositions: let $i, j, k \in \{1, 2, 3\}$, then $\mathcal{F}_{jik}^{z_{ijk}} = \pi_{ij} \circ \mathcal{F}_{ijk}^{z_{ijk}}$ and $\mathcal{F}_{ikj}^{z_{ijk}} = \pi_{jk} \circ \mathcal{F}_{ijk}^{z_{ijk}}$, where π is the flip map.

Theorem 3.24. (*Drinfel'd twist [25]*) Let $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \mathcal{Q} \otimes \mathcal{Q}$ be the p -rack universal \mathcal{R} -matrix. Let also $\hat{\mathcal{Q}}$ be the decorated p -rack algebra and $\mathcal{F}^{z_{ij}} \in \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$, such that $\mathcal{F}^{z_{ij}} = \sum_{b \in X} h_b \otimes (w_b^{z_{ij}})^{-1}$, for all $z_{i,j} \in Y$ and $\mathcal{F}_{ji}^{z_{ij}} \mathcal{R}_{ij}^{z_{ij}} = \mathcal{R}_{ij}^{Fz_{ij}} \mathcal{F}_{ij}^{z_{ij}}$. We also define:

$$\mathcal{F}_{1,23}^{z_{123}} := \sum_{a \in X} h_a \otimes (w_a^{z_{12}})^{-1} \otimes (w_a^{z_{13}})^{-1}, \quad \mathcal{F}_{12,3}^{*z_{123}} := \sum_{a,b \in X} h_a \otimes h_{\sigma_a^{z_{21}}(b)} \otimes (w_b^{z_{23}})^{-1} (w_a^{z_{13}})^{-1}. \quad (3.28)$$

Let also for every $a, b \in X$, $z_{i,j} \in Y$, $b \triangleright_{z_{ij}} a = \sigma_b^{z_{ji}}(\tau_{(\sigma_a^{z_{ij}})^{-1}(b)}^{z_{ij}}(a))$. Then, the following statements are true for $z_{1,2,3} \in Y$:

- (1) $\mathcal{F}_{12}^{z_{12}} \mathcal{F}_{12,3}^{*z_{123}} = \mathcal{F}_{23}^{z_{23}} \mathcal{F}_{1,23}^{z_{123}} =: \mathcal{F}_{123}^{z_{123}}$.
- (2) (i) $\mathcal{F}_{ikj}^{z_{ijk}} \mathcal{R}_{jk}^{z_{jk}} = \mathcal{R}_{jk}^{Fz_{jk}} \mathcal{F}_{ijk}^{z_{ijk}}$.
- (ii) $\mathcal{F}_{jik}^{z_{ijk}} \mathcal{R}_{ij}^{z_{ij}} = \mathcal{R}_{ij}^{Fz_{ij}} \mathcal{F}_{ijk}^{z_{ijk}}$.

The element $\mathcal{F}^{z_{ij}}$ is called an admissible Drinfel'd twist.

Proof. The proof is straightforward based on the underlying algebra $\hat{\mathcal{Q}}$.

- (1) Indeed, this is proved by a direct computation and use of the decorated rack algebra. In fact, $\mathcal{F}_{123}^{z_{123}} = \sum_{a,b \in X} h_a \otimes h_b (w_a^{z_{12}})^{-1} \otimes (w_b^{z_{23}})^{-1} (w_a^{z_{13}})^{-1}$.
- (2) Given the notation introduced before the theorem it suffices to show that $\mathcal{F}_{132}^{z_{132}} \mathcal{R}_{23}^{z_{23}} = \mathcal{R}_{23}^{Fz_{23}} \mathcal{F}_{123}^{z_{123}}$ and $\mathcal{F}_{213}^{z_{213}} \mathcal{R}_{12}^{z_{12}} = \mathcal{R}_{12}^{Fz_{12}} \mathcal{F}_{123}^{z_{123}}$.
- (i) Due to the fact that for all $a \in X$, $z_{i,j} \in Y$, $\Delta_{z_{ij}}(w_a^{z_{ki}}) \mathcal{R}^{z_{ij}} = \mathcal{R}^{z_{ij}} \Delta_{z_{ij}}(w_a^{z_{ki}})$ (see

Proposition 3.22) we arrive at $\mathcal{F}_{1,32}^{z_{132}} \mathcal{R}_{23}^{z_{23}} = \mathcal{R}_{23}^{z_{23}} \mathcal{F}_{1,23}^{z_{123}}$, then

$$\mathcal{F}_{132}^{z_{132}} \mathcal{R}_{23}^{z_{23}} = \mathcal{F}_{32}^{z_{32}} \mathcal{F}_{1,32}^{z_{132}} \mathcal{R}_{23}^{z_{23}} = \mathcal{F}_{32}^{z_{32}} \mathcal{R}_{23}^{z_{23}} \mathcal{F}_{1,23}^{z_{123}} = \mathcal{R}_{23}^{Fz_{23}} \mathcal{F}_{123}^{z_{123}}.$$

(ii) By using the relations of the decorated p -rack algebra $\hat{\mathcal{Q}}$ we compute:

$$\mathcal{F}_{21,3}^{*z_{213}} \mathcal{R}_{12}^{z_{12}} = \sum_{a,c \in X} h_a \otimes q_a^{z_{12}} h_{a \triangleright_{z_{12}} c} \otimes (w_c^{z_{23}} w_{(\sigma_c^{z_{12}})^{-1}(a)})^{-1},$$

$$\mathcal{R}_{12}^{z_{12}} \mathcal{F}_{12,3}^{*z_{123}} = \sum_{a,b \in X} h_a \otimes q_a^{z_{12}} h_{\sigma_a^{z_{21}}(b)} \otimes (w_a^{z_{13}} w_b^{z_{12}})^{-1}.$$

Using also, $b \triangleright_{z_{12}} a = \sigma_b^{z_{21}}(\tau_{(\sigma_a^{z_{12}})^{-1}(b)}(a))$ and $w_a^{z_{kj}} w_b^{z_{ij}} = w_{\sigma_a^{z_{ik}}(b)}^{z_{ij}} w_{\tau_b^{z_{ik}}(a)}^{z_{kj}}$ we conclude that $\mathcal{F}_{21,3}^{*z_{213}} \mathcal{R}_{12}^{z_{12}} = \mathcal{R}_{12} \mathcal{F}_{12,3}^{*z_{123}}$ and consequently (recall $\mathcal{F}_{213}^{z_{213}} = \mathcal{F}_{21}^{z_{21}} \mathcal{F}_{21,3}^{*z_{213}}$)

$$\mathcal{F}_{213}^{z_{213}} \mathcal{R}_{12}^{z_{12}} = \mathcal{F}_{21}^{z_{21}} \mathcal{F}_{21,3}^{*z_{213}} \mathcal{R}_{12}^{z_{12}} = \mathcal{F}_{21}^{z_{21}} \mathcal{R}_{12}^{z_{12}} \mathcal{F}_{12,3}^{*z_{123}} = \mathcal{R}_{12}^{Fz_{12}} \mathcal{F}_{123}^{z_{123}}. \quad \square$$

Corollary 3.25. [25, 26] *Let Y be a non-empty set and $z_{i,j} \in Y$. Let also $\mathcal{F}^{z_{ij}}$ be an admissible twist and $\mathcal{R}^{z_{ij}}$ be a solution of the Yang-Baxter equation. Then $\mathcal{R}^{Fz_{ij}} := (\mathcal{F}^{z_{ji}})^{(op)} \mathcal{R}^{z_{ij}} (\mathcal{F}^{z_{ij}})^{-1}$ ($(\mathcal{F}^{z_{ji}})^{(op)} = \pi \circ \mathcal{F}^{z_{ji}}$, π is the flip map) is also a solution of the Yang-Baxter equation.*

Proof. The proof is quite straightforward, [25, 26] (see also proof in [19] for set-theoretic solutions), we just give a brief outline here: if $\mathcal{F}^{z_{ij}}$ is admissible, then from the Yang-Baxter equation and due to Proposition 3.24, $z_{1,2,3} \in Y$:

$$\mathcal{F}_{321}^{z_{321}} \mathcal{R}_{12}^{z_{12}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{23}^{z_{23}} = \mathcal{F}_{321}^{z_{321}} \mathcal{R}_{23}^{z_{23}} \mathcal{R}_{13}^{z_{13}} \mathcal{R}_{12}^{z_{12}} \Rightarrow \mathcal{R}_{12}^{Fz_{12}} \mathcal{R}_{13}^{Fz_{13}} \mathcal{R}_{23}^{Fz_{23}} \mathcal{F}_{123}^{z_{123}} = \mathcal{R}_{23}^{Fz_{23}} \mathcal{R}_{13}^{Fz_{13}} \mathcal{R}_{12}^{Fz_{12}} \mathcal{F}_{123}^{z_{123}}.$$

But $\mathcal{F}_{123}^{z_{123}}$ is invertible, hence $\mathcal{R}^{Fz_{ij}}$ indeed satisfies the YBE. \square

Lemma 3.26. (The n -fold twist.) *Let $\hat{\mathcal{Q}}$ be the decorated p -rack algebra. Let also $\mathcal{R}^{z_{ij}} = \sum_{a \in X} h_a \otimes q_a^{z_{ij}} \in \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$ be a solution of the Yang-Baxter equation and $\mathcal{F}^{z_{ij}} \in \hat{\mathcal{Q}} \otimes \hat{\mathcal{Q}}$, such that $\mathcal{F}^{z_{ij}} = \sum_{a \in X} h_a \otimes (w_a^{z_{ij}})^{-1}$, $z_{i,j} \in X$. Define also for all $z_{1,2,\dots,n} \in Y$:*

$$\mathcal{F}_{1,23\dots n}^{z_{12\dots n}} := \sum_{a \in X} h_a \otimes \Delta_{z_{2\dots n}}^{(n-1)}((w_a^{z_{12}})^{-1}) = \sum_{a \in X} h_a \otimes (w_a^{z_{12}})^{-1} \otimes (w_a^{z_{13}})^{-1} \otimes \dots \otimes (w_a^{z_{1n}})^{-1},$$

$$\begin{aligned} \mathcal{F}_{12\dots n-1,n}^{*z_{12\dots n}} &:= \sum_{a_1, a_2, \dots, a_{n-1} \in X} h_{a_1} \otimes h_{\sigma_{a_1}^{z_{21}}(a_2)} \otimes h_{\sigma_{a_1}^{z_{31}}(\sigma_{a_2}^{z_{32}}(a_3))} \otimes \dots \\ &\otimes h_{\sigma_{a_1}^{z_{n-11}}(\sigma_{a_2}^{z_{n-12}}(\dots \sigma_{a_{n-2}}^{z_{n-1,n-2}}(a_{n-1}))\dots)} \otimes (w_{a_{n-1}}^{z_{n-1n}})^{-1} (w_{a_{n-2}}^{z_{n-2n}})^{-1} \dots (w_{a_1}^{z_{1n}})^{-1} \end{aligned}$$

Then,

$$(1) \mathcal{F}_{2\dots n}^{z_{2\dots n}} \mathcal{F}_{1,2\dots n}^{z_{12\dots n}} = \mathcal{F}_{12\dots n-1}^{z_{12\dots n-1}} \mathcal{F}_{12\dots n-1,n}^{*z_{12\dots n}} =: \mathcal{F}_{12\dots n}^{z_{12\dots n}}.$$

(2) *The explicit expression of the n -fold twist is given as*

$$\begin{aligned} \mathcal{F}_{12\dots n}^{z_{12\dots n}} &= \sum_{a_1, a_2, \dots, a_{n-1} \in X} h_{a_1} \otimes h_{a_2} (w_{a_1}^{z_{12}})^{-1} \otimes h_{a_3} (w_{a_2}^{z_{23}})^{-1} (w_{a_1}^{z_{13}})^{-1} \otimes \dots \otimes \\ &h_{a_{n-1}} (w_{a_{n-2}}^{z_{n-2,n-1}})^{-1} \dots (w_{a_1}^{z_{1,n-1}})^{-1} \otimes (w_{a_{n-1}}^{z_{n-1n}})^{-1} (w_{a_{n-2}}^{z_{n-2n}})^{-1} \dots (w_{a_1}^{z_{1n}})^{-1}. \quad (3.29) \end{aligned}$$

$$(3) \mathcal{F}_{1,23\dots j+1j\dots n}^{z_{12\dots j+1j\dots n}} \mathcal{R}_{jj+1}^{z_{jj+1}} = \mathcal{R}_{jj+1}^{z_{jj+1}} \mathcal{F}_{1,23\dots jj+1\dots n}^{z_{12\dots jj+1\dots n}}, \quad n-1 \geq j > 1,$$

$$\mathcal{F}_{12\dots j+1j\dots n-1,n}^{z_{12\dots j+1j\dots n-1,n}} \mathcal{R}_{jj+1}^{z_{jj+1}} = \mathcal{R}_{jj+1}^{z_{jj+1}} \mathcal{F}_{12\dots jj+1\dots n-1,n}^{z_{12\dots jj+1\dots n-1,n}}, \quad n-1 > j \geq 1,$$

$$\mathcal{F}_{12\dots j+1j\dots n}^{z_{12\dots j+1j\dots n}} \mathcal{R}_{jj+1} = \mathcal{R}_{jj+1}^{F_{jj+1}} \mathcal{F}_{12\dots jj+1\dots n}^{z_{12\dots jj+1\dots n}}, \quad n-1 \geq j \geq 1.$$

Proof. These statements are proven by iteration and direct computation by using the $\hat{\mathcal{Q}}$ algebra relations. Part (2) of Theorem 3.24 is also used in proving (3). \square

Corollary 3.27. *Let $\mathcal{F}_{12\dots n}^{z_{12\dots n}}$ be the n -fold twist 3.29. Let also (X, \circ) be a group, $z_i \circ z_j = z_j \circ z_i$ and $w_a^{z_{jk}} w_b^{z_{ik}} = w_{aob}^{z_{iojk}}$ for all $a, b \in X$, $z_{i,j,k} \in Y$, where z_{iojk} denotes dependence on (z_{ioj}, z_k) . Then for $z_{1,2,\dots,n} \in Y$,*

$$\mathcal{F}_{12\dots n}^{z_{12\dots n}} = \sum_{a_1, \dots, a_n \in X} h_{a_1} \otimes h_{a_2} (w_a^{z_{12}})^{-1} \otimes \dots \otimes h_{a_{n-1}} (w_{a_1 \circ a_2 \circ \dots \circ a_{n-1}}^{z_{1 \circ 2 \circ \dots \circ n-2n-1}})^{-1} \otimes (w_{a_1 \circ a_2 \circ \dots \circ a_n}^{z_{1 \circ 2 \circ \dots \circ n-1n}})^{-1}.$$

Proof. This is a consequence of the form of the n -twist (3.29) and relation $w_a^{z_{jk}} w_b^{z_{ik}} = w_{aob}^{z_{iojk}}$ for all $a, b \in X$, $z_{i,j,k} \in Y$. \square

Remark 3.28. *(Twisted universal \mathcal{R} -matrix) We recall the admissible twist $\mathcal{F}^{z_{12}} = \sum_{b \in X} h_b \otimes (w_b^{z_{12}})^{-1}$, and the universal p -rack \mathcal{R} -matrix is $\mathcal{R}^{z_{12}} = \sum_{a \in X} h_a \otimes q_a^{z_{12}}$, then we obtain:*

- The twisted \mathcal{R} -matrix:

$$\mathcal{R}^{F_{z_{12}}} = (\mathcal{F}^{z_{21}})^{(op)} \mathcal{R}^{z_{12}} (\mathcal{F}^{z_{12}})^{-1}$$

- The twisted coproducts: for $z_{12} \in Y$, $\Delta_{z_{12}}^F(y) = \mathcal{F}^{z_{12}} \Delta_{z_{12}}(y) (\mathcal{F}^{z_{12}})^{-1}$, $y \in \hat{\mathcal{Q}}$ and we recall, for $a \in X$, $z_{i,1,2} \in Y$

$$\Delta_{z_{12}}(w_a^{z_{i2}}) = w_a^{z_{i1}} \otimes w_a^{z_{i2}}, \quad \Delta_{z_{12}}(h_a) = \sum_{b,c \in X} h_b \otimes h_c \Big|_{b \bullet_{z_{12}} c = a}, \quad \Delta_{z_{12}}(q_a^{z_{i2}}) = q_a^{z_{i1}} \otimes q_a^{z_{i2}}.$$

Moreover it follows that $\mathcal{R}^{F_{z_{21}}} \Delta_{z_{12}}^F(y) = \Delta_{z_{12}}^{F(op)}(y) \mathcal{R}^{F_{z_{12}}}$, $y \in \hat{\mathcal{Q}}$, $z_{1,2} \in Y$.

Remark 3.29. Fundamental representation & the set-theoretic solution:

Let $\hat{\mathcal{Q}}$ be the p -set Yang-Baxter algebra and $\rho : \hat{\mathcal{Q}} \rightarrow \text{End}(V)$, such that

$$q_a^{z_{ij}} \mapsto \sum_{x \in X} e_{x, a \triangleright_{z_{ij}} x}, \quad h_a \mapsto e_{a,a}, \quad w_a^{z_{ij}} \mapsto \sum_{b \in X} e_{\sigma_a^{z_{ji}}(b), b}, \quad (3.30)$$

then $\mathcal{R}^{F_{z_{ij}}} \mapsto R^{F_{z_{ij}}} = \sum_{a,b \in X} e_{b, \sigma_a^{z_{ij}}(b)} \otimes e_{a, \tau_b^{z_{ij}}(a)}$, where we recall that

$\tau_b^{z_{ij}}(a) := \sigma_{(\sigma_a^{z_{ij}})^{-1}(b)}^{z_{ji}}(\sigma_a^{z_{ij}}(b) \triangleright_{z_{ij}} a)$. We note that $R^{F_{z_{ij}}}$ is the linearized version of the set-theoretic solution.

The n -fold twist 3.29 in the fundamental representation becomes:

$$\begin{aligned} \mathcal{F}_{12\dots n}^{z_{12}\dots z_n} \mapsto F_{12\dots n}^{z_{12}\dots z_n} &= \sum_{a_1, \dots, a_n \in X} e_{a_1, a_1} \otimes e_{a_2, \sigma_{a_1}^{z_{21}}(a_2)} \otimes \dots \otimes \\ &e_{a_{n-1}, \sigma_{a_1}^{z_{n-11}}(\sigma_{a_2}^{z_{n-12}}(\dots \sigma_{a_{n-2}}^{z_{n-1n-2}}(a_{n-1}) \dots))} \otimes e_{a_n, \sigma_{a_1}^{z_{n1}}(\sigma_{a_2}^{z_{n2}}(\dots \sigma_{a_{n-1}}^{z_{nn-1}}(a_n) \dots))}. \end{aligned} \quad (3.31)$$

In the absence of parameters (non-parametric case) expression (3.31) reduces to the expression derived in [19] and should also coincide with the linearized version of the non-local transformation introduced in [54].

Let the universal twisted \mathcal{R} -matrix of Remark 3.28 expressed in the compact form $\mathcal{R}^{Fz_{ij}} = \sum_{a, b \in X} f_{b,a}^{z_{ij}} \otimes g_{a,b}^{z_{ij}}$, then from the Yang-Baxter equation, and after recalling the representations (3.30): $(\rho \otimes \text{id})\mathcal{R}^{Fz_{ij}} := L^{Fz_{ij}} = \sum_{a \in X} e_{b, \sigma_a^{z_{ij}}(b)} \otimes g_{a,b}^{z_{ij}}$, $(\text{id} \otimes \rho)\mathcal{R}^{Fz_{ij}} := \hat{L}^{Fz_{ij}} = \sum_{a \in X} f_{b,a}^{z_{ij}} \otimes e_{a, \tau_b^{z_{ij}}(a)}$, and $(\rho \otimes \rho)\mathcal{R}^{Fz_{ij}} := R^{Fz_{ij}} = \sum_{a, b \in X} e_{b, \sigma_a^{z_{ij}}(b)} \otimes e_{a, \tau_b^{z_{ij}}(a)}$, the consistent algebraic relations (3.5) are satisfied (the interested reader is also referred to [21] for detailed computations). These lead to the p -set Yang-Baxter algebra and provide a consistency check on the associated algebraic relations.

Example 3.30. Recall the fundamental representation of the p -set Yang-Baxter algebra $\hat{\mathcal{Q}}$ (recall also Example 3.16), $\rho : \hat{\mathcal{Q}} \rightarrow \text{End}(V)$, $w_a^{z_{ij}} \mapsto \omega_a^{z_{ij}} := \sum_{b \in X} e_{\sigma_a^{z_{ji}}(b), b}$, $a \in X$, $z_{i,j} \in Y$, then:

$$\omega_a^{z_{jk}} \omega_b^{z_{ik}} = \sum_{c \in X} e_{\sigma_a^{z_{kj}}(\sigma_b^{z_{ki}}(c)), c}. \quad (3.32)$$

Recall also the map for $z_{i,j} \in Y$, $\sigma_a^{z_{ij}} : X \rightarrow X$, $\sigma_a^{z_{ij}}(b) = z_i^{-1} - a \circ z_i^{-1} \circ z_j + a \circ b \circ z_j$ and

$$a \circ b = \sigma_a^{z_{ij}}(b) \circ \tau_b^{z_{ij}}(a) \quad \text{and} \quad \sigma_a^{z_{kj}}(\sigma_b^{z_{ki}}(c)) = \sigma_{a \circ b}^{z_{kioj}}(c), \quad (3.33)$$

then via (3.32), we conclude that $\omega_a^{z_{jk}} \omega_b^{z_{ik}} = \omega_{a \circ b}^{z_{iojk}} := \omega_{\sigma_a^{z_{ij}}(b)}^{z_{ik}} \omega_{\tau_b^{z_{ij}}(a)}^{z_{jk}}$, where recall the shorthand notation z_{iojk} denotes dependence on $(z_i \circ z_j, z_k)$.

The n -fold twist in this case becomes (see also related Corollary 3.27):

$$F_{12\dots n}^{z_{12}\dots z_n} = \sum_{a_1, \dots, a_n \in X} e_{a_1, a_1} \otimes e_{a_2, \sigma_{a_1}^{z_{21}}(a_2)} \otimes \dots \otimes e_{a_{n-1}, \sigma_{a_1}^{z_{n-11} \circ a_2 \circ \dots \circ a_{n-2}}(a_{n-1})} \otimes e_{a_n, \sigma_{a_1}^{z_{n1} \circ a_2 \circ \dots \circ a_{n-1}}(a_n)}.$$

The special p -set algebra. Recall the decorated p -rack algebra and consider the special case, where for all $a, b \in X$, $z_{i,j} \in Y$ $a \triangleright_{z_{ij}} b = b$, and consequently $\sigma_{\sigma_a^{z_{ij}}(b)}^{z_{ji}}(\tau_b^{z_{ij}}(a)) = a$. Let also, $q_a^{z_{ij}} = 1_{\hat{\mathcal{Q}}}$, then the decorated p -rack algebra reduces to the *special p -set algebra*.

Definition 3.31. Let $\sigma_a^{z_{ij}}, \tau_b^{z_{ij}} : X \rightarrow X$, and $\sigma_a^{z_{ij}}$ be a bijection for all $a \in X$, $z_{i,j} \in Y$. We say that the unital, associative algebra $\hat{\mathcal{Q}}$ over k , generated by indeterminates $h_a, w_a^{z_{ij}}, (w_a^{z_{ij}})^{-1} \in \hat{\mathcal{Q}}$, $a \in X$, $1_{\hat{\mathcal{Q}}}$ (the unit element) and relations, for $a, b \in X$, $z_{i,j,k} \in Y$:

$$h_a h_b = \delta_{a,b} h_a, \quad w_a^{z_{ij}} (w_a^{z_{ij}})^{-1} = 1_{\hat{\mathcal{Q}}}, \quad w_a^{z_{ki}} w_b^{z_{ji}} = w_{\sigma_a^{z_{jk}}(b)}^{z_{ji}} w_{\tau_b^{z_{kj}}(a)}^{z_{ki}} \quad w_a^{z_{ji}} h_b = h_{\sigma_a^{z_{ij}}(b)} w_a^{z_{ji}}, \quad (3.34)$$

is a special p -set algebra.

In this case the p -rack universal \mathcal{R} -matrix reduces to the identity map, $\mathcal{R}^{z_{ij}} = \text{id}$, and the twisted \mathcal{R} -matrix is

$$\mathcal{R}^{Fz_{12}} = (\mathcal{F}^{z_{21}})^{(op)}(\mathcal{F}^{z_{12}})^{-1} = \sum_{a,b \in X} (w_a^{z_{ji}})^{-1} h_b \otimes h_a w_b^{z_{ij}}.$$

This is an reversible solution as it satisfies, $\mathcal{R}_{12}^{Fz_{12}} \mathcal{R}_{21}^{Fz_{21}} = \text{id}$. In the fundamental representation (Remark 3.29), $\mathcal{R}^{Fz_{ij}} \mapsto R^{Fz_{ij}} = \sum_{a,b \in X} e_{b, \sigma_a^{z_{ij}}(b)} \otimes e_{a, \tau_b^{z_{ij}}(a)}$, i.e. it reduces to the linearized version of the reversible set-theoretic solution.

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