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Monotonicity of savings function in Endogenous Gridpoint Method with stochastic portfolio returns

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ABSTRACT

This paper provides a comprehensive proof of monotonicity of the savings function in the application of the Method of Endogenous Gridpoints (EGM) to problems with stochastic portfolio returns. The proof contributes to the completeness of solutions by providing the sufficient condition for the application of EGM to problems with stochastic portfolio returns as seen in the literature.

1. Introduction

The method of endogenous gridpoints (EGM), introduced by Carroll (2006), provides significant computational efficiency in solving dynamic optimization problems compared to the traditional value function iteration methods (Iskhakov, 2015). These computational benefits have seen widespread use of EGM in the literature, including problems that involve stochastic asset returns. For instance, Love (2013) consider a dynamic model with stochastic asset returns and income shocks to solve for optimal consumption and portfolio allocation policy functions. Other works include Koijen et al. (2010) and Wu et al. (2023).

The monotonicity of pre-decision and post-decision state variables is a pivotal sufficient condition for the application of EGM (White, 2015). However, the application of EGM in problems with stochastic portfolio returns in the literature fails to provide a comprehensive monotonicity check. This omission represents a critical gap in the existing literature and may lead to potential inaccuracies in derived solutions (Iskhakov et al., 2017). Establishing a rigorous monotonicity proof not only strengthens the robustness of the method but also extends its applicability, offering insights that may facilitate further advancements in computational efficiency and solution accuracy (Fella, 2014). Therefore, this paper aims to provide the proofs for the monotonicity relationship between pre-decision and post-decision state variables when portfolio returns are stochastic.

The remainder of this paper is structured as follows. Section 2 introduces the specific problem setting and presents the corresponding monotonicity proofs. Section 3 discusses the key conclusions drawn from the proofs as well as suggestions for further research.

2. Monotonicity in stochastic dynamic optimization problems

Iskhakov et al. (2017) provide the proof of monotonicity of the savings function for dynamic optimization problems with deterministic portfolio returns. However, the literature has not extended this type of proof to include stochastic investment returns, a gap that we aim to fill. In Section 2.1 we formulate a representative problem with stochastic investment returns, derive the optimal policy functions and highlight the need for monotonicity in the EGM application. Section 2.2 then provides the proof of monotonicity.

2.1. An illustrative problem and the need for monotonicity

Consider a discrete-time, finite-horizon dynamic optimization problem with one state variable of pre-decision wealth M_t, for $t = 0, 1, \dots, T$, and two continuous decision variables: consumption choice C_t and proportion of portfolio allocated to the risky asset π_t, for $t = 0, 1, \dots, T-1$. Let the constant risk-free return and the stochastic risky asset return be designated as r and $\tilde{R}_{t+1}$, respectively, for $t = 0, 1, \dots, T-1$. The risky asset's return generates a natural filtration, denoted as $\mathbb{F} = \left\{\mathcal{F}_t\right\}_{t=0}^{T}$, where $\mathcal{F}_0$ is trivial. In particular, π_t acts as a dynamic decision variable constrained within $\mathbb{F}$-adapted bounds $\underline{\pi}_t$ and $\tilde{\pi}_t$ (i.e., $\pi_t \in [\underline{\pi}_t, \tilde{\pi}_t]$). For example, $\underline{\pi}_t$ and $\tilde{\pi}_t$ can be set deterministically as 0 and 1, respectively. The portfolio's investment return, R_{t+1}, for $t = 0, 1, \dots, T-1$, is then calculated as $\pi_t \times \tilde{R}_{t+1} + (1 - \pi_t) \times r$. Alongside the consumption choice C_t, constrained within $\mathbb{F}$-adapted bounds $\underline{C}_t$ and $\tilde{C}_t$ (i.e., $C_t \in [\underline{C}_t, \tilde{C}_t]$; for example, $\underline{C}_t = 0$ and $\tilde{C}_t = M_t$), the wealth dynamic is given by $M_{t+1} = (M_t - C_t) \times (1 + \pi_t \times \tilde{R}_{t+1} + (1 - \pi_t) \times r)$, for $t = 0, 1, \dots, T-1$.

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Furthermore, we assume that the consumption bounds $\underline{C}_t(M_t), \tilde{C}_t(M_t)$ is non-decreasing and 1-Lipschitz in M_t. We denote $A_t = M_t - C_t$ as the post-decision state variable.

To ensure non-negative individual wealth, we require the portfolio's investment return $\pi_t \times \tilde{R}_{t+1} + (1-\pi_t) \times r$ to be no less than -1 for all possible $\tilde{R}_{t+1}$ states. We define $\underline{R}_t = \text{ess inf}\tilde{R}_{t+1}$ and $\overline{R}_t = \text{ess sup}\tilde{R}_{t+1}$. Sufficient conditions to ensure this requirement include $\underline{R}_t < r < \overline{R}_t$ and $-\frac{1+r}{\overline{R}_t - r} = \underline{\pi}_t \le \pi_t \le \tilde{\pi}_t = \frac{1+r}{r-\underline{R}_t}$.

Consequently, by using the instantaneous non-decreasing concave differentiable utility function $u(\cdot)$ and the subjective discount factor $\beta \in (0,1]$, the Bellman equation is given by:

$$V_t(M_t) = \sup_{\substack{C_t \in [\underline{C}_t, \tilde{C}_t], \\ \pi_t \in [\underline{\pi}_t, \tilde{\pi}_t]}} \left\{ u(C_t) + \beta \times \mathbb{E}\left[V_{t+1}(M_{t+1}) \mid \mathcal{F}_t\right]\right\},$$
$$t = 0, 1, \dots, T-1, \tag{1}$$

with the terminal condition as $V_T(M_T) = u(M_T)$.

Applying the Karush–Kuhn–Tucker (KKT) conditions to Eq. (1), we derive the following optimality theorem.

Theorem 2.1. *Let $t = T-1, T-2, \dots, 1, 0$, and $M_t \ge 0$. Denote λ_t^1 and λ_t^2 as the Lagrangian multipliers for the lower and upper constraints for consumption, respectively; also, denote λ_t^3 and λ_t^4 for the lower and upper constraints for the risky asset allocation proportion, respectively. For any $C_t \in \mathbb{R}$ and $\pi_t \in \mathbb{R}$, define*

$$s_1(M_t; C_t, \pi_t) = \begin{cases} u'(C_{T-1}) - \beta \times \mathbb{E}\left[u'(M_T) \times \left(1 + (1-\pi_{T-1})r + \pi_{T-1}\tilde{R}_T\right) \mid \mathcal{F}_{T-1}\right] & t = T-1, \\ u'(C_t) - \beta \times \mathbb{E}\left[\left(u'(C_{t+1}^*(M_{t+1})) + \lambda_{t+1}^{1,*}(M_{t+1}) - \lambda_{t+1}^{2,*}(M_{t+1})\right) \times \left(1 + (1-\pi_t)r + \pi_t \tilde{R}_{t+1}\right) \mid \mathcal{F}_t\right] & t = T-2, \dots, 1, 0, \end{cases} \tag{2}$$

$$s_2(M_t; C_t, \pi_t) = \begin{cases} \beta \times \mathbb{E}\left[u'(M_T) \times (M_{T-1} - C_{T-1}) \times (\tilde{R}_T - r) \mid \mathcal{F}_{T-1}\right] & t = T-1, \\ \beta \times \mathbb{E}\left[\left(u'(C_{t+1}^*(M_{t+1})) + \lambda_{t+1}^{1,*}(M_{t+1}) - \lambda_{t+1}^{2,*}(M_{t+1})\right) \times (M_t - C_t) \times (\tilde{R}_{t+1} - r) \mid \mathcal{F}_t\right] & t = T-2, \dots, 1, 0. \end{cases} \tag{3}$$

The optimal consumption and risky allocation strategies, and the Lagrange multipliers, of the constrained optimization problem are given by:

$$\begin{aligned} C_t^*(M_t) &= \underline{C}_t \mathbb{1}_{\{\bar{C}_t(M_t) < \underline{C}_t\}} + \bar{C}_t(M_t)\mathbb{1}_{\{\underline{C}_t \le \bar{C}_t(M_t) \le \tilde{C}\}} + \tilde{C}_t \mathbb{1}_{\{\tilde{C}_t < \bar{C}_t(M_t)\}}, \\ \pi_t^*(M_t) &= \underline{\pi}_t \mathbb{1}_{\{\bar{\pi}_t(M_t) < \underline{\pi}_t\}} + \bar{\pi}_t(M_t)\mathbb{1}_{\{\underline{\pi}_t \le \bar{\pi}_t(M_t) \le \tilde{\pi}_t\}} + \tilde{\pi}_t \mathbb{1}_{\{\tilde{\pi}_t < \bar{\pi}_t(M_t)\}}, \\ \lambda_t^{1,*}(M_t) &= -s_1\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right)\mathbb{1}_{\{\bar{C}_t(M_t) < \underline{C}_t\}}, \\ \lambda_t^{2,*}(M_t) &= s_1\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right)\mathbb{1}_{\{\tilde{C}_t < \bar{C}_t(M_t)\}}, \\ \lambda_t^{3,*}(M_t) &= -s_2\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right)\mathbb{1}_{\{\bar{\pi}_t(M_t) < \underline{\pi}_t\}}, \\ \lambda_t^{4,*}(M_t) &= s_2\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right)\mathbb{1}_{\{\tilde{\pi}_t < \bar{\pi}_t(M_t)\}}, \end{aligned} \tag{4}$$

where $(\bar{C}_t(M_t), \bar{\pi}_t(M_t)) \in \mathbb{R}^2$ is a solution of the simultaneous equations:

$$s_1(M_t; C_t, \pi_t) = 0 \quad \text{and} \quad s_2(M_t; C_t, \pi_t) = 0. \tag{5}$$

Proof. See Appendix. □

At each time $t = T-1, T-2, \dots, 1, 0$, both equations in (5) are fully non-linear with respect to the two decision variables, C_t and π_t. The essential idea of the EGM is that, rather than solving two variables C_t and π_t simultaneously in (5) as in standard Value Function Iteration methods, EGM defines an endogenous post-decision state variable A_t which acts as a separator between the decision variables and significantly improves computational efficiency (Carroll, 2012).

Under the EGM framework, we define the endogenous grids as $\mathcal{G}^A = \{0, 1, \dots, T-1\} \times \{A^1, A^2, \dots, A^N\}$, where for each $t = 0, 1, \dots, T-1$, we set $0 = A^0 \le A^1 \le A^2 \le \cdots \le A^N$. The integer N is selected to be sufficiently large such that the mesh $\max_{n=1,2,\dots,N} |A^n - A^{n-1}|$ is sufficiently small while A^N is sufficiently large. For any $t = 0, 1, \dots, T-1$, the correspondence between M_t and A_t, given by $A_t = M_t - C_t$, facilitates the direct transformation of $s_1(M_t; C_t, \pi_t)$ and $s_2(M_t; C_t, \pi_t)$ into $s_1(A_t; C_t, \pi_t)$ and $s_2(A_t; C_t, \pi_t)$:

$$s_1(A_t; C_t, \pi_t) = \begin{cases} u'(C_{T-1}) - \beta \times \mathbb{E}\left[u'(M_T) \times \left(1 + (1-\pi_{T-1})r + \pi_{T-1}\tilde{R}_T\right) \mid \mathcal{F}_{T-1}\right] & t = T-1, \\ u'(C_t) - \beta \times \mathbb{E}\left[\left(u'(C_{t+1}^*(M_{t+1})) + \lambda_{t+1}^{1,*}(M_{t+1}) - \lambda_{t+1}^{2,*}(M_{t+1})\right) \times \left(1 + (1-\pi_t)r + \pi_t \tilde{R}_{t+1}\right) \mid \mathcal{F}_t\right] & t = T-2, \dots, 1, 0, \end{cases} \tag{6}$$

$$s_2(A_t; C_t, \pi_t) = \begin{cases} \beta \times \mathbb{E}\left[u'(M_T) \times A_{T-1} \times (\tilde{R}_T - r) \mid \mathcal{F}_{T-1}\right] & t = T-1, \\ \beta \times \mathbb{E}\left[\left(u'(C_{t+1}^*(M_{t+1})) + \lambda_{t+1}^{1,*}(M_{t+1}) - \lambda_{t+1}^{2,*}(M_{t+1})\right) \times A_t \times (\tilde{R}_{t+1} - r) \mid \mathcal{F}_t\right] & t = T-2, \dots, 1, 0. \end{cases} \tag{7}$$

where $M_{t+1} = A_t \times (1 + \pi_t \times \tilde{R}_{t+1} + (1-\pi_t) \times r)$. Thus, recursively and backwardly, at each time $t = T-1, T-2, \dots, 1, 0$, for any $n = 1, 2, \dots, N$, with $A_t = A_t^n$, we can numerically solve $\bar{\pi}_t(A_t^n)$ from the equation $s_2(A_t^n; C_t, \pi_t) = 0$ and then solve $\bar{C}_t(A_t^n)$ explicitly from the equation $s_1(A_t^n; C_t, \bar{\pi}_t) = 0$. As in Theorem 2.1, given the computed results $(\bar{C}_t, \bar{\pi}_t) \in \mathbb{R}^2$ and the admissible range $[\underline{C}_t, \tilde{C}_t] \times [\underline{\pi}_t, \tilde{\pi}_t]$, we obtain the corresponding optimal constrained decisions and the optimal Lagrange multipliers with respect to A_t^n:

$$\pi_t^*(A_t^n), C_t^*(A_t^n), \lambda_t^{1,*}(A_t^n), \lambda_t^{2,*}(A_t^n), \lambda_t^{3,*}(A_t^n), \lambda_t^{4,*}(A_t^n).$$

Given $C_t^*(A_t^n)$, we can compute the corresponding exogenous grids for pre-decision balance as $M_t = M_t^n$ for $n = 1, \dots, N$ through $A_t^n = M_t^n - C_t^*(A_t^n)$.

However, when solving the recursive problem at earlier periods, specifically at time $t = T-2, T-3, \dots, 1, 0$, the computation of $s_1(A_t^n; C_t, \bar{\pi}_t)$ and $s_2(A_t^n; C_t, \pi_t)$ requires the next-period optimal results of $C_{t+1}^*(M_{t+1}^n)$, $\lambda_{t+1}^{1,*}(M_{t+1}^n)$, and $\lambda_{t+1}^{2,*}(M_{t+1}^n)$. Thus, we require the decision variables to be with respect to M_t^n, i.e.,

$$\pi_t^*(M_t^n), C_t^*(M_t^n), \lambda_t^{1,*}(M_t^n), \lambda_t^{2,*}(M_t^n), \lambda_t^{3,*}(M_t^n), \lambda_t^{4,*}(M_t^n).$$

Given the savings function of $A_t = M_t - C_t^*(A_t)$, we can perform the 'relabeling' of the optimal policies *as long as* there is a one-to-one correspondence between A_t and M_t. White (2015) calls this the 'weak monotonicity' condition of the savings functions and notes that this ensures that the decision and value functions can be approximated using interpolation on the endogenous grid points.

2.2. Proofs of monotonicity under the EGM application

Inspired by Iskhakov et al. (2017), we can study the monotonicity of the savings function by modifying the Bellman equation (1), given an allocation to risky asset π_t and for $t = 0, 1, \dots, T-1$:

$$V_t(M_t | A_t, \pi_t) = \sup_{C_t \in [\underline{C}_t, \tilde{C}_t]} \left(u(C_t) + \beta \times \mathbb{E}\left[V_{t+1}(M_{t+1} | A_t, \pi_t) | \mathcal{F}_t\right]\right). \tag{8}$$

In the remaining of this paper, we make the following assumption.

Assumption 2.1. The problem is convex, i.e., the value function $V_t(\cdot)$ is globally concave. □

This assumption ensures that the KKT conditions are sufficient for generating a global optimum. Moreover, this assumption is in line with the literature, such as Love (2013), Love and Phelan (2015), and Wu et al. (2023), which study a similar optimality problem with risky asset allocation decisions; see also Hintermaier and Koeniger (2010), White (2015), and Ludwig and Schön (2018). We note that concavity typically exists in the value functions of models that have only continuous decision variables, as well as a concave and differentiable utility (Iskhakov et al., 2017).

To examine the monotonicity of the savings function, we present the following theorem.

Theorem 2.2. *Let $A_t\left(M_t,\pi_t\right) = M_t - C^*_t\left(M_t,\pi_t\right)$ denote the allocation-determined savings function at time t. When Assumption 2.1 holds, for wealth $M_t \geq 0$ and for all $\pi_t \in \left[\underline{\pi}_t, \tilde{\pi}_t\right]$ at time $t = T-1, T-2, \dots, 1, 0$, and given the existence of the concave and differentiable utility function $u(\cdot)$ as well as finite bounds for the stochastic risky asset return (i.e., $-\infty < \underline{R}_{t+1} < r < \overline{R}_{t+1} < \infty$), then the corresponding function $A^*_t\left(M_t,\pi_t\right)$ is weakly monotonic with respect to M_t.* □

Proof. As a function of M_t and A_t, the maximand in (8) is given by, where A_t is the decision variable,

$$f(A_t, M_t) = u(M_t - A_t) + \beta \times \mathbb{E}\left[V_{t+1}(M_{t+1}|A_t,\pi_t)|\mathcal{F}_t\right]. \tag{9}$$

When applying the KKT conditions to this optimization problem, the optimal post-decision balance A^* can either belong to the interior feasible set (i.e., $A^* \in (M - \tilde{C}, M - \underline{C})$) or sit on the boundary. If A^* sits on the boundary, it is obvious to see that weak monotonicity exists as $\frac{\partial A^*(M)}{\partial M} \geq 0$ due to the assumption that $\underline{C}$ and $\tilde{C}$ are 1-Lipschitz in M. Otherwise, if A^* is in the interior feasible set then, following Edlin and Shannon (1998) (Theorem 1), we need to show that:

1. $f(A, M)$ is differentiable and has continuous first-order partial derivatives (i.e., $f(A, M)$ is C^1);
2. the optimal result A^* lies within the interior of the interval $[M - \tilde{C}, M - \underline{C}]$ (i.e, $A^* \in \mathrm{int}[M - \tilde{C}, M - \underline{C}]$); and
3. the partial derivative of the function measuring the result of $f(A, M)$ with respect to A increases with M (i.e., $\frac{\partial f(A,M)}{\partial A}$ is increasing in M).

At time $t = T - 1$, partially differentiating Eq. (9) we get:

$$\begin{aligned}\frac{\partial f(A_{T-1}, M_{T-1})}{\partial A_{T-1}} &= -u'\left(M_{T-1} - A_{T-1}\right) + \beta \mathbb{E}\Big[u'\left(M_T\right) \\ &\quad \times \left(1 + r + \pi_{T-1}\left(\tilde{R}_T - r\right)\right)\Big|\mathcal{F}_{T-1}\Big],\end{aligned} \tag{10}$$

and

$$\begin{aligned}\frac{\partial f(A_{T-1}, M_{T-1})}{\partial M_{T-1}} &= u'\left(M_{T-1} - A_{T-1}\right) + \beta \mathbb{E}\Big[u'\left(M_T\right) \\ &\quad \times \left(1 + r + \pi_{T-1}\left(\tilde{R}_T - r\right)\right)\Big|\mathcal{F}_{T-1}\Big],\end{aligned} \tag{11}$$

where $M_T = A_{T-1} \times \left(1 + r + \pi_{T-1} \times \left(\tilde{R}_T - r\right)\right)$.

To prove that $f(A_{T-1}, M_{T-1})$ is C^1, we need to check whether both partial derivatives $\frac{\partial f(A_{T-1},M_{T-1})}{\partial A_{T-1}}$ and $\frac{\partial f(A_{T-1},M_{T-1})}{\partial M_{T-1}}$ exist and following equalities hold:

$$\begin{aligned}\lim_{A_{T-1}\to a} \frac{\partial f(A_{T-1}, M_{T-1})}{\partial A_{T-1}} &= \frac{\partial f(a, M_{T-1})}{\partial A_{T-1}}, \\ \lim_{M_{T-1}\to m} \frac{\partial f(A_{T-1}, M_{T-1})}{\partial M_{T-1}} &= \frac{\partial f(A_{T-1}, m)}{\partial M_{T-1}}.\end{aligned} \tag{12}$$

Given that $\pi_{T-1} \in \left[\underline{\pi}_{T-1}, \tilde{\pi}_{T-1}\right]$, $u(\cdot)$ is differentiable, and $\underline{R}_T$ and $\overline{R}_T$ are finite, the partial derivatives $\frac{\partial f(A_{T-1},M_{T-1})}{\partial A_{T-1}}$ and $\frac{\partial f(A_{T-1},M_{T-1})}{\partial M_{T-1}}$ exist, and the equalities in Eq. (12) hold. Thus, condition 1 is satisfied.

Further, differentiating Eq. (10) with respect to M_{T-1}, shows that the result of $\frac{\partial f(A,M)}{\partial A}$ at A^* is positively related to M when $M_{T-1} - A^*_{T-1} = C^*_{T-1} \geq \underline{C}_{T-1} \geq 0$. Hence, condition 3 is satisfied. Finally, since $\frac{\partial^2 f(A_{T-1},M_{T-1})}{\partial A^2_{T-1}}$ is negative, A^* is a local maximum and this ensures that condition 2 is satisfied.

Thus, all three conditions are satisfied, and the monotonicity theorem in Edlin and Shannon (1998) applies at time $t = T - 1$.

Next, at time $t = T-2, T-3, \dots, 1, 0$, partially differentiating Eq. (9) and building upon the solutions to the first-order conditions from Theorem 2.1, we obtain:

$$\begin{aligned}\frac{\partial f(A_t, M_t)}{\partial A_t} &= -u'\left(M_t - A_t\right) \\ &\quad + \beta \mathbb{E}\left[\frac{\partial V_{t+1}\left(M_{t+1}|A_t,\pi_t\right)}{\partial M_{t+1}} \times (1 + r \right. \\ &\quad \left. + \ \pi_t \times \left(\tilde{R}_{t+1} - r\right)\right)\Big|\mathcal{F}_t\Big],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\partial f(A_t, M_t)}{\partial M_t} &= u'\left(M_t - A_t\right) \\ &\quad + \beta \mathbb{E}\left[\frac{\partial V_{t+1}\left(M_{t+1}|A_t,\pi_t\right)}{\partial M_{t+1}} \times (1 + r \right. \\ &\quad \left. + \ \pi_t \times \left(\tilde{R}_{t+1} - r\right)\right)\Big|\mathcal{F}_t\Big],\end{aligned} \tag{14}$$

where $\frac{\partial V_{t+1}(M_{t+1}|A_t,\pi_t)}{\partial M_{t+1}}$ can be determined based on the following recursive calculation equation:

$$\begin{aligned}\frac{\partial V_{t+1}\left(M_{t+1}|A_t,\pi_t\right)}{\partial M_{t+1}} &= \beta \mathbb{E}\left[\frac{\partial V_{t+2}\left(M_{t+2}|A_{t+1},\pi_{t+1}\right)}{\partial M_{t+2}} \times (1 + r \right. \\ &\quad \left. + \ \pi^*_{t+1}(M_{t+1})\left(\tilde{R}_{t+2} - r\right)\right)\Big|\mathcal{F}_{t+1}\Big].\end{aligned} \tag{15}$$

For instance, at time $t = T - 2$, we have:

$$\begin{aligned}\frac{\partial V_{t+1}\left(M_{t+1}|A_t,\pi_t\right)}{\partial M_{t+1}} &= \beta \mathbb{E}\Big[u'\left(M_{t+2}\right) \times (1 + r \\ &\quad + \ \pi^*_{t+1}(M_{t+1})\left(\tilde{R}_{t+2} - r\right)\Big)\Big|\mathcal{F}_{t+1}\Big],\end{aligned}$$

and at all $t = T-3, \dots, 1, 0$ we work backwards using Eq. (15).

Similarly, to prove that $f(A_t, M_t)$ is C^1, we need to check whether both partial derivatives $\frac{\partial f(A_t,M_t)}{\partial A_t}$ and $\frac{\partial f(A_t,M_t)}{\partial M_t}$ exist and that the equalities in the time t version of Eq. (12) hold.

Using similar arguments to time $T - 1$ case, given that $\pi_t \in \left[\underline{\pi}_t, \tilde{\pi}_t\right]$, $u(\cdot)$ is differentiable, and $\underline{R}_{t+1}$ and $\overline{R}_{t+1}$ are finite, we can show that $f(A_t, M_t)$ is C^1. Thus, condition 1 is satisfied.

Further, by examining $\frac{\partial \frac{\partial f(A_t,M_t)}{\partial A_t}}{\partial M_t}$, we can show that $\frac{\partial f(A,M)}{\partial A}$ at A^* is positively related to M when we have $M_t - A^*_t = C^*_t \geq \underline{C}_t \geq 0$, thereby satisfying condition 3. Finally, we can confirm that $\frac{\partial^2 f(A_t,M_t)}{\partial A^2_t}$ is negative. Thus, A^* is a local maximum ensuring that condition 2 is satisfied.

Thus, all three conditions are satisfied, and the monotonicity theorem in Edlin and Shannon (1998) applies at time $t = T-2, T-3, \dots, 1, 0$, backward and recursively. □

3. Conclusions

In this paper, we have provided the monotonicity proof for a simple and fairly general life-cycle problem with stochastic portfolio returns. We show the need for the monotonicity proof in applying EGM in such problems, and hence provide an important sufficient condition for the application of EGM.

Our proof can easily be modified to dynamic optimization problems involving stochastic investment returns and deterministic portfolio choices. With a little effort, the proof can also be adopted to more sophisticated models as those presented in Love (2013) and Wu et al. (2023).[1] Through this proof, we expect further application of EGM in more sophisticated settings in the future.

Data availability

No data was used for the research described in the article.

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Appendix. Proof of Theorem 2.1

Based on the Bellman equation shown in Eq. (1) and the decision variable constraints, for any $C_t \in \mathbb{R}$, $\pi_t \in \mathbb{R}$, $\lambda_t^1 \in \mathbb{R}$, $\lambda_t^2 \in \mathbb{R}$, $\lambda_t^3 \in \mathbb{R}$, and $\lambda_t^4 \in \mathbb{R}$, we define the Lagrangian function of the constrained optimization problem by

$$\begin{aligned}\mathcal{L}\left(M_t; C_t, \pi_t; \lambda_t^1, \lambda_t^2, \lambda_t^3, \lambda_t^4\right) = u\left(C_t\right) + \beta \times \mathbb{E}\left[V_{t+1}\left(M_{t+1}\right) \middle| \mathcal{F}_t\right] \\ + \lambda_t^1\left(C_t - \underset{\sim}{C}_t\right) + \lambda_t^2\left(\tilde{C}_t - C_t\right) \\ + \lambda_t^3\left(\pi_t - \underset{\sim}{\pi}_t\right) + \lambda_t^4\left(\tilde{\pi}_t - \pi_t\right).\end{aligned} \tag{A.1}$$

The KKT conditions for the optimal consumption and investment strategies, and the Lagrange multipliers, of the constrained optimization problem are given by:

$$\begin{cases} \frac{\partial}{\partial C_t}\mathcal{L}\left(M_t; C_t^*(M_t), \pi_t^*(M_t); \lambda_t^{1,*}(M_t), \lambda_t^{2,*}(M_t), \lambda_t^{3,*}(M_t), \lambda_t^{4,*}(M_t)\right) = 0, \\ \frac{\partial}{\partial \pi_t}\mathcal{L}\left(M_t; C_t^*(M_t), \pi_t^*(M_t); \lambda_t^{1,*}(M_t), \lambda_t^{2,*}(M_t), \lambda_t^{3,*}(M_t), \lambda_t^{4,*}(M_t)\right) = 0, \\ \underset{\sim}{C}_t \le C_t^*(M_t) \le \tilde{C}_t, \quad \underset{\sim}{\pi}_t \le \pi_t^*(M_t) \le \tilde{\pi}_t, \\ \lambda_t^{1,*}(M_t) \ge 0, \quad \lambda_t^{2,*}(M_t) \ge 0, \quad \lambda_t^{3,*}(M_t) \ge 0, \quad \lambda_t^{4,*}(M_t) \ge 0, \\ \lambda_t^{1,*}(M_t)\left(C_t^*(M_t) - \underset{\sim}{C}_t\right) = 0, \quad \lambda_t^{2,*}(M_t)\left(\tilde{C}_t - C_t^*(M_t)\right) = 0, \\ \lambda_t^{3,*}(M_t)\left(\pi_t^*(M_t) - \underset{\sim}{\pi}_t\right) = 0, \quad \lambda_t^{4,*}(M_t)\left(\tilde{\pi}_t - \pi_t^*(M_t)\right) = 0. \end{cases} \tag{A.2}$$

where the first order conditions are simplified as:

$$\begin{cases} u'\left(C_t^*(M_t)\right) - \beta \times \mathbb{E}\left[\frac{\partial V_{t+1}\left(M_{t+1}\right)}{\partial M_{t+1}} \times \left(1 + \left(1 - \pi_t^*(M_t)\right) r \right.\right. \\ \qquad \left.\left. + \pi_t^*(M_t)\tilde{R}_{t+1}\right) | \mathcal{F}_t\right] \\ \qquad + \lambda_t^{1,*}(M_t) - \lambda_t^{2,*}(M_t) = 0, \\ \beta \times \mathbb{E}\left[\frac{\partial V_{t+1}\left(M_{t+1}\right)}{\partial M_{t+1}} \times \left(M_t - C_t^*(M_t)\right) \times \left(\tilde{R}_{t+1} - r\right) | \mathcal{F}_t\right] \\ + \lambda_t^{3,*}(M_t) - \lambda_t^{4,*}(M_t) = 0. \end{cases} \tag{A.3}$$

By the Envelope Theorem, for any $t = T-1, T-2, \ldots, 1, 0$, and $M_t \ge 0$, differentiating the Bellman equation (1) with respect to M_t on both sides yields

$$\begin{aligned}\frac{\partial V_t\left(M_t\right)}{\partial M_t} = \beta \times \mathbb{E}\left[\frac{\partial V_{t+1}\left(M_{t+1}\right)}{\partial M_{t+1}} \times \left(1 + \left(1 - \pi_t^*(M_t)\right) r \right.\right. \\ \left.\left. + \pi_t^*(M_t)\tilde{R}_{t+1}\right) | \mathcal{F}_t\right].\end{aligned} \tag{A.4}$$

By (A.3) and (A.4), for any $t = T-1, T-2, \ldots, 1, 0$, and $M_t \ge 0$, we obtain

$$\frac{\partial V_t\left(M_t\right)}{\partial M_t} = u'\left(C_t^*(M_t)\right) + \lambda_t^{1,*}(M_t) - \lambda_t^{2,*}(M_t). \tag{A.5}$$

Hence, at time $t = T-1, T-2, \ldots, 1, 0$, for any $M_t \ge 0$, the first order conditions in (A.3) of the KKT conditions in (A.2) are simplified as:

$$\begin{cases} s_1\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right) + \lambda_t^{1,*}(M_t) - \lambda_t^{2,*}(M_t) = 0, \\ s_2\left(M_t; C_t^*(M_t), \pi_t^*(M_t)\right) + \lambda_t^{3,*}(M_t) - \lambda_t^{4,*}(M_t) = 0, \end{cases} \tag{A.6}$$

where s_1 and s_2 are exactly the same as in (2) and (3).

Solving the system of equations in (A.6), we obtain the constrained optimization solutions as shown in (4).

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[1] The authors have adopted the monotonicity proof to Love (2013). This is available upon request.