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S. Maniscalco,[1,6,*] J. Piilo,[2,3,6] F. Intravaia,[4] F. Petruccione,[5,6] and A. Messina[1]

[1]*INFM, MIUR and Dipartimento di Scienze Fisiche ed Astronomiche dell'Università di Palermo, via Archirafi 36,*
90123 Palermo, Italy
[2]*Department of Physics, University of Turku, FIN-20014 Turun yliopisto, Finland*
[3]*Helsinki Institute of Physics, PL 64, FIN-00014 Helsingin yliopisto, Finland*
[4]*Laboratoire Kastler Brossel, Case 74, 4 place Jussieu, F-75252 Paris Cedex 05, France*[†]
[5]*Physikalisches Institut, Albert-Ludwigs-Universität, Hermann-Herder Straße 3, D-79104 Freiburg im Breisgau, Germany*
[6]*School of Pure and Applied Physics, University of KwaZulu-Natal, Durban 4041, South Africa*

We study the open system dynamics of a harmonic oscillator coupled with an artificially engineered reservoir. We single out the reservoir and system variables governing the passage between Lindblad-type and non-Lindblad-type dynamics of the reduced system's oscillator. We demonstrate the existence of conditions under which virtual exchanges of energy between system and reservoir take place. We propose to use a single trapped ion coupled to engineered reservoirs in order to simulate quantum Brownian motion.

I. INTRODUCTION

The dynamics of closed systems may be calculated exactly by solving directly the Schrödinger equation. In realistic physical conditions, however, the system one is interested in is coupled to its surrounding. In this case the dynamics of the total closed system can be extremely complicated. For this reason, since the very early days of quantum mechanics, a huge deal of attention has been devoted to the study of the dynamics of open quantum systems [1].

Nowadays the interest in such a broad field has notably increased mainly for two reasons. On the one hand experimental advances in the coherent control of single or few atom systems have paved the way to the realization of the first basic elements of quantum computers, CNOT [2] and phase quantum gates [3]. Moreover, the first quantum cryptographic [4] and quantum teleportation [5] schemes have been experimentally implemented. These technological applications rely on the persistence of quantum coherence. Thus, understanding decoherence and dissipation arising from the unavoidable interaction between the system and its surrounding is necessary in order to implement real-size quantum computers [6] and quantum technologies. On the other hand, one of the most debated aspects of quantum mechanics, namely, the quantum measurement problem, can be interpreted in terms of environment induced decoherence [7]. According to this interpretation the emergence of the classical world from the quantum world can be seen as a decoherence process due to the interaction between system and environment [8].

A paradigmatic model of the theory of open systems is a harmonic oscillator linearly coupled with a reservoir modeled as an infinite set of noninteracting oscillators. Indeed this model is central in many physical contexts, e.g., quantum field theory [9], quantum optics [1,10,11], and solid-state physics [12].

In order to describe quantitatively and qualitatively how the reservoir affects the system dynamics one needs to make some assumptions on its nature and properties. Some of these properties, as, for example, the temperature of the reservoir, can be experimentally measured. Other parameters, as the reservoir spectral density or the system-reservoir coupling, are assumed on the basis of physical reasonableness and deduced by the comparison with experimental data. In this sense the model is a phenomenological one.

The importance of the damped harmonic oscillator is also due to the fact that it is one of the few exactly solvable nontrivial systems. In fact the Heisenberg equations of motion for the total system (oscillator plus reservoir) can easily be solved. The solution of the Heisenberg equations of motion is indeed the most straightforward method for the description of the dynamics of the expectation values of observables of interest, e.g., the mean energy of the system. An example showing the easiness and conceptual transparency of the method is given in Ref. [1].

Another way to describe the time evolution of the system is to look at the reduced density matrix which is obtained by tracing the density matrix of the total system over the environmental degrees of freedom. This procedure is motivated by the fact that, in general, one is interested only in the dynamics of the system and not in the state of the reservoir. An exact master equation for the reduced density matrix can be formulated and exactly solved [13–19].

During the last decade huge advances in laser cooling and trapping experimental techniques have made it possible to confine harmonically a single ion and cool it down to very low temperatures where purely quantum manifestations begin to play an important role [22]. A single laser cooled ion is theoretically equivalent to a particle moving in a harmonic potential, whose center of mass motion is quantized as a harmonic oscillator. Such a system is a unique experimental system since it approximates very well a closed system [22]. Indeed unwanted dissipation, which in this case manifests itself as a heating process depopulating the vibrational

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*Electronic address: sabrina@fisica.unipa.it

†École Normale Supérieure, Centre National de la Recherche Scientifique, Université Pierre et Marie Curie.

ground state of the ion, is negligible for times much longer than the usual times in which experiments take place [23,24]. Moreover, arbitrary states of the ion motion can be prepared and coherently manipulated using proper laser pulses [25,26]. Even extremely fragile states as Schrödinger cat states have been realized and detected [27]. Quite recently, by using multiple ions in a linear Paul trap, experiments on quantum nonlocality have been performed [28] and many-particle entangled states have been realized [29]. Furthermore cold trapped ions are the favorite candidates for a physical implementation of quantum computers [2,6].

The aim of this paper is to study the interaction of a quantum harmonic oscillator with engineered reservoirs. In the context of trapped ions it is possible not only to engineer experimentally an "artificial" reservoir but also to synthesize both its spectral density and the coupling with the system oscillator [30,31]. This makes it possible to think of new types of experiments aimed at testing the predictions of fundamental models as the one of quantum Brownian motion (QBM) (or its high-T limit: the famous Caldeira-Leggett model [15]).

The mean vibrational quantum number of the ion, also called heating function [23], is the central quantity we investigate. We study the heating dynamics of the single oscillator in correspondence to different reservoir characteristic parameters. We analyze the influence of the variations of the engineered reservoir parameters, e.g., the cutoff frequency of the reservoir spectral density, on the heating process. The idea of looking at the variation of the open system dynamics induced by the changes of the relevant reservoir parameters is in fact rather unusual. Up to recently, indeed, only dissipation and/or decoherence due to interaction with the "natural" reservoir were studied [1,10,12,13,32,33].

We compare our analytic results for the observable quantities of interest, as the heating function, with the non-Markovian wave function (NMWF) simulations [1,34], finding a very good agreement. The main result of the paper is the experimental proposal for observing features of quantum Brownian motion with single trapped ions. We demonstrate that, with currently available technology, a regime of the open system dynamics characterized by virtual phonon exchanges between the system and the reservoir may be explored.

The paper is organized as follows. In Sec. II we introduce the master equation for QBM and its solution obtained by means of a superoperatorial approach. In Sec. III we study the behavior of the heating function for different values of the reservoir parameters. In Sec. IV we review the basic ingredients of the experimental procedures for engineering artificial amplitude reservoir, as the one we study in the paper, and for measuring the heating function. In Sec. V we describe our experimental proposal for revealing non-Markovian dynamics of a quantum Brownian particle, simulated with a single trapped ion. Finally in Sec. VI conclusions are presented.

II. EXACT DYNAMICS OF A QUANTUM BROWNIAN PARTICLE

A. Generalized master equation

The dynamics of a harmonic oscillator linearly coupled with a quantized reservoir, modeled as an infinite chain of quantum harmonic oscillators, may be described exactly by means of a generalized master equation of the form [1,16,35,36]

$$\frac{d\rho_S(t)}{dt} = \frac{1}{i\hbar}\mathbf{H}_0^S\rho_S(t) - \left[\Delta(t)(\mathbf{X}^S)^2 - \Pi(t)\mathbf{X}^S\mathbf{P}^S \right.$$
$$\left. - \frac{i}{2}r(t)(\mathbf{X}^2)^S + i\gamma(t)\mathbf{X}^S\mathbf{P}^\Sigma \right]\rho_S(t). \quad (1)$$

We indicate with $\mathbf{X}^{S(\Sigma)}$ and $\mathbf{P}^{S(\Sigma)}$ the commutator (anticommutator) position and momentum superoperators, respectively, and with $\mathbf{H}_0^S$ the commutator superoperator relative to the system Hamiltonian. The time dependent coefficients appearing in the master equation can be written, to the second order in the coupling strength, as follows:

$$\Delta(t) = \int_0^t \kappa(\tau)\cos(\omega_0\tau)d\tau, \quad (2)$$

$$\gamma(t) = \int_0^t \mu(\tau)\sin(\omega_0\tau)d\tau, \quad (3)$$

$$\Pi(t) = \int_0^t \kappa(\tau)\sin(\omega_0\tau)d\tau, \quad (4)$$

$$r(t) = 2\int_0^t \mu(\tau)\cos(\omega_0\tau)d\tau, \quad (5)$$

where

$$\kappa(\tau) = \alpha^2\langle\{E(\tau),E(0)\}\rangle, \quad (6)$$

and

$$\mu(\tau) = i\alpha^2\langle[E(\tau),E(0)]\rangle, \quad (7)$$

are the noise and dissipation kernels, respectively. In the previous equations we indicate with α the system-reservoir coupling constant, with ω_0 the frequency of the system oscillator and with E the generalized reservoir position operator.

The master equation (1) is local in time, even if non-Markovian. This feature is typical of all the generalized master equations derived by using the time-convolutionless projection operator technique [1,34] or equivalent approaches such as the superoperatorial one presented in Refs. [19,20].

The time dependent coefficients appearing in Eq. (1) contain all the information about the short time system-reservoir correlation. The coefficient $r(t)$ gives rise to a time dependent renormalization of the frequency of the oscillator. The term proportional to $\gamma(t)$ is a classical damping term while the coefficients $\Delta(t)$ and $\Pi(t)$ are diffusive terms.

In what follows we study the time evolution of the heating function $\langle n(t)\rangle$ with n quantum number operator. The dynamics of $\langle n(t)\rangle$ depends only on the diffusion coefficient $\Delta(t)$ and on the classical damping coefficient $\gamma(t)$ [19]. Furthermore, the quantum number operator n belongs to a class of observables not influenced by a secular approximation which takes the master equation (1) into the form [19,37]

$$\frac{d\rho_S(t)}{dt} = \frac{\Delta(t) + \gamma(t)}{2}[2a\rho_S(t)a^\dagger - a^\dagger a\rho_S(t) - \rho_S(t)a^\dagger a]$$

$$+ \frac{\Delta(t) - \gamma(t)}{2}[2a^\dagger\rho_S(t)a - aa^\dagger\rho_S(t) - \rho_S(t)aa^\dagger].$$

$$(8)$$

For this reason, in order to calculate the exact time evolution of the heating function, one can use the solution of the approximated master equation (8). In the previous equation we have introduced the bosonic annihilation and creation operators $a = (X + iP)/\sqrt{2}$ and $a^\dagger = (X - iP)/\sqrt{2}$, with X and P dimensionless position and momentum operator. Note that the above master equation is of Lindblad type as far as the coefficients $\Delta(t) \pm \gamma(t)$ are positive [21].

B. Analytic solution

The superoperatorial master equation (1) can be exactly solved by using specific algebraic properties of the superoperators [19]. The solution for the density matrix of the system is derived in terms of the quantum characteristic function (QCF) $\chi_t(\xi)$ at time t, defined through the equation [9]

$$\rho_S(t) = \frac{1}{2\pi}\int \chi_t(\xi)e^{(\xi a^\dagger - \xi^* a)}d^2\xi. \qquad (9)$$

It is worth noting that one of the advantages of the superoperatorial approach is the relative easiness in calculating the analytic expression for the mean values of observables of interest by using the relation

$$\langle a^{\dagger m}a^n\rangle = \left(\frac{d}{d\xi}\right)^m\left(-\frac{d}{d\xi^*}\right)^n e^{|\xi|^2/2}\chi(\xi)\bigg|_{\xi=0}. \qquad (10)$$

The exact analytic expression for the time evolution of the heating function can be obtained from the secular solution. In the secular approximation the QCF is [19]

$$\chi_t(\xi) = e^{-\Delta_\Gamma(t)|\xi|^2}\chi_0[e^{-\Gamma(t)/2}e^{-i\omega_0 t}\xi] \qquad (11)$$

with χ_0 QCF of the initial state of the system. The quantities $\Delta_\Gamma(t)$ and $\Gamma(t)$ appearing in Eq. (11) are defined in terms of the diffusion and dissipation coefficients $\Delta(t)$ and $\gamma(t)$, respectively, as follows:

$$\Gamma(t) = 2\int_0^t \gamma(t_1)dt_1, \qquad (12)$$

$$\Delta_\Gamma(t) = e^{-\Gamma(t)}\int_0^t e^{\Gamma(t_1)}\Delta(t_1)dt_1. \qquad (13)$$

Equation (11) shows that the QCF is the product of an exponential factor, depending on both the diffusion $\Delta(t)$ and the dissipation $\gamma(t)$ coefficients, and a transformed initial QCF. The exponential term accounts for energy dissipation and is independent of the initial state of the system. Information on the initial state is given by the second term of the product, the transformed initial QCF.

Having in mind Eq. (11) and using Eq. (10), one gets the following expression for the heating function

$$\langle n(t)\rangle = e^{-\Gamma(t)}\langle n(0)\rangle + \tfrac{1}{2}(e^{-\Gamma(t)} - 1) + \Delta_\Gamma(t). \qquad (14)$$

The asymptotic long-time behavior of the heating function is readily obtained by using the Markovian stationary values for $\Delta(t)$ and $\gamma(t)$. For a thermal reservoir, one gets

$$\langle n(t)\rangle = e^{-\Gamma t}\langle n(0)\rangle + n(\omega_0)(1 - e^{-\Gamma t}). \qquad (15)$$

In the following section we will discuss in detail the dynamics of the heating process and we will show the changes in the short-time dynamics due to the variations of typical reservoir parameters such as its temperature and cutoff frequency.

III. NON-MARKOVIAN DYNAMICS OF LINDBLAD AND NON-LINDBLAD TYPE

In a previous paper we have presented a theory of heating for a single trapped ion interacting with a natural reservoir able to describe both its short-time non-Markovian behavior and the asymptotic thermalization process [38]. In this paper we focus instead on the case of interaction with engineered reservoirs. In the trapped ion context, it is experimentally possible to engineer artificial reservoirs and couple them to the system in a controlled way. Since the coupling with the natural reservoir is negligible for long-time intervals [22,23], this allows to test fundamental models of open system dynamics as the one for QBM we are interested in. In a sense this extends the idea of using trapped ions for simulating the closed dynamics of quantum optical systems [24,39,40] to the possibility of simulating the dynamics of an ubiquitous open system as the damped harmonic oscillator. In particular, by using the analytic solution, one can look for ranges of the relevant parameters of both the reservoir and the system in correspondence of which deviations from Markovian dissipation become experimentally observable.

In the experiments on artificially engineered amplitude reservoirs [31] the high-temperature condition $\hbar\omega_0/KT \ll 1$ is always satisfied. For this reason here we concentrate on this regime of the parameters. For an Ohmic reservoir spectral density with Lorentz-Drude cutoff

$$J(\omega) = \frac{2\omega}{\pi}\frac{\omega_c^2}{\omega_c^2 + \omega^2}, \qquad (16)$$

the dissipation and damping coefficients $\gamma(t)$ and $\Delta(t)$, appearing in the master equation (8), to second order in the coupling constant, take the form

$$\gamma(t) = \frac{\alpha^2\omega_0 r^2}{1 + r^2}[1 - e^{-\omega_c t}\cos(\omega_0 t) - re^{-\omega_c t}\sin(\omega_0 t)],$$

$$(17)$$

and

$$\Delta(t) = 2\alpha^2 kT\frac{r^2}{1 + r^2}\{1 - e^{-\omega_c t}[\cos(\omega_0 t) - (1/r)\sin(\omega_0 t)]\}$$

$$(18)$$

with $r = \omega_c/\omega_0$. Equation (18) has been derived in the high-T limit, while $\gamma(t)$ does not depend on temperature. Comparing

Eq. (18) with Eq. (17), one notices immediately that in the high-temperature regime, $\Delta(t) \gg \gamma(t)$. Having this in mind it is easy to prove that the heating function, given by Eq. (14), may be written in the following approximated form

$$\langle n(t) \rangle \simeq \Delta_\Gamma(t), \tag{19}$$

where we have assumed that the initial state of the ion is its vibrational ground state, as it is actually the case at the end of the resolved sideband cooling process [25,26,41]. For times much bigger than the reservoir correlation time $\tau_R = 1/\omega_c$ the asymptotic behavior of Eq. (19) is given by Eq. (15). This equation gives evidence for a second characteristic time of the dynamics, namely, the thermalization time $\tau_T = 1/\Gamma$, with $\Gamma = \alpha^2 \omega_0 r^2 /(r^2+1)$. The thermalization time depends both on the coupling strength and on the ratio $r = \omega_c/\omega_0$ between the reservoir cutoff frequency and the system oscillator frequency. Usually, when one studies QBM, the condition $r \gg 1$, which corresponds to a natural flat reservoir, is assumed. In this case the thermalization time is simply inversely proportional to the coupling strength. For an "out of resonance" engineered reservoir with $r \ll 1$, τ_T is notably increased and therefore the thermalization process is slowed down.

A further approximation to the heating function of Eq. (19) can be obtained for times $t \ll \tau_T$:

$$\langle n(t) \rangle \simeq \int_0^t \Delta(t_1) dt_1 = \frac{2\alpha^2 KT}{\omega_c} \frac{r^2}{(r^2+1)^2} \{ \omega_c t (r^2+1) \tag{20}$$

$$- (r^2-1)[1 - e^{-\omega_c t} \cos(\omega_0 t)] - r e^{-\omega_c t} \sin(\omega_0 t) \}. \tag{21}$$

This approximation shows a clear connection between the sign of the diffusion coefficient $\Delta(t)$ and the time evolution of the heating function before thermalization. The diffusion coefficient is indeed the time derivative of the heating function. We remind that, since $\Delta(t) \gg \gamma(t)$ for the case considered here, whenever $\Delta(t) > 0$ the master equation (8) is of Lindblad type, whilst the case $\Delta(t) < 0$ corresponds to a non-Lindblad-type master equation. From Eq. (20) one sees immediately that while for $\Delta(t) > 0$ the heating function grows monotonically, when $\Delta(t)$ assumes negative values it can decrease and present oscillations.

To better understand such a behavior we study in more details the dynamics for three exemplary values of the ratio r between the reservoir cutoff frequency and the system oscillator frequency: $r \gg 1$, $r = 1$, and $r \ll 1$. As we have already noticed the first case corresponds to the assumption commonly done when dealing with natural reservoir while the last case corresponds to an engineered out of resonance reservoir.

For $r \gg 1$ the diffusion coefficient, given by Eq. (18), is positive for all t and r since

$$\Delta(t) \propto 1 - e^{-\omega_c t} \cos(\omega_0 t) \geqslant 0. \tag{22}$$

Therefore the master equation is always of Lindblad type and the heating function grows monotonically from its initial null value. Equation (20) shows that, for times $t \ll \tau_R$ and for $r \gg 1$, $\langle n(t) \rangle \simeq (\alpha^2 \omega_c kT) t^2$, that is the initial non-Markovian behavior of the heating function is quadratic in time.

For $r = 1$, a similar behavior is observed since also in this case $\Delta(t)$ is positive at all times.

Finally, in the case $r \ll 1$, Eq. (18) shows that, if r is sufficiently small, $\Delta(t)$ oscillates acquiring also negative values. It is worth noting, however, that the long-time asymptotic value of $\Delta(t)$ is always positive. Whenever the diffusion coefficient is negative, the heating function decreases, so the overall heating process is characterized by oscillations of the heating function. The decrease in the population of the ground state of the system oscillator, after an initial increase due to the interaction with the high-T reservoir, is due to the emission and subsequent reabsorption of the same quantum of energy. Such an event is possible since the reservoir correlation time $\tau_R = 1/\omega_c$ is now much longer than the period of oscillation $\tau_s = 1/\omega_0$. We underline that, although the master equation in this case is not of Lindblad type, it conserves the positivity of the reduced density matrix. This of course does not contradict the Lindblad theorem since the semigroup property is clearly violated for the reduced system dynamics [1].

IV. EXPERIMENTAL TECHNIQUES

This section gives a brief review of the experimental procedures for engineering artificial reservoirs and for measuring the heating function of the trapped ion. Starting from the careful analysis of the recent experiments presented in this section, we will describe, in the Sec. V, our experimental proposal for simulating QBM with single trapped ions.

A. Engineering reservoirs

Let us begin discussing the technique used to engineer an artificial reservoir coupled to a single trapped ion. Reference [31] presents recent experimental results showing how to couple a properly engineered reservoir with a quantum oscillator, namely, the quantized center of mass (c.m.) motion of the ion. These experiments aim at measuring the decoherence of a quantum superposition of coherent states and Fock states due to the presence of the reservoir. Several types of engineered reservoirs are demonstrated, e.g., thermal amplitude reservoirs, phase reservoirs, and zero-temperature reservoirs.

A high-T amplitude reservoir is obtained by applying a random electric field $\vec{E}$ whose spectrum is centered on the axial frequency $\omega_z/2\pi = 11.3$ MHz of oscillation of the ion [31]. The trapped ion motion couples to this field due to the net charge q of the ion: $H_{int} = -q\vec{x} \cdot \vec{E}$, with $\vec{x} = (X, Y, Z)$ displacement of the c.m. of the ion from its equilibrium position. Remembering that $\vec{E} \propto \sum_i \vec{\varepsilon}_i (b_i + b_i^\dagger)$, with b_i and $b_i^\dagger$ annihilation and creation operators of the fluctuating field modes, and that $X \propto (a + a^\dagger)$ one realizes that this coupling is equiva-

lent to the bilinear one assumed to derive Eq. (1).

The random electric field is applied to the endcap electrodes through a network of properly arranged low-pass filters limiting the natural environmental noise but allowing deliberately large applied fields to be effective. This type of drive simulates an infinite-bandwidth amplitude reservoir [31]. It is worth stressing that, for the times of duration of the experiment, namely, $\Delta t = 20$ μs, the heating due to the natural reservoir is definitively negligible [31].

The reservoir considered in our paper is a thermal reservoir with spectral distribution given by

$$I(\omega) = J(\omega)[n(\omega) + 1/2]$$

$$= \frac{\omega}{\pi} \frac{\omega_c^2}{\omega_c^2 + \omega^2} \coth(\omega/KT), \qquad (23)$$

where Eq. (16) has been used. For high T, Eq. (23) becomes

$$I(\omega) = \frac{2KT}{\pi} \frac{\omega_c^2}{\omega_c^2 + \omega^2}. \qquad (24)$$

The infinite-bandwidth amplitude reservoir realized in the experiments corresponds to the case $\omega_c \to \infty$ in the previous equation. Therefore, for high T, the reservoir discussed in the paper can be realized experimentally by filtering the random field, used in the experiments for simulating an infinite-bandwidth reservoir, with a Lorentzian shaped low-pass filter at frequency ω_c. The change of the ratio r thus would be accomplished simply by changing the low-pass filter.

B. Measurements of the heating function

In this section we focus on two experimental methods for measuring the heating function of a single trapped ion. The first method is based on the asymmetry in the sideband motional spectrum of the ion and it has been used in Ref. [23] for measuring the process of thermalization of an ion, initially cooled down to its ground vibrational state, due to the interaction with the natural reservoir. The same method is used in Ref. [26] for measuring the cooling dynamics of an ion subjected to sideband cooling lasers. The second technique allows to measure the populations of the vibrational density matrix of the ion, from which the heating function can be obtained. This last method has been used in Ref. [31] in the case of interaction with an artificial amplitude reservoir.

For both techniques the first step is the preparation of the initial vibrational and electronic ground state, $|n=0,-\rangle \equiv |n=0\rangle \otimes |-\rangle$, obtained by laser cooling and optical pumping to the state $|-\rangle$. The mean vibrational number is then measured after fixed delay time intervals. During the delay artificial noise, which simulates the amplitude reservoir, may be applied. In this way, the time evolution of $\langle n(t) \rangle$ is obtained.

Let us begin with the first technique. At each fixed delay time, the ion is in its electronic ground state $|-\rangle$ and in a certain vibrational state. A laser pulse tuned to a vibrational sideband is then used to transfer the population to the upper electronic level $|+\rangle$. After this, by means of an electron shelving technique, the electronic state of the ion is detected in order to check wether a transition to $|+\rangle$ has occurred or

not. Repeating this procedure one gets the electronic excited state occupation probability P_+. The amplitude of the blue and red vibrational sidebands is defined as the probability of making a transition $|-\rangle \to |+\rangle$ due to a laser pulse tuned to the blue or red sideband, respectively, and therefore is given by P_+. This quantity depends on the mean occupation number $\langle n \rangle$. For $|n=0\rangle$, only the blue sideband can be exited while the red one is absent. In general the asymmetry in the amplitude of the kth red (I_{red}^k) and blue (I_{blue}^k) vibrational sidebands, allows to extract $\langle n \rangle$ [23]:

$$I_{red}^k = \left(\frac{\langle n \rangle}{\langle n \rangle + 1}\right)^k I_{blue}^k. \qquad (25)$$

A limitation of this method is given by off-resonant excitation via the carrier transition. If the driving field is tuned to the first lower vibrational sideband, in the resolved sideband condition $\omega_0 \gg \Omega$, with Ω Rabi frequency of the laser pulse, processes involving off-resonant transitions go as $(\Omega/\omega_0)^2$. In order to have a sensible measurement of the heating function, the population due to off-resonant transitions have to be much smaller than the scale over which $\langle n(t) \rangle$ varies.

We now focus on the second experimental method for measuring the heating function. This method actually allows to measure the diagonal elements P_n of the vibrational density matrix and it has been used to observe their decay due to the interaction with an artificial amplitude reservoir, as the one described in the preceding section. For this type of reservoir and in the experimental conditions of Ref. [31], the time evolution of $P_n(t) \equiv \rho_{nn}(t)$ is well approximated by the law

$$\rho_{nn}(t) \simeq \frac{1}{1+\bar{n}\gamma t} \sum_{j=0}^{n} \left(\frac{\bar{n}\gamma t}{1+\bar{n}\gamma t}\right)^j \left(\frac{1}{1+\bar{n}\gamma t}\right)^{2n-2j}$$

$$\times \sum_{l=0}^{\infty} \left(\frac{\bar{n}\gamma t}{1+\bar{n}\gamma t}\right)^l \binom{n+l-j}{n-j}\binom{n}{j} \rho_{n+l-j,n+l-j}(0), \qquad (26)$$

where the phenomenological parameters $\bar{n}$ and γ are the mean reservoir quantum number and the heating rate, respectively. Equation (26) is valid under the assumptions of high-T reservoir and for times much smaller than the thermalization time, $\gamma t \ll 1$. Such conditions turn out to be verified in the experiments described in Ref. [31]. In this experimental situation, the Markovian behavior of the heating function, before thermalization takes place, is simply given by $\langle n(t) \rangle = \bar{n}\gamma t$. Note that in the limit $r \gg 1$, corresponding to an infinite bandwidth Markovian reservoir, Eq. (20) becomes $\langle n(t) \rangle \simeq 2\alpha^2 KTt/\pi$. Comparing this expression of the heating function with the one of the experiments one can identify $\bar{n} = KT/\omega_0$ and $\gamma = 2\alpha^2 \omega_0/\pi$. It is worth underlining, however, that according to Eq. (26), only the product $\bar{n}\gamma$ may be deduced from the experimental data.

In order to describe our experimental proposal for simulating QBM, it is important to look in more detail at the procedure used in Ref. [31] to obtain the time evolution of the populations P_n. This allows to deduce from the experi-

mental data the characteristic parameters of the amplitude reservoir used in the experiments. In Ref. [31], after preparing a superposition of Fock states, the amplitude noise is applied for a fixed time of $\bar{t}=3$ μs after which the populations P_n are measured. In order to measure P_n the ion is irradiated with a pair of Raman beams tuned to the first blue sideband for various probe times t_p, and $P_-(t_p)$ is measured from the fluorescence signal. From the $P_-(t_p)$ data the populations $P_n(\bar{t})$ are finally extracted with a single value decomposition [31]. In order to observe the time evolution of P_n one can proceed in two equivalent ways. Either one changes the interval of time $\bar{t}$ during which the amplitude noise is applied, or one fixes $\bar{t}$ and varies the variance of the applied noise $\langle V^2 \rangle$. In fact, as shown in Ref. [31], the variance of the random noise used to simulate an amplitude reservoir is $2\langle V^2 \rangle = \bar{n}\gamma t$. Practically, increasing the fluctuations of the random electric field applied at the trap electrodes is equivalent to an increase in the heating function $\langle n \rangle = \bar{n}\gamma t$. Since $P_n(t)$, as given by Eq. (20), depends only on $\bar{n}\gamma t$, one can obtain the time evolution of the populations simply by changing $\langle V^2 \rangle$. This is the method used in Ref. [31] for measuring the populations, so, in fact, in the experiment $P_n(\langle V^2 \rangle)$ is recorded. In principle, by using the relation $\bar{n}\gamma = \langle V^2 \rangle / \bar{t}$, with $\bar{t}=3$ μs, one could directly obtain the characteristic parameter of the reservoir from the value of the noise voltage applied to the trap electrodes. However, unknown geometrical factors in converting the voltage to variations in the secular frequency prevent a direct comparison. If we indicate with $\langle V^2 \rangle_{appl} = c\langle V^2 \rangle$ the fluctuations of the voltage applied to the electrodes, with $c \in \mathbb{R}$, fitting the experimental data according to the theoretical law given by Eq. (26), allows to extrapolate the factor c and therefore $\bar{n}\gamma$. It is not difficult to show that, in the experiment on the decay (heating) of a Fock state due to interaction with engineered amplitude reservoir (see Fig. [15] of Ref. [31]), $c \simeq 10$ and hence, for $\bar{t}=3$ μs and $\langle V^2 \rangle = 0.25V^2$, $\bar{n}\gamma \simeq 0.84 \cdot 10^7$ Hz. Note that, in the experiment, $\langle V^2 \rangle$ is varied from 0 to $0.3V^2$.

At this point we are ready to describe our experimental proposal for observing the non-Markovian dynamics of the heating function and, in general, for simulating the dynamics of a quantum Brownian particle.

V. EXPERIMENTAL PROPOSAL FOR SIMULATING QBM

It is well known that non-Markovian features usually occur in the dynamics for times $t \ll \tau_R = 1/\omega_c$. In general, since $\omega_c \gg \omega_0$ and typically $\omega_0 \simeq 10^7$ Hz for trapped ions, this means that deviations from the Markovian dynamics appear for times $t \ll 0.1$ μs. This is the reason why the initial quadratic behavior of the heating function is not observed in the experiments, wherein the typical time scales go from 1 to 100 μs.

A way to force non-Markovian features to appear is to "detune" the trap frequency from the reservoir spectral density. This corresponds, for example, to the case in which $r = \omega_c/\omega_0 = 0.1$. In this case the reservoir correlation time is bigger than the period of oscillation of the ion and this leads to the oscillatory behavior of the heating function predicted
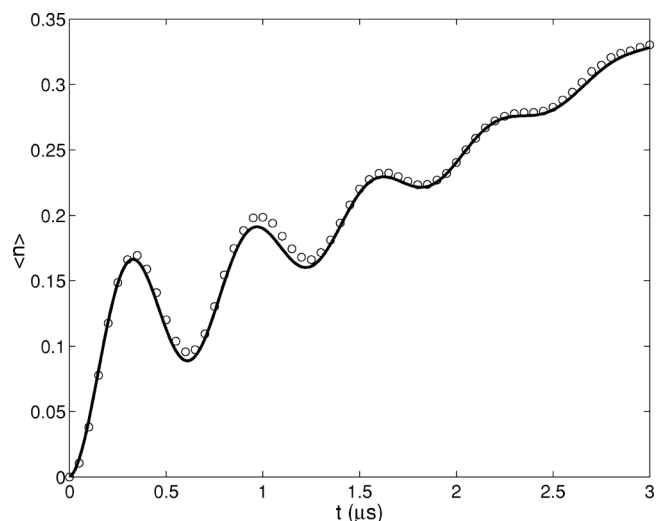

FIG. 1. Time evolution of the heating function for $2\alpha^2 KT/\pi = 0.84 \times 10^9$ Hz, $\omega_c = 1$ MHz, and $r = 0.1$. Solid line is the analytical and circles the simulation result.

by Eq. (20) for $r \ll 1$. Under this condition $\tau_c = 1$ μs, and therefore the non-Markovian features show up in the time evolution and can be measured. Detuning the trap frequency from the reservoir, however, decreases the effective coupling between the system and the environment and, for this reason, in order to obtain values of the heating function big enough to be measured we need to increase either the coupling constant α^2, which correspond to an increase in the intensity of the voltage applied to the electrodes, or the strength of the fluctuations $\langle V^2 \rangle$, which correspond to an increase in the effective temperature of the reservoir.

Let us look in more detail to Eq. (20). For $r \ll 1$ this equation becomes

$$\langle n(t) \rangle \simeq \frac{2\alpha^2 KT}{\pi\omega_c} r^2 \{\omega_c t + [1 - e^{-\omega_c t}\cos(\omega_0 t)]$$
$$- re^{-\omega_c t}\sin(\omega_0 t)\}. \quad (27)$$

In the comparison with the experiment done in the preceding section we have seen that the front factor $2\alpha^2 KT/\pi = \bar{n}\gamma \simeq 0.84 \times 10^7$ Hz. Increasing of two order of magnitude this front factor, and for $r = 0.1$, Eq. (27) predicts the behavior for the heating function shown in Fig. 1. We believe that for this range of $\langle n \rangle$ and of times the oscillations of the heating function could be experimentally measurable. For example, with the first technique described in Sec. IV B, using $\Omega = 10^6$ Hz and for $\omega_0 = 10^7$ Hz, the ground-state population transferred to the excited level due to off resonant excitation is of the order of 10^{-2}, which is one order of magnitude smaller than the variation of the heating function we want to measure (see Fig. 1). Also the other method described in the preceding section seems to be enough accurate to reveal the oscillatory behavior of the heating function in the conditions here examined. However, it is worth noting that, since the heating function does not depend now on the dimensionless variable $\bar{n}\gamma t$, but rather on $\omega_c t$, the time evolution can be obtained only
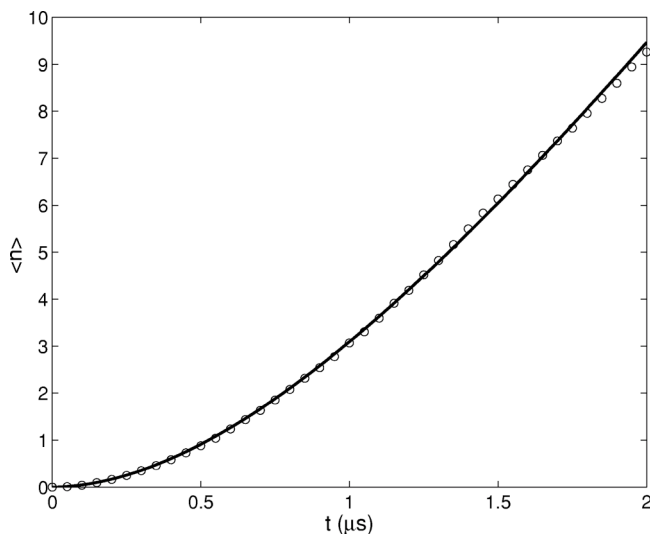
FIG. 2. Time evolution of the heating function for $2\alpha^2 KT/\pi$ $=0.84\times10^9$ Hz, $\omega_c=1$ MHz, and $r=10$. Solid line is the analytical and circles the simulation result.

by varying the duration of the time of application of the amplitude reservoir. This is not equivalent to a change in the applied voltage fluctuations, as it was in the Markovian case discussed in Sec. IV B.

Summarizing, in order to observe the virtual exchanges of phonons between the system and the reservoir, leading to the oscillations of the heating function, one needs to increase of two order of magnitude the coefficient $2\alpha^2 KT/\pi=\bar{n}\gamma$. This can be done either increasing the intensity or increasing the fluctuations of the applied noise, or combining an increase in the intensity with an increase in the fluctuations. Moreover one needs to use a low pass filter for the applied noise, as described in Sec. IV A, having cutoff frequency $\omega_c=0.1\omega_0$.

We now examine briefly the conditions for which the quadratic behavior of the heating function, could be observed. We remind that this is the case in which $r\gg1$ and the time evolution of the density matrix is of Lindblad type. In view of the considerations done at the beginning of this section, in order to reveal non-Markovian dynamics in a time scale of $1-100$ μs we need to have $\tau_R\gtrsim1$ μs, that is, for $r=10$, ω_0 $\lesssim0.1$ MHz. This means that one actually needs a "loose" trap. For example, for $\omega_0=100$ kHz, with the same applied noise used in Ref. [31]), i.e., $\bar{n}\gamma=0.84\times10^7$ Hz, the time evolution of the heating function is the one shown in Fig. 2.

If one wants to perform a fast measurement of $\langle n\rangle$, e.g., using the first method and assuming $\Omega=10^6$ Hz, the small value of the trap frequency makes it difficult not only to implement one of the two methods for measuring the heating function, but also to reach the initial ground state since the sidebands are not clearly resolved, and therefore resolved sideband cooling technique cannot be applied. It is worth

stressing, however, that contrarily to the case of a natural reservoir, which is always "in action," in the case of an engineered reservoir one can switch off the applied noise after a certain delay time and, assuming that the effect of the natural reservoir is negligible, measure the heating function without the severe requirement of big values of Ω. If one assumes that after an amplitude noise pulse the state of the ion does not change, then it is not necessary to perform a fast measurement of $\langle n\rangle$. Therefore one can work with smaller values of Ω, such that $\Omega/\omega_0\ll1$.

Concluding, while measuring the quadratic behavior of the heating function ($r\gg1$) could be a more challenging task from the experimental point of view, revealing the oscillatory non-Markovian behavior ($r\ll1$) appears to us in the grasp of the experimentalists, in the conditions we have analyzed in this section.

VI. CONCLUSIONS

In this paper we have studied the dynamics of a single harmonic oscillator coupled to a quantum reservoir at generic temperature T. In our analysis we have used both the analytic solution for the reduced density matrix and the NMWF method.

We have paid special attention to the non-Markovian heating dynamics typical of short times. In this regime the system time evolution is influenced by correlations between the system and the reservoir. For certain values of the system and reservoir parameters, virtual exchanges of energy between the system and its environment become dominant. These virtual processes strongly affect the short time dynamics and are responsible for the appearance of oscillations in the heating function (non-Lindblad-type dynamics).

Extending the ideas of using trapped ions for simulating quantum optical systems, we have proposed to simulate QBM with single trapped ions coupled to artificial reservoirs. We have carefully analyzed the possibility of revealing, by using present technologies, the non-Markovian dynamics of a single trapped ion interacting with an engineered reservoir and we have underlined the conditions under which non-Markovian features become observable.

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