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Calculus of Variations

Extension of convex functions from a hyperplane to a half-space

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Abstract
It is shown that a possibly infinite-valued proper lower semicontinuous convex function on $\mathbb{R}^n$ has an extension to a convex function on the half-space $\mathbb{R}^n \times [0, \infty)$ which is finite and smooth on the open half-space $\mathbb{R}^n \times (0, \infty)$. The result is applied to nonlinear elasticity, where it clarifies how the condition of polyconvexity of the free-energy density $\psi(Dy)$ is best expressed when $\psi(A) \to \infty$ as $\det A \to 0+$.

Mathematics Subject Classification 26B25 · 49J45 · 74B20

1 Introduction

The main purpose of this paper is to prove the following theorem, giving an extension of a possibly infinite-valued proper lower semicontinuous convex function on $\mathbb{R}^n$ to a convex function on the half-space $\mathbb{R}^n \times [0, \infty)$ which is finite and smooth on the open half-space $\mathbb{R}^n \times (0, \infty)$.

Theorem 1 *Let $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then there exists a lower semicontinuous convex function*

$$\varphi : [0, \infty) \times \mathbb{R}^n \to (-\infty, \infty], \ \varphi = \varphi(x, y),$$

such that

(i) $\varphi(0, y) = \Phi(y)$ *for all* $y \in \mathbb{R}^n$,
(ii) $\lim_{x\to 0+} \varphi(x, y) = \Phi(y)$ *for each* $y \in \mathbb{R}^n$.
(iii) $\varphi : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ *is smooth.*

If $\Phi \geq 0$*, then* φ *can be chosen so that* $\varphi \geq 0$*, and if* $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ *is continuous,* $\varphi : [0, \infty) \times \mathbb{R}^n \to (-\infty, \infty]$ *can be chosen to be continuous. If* Φ *is strictly convex on* $\operatorname{dom} \Phi := \{y \in \mathbb{R}^n : \Phi(y) < \infty\}$ *then* φ *can be chosen to be strictly convex on* $(0, \infty) \times \mathbb{R}^n$.

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The following result is an immediate consequence (setting $\Phi^{(j)}(y) = \varphi(j^{-1}, y)$).

Corollary 1 *Let $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Then there exists a sequence $\Phi^{(j)}$ of smooth convex functions on $\mathbb{R}^n$ such that $\lim_{j\to\infty} \Phi^{(j)}(y) = \Phi(y)$ for each $y \in \mathbb{R}^n$.*

The theorem applies, for example, to the case when Φ is the indicator function i_K of a nonempty closed convex subset $K \subset \mathbb{R}^n$, defined by

$$i_K(y) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \notin K. \end{cases}$$

With $K = \{0\}$ a suitable smooth strictly convex extension is then given by $\varphi(x, y) = \theta(x, y) - \frac{x}{x+1}$, where

$$\theta(x, y) = \begin{cases} \frac{|y|^2}{x}, & \text{if } x > 0, y \in \mathbb{R}^n, \\ 0, & \text{if } x = 0, y = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{1}$$

which follows as a special case of (4) (or (10)) below. (We note that with y momentum and x density the convexity of θ plays an important role in optimal transport, as noted in [5].)

The theorem was motivated by the problem of proving the existence of energy minimizers in 3D nonlinear elasticity under the assumption of polyconvexity of the free-energy density. In [3] an apparently weaker version of the polyconvexity condition given in [2] was used. That this version is indeed weaker follows from Theorem 1, and this is explained in Sect. 3.

2 Proof of theorem 1

We first show the existence of an extension $\tilde{\varphi}$ satisfying (i), (ii), which in addition is (strictly) decreasing in x, giving two different proofs. The first proof is the more direct and provides a wide range of possible extensions, while the second uses infimal convolution and is convenient for proving the assertion in the theorem regarding strict convexity.

Proposition 1 *Under the assumptions of Theorem 1 there exists a lower semicontinuous convex extension $\tilde{\varphi} = \tilde{\varphi}(x, y)$ of Φ to $[0, \infty) \times \mathbb{R}^n$ that is finite for $x > 0$, decreasing in x, and such that $\lim_{x\to 0+} \tilde{\varphi}(x, y) = \Phi(y)$ for each $y \in \mathbb{R}^n$.*

1st Proof. We first note that Φ is the supremum of a family of affine functions:

$$\Phi(y) = \sup_{(\alpha,b)\in S} (\alpha + b \cdot y), \text{ for all } y \in \mathbb{R}^n, \tag{2}$$

for some nonempty set $S \subset \mathbb{R}^{n+1}$. This is a standard result; see, for example, [8, Proposition 3.1], [12, Theorem 12.1]. (In Remark 1 below we note that we can take the family of affine functions to consist of exact affine minorants, but this is not needed for the proof.)

Let $\psi : [0, \infty) \to [0, \infty)$ satisfy

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty. \tag{3}$$

We claim that

$$\tilde{\varphi}(x, y) := \sup_{(\alpha,b)\in S} (\alpha + b \cdot y - \psi(|\alpha| + |b|)\, x) \tag{4}$$

provides a suitable convex extension. Indeed by (2) $\tilde\varphi(0,y)=\Phi(y)$ for all $y\in\mathbb{R}^n$, and since it is the supremum of continuous affine functions $\tilde\varphi$ is convex and lower semicontinuous.

Given $x>0$, $y\in\mathbb{R}^n$, by (3) there exists $M(x,y)>0$ such that

$$\frac{\psi(|\alpha|+|b|)}{|\alpha|+|b|}>x^{-1}\max(1,|y|)\quad\text{if } |\alpha|+|b|>M(x,y). \tag{5}$$

Hence for $|\alpha|+|b|>M(x,y)$ we have

$$\alpha+b\cdot y-\psi(|\alpha|+|b|)x\le 0. \tag{6}$$

Therefore, since $\psi\ge 0$, $\tilde\varphi(x,y)\le\max(1,|y|)M(x,y)<\infty$, as required.

$\tilde\varphi(x,y)$ is nonincreasing in x, and can be made decreasing by adding $-x$ to $\tilde\varphi$.

Since $\tilde\varphi$ is lower semicontinuous

$$\begin{aligned}\Phi(y)=\tilde\varphi(0,y)&\le\liminf_{x\to0+}\tilde\varphi(x,y)\le\limsup_{x\to0+}\tilde\varphi(x,y)\\&\le\sup_{(\alpha,b)\in S}(\alpha+b\cdot y)=\Phi(y),\end{aligned} \tag{7}$$

so that

$$\lim_{x\to0+}\tilde\varphi(x,y)=\Phi(y). \tag{8}$$

2nd Proof. Define $\theta:\mathbb{R}^{n+1}\to[0,\infty]$ by (1). Note that θ is convex and lower semicontinuous on $\mathbb{R}^{n+1}$; the convexity follows, for example, from the identity

$$\begin{aligned}&\lambda\theta(x_1,y_1)+(1-\lambda)\theta(x_2,y_2)-\theta(\lambda(x_1,y_1)+(1-\lambda)(x_2,y_2))\\&\quad=\tfrac{\lambda(1-\lambda)}{\lambda x_1+(1-\lambda)x_2}\left|\sqrt{\tfrac{x_1}{x_2}}y_2-\sqrt{\tfrac{x_2}{x_1}}y_1\right|^2\ge0\end{aligned} \tag{9}$$

for $\lambda\in[0,1]$ and $(x_1,y_1),(x_2,y_2)\in(0,\infty)\times\mathbb{R}^n$, and examining the behaviour of θ along lines in $\mathbb{R}^{n+1}$.

Let $\tilde\varphi=\Phi\,\square\,\theta$ be the infimal convolution of Φ and θ with respect to $y\in\mathbb{R}^n$ defined by

$$(\Phi\,\square\,\theta)(x,y)=\inf_{y'\in\mathbb{R}^n}\left(\Phi(y')+\theta(x,y-y')\right). \tag{10}$$

The convexity of Φ and θ implies that the function

$$h(x,y,y'):=\Phi(y')+\theta(x,y-y') \tag{11}$$

is convex on $\mathbb{R}^{2n+1}$. Hence by [4, Prop. 8.26] $\tilde\varphi(x,y)=\inf_{y'}h(x,y,y')$ is convex in (x,y). Since Φ is proper, there exists $\bar y\in\mathbb{R}^n$ with $\Phi(\bar y)<\infty$. Therefore for $x>0$ we have that $\tilde\varphi(x,y)\le\Phi(\bar y)+\frac{|y-\bar y|^2}{x}<\infty$. Also $\tilde\varphi(0,y)=\min(\Phi(y),\infty)=\Phi(y)$, so that $\tilde\varphi$ is an extension of Φ. Furthermore,

$$\Phi(y)\ge\alpha+b\cdot y\text{ for all }y\in\mathbb{R}^n\text{ and some }\alpha\in\mathbb{R},b\in\mathbb{R}^n. \tag{12}$$

Hence for $x>0$

$$\tilde\varphi(x,y)\ge\inf_{y'\in\mathbb{R}^n}\left(\alpha+b\cdot y'+\frac{|y-y'|^2}{x}\right)=\alpha+b\cdot y-\frac{|b|^2x}{4}>-\infty, \tag{13}$$

so that $\tilde\varphi(x,y)$ is finite, and thus by convexity continuous on $(0,\infty)\times\mathbb{R}^n$. If $\tilde\varphi$ were not lower semicontinuous there would exist a sequence $(x_j,y_j)\to(0,y)$ and y'_j with

$$\sup_j\left(\Phi(y'_j)+\frac{|y_j-y'_j|^2}{x_j}\right)<\Phi(y). \tag{14}$$

In particular the left-hand side of (14) is bounded, and so, using (12), $y_j' \to y$. Thus by the lower semicontinuity of Φ the left-hand side is greater than or equal to $\Phi(y)$, a contradiction. If $x_j \to 0+$ and $y \in \mathbb{R}^n$ then by the lower semicontinuity $\Phi(y) \le \liminf_{j\to\infty} \tilde{\varphi}(x_j, y) \le \limsup_{j\to\infty} \tilde{\varphi}(x_j, y) \le \Phi(y)$, so that $\lim_{x\to 0+} \tilde{\varphi}(x, y) = \Phi(y)$ as required.

Clearly $\tilde{\varphi}(x, y)$ is nonincreasing in x. As defined it may not be decreasing (consider the case $\Phi \equiv 0$), but $\tilde{\varphi}(x, y) - x$ is decreasing in x and satisfies the other requirements. □

Corollary 2 *Assume in addition to the hypotheses of Proposition 1 that Φ is strictly convex on $\operatorname{dom}\Phi$. Then $\tilde{\varphi}$ can be chosen so that in addition it is strictly convex on $(0,\infty)\times\mathbb{R}^n$.*

Proof We use the construction in the second proof of Proposition 1. Fix $x > 0$. Given $y \in \mathbb{R}^n$, by (12) the minimum of $h(y, z) := \Phi(z) + \frac{|y-z|^2}{x}$ for $z \in \mathbb{R}^n$ is attained by some $z = y' \in \operatorname{dom}\Phi$, and the strict convexity of Φ on $\operatorname{dom}\Phi$ implies that y' is unique. Given distinct $y, \bar{y} \in \mathbb{R}^n$ let the corresponding unique minimizers be $y', \bar{y}'$ respectively. For $\lambda \in (0, 1)$ the strict convexity of h on $\mathbb{R}^n \times \operatorname{dom}\Phi$ implies that

$$\begin{aligned}\tilde{\varphi}(x, \lambda y + (1-\lambda)\bar{y}) &\le h(\lambda y + (1-\lambda)\bar{y}, \lambda y' + (1-\lambda)\bar{y}') \\ &< \lambda h(y, y') + (1-\lambda)h(\bar{y}, \bar{y}') \qquad (15)\\ &= \lambda\tilde{\varphi}(x, y) + (1-\lambda)\tilde{\varphi}(x, \bar{y}). \qquad (16)\end{aligned}$$

Hence $\tilde{\varphi}(x, y)$ is strictly convex in y.

To complete the proof we use the following lemma. □

Lemma 1 *Let $f : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be convex with $f(x, y)$ strictly convex in y for each x. If $\psi : (0,\infty) \to \mathbb{R}$ is strictly convex then $g(x, y) := f(x, y) + \psi(x)$ is strictly convex in (x, y).*

Proof g is convex. If g were not strictly convex then there would exist distinct pairs $(x_1, y_1), (x_2, y_2)$ and $\lambda \in (0, 1)$ with

$$\begin{aligned}&f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) + \psi(\lambda x_1 + (1-\lambda)x_2) \\ &\quad = \lambda f(x_1, y_1) + (1-\lambda) f(x_2, y_2) + \lambda\psi(x_1) + (1-\lambda)\psi(x_2). \qquad (17)\end{aligned}$$

It follows from (17) and the convexity of ψ that $\psi(\lambda x_1+(1-\lambda)x_2) = \lambda\psi(x_1)+(1-\lambda)\psi(x_2)$, and since ψ is strictly convex we must have $x_1 = x_2$. But then (17) contradicts the strict convexity of $f(x, y)$ in y. □

Now let $\psi : [0,\infty) \to \mathbb{R}$ be strictly convex and decreasing with $\psi(0) = 0$ (for example, $\psi(x) = -\frac{x}{x+1}$). Then, by Lemma 1, $\tilde{\varphi}(x, y) + \psi(x)$ is a suitable strictly convex extension. □

To complete the proof of Theorem 1 we mollify $\tilde{\varphi}$ as constructed in Proposition 1 with an x-dependent mollifier. Let $\rho = \rho(x, y) \ge 0$, $\rho \in C_0^\infty(\mathbb{R}^{n+1})$, $\operatorname{supp}\rho \subset (0, 1)\times\mathbb{R}^n$, $\int_{\mathbb{R}^{n+1}} \rho\, dx\, dy = 1$, and define for $(x, y) \in [0,\infty)\times\mathbb{R}^n$

$$\varphi(x, y) = \int_{\mathbb{R}^n}\int_0^1 \rho(x', y')\tilde{\varphi}(x(1-x'), y - xy')\, dx' dy'. \qquad (18)$$

The integral is well defined since $\tilde{\varphi}$ is convex on $[0,\infty)\times\mathbb{R}^n$ and thus bounded below by a linear function, and the convexity of $\tilde{\varphi}$ also implies that φ is convex. Since $\tilde{\varphi}$ is lower semicontinuous, by Fatou's Lemma (valid because $\tilde{\varphi}$ is bounded below by a linear function) φ is lower semicontinuous. Furthermore

$$\varphi(0, y) = \Phi(y). \qquad (19)$$

Making the change of variables $u = x(1-x')$, $v = y - xy'$ we have that for $x > 0$

$$\varphi(x, y) = x^{-(n+1)} \int_{\mathbb{R}^n} \int_0^\infty \rho\left(\frac{x-u}{x}, \frac{y-v}{x}\right) \tilde{\varphi}(u, v)\, du\, dv, \tag{20}$$

from which it follows that φ is smooth for $x > 0$.

We next note that for any $y \in \mathbb{R}^n$, the convexity of φ implies that

$$\varphi(x, y) \le (1-x)\varphi(0, y) + x\varphi(1, y), \tag{21}$$

so that by (19)

$$\limsup_{x \to 0+} \varphi(x, y) \le \Phi(y). \tag{22}$$

But also, since φ is lower semicontinuous,

$$\Phi(y) \le \liminf_{x \to 0+} \varphi(x, y). \tag{23}$$

Combining (22), (23) we see that

$$\lim_{x \to 0+} \varphi(x, y) = \Phi(y). \tag{24}$$

If $\Phi \ge 0$, then we can replace $\tilde{\varphi}$ by $\max(\tilde{\varphi}, 0)$, so that $\varphi \ge 0$ also.

Suppose that $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ is continuous, and let $x^{(j)} \to 0+$, $y^{(j)} \to y$ in $\mathbb{R}^n$. If $\Phi(y) = \infty$ then the lower semicontinuity of φ implies that $\varphi(x^{(j)}, y^{(j)}) \to \varphi(0, y) = \infty$. If $\Phi(y) < \infty$ then the continuity of Φ implies that $\Phi(z) < \infty$ for $|z - y|$ sufficiently small. By what we have proved the sequence $\Phi^{(j)}(z) := \varphi(x^{(j)}, z)$ of convex functions converges pointwise to Φ, and hence by [12, Theorem 10.8] the convergence is uniform on a neighbourhood of y, so that again $\varphi(x^{(j)}, y^{(j)}) \to \varphi(0, y) = \Phi(y)$. Hence $\varphi : [0, \infty) \times \mathbb{R}^n \to (-\infty, \infty]$ is continuous.

Finally, if Φ is strictly convex on $\mathrm{dom}\, \Phi$ then by Corollary 2 we can suppose that $\tilde{\varphi}$ is strictly convex on $(0, \infty) \times \mathbb{R}^n$, so that φ is strictly convex on $(0, \infty) \times \mathbb{R}^n$ by (18). □

Remark 1 In (2) we can take S to consist of all points $(\Phi(y_0) - b(y_0) \cdot y_0, b(y_0))$ where y_0 belongs to the domain $\mathrm{dom}\, \partial\Phi$ of the subdifferential $\partial\Phi$ of Φ and $b(y_0) \in \partial\Phi(y_0)$. That is Φ is the supremum of all its exact affine minorants. This fact is not typically given in standard texts on convex analysis, although [11, Corollary 3.21] gives such a result for points y where $\Phi(y) < \infty$. The result is stated (for Hilbert spaces) in the paper of Moreau [9, Section 8.c] (see also [10, Section 13]), and follows from his theorem [9, Section 8.b] (see also [12, Theorem 24.9]) that if Φ, Ψ are proper lower semicontinuous convex functions with $\partial\Phi(y) \subset \partial\Psi(y)$ for all $y \in \mathbb{R}^n$ then $\Phi = \Psi + c$ for some constant c. Indeed if we define

$$\Psi(y) = \sup_{y_0 \in \mathrm{dom}\, \partial\Phi, b(y_0) \in \partial\Phi(y_0)} \Phi(y_0) + b(y_0) \cdot (y - y_0), \tag{25}$$

then $\Phi \ge \Psi$ and for any $y_0 \in \mathrm{dom}\, \partial\Phi$ and $b(y_0) \in \partial\Phi(y_0)$ we have for all $y \in \mathbb{R}^n$

$$\Psi(y) \ge \Phi(y_0) + b(y_0) \cdot (y - y_0) \ge \Psi(y_0) + b(y_0) \cdot (y - y_0). \tag{26}$$

Hence $\Psi(y_0) = \Phi(y_0)$ and therefore $b(y_0) \in \partial\Psi(y_0)$. Hence by the result of Moreau $\Psi = \Phi + c$ for some constant c. But $\mathrm{dom}\, \partial\Phi$ is nonempty (for example because $\partial\Phi$ is maximal monotone) and so $c = 0$ and $\Psi = \Phi$.

Remark 2 It does not seem obvious how to construct a smooth extension $\varphi(x, y)$ that is decreasing in x. This does not immediately follow from the fact that $\tilde{\varphi}(x, y)$ is decreasing in x because the mollification (18) averages $\tilde{\varphi}$ over a range of values of y' that grows with x.

Remark 3 If Φ is not strictly convex on $\mathrm{dom}\,\Phi$ then the function φ cannot in general be chosen to be strictly convex on $\mathbb{R}^n \times (0, \infty)$. Indeed if $\Phi = 0$ then φ can only depend on x. To see this let $x > 0$, $y, y' \in \mathbb{R}^n$ and for $\varepsilon > 0$ note that

$$(x-\varepsilon, y') = \frac{x-\varepsilon}{x}(x, y) + \left(1 - \frac{x-\varepsilon}{x}\right)(0, z), \tag{27}$$

where $z := \varepsilon^{-1}(xy' - (x-\varepsilon)y)$, so that by convexity

$$\begin{aligned}\varphi(x-\varepsilon, y') &\leq \frac{x-\varepsilon}{x}\varphi(x, y) + \left(1 - \frac{x-\varepsilon}{x}\right)\varphi(0, z) \\ &= \frac{x-\varepsilon}{x}\varphi(x, y).\end{aligned} \tag{28}$$

Letting $\varepsilon \to 0$ we obtain $\varphi(x, y') \leq \varphi(x, y)$. Interchanging y, y' we deduce that $\varphi(x, y) = \varphi(x, y')$ as required.

Remark 4 An interesting open problem is to determine the pairs Φ_0 and Φ_1 of proper lower semicontinuous convex functions on $\mathbb{R}^n$ which are such that there is a convex function $\varphi : [0, 1] \times \mathbb{R}^n \to (-\infty, \infty]$ that is finite on $(0, 1) \times \mathbb{R}^n$ and interpolates between Φ_0 and Φ_1 in the sense that $\varphi(0, y) = \Phi_0(y)$, $\varphi(1, y) = \Phi_1(y)$ for all $y \in \mathbb{R}^n$ and

$$\lim_{x\to 0+} \varphi(x, y) = \Phi_0(y), \ \lim_{x \to 1-} \varphi(x, y) = \Phi_1(y) \text{ for each } y \in \mathbb{R}^n. \tag{29}$$

The set of such pairs (Φ_0, Φ_1) is clearly convex. In the case $\Phi_0 = 0$, Remark 3 shows that the only possibility is that Φ_1 is constant, while in the case $\Phi_0 = i_{\{0\}}$ Example 1 below shows that any convex $\Phi_1 : \mathbb{R}^n \to \mathbb{R}$ satisfying $\lim_{|y|\to\infty} \frac{\Phi_1(y)}{|y|} = \infty$ is possible.

Setting $C = \{0, 1\} \times \mathbb{R}^n$ the problem is seen to be related to that of extending a convex function on $C \subset \mathbb{R}^s$ to a convex function on its convex hull $\mathrm{co}(C)$. This is studied for C compact in [6] and for general C in [13] (but without any assertion of continuity of the extension as C is approached as in (29)). When C is compact and convex the question of extending a smooth convex function on C to a smooth convex function on $\mathbb{R}^s$ is discussed in [1].

We give two examples of explicit constructions of convex extensions, using the two methods in the different proofs of Proposition 1. In neither example do we need to mollify $\tilde{\varphi}$ since it is already smooth.

Example 1 Let $\Phi = i_{\{0\}}$ be the indicator function of 0 as described in the introduction. Then $\mathrm{dom}\,\partial\Phi = \{0\}$ and $\partial\Phi(0) = \mathbb{R}^n$, so that (4) with S given by $\{(0, b) : b \in \mathbb{R}^n\}$ and $\psi(t) = c_p t^{\frac{p}{p-1}}$, $p > 1$, $c_p = (p-1)p^{\frac{-p}{p-1}}$ gives $\tilde{\varphi}(x, y) = \sup_{b\in\mathbb{R}^n}(b \cdot y - c_p|b|^{\frac{p}{p-1}}x)$. An elementary calculation then shows that

$$\tilde{\varphi}(x, y) = \begin{cases} \frac{|y|^p}{x^{p-1}}, & x > 0 \\ i_{\{0\}}(y), & x = 0 \end{cases}, \tag{30}$$

which is smooth for $x > 0$ if p is an even integer. In fact it is not hard to check that a more general convex extension which is smooth for $x > 0$ is given by

$$\tilde{\varphi}(x, y) = \begin{cases} x\eta(\frac{y}{x}), & x > 0 \\ i_{\{0\}}(y), & x = 0 \end{cases}, \tag{31}$$

where $\eta : \mathbb{R}^n \to \mathbb{R}$ is convex, smooth, and such that $\lim_{|y|\to\infty} \frac{\eta(y)}{|y|} = \infty$.

Example 2 Let $n = 1$ and

$$\Phi(y) = \begin{cases} -\ln y, & y > 0 \\ \infty, & y \leq 0 \end{cases}. \tag{32}$$

Then an elementary calculation shows that

$$\begin{aligned} \tilde{\varphi}(x, y) &:= (\Phi \,\square\, \theta)\,(x, y) \\ &= \begin{cases} -\ln\left(\frac{1}{2}(y + \sqrt{y^2 + 2x})\right) + \frac{1}{4x}(\sqrt{y^2 + 2x} - y)^2, & x > 0 \\ \Phi(y), & x = 0 \end{cases}. \end{aligned} \tag{33}$$

3 Polyconvexity conditions

In this section we give an application of Theorem 1 to 3D nonlinear elasticity. Denote by $M^{3\times3}$ the space of real 3×3 matrices. Consider an elastic body occupying a bounded open set $\Omega \subset \mathbb{R}^3$ in a reference configuration. The total free energy at a constant temperature corresponding to a deformation $y : \Omega \to \mathbb{R}^3$ is given by

$$I(y) = \int_\Omega \psi(Dy(x))\,dx, \tag{34}$$

where the free-energy density $\psi : M_+^{3\times3} \to [0, \infty)$, and $M_+^{3\times3} := \{A \in M^{3\times3} : \det A > 0\}$.

To help prevent interpenetration of matter it is usually assumed that

$$\psi(A) \to \infty \text{ as } \det A \to 0+, \tag{35}$$

which implies that if $I(y) < \infty$ then $\det Dy(x) > 0$ for a.e. $x \in \Omega$.

In order to prove existence of an absolute minimizer of I it is necessary to suppose, among other hypotheses, that ψ satisfies a suitable convexity condition. The convexity condition assumed in [2] (see [7] for a clear and more recent exposition) is that ψ is *polyconvex*, that is there is a convex function $g : M^{3\times3} \times M^{3\times3} \times [0, \infty) \to \mathbb{R} \cup \{\infty\}$ such that

$$\psi(A) = g(A, \operatorname{cof} A, \det A) \text{ for all } A \in M_+^{3\times3}, \tag{36}$$

where $\operatorname{cof} A$ denotes the matrix of cofactors of A. Given $\delta > 0$, define $E_\delta = \{(A, \operatorname{cof} A, \delta) : \det A = \delta\}$. Since, as is proved in [2, Theorem 4.3], the convex hull of E_δ in $M^{3\times3} \times M^{3\times3} \times \mathbb{R} \cong \mathbb{R}^{19}$ is equal to $M^{3\times3} \times M^{3\times3} \times \{\delta\}$, it follows from (36) that $g(A, H, \delta) < \infty$ for all $A, H \in M^{3\times3}$ and $\delta > 0$.

In [2] it was further assumed that g is continuous with $g(A, H, 0) = \infty$ for all $A, H \in M^{3\times3}$. Provided that $\psi(A) \to \infty$ as $|A| \to \infty$ this implies that (35) holds.

Later, in [3] (see also [7]) it was observed that existence could be proved if one only assumes (35) and that (36) holds for a convex $g : M^{3\times3} \times M^{3\times3} \times (0, \infty) \to \mathbb{R}$. But it is not immediately obvious that this really is a weaker hypothesis. Applying Theorem 1 we see that it is.

Theorem 2 *There exists a smooth polyconvex function $\psi : M_+^{3\times3} \to [0, \infty)$ satisfying (35) for which the corresponding g is continuous but does not satisfy $g(A, H, 0) = \infty$ for all $A, H \in M^{3\times3}$.*

Proof Let $V = \{(A, \operatorname{cof} A) : \det A = 0\}$. Then V is a closed subset of $M^{3\times3} \times M^{3\times3}$ and $(\mathbf{1}, \mathbf{1}) \notin V$, where $\mathbf{1}$ is the identity 3×3 matrix. Let $r > 0$ be such that $\det A > 0$ if $|A - \mathbf{1}| \leq r$, where $|\cdot|$ denotes the Euclidean norm on $M^{3\times 3}\cong\mathbb{R}^9$. Define $\Phi : M^{3\times 3}\times M^{3\times 3}\to[0,\infty]$ by

$$\Phi(A,H)=\begin{cases}\frac{1}{r^2-|A-\mathbf{1}|^2}+\frac{1}{r^2-|H-\mathbf{1}|^2} & \text{if } |A-\mathbf{1}|<r, |H-\mathbf{1}|<r\\ \infty & \text{otherwise.}\end{cases} \tag{37}$$

Then Φ is convex and continuous, so that by Theorem 1 there exists a continuous convex function $g : M^{3\times 3}\times M^{3\times 3}\times[0,\infty)\to[0,\infty]$ such that $g(A,H,0)=\Phi(A,H)$ for all $A,H\in M^{3\times 3}$ and $g(A,H,\delta)$ is smooth for $\delta>0$. Define $\psi(A)=g(A,\operatorname{cof}A,\det A)+|A|^2$. Then $\psi : M^{3\times 3}_+\to[0,\infty)$ is smooth and polyconvex, $\psi(A)\geq|A|^2$ and $g(\mathbf{1},\mathbf{1},0)<\infty$. If $\det A^{(j)}\to 0+$ then we may assume either that $|A^{(j)}|\to\infty$, in which case $\psi(A^{(j)})\to\infty$, or that $A^{(j)}\to A\in M^{3\times 3}$ with $\det A=0$, when $\psi(A^{(j)})=g(A^{(j)},\operatorname{cof}A^{(j)},\det A^{(j)})\to g(A,\operatorname{cof}A,0)=\infty$. Hence (35) holds. □

From the point of view of mechanics, Theorem 2 is unsatisfactory because the ψ constructed does not satisfy the physically necessary *frame-indifference condition*

$$\psi(RA)=\psi(A) \text{ for all } R\in SO(3), A\in M^{3\times 3}_+, \tag{38}$$

which is not used for the proofs of existence in [2, 3]. In addition one would like an example which is also *isotropic*, so that

$$\psi(AQ)=\psi(A) \text{ for all } Q\in SO(3), A\in M^{3\times 3}_+. \tag{39}$$

However we can adapt Example 1 to give a frame-indifferent and isotropic example.

Example 3 The frame-indifferent and isotropic function

$$\psi(A)=\frac{|A|^2}{\det A} \tag{40}$$

is polyconvex with corresponding $g : M^{3\times 3}\times M^{3\times 3}\times[0,\infty)\to[0,\infty]$ given by

$$g(A,H,\delta)=\begin{cases}\frac{|A|^2}{\delta}, & A,H\in M^{3\times 3}, \delta>0\\ 0, & (A,H,\delta)=(0,0,0)\\ \infty, & \text{otherwise}\end{cases} \tag{41}$$

and $\psi(A)\to\infty$ as $\det A\to 0+$.
That g is convex and lower semicontinuous follows as for θ (see (9)), while Hadamard's inequality $|A|^3\geq 3^{\frac{3}{2}}\det A$ implies that $\psi(A)\geq 3(\det A)^{-\frac{1}{3}}$.

If ψ is polyconvex and frame-indifferent, we can without loss of generality suppose that the corresponding g satisfies the invariance condition

$$g(RA,RH,\delta)=g(A,H,\delta) \text{ for all } R\in SO(3), A,H\in M^{3\times 3}, \delta\in[0,\infty). \tag{42}$$

Indeed we can replace g by

$$\tilde{g}(A,H,\delta)=\fint_{SO(3)} g(RA,RH,\delta)\,d\mu(R), \tag{43}$$

where

$$\fint_{SO(3)} f(R)\,d\mu(R):=\frac{\int_{SO(3)} f(R)\,d\mu(R)}{\mu(SO(3))}$$

and μ denotes Haar measure on $SO(3)$. Then $\tilde{g}$ satisfies (42), is convex, and by (38) and the relation $\operatorname{cof}(RA) = R\operatorname{cof}A$ we have

$$\begin{aligned}\psi(A) &= g(A, \operatorname{cof}A, \det A)\\ &= \fint_{SO(3)} g(RA, R\operatorname{cof}A, \det A)\, d\mu(R)\\ &= \tilde{g}(A, \operatorname{cof}A, \det A).\end{aligned} \tag{44}$$

But, as is well known, 0 belongs to the convex hull of $SO(3)$. Explicitly, $0 = \frac{1}{4}\sum_{i=0}^{3} R_i$, where $R_0 = \mathbf{1}$, $R_i = -\mathbf{1} + e_i \otimes e_i$ for $i = 1, 2, 3$, and e_i is the unit vector in the i^{th} coordinate direction. So for any A, H

$$\begin{aligned}\tilde{g}(0,0,0) &= \tilde{g}\left(\sum_{i=0}^{3}\frac{1}{4}R_iA, \sum_{i=0}^{3}\frac{1}{4}R_iH, 0\right)\\ &\leq \sum_{i=0}^{3}\frac{1}{4}\tilde{g}(R_iA, R_iH, 0)\\ &= \left(\sum_{i=0}^{3}\frac{1}{4}\right)\tilde{g}(A, H, 0) = \tilde{g}(A, H, 0),\end{aligned} \tag{45}$$

so that $\tilde{g}(0,0,0) = \infty$ implies $\tilde{g}(A, H, 0) = \infty$ for all A, H. Thus to construct an example we need $\tilde{g}(0,0,0) < \infty$, as in Example 3.

In Theorem 2 $g(\cdot,\cdot,0)$ is finite on an open subset of $M^{3\times3} \times M^{3\times3}$. However no such example is possible for $\tilde{g}$. Indeed, if $\tilde{g}(\cdot,\cdot,0) < \infty$ on an open set $U \subset M^{3\times3} \times M^{3\times3}$ then $\tilde{g}(A, H, 0) < \infty$ for (A, H) in the open set $\{(RA, RH) : (A, H) \in U, R \in SO(3)\}$, the convex hull of which is therefore open, and which contains $(0, 0)$ by (45). Therefore $g(A, H, 0) < \infty$ for (A, H) in some open ball $B(0, r)$ with centre $0 \in M^{3\times3} \times M^{3\times3}$. Since $g(\cdot,\cdot,0)$ is convex, it is continuous and bounded on $B(0, r/2)$. Similarly $\tilde{g}(\cdot,\cdot,1)$ is bounded on $B(0, r/2)$, so that by convexity $\tilde{g}$ is bounded on $B(0, r/2) \times [0, 1]$. But then $\psi(A) = \tilde{g}(A, \operatorname{cof}A, \det A)$ is bounded as $A \to 0$ with $\det A > 0$.

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Declarations

Conflict of interest The authors have no Conflict of interest.

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