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**Citation for published version:**
Chew, CY, Teng, G & Lai, YS 2024, 'Simulation of Erlang and negative binomial distributions using the generalized Lambert W function', *Journal of Computational Mathematics and Data Science*, vol. 10, 100092. https://doi.org/10.1016/j.jcmds.2024.100092

**Digital Object Identifier (DOI):**
10.1016/j.jcmds.2024.100092



**Published In:**
Journal of Computational Mathematics and Data Science

Journal of Computational Mathematics and Data Science 10 (2024) 100092

Contents lists available at ScienceDirect

Journal of Computational Mathematics and Data Science

journal homepage: www.elsevier.com/locate/jcmds

# Simulation of Erlang and negative binomial distributions using the generalized Lambert $W$ function

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ARTICLE INFO

*MSC:*
65C99
33E99

*Keywords:*
Inverse transform method
Lambert $W$ function

ABSTRACT

We present a simulation method for generating random variables from Erlang and negative binomial distributions using the generalized Lambert $W$ function. The generalized Lambert $W$ function is utilized to solve the quantile functions of these distributions, allowing for efficient and accurate generation of random variables. The simulation procedure is based on Halley's method and is demonstrated through the generation of 100,000 random variables for each distribution. The results show close agreement with the theoretical mean and variance values, indicating the effectiveness of the proposed method. This approach offers a valuable tool for generating random variables from Erlang and negative binomial distributions in various applications.

## 1. Introduction

Random variables from Erlang and negative binomial distributions are widely used in various fields, including engineering, finance, and telecommunications, due to their ability to model real-world phenomena with positive integer values and overdispersion, respectively. Generating random variables from these distributions is essential for conducting statistical analyses, simulations, and decision-making processes. However, obtaining random variables from these distributions can be challenging, especially when closed-form expressions for their quantile functions are not readily available.

The Lambert $W$ function, named after Johann Heinrich Lambert, holds a special place in various mathematical and engineering domains. Defined as the inverse function of $f(x) = xe^x$, it plays a vital role in fields such as physics [1], delay differential equations [2,3], and mathematics [4], among others. Further details about the Lambert $W$ function can be found in [5,6].

In recent years, the Lambert W function has also gained attention for its role in probability and statistics. The Erlang distribution, which is characterized by its shape parameter $k = 2$ and rate parameter $\lambda$, and the negative binomial distribution, which is characterized by its shape parameter $k = 2$ and success probability $p$, have been shown to be simulatable using the Lambert $W$ function [7]. Jodrá [8] has also proposed the utilization of the Lambert $W$ function in simulating random variables from the Lindley distribution, Poisson-Lindley distribution, or zero-truncated Poisson-Lindley distribution.

Additionally, a new family of skewed distributions was introduced by incorporating the Lambert $W$ function [9]. The application of the Lambert $W$ function in analyzing heavy-tailed data was also discussed [10]. Moreover, the simulation of Hawkes processes and survival times has been investigated in previous studies [11,12]. Apart from simulation or inverse transform methods, another area where the Lambert $W$ function could prove useful is in analyzing the distribution of the likelihood ratio test statistic [13–16].

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https://doi.org/10.1016/j.jcmds.2024.100092
Received 20 September 2023; Received in revised form 21 December 2023; Accepted 4 February 2024

While the Lambert $W$ function may not provide solutions to all problems, its generalization is imperative. Numerous works have explored the generalized Lambert $W$ function and its applications, including physical applications [17–19], delay differential equations, and modeling [20–22], as well as mathematics [23].

In this research article, we propose a simulation method based on the generalized Lambert $W$ function for generating random variables from Erlang and negative binomial distributions with shape parameter $k = 3$.

The main objective of this study is to develop a practical and efficient simulation procedure that leverages the properties of the generalized Lambert $W$ function to generate random variables accurately. To achieve our goal, we first analyze the real solutions of the equation $(w^2 - r)e^w = x$ and derive the quantile functions for the Erlang and negative binomial distributions in terms of the generalized Lambert $W$ function. Subsequently, we introduce a computational approach based on Halley's method to approximate the values of the generalized Lambert $W$ function efficiently.

To assess the effectiveness of our simulation method, we generate 100,000 random variables for each distribution and compare the results with the theoretical mean and variance values. The simulation results demonstrate the accuracy and reliability of our approach, as the generated random variables closely match the expected theoretical values.

In summary, this research article provides a novel simulation method based on the generalized Lambert $W$ function for generating random variables from Erlang and negative binomial distributions. The proposed approach offers an efficient and accurate solution to the challenge of obtaining random variables from these distributions, enabling researchers and practitioners to conduct robust statistical analyses and simulations in various domains.

## 2. Background and theory

In this section, we provide an overview of the Erlang and negative binomial distributions, as well as the Lambert $W$ function.

### 2.1. Erlang distribution

The Erlang distribution is a continuous probability distribution that is commonly used to model the waiting times between events in a Poisson process. It is characterized by two parameters: shape parameter $k$ and rate parameter $\lambda$. The probability density function (PDF) of the Erlang distribution is given by:

$$f(y; k, \lambda) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

where $y \geq 0$, $k$ is a positive integer, and $\lambda > 0$. The cumulative distribution function (CDF) of the Erlang distribution is expressed as [24]:

$$F(y; k, \lambda) = 1 - \sum_{t=0}^{k-1} \frac{(\lambda y)^t e^{-\lambda y}}{t!}. \tag{1}$$

The Erlang distribution is often employed to model the time required for $k$ events to occur in a system with an average event rate of $\lambda$.

### 2.2. Negative binomial distribution

The negative binomial distribution is a discrete probability distribution that is frequently used to model the number of successes in a sequence of independent and identically distributed Bernoulli trials before a specified number of failures occur. It is characterized by two parameters: the number of failures $r$ and the success probability $p$. The probability mass function (PMF) of the negative binomial distribution is given by:

$$P(N = n; r, p) = \binom{n + r - 1}{n} p^n (1-p)^r$$

where $n$ is a non-negative integer and $p \in (0, 1)$. The CDF of the negative binomial distribution is expressed as [25]:

$$F(n; r, p) = 1 - \sum_{i=0}^{r-1} \binom{r+n}{t} (1-p)^{r+n-t} p^t. \tag{2}$$

The negative binomial distribution is often used to model count data with overdispersion, where the variance is greater than the mean.

### 2.3. The Lambert $W$ function

The Lambert $W$ function, denoted as $W_k(x)$, is defined as the inverse function of the equation $we^w = x$. Here, $k \in \mathbb{Z}$ represents the solution in the $k$th branch. It has been established that there are two possible real branches: $W_0(x)$ and $W_{-1}(x)$. A comprehensive review of this function can be found in [5]. Additionally, Mező [26] conducted an in-depth analysis of the Lambert $W$ function and its generalization.

## 3. Main results

### 3.1. Transformation of CDFs

The quantile functions of the Erlang and negative binomial distributions can be obtained by solving specific equations. For the Erlang distribution with $k = 3$, Eq. (1) can be rewritten as:

$$1 - \left(1 + \lambda y + \frac{1}{2}\lambda^2 y^2\right) e^{-\lambda y} = u' = 1 - u, \tag{3}$$

where $y$ is the quantile, $\lambda$ is a parameter, and $u$ is a random variable. By introducing the variable

$$w = -\lambda y - 1, \tag{4}$$

the equation can be rewritten as:

$$\left(w^2 - r\right) e^{w} = x, \tag{5}$$

where $r = -1$ and $x = 2ue^{-1}$.

For the negative binomial distribution with $k = 3$, Eq. (2) can be simplified to:

$$1 - \sum_{t=0}^{2} \binom{3+n}{t} q^{3+n-t} p^t = u' = 1 - u, \tag{6}$$

where $n$ is the quantile, $p = 1 - q$ is the probability of success, and $u$ is a random variable. By introducing the variables $\alpha = qp^{-1}$ and

$$w = (n + \alpha + 2.5)\log q, \tag{7}$$

the equation can be written as:

$$(w^2 - r)e^{w} = x, \tag{8}$$

where $r = (0.25 - \alpha - \alpha^2)\log^2 q$ and $x = 2uq^{\alpha+1.5}p^{-2}\log^2 q$.

### 3.2. The generalized Lambert $W$ function

The generalized Lambert $W$ function emerges as the inverse of the equation

$$(w^2 - r)e^{w} = x, \tag{9}$$

where $r, x \in \mathbb{R}$ and $r \geq -1$. Considering the complex solutions reveals an infinite solutions, denoted as $W_k^{(r)}(x)$, where $k \in \mathbb{Z}$ represents the branch index.

However, in our study, our focus is primarily on the quantile functions of the Erlang and negative binomial distributions, emphasizing the investigation of real solutions exclusively.

A prior demonstration [27] established that when $e^{w} = P(w)$, where $P(w)$ is polynomial with real coefficients of degree $n$, it exhibits at most $n + 1$ real roots [27]. This finding leads us to the subsequent lemma.

**Lemma 3.1.** *Given $P(w)$ to be polynomial of degree $n$, the equation*

$$P(w)e^{w} = x$$

*where $x \neq 0$ has a maximum of $n + 1$ real roots.*

**Proof.** Since $e^{w} = P(w)$ has at most $n + 1$ real roots, we can transform the equation to $P(w)e^{-w} = 1$, or equivalently, without loss of generality, $P(w)e^{-w} = x' = \frac{1}{x}$.

Hence, the equation $P(w)e^{-w} = \frac{1}{x}$ having at most $n + 1$ real roots can be rewritten by reformatting the polynomial in terms of $-w$:

$$\begin{aligned} P(w) &= c_0 + c_1 w + c2w^2 + \cdots + c_n w^n \\ &= c_0 + (-c_1)(-w) + c_2(-w)^2 + \cdots + (-1)^n c_n(-w)^n. \end{aligned}$$

Let $d_n = (-1)^n c_n$. Thus, we have

$$P(w) = d_0 + d_1'(-w) + d_2'(-w)^2 + \cdots + d_n'(-w)^n.$$

Consequently, $P(w)e^{-w} = Q(-w)e^{-w} = Q(w')e^{w'} = x'$ has at most $n + 1$ roots. □

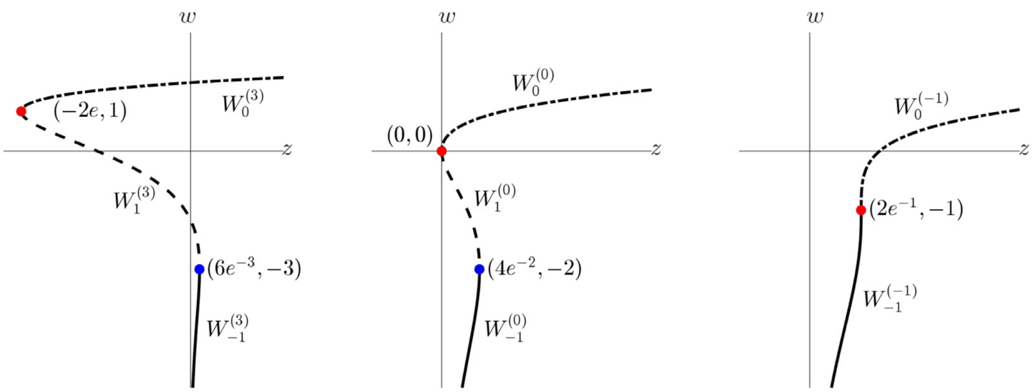


**Fig. 1.** Real branches for $r = 3$, $r = 0$ and $r = -1$.

From Lemma 3.1, we know that for all $x \neq 0$, Eq. (9) has at most three real roots.

Differentiating both sides of Eq. (9) with respect to $x$, we obtain the first derivative:

$$
\begin{aligned}
e^w \frac{d}{dx}(w^2 - r) + (w^2 - r)\frac{d}{dx}e^w &= 1 \\
\left\{2we^w + (w^2 - r)e^w\right\} \frac{dw}{dx} &= 1 \\
\frac{dw}{dx} &= \frac{e^{-w}}{w^2 + 2w - r},
\end{aligned}
$$

Equating the denominator to be zero, it is determined that when $r > -1$, two distinct real branch points $w_2$ and $w_1$ exist, given by

$$
w_j = \frac{-2 \pm \sqrt{4 + 4r}}{2} = -1 + (-1)^j \sqrt{1 + r} \quad (j = 1, 2).
$$

In cases where $r \leq -1$, there is at most one real solution. The behavior of the real branches can be observed in Fig. 1.

It is important to note that the branch point on the left is denoted as $(z_2, w_2)$, while the branch point on the right is denoted as $(z_1, w_1)$. Here, $z_j = (w_j^2 - r)e^{w_j}$.

In the case of $r > -1$, if $\max(z_2, 0) < x < z_1$, there are three potential real values for $W^{(r)}(x)$. The branch satisfying $w_2 \leq w$ is denoted as $W_0^{(r)}(x)$, $W_1^{(r)}(x)$ represents the range where $w_2 \leq w \leq w_1$, and $W_{-1}^{(r)}(x)$ corresponds to the scenario when $w \leq w_1$.

When $r = -1$, the two branch points converge, resulting in only one real solution. However, there are still two branches: $k = -1$ and $k = 0$.

### 3.3. *Quantile functions in terms of* $W^{(r)}(x)$

From Eq. (5), we know that for $r = -1$ and $x \leq 2e^{-1} = z_1$, Eq. (5) has only one real solution. Solving Eq. (5), we have

$$
w = W_{-1}^{(r)}(x),
$$

where $r = -1$ and $x = 2ue^{-1}$. From Eq. (4), we have that $y = -\frac{1}{\lambda}(w + 1)$, or

$$
y = -\frac{1}{\lambda}\left[W_{-1}^{(-1)}(2ue^{-1}) + 1\right]. \tag{10}
$$

For the negative binomial distribution, we know that $n$ is negative binomial distributed random variable and $n \geq 0$, which leads to $w = (n + \alpha + 2.5)\log q < 0$.

Since $w < 0$ and $F(n; 3, p) = 1 - u$ implies that $n$ is inversely proportional to $u$, we know that $w \propto x = 2uq^{\alpha+1.5}p^{-2}\log^2 q$. The only real branch that is increasing and negative is $W_{-1}^{(r)}$.

Let $u_0 = 1 > u_1 > \cdots > u_{n-1} > u_n > \cdots > 0$ such that

$$
w_n = (n + \alpha + 2.5)\log q,
$$

where $(w_n^2 - r)e^{w_n} = x_n$ and $x_n = 2u_nq^{\alpha+1.5}p^{-2}\log^2 q$. Thus, for $u_{n-1} < u \leq u_n$, we have $w_n \leq w < w_{n-1}$. Solving Eq. (7) for inverse transform method, we obtain:

$$
n = \left\lceil \frac{W_{-1}^{(r)}(x)}{\log q} - \alpha - 2.5 \right\rceil. \tag{11}
$$

### 3.4. Computation of $W_{-1}^{(r)}(z)$

To compute $W_{-1}^{(r)}(x)$, we utilize the Lagrange's inversion method. According to Lagrange's inversion method [28, Page 14], if $x = f(w)$, $x_0 = f(w_0)$, and $f'(w_0) \neq 0$, then we have:

$$w = w_0 + \sum_{k=1}^{\infty} \frac{(x-x_0)^k}{k!} \lim_{w\to w_0} \left[ \frac{d^{k-1}}{dw^{k-1}} \left\{ \frac{w-w_0}{f(w)-x_0} \right\}^k \right].$$

For $r = 0$, we have $W_k^{(0)}(x) = 2W_k\left(\pm\frac{\sqrt{x}}{2}\right)$, where $W_k(x)$ is the $k$th branch of the Lambert $W$ function. Using Lagrange's inversion method, we can obtain the series expansion of $w$ based on Eq. (12), where $w_0 = 2W_{-1}\left(-\frac{\sqrt{x}}{2}\right)$:

$$\begin{aligned} W_{-1}^{(r)}(x) &= 2\ W \\ &\quad + \sum_{t=1}^{\infty} \frac{r^t e^{2tW}}{t!} \lim_{w\to 2W} \left[ \frac{d^{t-1}}{dw^{t-1}} \left\{ \frac{w-2W}{(w^2-r)e^w - (4W^2-r)e^{2W}} \right\}^t \right] \\ &= 2\ W + \frac{4W^2}{z(4W^2+4\ W-r)} - \frac{16W^4(4W^2+8\ W-r+2)}{z^2(4W^2+4\ W-r)^3} + \dots . \end{aligned} \tag{12}$$

where $W = W_{-1}\left(-\frac{\sqrt{x}}{2}\right)$.

### 3.5. Simulation procedure

For the computation of $W_{-1}^{(r)}(x)$, we employ Halley's method. The iterative formula for Halley's method is given by:

$$w_{j+1} = w_j - \frac{f(w_j)}{f'(w_j) - \frac{1}{2}\frac{f''(w_j)}{f'(w_j)} f(w_j)}, \tag{13}$$

where $j = 0, 1, 2, \dots$. In our case, we can use Eq. (12) as the initial point for Halley's method. However, note that $W$ is not real when $x > 4e^{-2}$. Therefore, we define $w_0$ as the real part of first three terms of Eq. (12):

$$w_0 = \Re\left\{ 2\ W + \frac{4W^2}{z(4W^2+4\ W-r)} - \frac{16W^4(4W^2+8\ W-r+2)}{z^2(4W^2+4\ W-r)^3} \right\}. \tag{14}$$

We present the following pseudocode for simulating random variables from the Erlang and negative binomial distributions, given a known parameter $\lambda$ or $q$:

1. Generate a random variable $u$ from a uniform distribution in the interval $[0,1]$.
2. Set $r = -1$ for the Erlang distribution or $r = (0.25 - \alpha - \alpha^2)\log^2 q$ for the negative binomial distribution.
3. Compute the value of $x$ using $x = 2ue^{-1}$ or $x = 2uq^{\alpha+1.5}p^{-2}\log^2 q$.
4. Obtain $w_0$ using Eq. (14).
5. Use Halley's method to obtain $w_j$ with the desired precision.
6. Obtain the Erlang or negative binomial random variable using Eq. (10) or Eq. (11).
7. Repeat steps above.

## 4. Conclusion

In this study, we conducted a simulation of 100,000 random variables from the Erlang and negative binomial distributions based on the provided pseudocode. The results are summarized below.

For the Erlang distribution with parameters $k = 3$ and $\lambda = 0.5$, the simulation yielded a mean of 5.9991 and a variance of 11.9842, which closely align with the theoretical mean of 6 and variance of 12 (see Fig. 2).

For the negative binomial distribution with parameters $n = 3$ and $p = 3/13$, the simulated mean and variance were found to be 9.9993 and 43.4553, respectively. The theoretical mean and variance for this case are 10 and 43.3333 (see Fig. 3).

Given the wide-ranging applications of Erlang and negative binomial distributions, we believe that our exploration into the generalized Lambert $W$ functions could offer valuable insights. We hope this encourages other researchers to investigate and apply these findings within the domain of probability and statistics.

For those interested, the MATLAB code used for computing $W_{-1}^{(r)}(x)$ and performing the simulations is available in our GitHub repository: https://github.com/my-arch/simulation.

## CRediT authorship contribution statement

**C.Y. Chew:** Conceptualization, Methodology, Writing – original draft, Supervision, Funding acquisition. **G. Teng:** Validation, Writing – review & editing, Visualization, Project administration. **Y.S. Lai:** Software, Formal analysis.

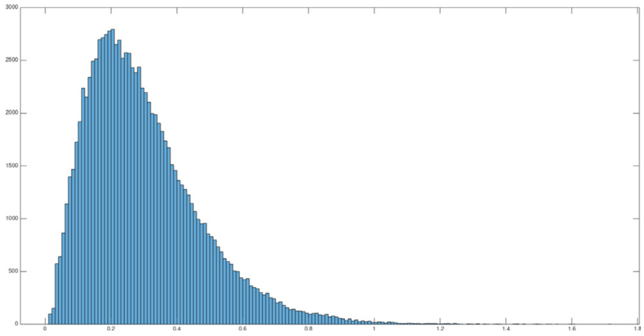

**Fig. 2.** 100,000 random variables generated from Erlang(3, $\lambda = 0.5$).

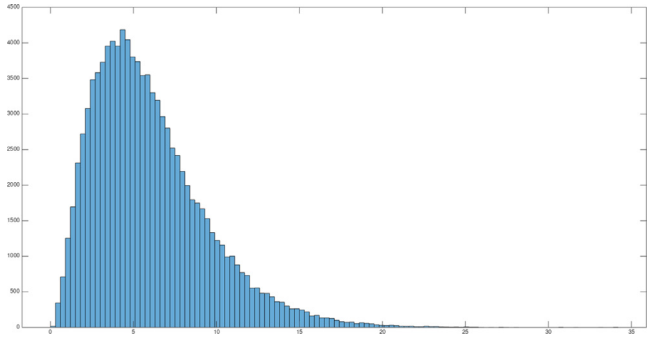

**Fig. 3.** 100,000 random variables generated from NB(3, $p = 3/13$).

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgments

We would like to acknowledge the support and funding provided by the MACS EmPOWER Research Grant Scheme (EmRGS), Malaysia 2022/2023, which made this work possible.

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