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A context for financial mathematics: ethics in the face of uncertainty.*

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Abstract

This paper identifies the mathematical Fundamental Theorem of Asset Pricing with the ethical statement that equality is a sufficient and necessary criterion for justice. The consequence of this identification is that it highlights the importance of ethics and social cohesion when society faces uncertainty, rather than scarcity, and it challenges the assumption that economic decisions should be based exclusively on utility maximisation.

Keywords: Arbitrage, asset pricing, probability, ethics, ultimatum game

2010 Mathematics Subject Classifications: 91G99, (91D10, 00A09, 60A99)

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This paper considers the development of mathematical probability before 1713 in the context of the ethics of fair exchange and relates this to the contemporary Fundamental Theorem

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of Asset Pricing of financial mathematics (e.g. [Shreve, 2004, Theorem 5.4.7]). The result of this comparison is the identification of the Theorem as an expression of the ethical statement that equality in exchange is a sufficient and necessary criterion for justice in exchange. The paper concludes that when an individual is faced with scarcity, decisions based on maximising utility, the dominant method in neo-classical economics, may be optimal, however when society is faced with uncertainty, ethics and social cohesion become paramount.

One experiment that undermines the assumption that economic agents are rational utility maximisers is the *Ultimatum Game*. The game involves two participants and a sum of money. The first player proposes how to share the money with the second participant. The division is made if the second participant accepts the split, but no-one gets anything if the first player's proposal is rejected. The key result is that if the money is not split 'fairly' (approximately equally) then the second player rejects the offer. This contradicts the assumption that people are rational utility maximising agents, since if they were the second player would accept any positive payment.

Research has shown that chimpanzees are rational maximisers while the willingness of the second player to accept an offer is dependent on age and culture. Older people from societies where exchange plays a significant role are more likely to demand a fairer split of the pot than young children or adults from isolated communities ([Murnighan and Saxon, 1998], [Henrich et al., 2006], [Jensen et al., 2007]). Fair exchange appears to be learnt behaviour developed in a social context and is fundamental to human society.

The idea that reciprocity in market exchange is important in creating social cohesion was discussed by Aristotle in *Nicomachean Ethics* [Kaye, 1998, p 51] and studied in the twentieth century by, amongst others, Malinowski, Mauss, and Sahlins. *Ethics* focuses on an individual's actions and presents morality in the context of balancing the four cardinal virtues, justice, courage, temperance and prudence, so-called 'virtue ethics'. Exchange is considered in Book 5 and Aristotle argues that it should be ruled by justice, and justice is founded on equality.

Given that there is a modern debate on whether the certainty of mathematics is relevant to economics it is ironic that Aristotle discussed the equality necessary for fair exchange in geometrical terms, while he was reluctant to employ mathematics in physics ([von Neumann and Morgenstern, 1967, p 3], [Hadden, 1994, Ch 3], [Crosby, 1997, pp 12–16], [Kaye, 1998, Ch 7]). Richard Hadden and Joel Kaye have argued that it was through an analysis of *Ethics* that scholastics, such as Albert the Great, in the mid thirteenth century, and then the Merton Calculators, in the fourteenth century, began the mathematisation of physics, which became the key distinguishing feature of western science ([Dear, 2001, Chapters 5–6], [Henry, 2008, Chapter 5]). However, on its own, analysis of Aristotle’s *Ethics* would not, automatically, lead to the mathematisation of physics, since it was not undertaken by Islamic scholars who had not lost the Greek texts in the Early Middle Ages.

Western Europe undertook rapid economic development between 1000–1300 CE, development that was hindered by the heterogeneity of currency, individual local magnates minted their own coin, and a distinction between illicit *usury*, the charging for the use of money and *interest*, a compensation for the opportunity-loss of lending money. It was in overcoming these twin financial obstacles, unique to Latin Europe, that European mathematics built its status.

To appreciate the consequences of the heterogeneity of currency, say an Italian merchant who had 100 Pisan Lira knew that they could change

100 Pisan Lira into 150 Florins

100 Pisan Lira into 180 Venetian Ducats

while their cousin in Florence could change

100 Florins into 110 Ducats.

The merchant could use 100 Pisan Lira to buy 180 Ducats. They could simultaneously arrange for their cousin to sell the 180 Ducats for 163 Florins, and the merchant, using a ‘fictitious Bill of Exchange’, would agree to sell the 163 Florins to buy 108 Lira, making 8 Lira, at no risk. Using a third currency to arbitrate a rate of exchange between two other currencies was

known as *arbitration of exchange* [Poitras, 2000, p 244], or simply *arbitrage*.

The impact on usury prohibitions was three-fold. Firstly most investments were asset backed; investment income had to be linked to ‘fruitful property’, which could range from the agricultural produce of a great estate down to a craftsman’s labour. Secondly, financial contracts became complex to disguise usury. For example the triple contract was constructed to enable entrepreneurs, who were unconnected by family ties to capital, to borrow money. The three contracts were a loan, an agreement to repay the lender a fixed amount, rather than as a share of the profit, and an insurance contract protecting the repayment [Poitras, 2000, p 38], providing the credit-enhancement and cashflow-transformation that are central to modern securitisation. Finally, commercial contracts came to the attention of canon lawyers and scholastic theologians.

Faced with these two obstacles, medieval merchants turned to the Islamic mathematical thinking that Fibonacci collated into the *Liber Abaci* in 1202 CE. The most obvious innovation of Fibonacci was the use of Indian numbers, which enabled merchants to record algorithms for performing the complex calculations they needed to analyse transactions that could span years and currencies. More intangible, but equally significant, were the *abaci* schools that emerged across Latin Europe to teach the techniques, with its descriptions of how arbitrage can be used to price assets [Sigler, 2002, p 180].

Canon lawyers scrutinised mercantile activity and by 1250 had established that usury was not present if there was uncertainty, risk premia were licit ([Rothbard, 1996, p 41], [Franklin, 2001, p 263]), and as a result the triple-contract was outlawed since the lender bore no risk. After 1250, theologians studied the newly translated *Nicomachean Ethics*, integrating it into Catholic doctrine by supplementing the four Cardinal virtues with the three Christian virtues of hope, faith and charity. While Albert the Great considered the implications on mathematics of regarding money as a measure of need, his student, Thomas Aquinas, considered the justice of exchange and “Whether it is lawful to sell something for more than it is worth?” through

a case presented by Stoic philosophers:

A grain merchant from Alexandria arrives at Rhodes, which is gripped by famine. The merchant knows that other merchants are following him with plentiful supplies of grain, though the town's inhabitants do not know this. How should the merchant price the grain he has?

The Roman jurist Cicero was typical in arguing that the merchant should charge a lower price based on the knowledge of the coming relief [Cicero, 1913, Book 1, XII]. Aquinas disagreed, the merchant may *think* there are more grain shipments on the way, but does not *know* [Aquinas, 1947, Second part of the second part, Q77, 3]. Since there was a risk of a loss, by under-pricing the grain when no more supply was coming, a premium price could be charged [Rothbard, 1996, p 53].

Aquinas' opinion was challenged by a leader of the Spiritual Franciscans, Pierre Jean Olivi. As a Spiritual Franciscan, Olivi argued that monks should limit their use property, not just its possession. As a Franciscan, Olivi preached to the urban communities and appears to have been a close observer of mercantile activity and recognised that merchants could demonstrate the virtues as much as a prince or a priest ([Rothbard, 1996, p 81], [Kaye, 1998, p 118]). In particular, Olivi combined his *fideism* with his knowledge of markets to argue that the probability of more grain appearing at Rhodes had certain reality beyond the empirical fact of market prices and he introduces the idea that market exchange is about equating expectations, not quoted prices ([Kaye, 1998, p 119], [Franklin, 2001, pp 265–267]). What united both Olivi and Aquinas was a realisation that if a just price was simply the market price it removed personal responsibility from economic activity [Kaye, 1998, p 98].

Olivi made these observations around 1280, in the centuries that followed, probability theory developed in a legal and theological context but there was no absolute or mathematical conception of probability, just as there was no absolute conception of space, which would come with Descartes, or time, which Newton introduced. The first record of a mathemati-

cal treatment of probability comes in Cardano's 1564 *Liber de Ludo Alea* where the critical observation, that the chance of rolling a six with a fair dice is 1 in 6, is made.

Up until the 1950s and Ore's re-assessment of his work [Ore, 1953], Cardano's contribution to probability theory had been generally ignored. In the context frequentist interpretations of probability, that dominated the nineteenth and early twentieth centuries, it was seen as incoherent. More recently, David Bellhouse [Bellhouse, 2005] has re-evaluated the *Liber* not as a mathematical document but as a humanist philosophical text, based on the fact that Cardano, himself, did not list it as one of his mathematical works. Bellhouse's hypothesis is that in the *Liber* Cardano is trying to establish under what grounds gambling can be considered ethical, and as a Renaissance scholar he would have done this in the context of *Nicomachean Ethics*. Cardano, following Aristotle, observes that

The most fundamental principle of all in gambling is simply equal conditions, e.g., of opponents, of bystanders, of money, of situation, of the dice box, and of the die itself. To the extent to which you depart from that equality, if it is in your opponent's favour, you are a fool, and if in your own, you are unjust. [Bellhouse, 2005]

One problem Cardano considered was the so-called *Problem of Points*, which comes out of the abaco tradition of using 'stories' to give examples of how to solve problems in commercial arithmetic. In this case the problem is how the capital tied up in a business partnership should be divided up if the venture has to finish prematurely [Sylla, 2003]. The *Problem* was eventually solved in 1654 in a correspondence between Pascal and Fermat, a result that is widely regarded as the start of mathematical probability because an algorithm for solving a general version of the problem was deduced, an algorithm that is recognisable today as a special case of the Cox-Ross-Rubenstein option pricing formula [Cox et al., 1979].

Christian Huygens followed Pascal and Fermat writing *On the Reckoning at Games of Chance* in 1656–1657. Huygens argues, much like Cardano, that

I take as fundamental for such [fair] games that the chance to gain something is worth so much that, if one had it, one could get the same in a fair game, that is a game in which nobody stands to lose. [Hald, 1990, p 69]

Thirty years later James Bernoulli expanded Huygens work in *The Art of Conjecturing*, published posthumously in 1713. The *Art* is made up of four parts, a commentary on Huygens' book, original work on calculating probabilities, applications of these ideas to games of chance and finally the application of the ideas to "civil, moral and economic affairs" [Hald, 1990, p 224]. While the first three sections are un-controversial, the final section has been problematic because Bernoulli considered situations where the sum of probabilities could be greater than one [Sylla, 2006, p 27], incoherent in a mathematical approach to probability. Edith Dudley Sylla recently re-evaluated the text and comparing Bernoulli's work to that of Huygens' and other contemporaries, de Witt and de Moivre, has concluded that

equity among associates or partners rather than probabilities in the sense of relative frequencies provided the foundation for the earliest mathematical probability theory. [Sylla, 2006, p 13]

The frequentist approach began to dominate the mathematical treatment of probability following Montmort's and de Moivre's simultaneous "defeat of chance" in the second decade of the eighteenth century [Bellhouse, 2008]. By the time Laplace published his two texts on probability (1812–1819), the approach taken by Huygens and Bernoulli had all but been forgotten. The relative frequencies of Quetelet's 'social physics' and the mechanistic models of Walras and Jevons came to dominate economics. Utility theory, advocated by Say and Bentham, became popular and Mill argued that (political) economics

is concerned with [man] solely as a being who desires to possess wealth, and who is capable of judging the comparative efficacy of means for obtaining that end. [Per-sky, 1995, quoting Mill, p 223]

Marshall, synthesised Mill's approach to economics with Darwinian metaphors of competition ([Backhouse, 1985, 10.1], [Thomas, 1991]) to lay the foundations of neo-classical economics, which, by the late 1960s, centred on rational individuals maximising expected utility in the face of scarcity.

Out of this tradition, Fischer Black and Myron Scholes, assisted by Robert C. Merton, considered the problem of pricing stock options. The approach they took was to dynamically hedge their position, making it theoretically riskless, and the consequence was that "it should not be possible to make sure profits" [Black and Scholes, 1973]. Within ten years of this work, Harrison and Kreps synthesised the approach with the abstract mathematics of measure theoretic probability to produce the Fundamental Theorem of Asset Pricing [Harrison and Kreps, 1979].

Measure theoretic probability was developed in 1933 by Kolmogorov [Kolmogorov, 1933 (1956)] to provide an axiomatic basis for mathematical probability. Kolmogorov's achievement was that he provided a framework for probability that incorporated both traditional frequentist interpretations and subjective, or Bayesian, approaches which were becoming significant at the time and relate to the Huygens-Bernoulli conception of probability. This was done by associating a probability with any, abstract, measure on a set, not simply the count of elements in a set and came out of the mathematical revolution of the late nineteenth century that saw the introduction of set theory and transfinite numbers.

The key statement of the Fundamental Theorem is that

A market is arbitrage free if and only if an equivalent martingale measure exists.

This is to the Black-Scholes equation what Newton's Laws of Motion are to his Law of Gravity, it is the theorem that explains an equation, and is the foundation of all mathematical approaches to pricing assets in a market.

To appreciate the meaning of the Theorem, recall that if an arbitrage exists a trader could carry out a series of transactions to generate a possible profit with no risk of a loss. A

martingale measure is a system of valid probabilities, summing to 1, such that asset prices now are the mathematical expectation of their value in the future. It is synthetically constructed on the basis of a known model of future asset price moves. If the model for future asset prices is not known with certainty, as assumed in Black-Scholes but impossible in reality, the market is incomplete, and there can be a range of no-arbitrage, or fair, prices.

For example, consider the bet where by you pay \$1.01 and, on the toss of a coin, you win nothing or \$1. There is no system of, valid, probabilities that mean the price of the bet is the expected winnings, and so there is no martingale measure. The structure of the bet also means that the seller of the bet will make a certain profit, there is an arbitrage.

In the context of virtue-ethics, scholastic theologians and philosophers argued that a risk-less profit was illicit. The case of the Merchant of Rhodes showed that in the face of a possible loss merchants could demand excess profits and Olivi argued that in such situations, expectations were as ‘real’ as market prices. From Cardano to Bernoulli, mathematical probability emerged in the context of establishing an equality between a current price and expected value, ‘probability’ created this equality independent of any concept of relative frequency. Mathematics relaxed the linkage between probability and frequency in the 1930s and was applied to pricing assets in uncertain environments in the 1970s. The resulting Fundamental Theorem established that for fairness to be present, no-arbitrage prices, there needed to be equality between the current price and expected value based on a synthetic probability measure. The Theorem is essentially restating the argument that justice in commercial exchange is based on equality. Mathematics had transformed the Black-Scholes equation, developed in the tradition of individualistic utility maximisation, into a Theory that places the justice of equitable exchange at its heart.

The Fundamental Theorem does not replace utility maximisation. It applies in the context of market participants who are both giving and taking prices, but is irrelevant to an individual agent who is a price taker, just as utility maximisation, alone, will fail to give an asset’s

price in a market. To understand the relevance of the Fundamental Theorem we turn to a contemporary of Fibonacci, Moses ben Maimon, (Maimonides). In the Christian tradition, building on Hesiod's *Works and Days*, humans were expelled from Eden into a world of scarcity. However, ben Maimon argued that God's punishment is not so much about scarcity but uncertainty. In Eden humans had perfect knowledge and it is the loss of this knowledge which is at the root of suffering: If we know what will happen we can manage scarcity [Perlman, 1997].

We note that the 'ethical' approach to pricing contracts discussed by Bernoulli and his predecessors, disappeared after the "defeat of chance" by Montmort and de Moivre and this removed the need for justice in exchange, which promotes social cohesion and risk-sharing. Neo-classical economics developed largely in a world, from Laplace to Knight and Keynes, considered deterministic while between 1950-1970 there were fixed exchange rates, stable interest rates and energy prices set by cartels. The Fundamental Theorem emerged only after the collapse of Bretton-Woods, when assets were traded in stochastic markets. We conclude that when society is faced by scarcity, utility maximisation is the preferred method, but when society faces uncertainty, the ethical basis of the Fundamental Theorem is essential.

References

- Thomas Aquinas. *Summa Theologica*. Benziger Bros, 1947.
- R. Backhouse. *A History of Modern Economic Analysis*. Blackwell, 1985.
- D. Bellhouse. Banishing Fortuna: Montmort and De Moivre. *Journal of the History of Ideas*, 69(4):pp. 559–581, 2008.
- D. Bellhouse. Decoding Cardano's Liber de Ludo Aleae. *Historia Mathematica*, 32:180–202, 2005.

- F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.
- Cicero. *On duties, Philosophical Treatises Volume XXI*, translated by W. Miller. Loeb Classical Library, Harvard University Press, 1913.
- J. C. Cox, S. A. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–263, 1979.
- A. W. Crosby. *The Measure of Reality*. Cambridge University Press, 1997.
- P. Dear. *Revolutionizing the Sciences*. Palgrave, 2001.
- J. Franklin. *The Science of Conjecture: Evidence and Probability before Pascal*. Johns Hopkins University Press, 2001.
- R. W. Hadden. *On the Shoulders of Merchants: Exchange and the Mathematical Conception of Nature in Early Modern Europe*. State University of New York Press, 1994.
- A. Hald. *A History of Probability and Statistics and their Applications before 1750*. Wiley, 1990.
- J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–401, 1979.
- J. Henrich et al. Costly punishment across human societies. *Science*, 312:1767–1770, 2006.
- J. Henry. *The Scientific Revolution and the Origins of Modern Science*. Palgrave, 2008.
- K. Jensen, J. Call, and M. Tomasello. Chimpanzees are rational maximizers in an ultimatum game. *Science*, 318:107–108, 2007.
- J. Kaye. *Economy and Nature in the Fourteenth Century*. Cambridge University Press, 1998.

- A. N. Kolmogorov. *Foundations of the Theory of Probability*. Julius Springer (Chelsea), 1933 (1956).
- J. K. Murnighan and M. S. Saxon. Ultimatum bargaining by children and adults. *Journal of Economic Psychology*, 19:415–445, 1998.
- Ø. Ore. *Cardano – The Gambling Scholar*. Princeton University Press, 1953.
- M. Perlman. Looking for ourselves in the mirror of the past. In B. B. Price, editor, *Ancient Economic Thought*, chapter 3, pages 61–75. Routledge Studies in the History of Economics, 1997.
- J. Persky. Retrospectives: The ethology of *Homo economicus*. *The Journal of Economic Perspectives*, 9(2):221–231, 1995.
- G. Poitras. *The Early History of Financial Economics, 1478–1776*. Edward Elgar, 2000.
- M. N. Rothbard. *Economic Thought before Adam Smith*. Edward Elgar, 1996.
- S. E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer, 2004.
- L. E. Sigler. *Fibonacci’s Liber Abaci*. Springer-Verlag, 2002.
- E. D. Sylla. Business ethics, commercial mathematics, and the origins of mathematical probability. *History of Political Economy*, 35:309–337, 2003.
- E. D. Sylla. Commercial arithmetic, theology and the intellectual foundations of Jacob Bernoulli’s *Art of Conjecturing*. In G. Poitras, editor, *Pioneers of Financial Economics: contributions prior to Irving Fisher*, pages 11–45. Edward Elgar, 2006.
- B. Thomas. Alfred Marshall on economic biology. *Journal of Financial Intermediation*, 3(1): 1–14, 1991.

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Wiley, 3rd edition, 1967.