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Characterising and Verifying the Core in Concurrent Multi-Player Mean-Payoff Games

Julian Gutierrez
Monash University, Clayton, Australia

Anthony W. Lin
University of Kaiserslautern-Landau, Germany
Max-Planck Institute for Software Systems, Kaiserslautern, Germany

Muhammad Najib
Heriot-Watt University, Edinburgh, UK

Thomas Steeples
University of Oxford, UK

Michael Wooldridge
University of Oxford, UK

Abstract
Concurrent multi-player mean-payoff games are important models for systems of agents with individual, non-dichotomous preferences. Whilst these games have been extensively studied in terms of their equilibria in non-cooperative settings, this paper explores an alternative solution concept: the core from cooperative game theory. This concept is particularly relevant for cooperative AI systems, as it enables the modelling of cooperation among agents, even when their goals are not fully aligned. Our contribution is twofold. First, we provide a characterisation of the core using discrete geometry techniques and establish a necessary and sufficient condition for its non-emptiness. We then use the characterisation to prove the existence of polynomial witnesses in the core. Second, we use the existence of such witnesses to solve key decision problems in rational verification and provide tight complexity bounds for the problem of checking whether some/every equilibrium in a game satisfies a given LTL or $\text{GR}(1)$ specification. Our approach is general and can be adapted to handle other specifications expressed in various fragments of LTL without incurring additional computational costs.

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1 Introduction

Concurrent games, where agents interact over an infinite sequence of rounds by choosing actions simultaneously, are one of the most important tools for modelling multi-agent systems. This model has received considerable attention in the research community (see,
In these games, the system evolves based on the agents’ choices, and their preferences are typically captured by associating them with a Boolean objective (e.g., a temporal logic formula) representing their goal. Strategic issues arise as players seek to satisfy their own goals while taking into account the goals and rational behaviour of other players. Note that the preferences induced by such goals are dichotomous: a player will either be satisfied or unsatisfied. However, many systems require richer models of preferences that capture issues such as resource consumption, cost, or system performance [19, 18, 22].

Mean-payoff games [29, 62] are widely used to model the quantitative aspects of systems. Whilst much research has been conducted on non-cooperative mean-payoff games and solution concepts such as Nash equilibrium (NE) and subgame perfect equilibrium (e.g., [58, 14, 15]), this paper focuses on a cooperative setting. In this setting, players can reach binding agreements and form coalitions to collectively achieve better payoffs or eliminate undesirable outcomes. As a result, NE and its variants may not be suitable for examining the stable behaviours that arise in these types of games. For example, in the Prisoner’s Dilemma game, players can avoid mutual defection, which is the unique NE, by establishing binding agreements [53]. Thus, analysing games through the lens of cooperative game theory poses distinct challenges and is important in and of itself. This paradigm is particularly relevant for modelling and analysing cooperative AI systems, which have recently emerged as a prominent topic [26, 25, 24, 7]. In these systems, agents are able to communicate and benefit from cooperation, even when their goals are not fully aligned. We illustrate that this is also the case in the context of mean-payoff games in Example 1.

We focus on a solution concept from cooperative game theory known as the core [5, 54, 40], which is the most widely-studied solution concept for cooperative games. Particularly, we study the core of mean-payoff games where players have access to finite but unbounded memory strategies. The motivation is clear, as finite-memory strategies are sufficiently powerful for implementing LTL objectives while being realisable in practice. Our main contribution is twofold: First, we provide a characterisation of the core using techniques from discrete geometry (cf. logical characterisation in [40, 39]) and establish a necessary and sufficient condition for its non-emptiness. We believe that our characterisation holds value in its own right, as it connects to established techniques used in game theory and economics. This has the potential to enable the application of more sophisticated methods and computational tools (e.g., linear programming solvers) in the area of rational verification [1, 38]. Second, we provide tight complexity bounds for key decision problems in rational verification with LTL and \( \text{GR}(1) \) specifications (see Table 1). \( \text{GR}(1) \) is a LTL fragment that has been used in various domains [35, 17, 31, 47] and covers a wide class of common LTL specification patterns [46]. Our approach to solving rational verification problems is very general and can be easily adapted for different LTL fragments beyond \( \text{GR}(1) \). This is the first work to study the core of mean-payoff games with finite but unbounded memory strategies, and to explore the complexity of problems related to the rational verification of such games in this setting.

**Related Work.** The game-theoretical analysis of temporal logic properties in multi-agent systems has been studied for over a decade (see e.g., [32, 23, 49, 45, 37, 61]). However, most of the work has focused on a non-cooperative setting. Recently, there has been...
an increased interest in the analysis of concurrent games in a cooperative setting. The core has been studied in the context of deterministic games with dichotomous preferences by [40, 39] using the logics ATL* [4] and SL [49]. However, as far as we are aware, there are no extensions of these logics that adopt mean-payoff semantics. Quantitative extensions exist [16, 12], and the core is studied in [12] using the logic SL[F] that extends SL with quantitative satisfaction value, but the semantics of these logics are not defined on mean-payoff conditions and thus cannot be used to reason about the core of mean-payoff games. In the stochastic setting, [36] examines the core in stochastic games with LTL objectives under the almost-sure satisfaction condition. The approach relies on qualitative parity logic [6] and is not applicable to mean-payoff objectives. Closer to our work is [57] which studies the core of multi-player mean-payoff games with Emerson-Lei condition [30] in the memoryless setting. Whilst memoryless strategies are easy to implement, finite-memory and arbitrary mathematical strategies offer greater richness. For instance, players can achieve higher payoffs and implement LTL properties with finite-memory strategies, which may not be possible with memoryless ones (see Example 1). The approach proposed in [57], which involves using a non-deterministic Turing machine to guess the correct strategies, is not applicable in the present work’s setting. This is because players may have finite but unbounded memory strategies, and as such, strategies may be arbitrarily large. To address this limitation, we propose a new approach that can handle such scenarios.

2 Preliminaries

Given any set $X$, we use $X^*$, $X^\omega$, and $X^+$ for, respectively, the sets of finite, infinite, and non-empty finite sequences of elements in $X$. For $Y \subseteq X$, we write $X - Y$ for $X \setminus Y$ and $X_{-i}$ if $Y = \{i\}$. We extend this notation to tuples $w = (x_1, \ldots, x_k, \ldots, x_n) \in X_1 \times \cdots \times X_n$, and write $w_{-k}$ for $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. Similarly, for sets of elements, we write $w_{-Y}$ to denote $w$ without each $x_k$, for $k \in Y$. For a sequence $v$, we write $v[t]$ or $v^t$ for the element in position $t + 1$ in the sequence; for example, $v[0] = v_0$ is the first element of $v$.

Mean-Payoff. For an infinite sequence of real numbers, $r^0r^1r^2 \cdots \in \mathbb{R}^\omega$, we define the mean-payoff value of $r$, denoted $\text{mp}(r)$, to be the quantity, $\text{mp}(r) = \lim \inf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} r^i$.

Temporal Logics. We use LTL [52] with the usual temporal operators, $X$ (“next”) and $U$ (“until”), and the derived operators $G$ (“always”) and $F$ (“eventually”). We also use GR(1) [8], a fragment of LTL given by formulae written in the following form:

$$(G F \psi_1 \land \cdots \land G F \psi_m) \to (G F \varphi_1 \land \cdots \land G F \varphi_n),$$

where each subformula $\psi_i$ and $\varphi_i$ is a Boolean combination of atomic propositions. Additionally, we also utilise an extension of LTL known as LTL$_{lim}$[9] that allows mean-payoff assertion such as $\text{mp}(v) \geq c$ for a numeric variable $v$ and a constant number $c$, which asserts
that the mean-payoff of $v$ is greater than or equal to $c$ along an entire path. The satisfaction of temporal logic formulae is defined using standard semantics. We use the notation $\alpha \models \varphi$ to indicate that the formula $\varphi$ is satisfied by the infinite sequence $\alpha$.

**Arenas.** An arena is a tuple $A = (N, \{A_c\}_{i \in N}, St, s_{init}, tr, lab)$ where $N$, $A_c$, and $St$ are finite non-empty sets of players, actions for player $i$, and states, respectively; $s_{init} \in St$ is the initial state; $tr : St \times \bar{A}c \rightarrow St$ is a transition function mapping each pair consisting of a state $s \in St$ and an action profile $\bar{a}c \in \bar{A}c = A_c \times \cdots \times A_c$, with one action for each player, to a successor state; and $lab : St \rightarrow 2^{AF}$ is a labelling function, mapping every state to a subset of atomic propositions.

A run $\rho = (s^0, \bar{a}c^0), (s^1, \bar{a}c^1) \cdots$ is an infinite sequence in $(St \times \bar{A}c)^\omega$ such that $tr(s^k, \bar{a}c^k) = s^{k+1}$ for all $k$. Runs are generated in the arena by each player $i$ selecting a strategy $\sigma_i$ that will define how to make choices over time. A strategy for $i$ can be understood abstractly as a function $\sigma_i : St^+ \rightarrow A_c$ which maps sequences (or histories) of states into a chosen action for player $i$. A strategy $\sigma_i$ is a finite-memory strategy if it can be represented by a finite state machine $\sigma_i = (Q_i, q_i^0, \delta_i, \tau_i)$, where $Q_i$ is a finite and non-empty set of internal states, $q_i^0$ is the initial state, $\delta_i : Q_i \times St \rightarrow Q_i$ is a deterministic internal transition function, and $\tau_i : Q_i \rightarrow A_c$ an action function. A memoryless strategy $\sigma_i : St \rightarrow A_c$ chooses an action based only on the current state of the environment. We write $\Sigma_i$ for the set of strategies for player $i$.

A strategy profile $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is a vector of strategies, one for each player. Once a state $s$ and profile $\vec{\sigma}$ are fixed, the game has an outcome, i.e., a path in $A$, denoted by $\pi(\vec{\sigma}, s)$. In this paper, we assume that players’ strategies are finite-memory and deterministic, as such, $\pi(\vec{\sigma}, s)$ is the unique path induced by $\vec{\sigma}$, that is, the sequence $s^0 s^1 s^2 \cdots$ such that $s^0 = s$, $s^{k+1} = tr(s^k, (\tau_i(q_i^k), \ldots, \tau_n(q_n^k)))$, and $q_i^{k+1} = \delta_i(q_i^k, s^k)$, for all $k \geq 0$. Note that such a path is ultimately periodic (i.e., a lasso). We simply write $\pi(\vec{\sigma})$ for $\pi(\vec{\sigma}, s_{init})$. We extend this to run induced by $\vec{\sigma}$ in a similar way, i.e., $\rho(\vec{\sigma}) = (s^0, \bar{a}c^0), (s^1, \bar{a}c^1), \ldots$. For an element of a run $\rho(\vec{\sigma})[k] = (s^k, \bar{a}c^k)$, we associate the configuration $\text{cfg}(\vec{\sigma}, k) = (s^k, q_1^k, \ldots, q_n^k)$ with $\tau_i(q_i^k) = \bar{a}c_i^k$ for each $i$.

**Multi-Player Games.** A multi-player game is obtained from an arena $A$ by associating each player with a goal. We consider multi-player games with mean-payoff goals. A multi-player mean-payoff game (or simply a game) is a tuple $G = (A, (w_i)_{i \in N})$, where $A$ is an arena and $w_i : St \rightarrow \mathbb{Z}$ is a function mapping, for every player $i$, every state of the arena into an integer number. Given a game $G = (A, (w_i)_{i \in N})$ and a strategy profile $\vec{\sigma}$, an outcome $\pi(\vec{\sigma})$ in $A$ induces a sequence $\text{lab}(\pi(\vec{\sigma})) = \text{lab}(s^0) \cdots$ of sets of atomic propositions, and for each player $i$, the sequence $w_i(\pi(\vec{\sigma})) = w_i(s^0)w_i(s^1) \cdots$ of weights. The payoff of player $i$ is $pay_i(\pi(\vec{\sigma})) = mp(w_i(\pi(\vec{\sigma})))$. By a slight abuse of notation, we write $pay_i(\vec{\sigma})$ for $pay_i(\pi(\vec{\sigma}))$, and $\pi(\vec{\sigma}) \models \varphi$ or $\vec{\sigma} \models \varphi$ for $\text{lab}(\pi(\vec{\sigma})) \models \varphi$ for some temporal logic formula $\varphi$.

**Solution Concept.** We focus on a solution concept known as the core [5, 54, 40]. To understand the concept of the core, it might be helpful to compare it with the NE and how each can be characterised by deviations. Informally, a NE is a strategy profile from which no player has any incentive to unilaterally deviate. On the other hand, the core comprises strategy profiles from which no coalitions of agents can deviate such that every agent in the coalition is strictly better off, regardless of the actions of the remaining players.

Formally, we say that a strategy profile $\vec{\sigma}$ is in the core if for all coalitions $C \subseteq N$, and strategy profiles $\vec{\sigma}_C$, there is some counter-strategy profile $\vec{\sigma}_{-C}$ such that $pay_i(\vec{\sigma}) \geq pay_i(\vec{\sigma}_C, \vec{\sigma}_{-C})$, for some $i \in C$. Alternatively, as we already discussed above, we can...
characterise the core by using the notion of beneficial deviations: Given a strategy profile \( \vec{\sigma} \) and a coalition \( C \subseteq \mathbb{N} \), we say that the strategy profile \( \vec{\sigma}^*_C \) is a beneficial deviation if for all counter-strategies \( \vec{\sigma}^C \), we have \( \text{pay}_i((\vec{\sigma}^*_C, \vec{\sigma}^C)) > \text{pay}_i(\vec{\sigma}) \) for all \( i \in C \). The core then consists of those strategy profiles which admit no beneficial deviations; note that these two definitions are equivalent. For a given game \( \mathcal{G} \), let \( \text{Core}(\mathcal{G}) \) denote the set of strategy profiles in the core of \( \mathcal{G} \).

\[ \begin{array}{c}
(1,0) & (0,0) & (0,1) \\
\hline
(l,l) & m & (l,r) \\
(r,l) & (r,r) \\
\end{array} \]

Figure 1 Arena for Example 1. The symbol * is a wildcard that matches all possible actions.

Example 1. We illustrate how the core differs from NE, and how cooperation and memory affect the outcome of a game. Consider a game consisting of two players \{1, 2\}. The arena is depicted in Figure 1, and the players are initially in \( m \). Each player has two actions: \( L \) and \( R \). Player 1 (resp. 2) gets 1 when the play visits \( l \) (resp. \( r \)) — e.g., tasks assigned to the players, for which they are rewarded upon completion. However, these states can only be visited by agreeing on the actions (e.g., tasks that must be carried out by multiple robots). Observe that player 1 (resp. 2) always choosing \( L \) (resp. \( R \)) is a NE, and a “bad” one since each player receives a payoff of 0. On the other hand, this bad equilibrium is not included in the core: the players can coordinate/cooperate to alternately visit \( l \) and \( r \) and obtain higher payoffs (i.e., each receives \( \frac{1}{2} \)). Furthermore, observe that to execute this plan, the players must remember previously visited states (i.e., finite-memory strategies are necessary). This outcome also corresponds to the liveness property \( GF^I \land GF^r \) (“the tasks will be completed infinitely often”), which cannot be realised using memoryless strategies.

Vectors and Inequations. Given two vectors \( \vec{a}, \vec{b} \in \mathbb{Q}^d \) the notation \( \vec{a} \geq \vec{b} \) corresponds to the component-wise inequality, and let \( ||\vec{a}|| = d + \sum_{i \in [1,d]} ||a_i|| \), \( a_i \) is represented using the usual binary encoding of numerators/denominators. The linear function \( f_\vec{a} : \mathbb{R}^d \to \mathbb{R} \) is the function \( f_\vec{a}(\vec{x}) = \sum_{i \in [1,d]} a_i \cdot x_i \). A linear inequation is a pair \((\vec{a}, b)\) where \( \vec{a} \in \mathbb{Q}^d \setminus \{\vec{0}\} \) and \( b \in \mathbb{Q} \). The size of \((\vec{a}, b)\) is \( ||(\vec{a}, b)|| = ||\vec{a}|| + ||b|| \). The half-space corresponding to \((\vec{a}, b)\) is the set \( \text{halfspace}(\vec{a}, b) = \{ \vec{x} \in \mathbb{R}^d \mid f_\vec{a}(\vec{x}) \leq b \} \). A linear inequality system is a set \( \lambda = \{ (\vec{a}_1, b_1), \ldots, (\vec{a}_i, b_i) \} \) of linear inequations. A polyhedron generated by \( \lambda \) is denoted by \( \text{poly}(\lambda) = \bigcap_{(\vec{a}, b) \in \lambda} \text{halfspace}(\vec{a}, b) \). Let \( P \) be a polyhedron in \( \mathbb{R}^d \) and \( C \subseteq D = \{1, \ldots, d\} \), and let \( c = |C| \). The projection of \( P \subseteq \mathbb{R}^d \) on variables with indices in \( C \) is the set \( \text{proj}_C(P) = \{ \vec{x} \in \mathbb{R}^c \mid \exists \vec{y} \in P \land \forall i \in C, y_i = x_i \} \).

3 Characterising the core

In this section, we provide a characterisation of the core and other important concepts which we will use to prove our complexity results.

Multi-Mean-Payoff Games. Multi-mean-payoff games (MMPGs) [21, 60] are similar to two-player, turn-based, zero-sum mean-payoff games, except the states of the game graph are labelled with \( k \)-dimensional integer vectors representing the weights. Player 1’s objective is to maximise the mean-payoff of the \( k \)-dimensional weight function. Note that since the weights are multidimensional, there is not a unique maximal value in general.
Formally, a multi-mean-payoff game \( G \) is a tuple, \( G = (V_1, V_2, E, w) \), where \( V_1, V_2 \) are the states controlled by player 1 and 2 respectively, with \( V := V_1 \cup V_2 \) and \( V_1 \cap V_2 = \emptyset \); 
\( E \subseteq V \times V \) is a set of edges; \( w : V \rightarrow \mathbb{Z}_k \) is a weight function with \( k \in \mathbb{N} \). Given a start state \( v^0 \in V_1 \), player \( i \) chooses an edge \((v^0, v^1) \in E\), and the game moves to state \( v^1 \in V_j \).

Then player \( j \) chooses an edge and the game moves to the specified state, and this continues forever. Paths are defined in the usual way and for a path \( \pi \), the payoff \( \text{pay}(\pi) \) is the vector \((\text{mp}(w_1(\pi)), \ldots, \text{mp}(w_k(\pi)))\). It is shown in [60] that memoryless strategies suffice for player 2 to act optimally, and that the decision problem which asks if player 1 has a strategy that ensures \( \text{pay}(\pi) \geq \vec{x} \) from a given state and for some \( \vec{x} \in \mathbb{R}^k \) is \( \text{coNP} \)-complete.

We consider a sequentialisation of a game where players are partitioned into two coalitions, \( C \subseteq \mathbb{N} \) and \( -C = \mathbb{N} \setminus C \). This game is modelled by a MMPG where coalition \( C \) acts as player 1 and \(-C\) as player 2. The \( k \)-dimensional vectors represent the weight functions of players in \( C \). In the case \( C = \mathbb{N} \), player 2 is a “dummy” player with no influence in the game.

**Definition 2.** Let \( \mathcal{G} = (A, (w_i)_{i \in N}) \) be a game with \( A = (N, \Sigma, S, s_{init}, \text{tr}, \text{lab}) \) and let \( C \subseteq \mathbb{N} \). The sequentialisation of \( \mathcal{G} \) with respect to \( C \) is the (turn-based two-player) MMPG \( \mathcal{G}^C = (V_1, V_2, E, w) \) where \( V_1 = St, V_2 = St \times \bar{\mathcal{A}}_C; w : V_1 \cup V_2 \rightarrow \mathbb{Z}_k \) is such that \( w_i(s) = w_i(s, \vec{a}_C) = w_i(s); \) and \( E = \{(s, \vec{a}_C) \in St \times (St \times \bar{\mathcal{A}}_C) \} \cup \{(s, \vec{a}_C), (s') \in (St \times \bar{\mathcal{A}}_C) \times St : \exists \vec{a}_C \in \bar{\mathcal{A}}_C, s' = \text{tr}(s, (\vec{a}_C, \vec{a}_{-C}))\} \).

The construction above is clearly polynomial in the size of the original game \( \mathcal{G} \).

Let \( \Sigma_2^M \) be the set of memoryless strategies\(^4\) for player 2. For a strategy \( \sigma_2 \in \Sigma_2^M \), the game induced by applying such strategy is given by \( \mathcal{G}^c[\sigma_2] = (V_1, V_2, E', (w_i)_{i \in C}) \) where \( E' = \{(s, s') \in E \mid s \in V_1 \lor s \in V_2 \land \sigma_2(s) = s'\} \). That is, a subgame in which player 2 plays according to the memoryless strategy \( \sigma_2 \).

**Enforceable Values and Pareto Optimality.** We present the definitions of enforceable values and Pareto optimality in MMPGs [13] below, which we will use for our characterisation of the core.

**Definition 3.** For a MMPG \( \mathcal{G}^C \) and a state \( s \in V_1 \cup V_2 \), define the set of enforceable values that player 1 can guarantee from state \( s \) as:

\[
\text{val} \left( \mathcal{G}^C, s \right) = \{ \vec{x} \in \mathbb{R}^c \mid \exists \sigma_1 \forall \sigma_2 \forall j \in C : x_j \leq \text{mp}_j \left( (\sigma_1, \sigma_2, s) \right) \}.
\]

A vector \( \vec{x} \in \mathbb{R}^c \) is \( C \)-Pareto optimal from \( s \) (or simply Pareto optimal when \( C = \mathbb{N} \) or \( C \) is clear from the context, and \( s = s_{init} \)) if it is maximal in the set \( \text{val}(\mathcal{G}^C, s) \). The set of Pareto optimal values is called Pareto set, formally defined as:

\[
\text{PO} \left( \mathcal{G}^C, s \right) = \{ \vec{x} \in \text{val}(\mathcal{G}^C, s) \mid \nexists \vec{x} \in \text{val}(\mathcal{G}^C, s) : \vec{x} \geq \vec{x} \land \exists i \text{ s.t. } x'_i > x_i \}.
\]

When \( s = s_{init} \), we simply write \( \text{val}(\mathcal{G}^C) \) and \( \text{PO}(\mathcal{G}^C) \). We naturally extend Pareto optimality to strategy profiles: a strategy profile \( \vec{\sigma} \) is \( C \)-Pareto optimal if \( \left( \text{pay}_i(\vec{\sigma}) \right)_{i \in C} \in \text{PO}(\mathcal{G}^C) \).

A notable aspect of the core in mean-payoff games is that it generally does not coincide with Pareto optimality, as shown in Propositions 4 and 5 below (the proofs are provided in [41]). This stands in sharp contrast to conventional cooperative (transferable utility, superadditive) games in which the core is always included in the Pareto set [20, pp. 24–25].

---

\(^3\) For \( C = \mathbb{N} \), the set \( \bar{\mathcal{A}}_{-C} \) is empty, and the transition is fully characterised by \( \bar{\mathcal{A}}_C \). We keep the current notation to avoid clutters.

\(^4\) Here we define a strategy as a mapping from sequences of states to a successor state \( \sigma_i : V^* \rightarrow V \) for \( i \in \{1, 2\} \). A strategy is memoryless when it chooses a successor based on the current state \( \sigma_i : V_i \rightarrow V \).
Proposition 4. There exist games $G$ such that $\vec{a} \in \text{Core}(G)$ and $\vec{a}$ is not Pareto optimal.

Proposition 5. There exist games $G$ such that $\vec{a}$ is Pareto optimal and $\vec{a} \notin \text{Core}(G)$.

Discrete Geometry and Values. To characterise the core, we utilise techniques from discrete geometry. First, we provide the definitions of two concepts: convex hull and downward closure. The convex hull of a set $X \subseteq \mathbb{R}^d$ is the set $\text{conv}(X) = \{\sum_{x \in X} a_x \cdot x \mid \forall x \in X, a_x \in [0, 1] \land \sum_{x \in X} a_x = 1\}$. The downward closure of a set $X \subseteq \mathbb{R}^d$ is the set $\downarrow X = \{\vec{x} \in \mathbb{R}^d \mid \exists \vec{x}' \in X, \forall i \in [1, d], x_i \leq x_i'\}$. Note that if the set $X$ is finite, then $\text{conv}(X)$ and $\downarrow \text{conv}(X)$ are convex polyhedra, thus can be represented by intersections of some finite number of half-spaces [34, Theorem 3.1.1].

Now, observe that the downward closure of the Pareto set is equal to the set of values that player 1 can enforce, that is, $\downarrow \text{PO}(G^C, s) = \text{val}(G^C, s)$. The set $\text{val}(G^C)$ can also be characterised by the set of simple cycles and strongly connected components (SCCs) in the arena of $G^C$ [3]. A simple cycle within $S \subseteq (V_1 \cup V_2)$ is a finite sequence of states $o = s^0 s^1 \cdots s^k \in S^*$ with $s^0 = s^k$ and for all $i$ and $j, 0 \leq i < j < k, s^i \neq s^j$. Let $\mathcal{C}(S)$ be the set of simple cycles in $S$, and SCC($G^C[\sigma_2]$) the set of SCCs reachable from $s_{init}$ in $G^C[\sigma_2]$. The set of values that player 1 can enforce is characterised by the intersection of all sets of values that it can achieve against memoryless strategies of player 2. Formally, we have the following [13, Theorem 4]:

$$\text{val}(G^C) = \bigcap_{\sigma_2 \in \Sigma^H} \bigcup_{S \in \text{SCC}(G^C[\sigma_2])} \downarrow \text{conv}\left(\left\{\left(\sum_{j=0}^k w_i(o_j) \right)_{i \in \mathcal{C}} \mid o \in \mathcal{C}(S)\right\}\right).$$

With these definitions in place, we first obtain the following lemma, which shows that the set of enforceable values has polynomial representation.

Lemma 6. The set $\text{val}(G^C)$ can be represented by a finite union of polyhedra $P^C_1, \ldots, P^C_k$, each of them definable by a system of linear inequations $\lambda^C_j$. Moreover, each linear inequation $(a, b) \in \lambda^C_j$ can be represented polynomially in the size of $G^C$.

Proof. Let $X = \{\vec{x}_1, \ldots, \vec{x}_m\}$ be the set of extreme points of $\text{conv}(\{(\sum_{i=0}^{|o|} w_i(o_j))_{i \in \mathcal{C}} \mid o \in \mathcal{C}(S)\})$ for a given $S \in \text{SCC}(G^C[\sigma_2])$. Observe that $X$ corresponds to the set of simple cycles in $S$, as such, for each $\vec{x} \in X$ we have $|\vec{x}|$ that is of polynomial in the size of $G^C$. As shown in [13, Theorem 3], $\downarrow \text{conv}(X)$ has a system of inequalities $\lambda$ whose each inequation has representation polynomial in $c$ and $\log_2(\max(|\vec{x}| \mid \vec{x} \in X))$. Since this holds for each $\sigma_2 \in \Sigma^H$ and for each SCC in $G^C[\sigma_2]$, we obtain the lemma. ▶

Let $\text{PS}(G^C)$ denote the set of polyhedrons whose union represents $\text{val}(G^C)$, and for a polyhedron $P^C_j \in \text{PS}(G^C)$, we denote by $\mathcal{H}^C_j$ the set of half-spaces whose intersection corresponds to $P^C_j$.

Polynomial Witness in the Core. A polynomial witness in the core of $G$ is a vector $\vec{x} \in \mathbb{Q}^n$ such that there exists $\vec{a} \in \text{Core}(G)$ where $(\text{pay}_j(\vec{a}))_{j \in N} = \vec{x}$ and $\vec{x}$ has a polynomial representation with respect to $G$. The rest of this section focuses on characterising the core (Theorem 12) and showing the existence of a polynomial witness in a non-empty core (Theorem 13). We start by introducing some concepts and proving a couple of lemmas.

Definition 7. Given a set of player $N$ and a coalition $C \subseteq N$. The inclusion mapping of $X \subseteq \mathbb{R}^c$ to subsets of $\mathbb{R}^n$ is the set $\mathcal{F}(X) = \{\vec{y} \in \mathbb{R}^n \mid \exists \vec{x} \in X, \forall j \in C, x_j = y_j\}$. 
Lemma 10. ▶

Proof. Suppose, for the sake of contradiction, that there is a strategy profile $\vec{\sigma} \in \text{Core}(G)$, coalition $C \subseteq N$, and polyhedron $P^C_j$ such that for every half-space $H \in H^C_j$ we have $F((\text{pay}_i(\vec{\sigma}))_{i \in C}) \subseteq F(\overline{F(P^C_j)})$. Thus, it follows that $F((\text{pay}_i(\vec{\sigma}))_{i \in C}) \subseteq F(\text{val}(G^C))$ and there exists a vector $\vec{x} \in F(\text{val}(G^C))$ such that for every player $i \in C$, we have $x_i > \text{pay}_i(\vec{\sigma})$. This implies that there exists a strategy profile $\vec{\sigma}_C$ such that for all counter-strategies $\vec{\sigma}_C$ and players $i \in C$, we have $\text{pay}_i((\vec{\sigma}_C, \vec{\sigma}_C)) > \text{pay}_i(\vec{\sigma})$. In other words, there is a beneficial deviation by the coalition $C$. Therefore, $\vec{\sigma}$ cannot be in the core, leading to a contradiction.

In essence, Lemma 9 states that the absence of a beneficial deviation from a strategy profile $\vec{\sigma}$ can be expressed in terms of polyhedral representations and closed complementary half-spaces. The next lemma, asserts that any value $\vec{x} \in \mathbb{R}^c$ enforceable by a coalition $C$ can also be achieved by the grand coalition $N$.

Lemma 10. ▶

Proof. Suppose, for the sake of contradiction, that there is a vector $\vec{x} = (x_1, \ldots, x_c) \in \text{val}(G^C)$ such that $\vec{x} \notin \text{proj}_C(\text{val}(G^N))$. This means that there is some strategy profile $(\vec{\sigma}_C, \vec{\sigma}_{-C})$ and a player $i \in C$ with $\text{pay}_i((\vec{\sigma}_C, \vec{\sigma}_{-C})) > \text{pay}_i(\vec{\sigma})$ for all $\vec{\sigma} \in \Sigma_{C \cup -C}$. This implies that there is a strategy profile $(\vec{\sigma}_C, \vec{\sigma}_{-C}) \notin \Sigma_{C \cup -C}$, i.e., there is a strategy profile that is not included in the set of all strategy profiles, which is a contradiction.

Example 11. Consider a game with $N = \{1, 2, 3\}$. The arena is depicted in Figure 2. Observe that the game has an empty core: if the players stay in $s$ forever, then $\{1, 2\}$ can beneficially deviate to $t$. If the play goes to $t$, then $\{2, 3\}$ can beneficially deviate to $m$. Similar arguments can be used for $m$ and $b$; thus, no strategy profile lies in the core. We can show this using (the contrapositive of) Lemma 9: for instance, take a strategy profile $\vec{\sigma}$ that goes to $t$, and let $C = \{2, 3\}$. Then $\text{val}(G^C)$ can be represented by the intersection of half-spaces.

\textbf{Definition 8.} Let $H = \text{hspace}(\vec{a}, b)$ be a half-space, the closed complementary half-space $\overline{H}$ is given by $\overline{H} = \{\vec{x} \in \mathbb{R}^d \mid f_2(\vec{x}) \geq b\}$.

\textbf{Lemma 9.} If $\vec{\sigma} \in \text{Core}(G)$ then for all coalitions $C \subseteq N$ and for all polyhedra $P^C_j \in \text{PS}(G^C)$, there is a half-space $H \in H^C_j$ such that $F((\text{pay}_i(\vec{\sigma}))_{i \in C}) \subseteq F(\overline{H})$.

Proof. Suppose, for the sake of contradiction, that there is a strategy profile $\vec{\sigma} \in \text{Core}(G)$, coalition $C \subseteq N$, and polyhedron $P^C_j$ such that for every half-space $H \in H^C_j$ we have $F((\text{pay}_i(\vec{\sigma}))_{i \in C}) \subseteq F(H^C_j)$. Thus, it follows that $F((\text{pay}_i(\vec{\sigma}))_{i \in C}) \subseteq F(\text{val}(G^C))$ and there exists a vector $\vec{x} \in F(\text{val}(G^C))$ such that for every player $i \in C$, we have $x_i > \text{pay}_i(\vec{\sigma})$. This implies that there exists a strategy profile $\vec{\sigma}_C$ such that for all counter-strategies $\vec{\sigma}_C$ and players $i \in C$, we have $\text{pay}_i((\vec{\sigma}_C, \vec{\sigma}_{-C})) > \text{pay}_i(\vec{\sigma})$. In other words, there is a beneficial deviation by the coalition $C$. Therefore, $\vec{\sigma}$ cannot be in the core, leading to a contradiction.

\textbf{Definition 11.} Consider a game with $N = \{1, 2, 3\}$. The arena is depicted in Figure 2. Observe that the game has an empty core: if the players stay in $s$ forever, then $\{1, 2\}$ can beneficially deviate to $t$. If the play goes to $t$, then $\{2, 3\}$ can beneficially deviate to $m$. Similar arguments can be used for $m$ and $b$; thus, no strategy profile lies in the core. We can show this using (the contrapositive of) Lemma 9: for instance, take a strategy profile $\vec{\sigma}$ that goes to $t$, and let $C = \{2, 3\}$. Then $\text{val}(G^C)$ can be represented by the intersection of half-spaces.
\[ H_2 = \{ \vec{x} \in \mathbb{R}^2 \mid x_2 \leq 2 \} \] and \[ H_3 = \{ \vec{x} \in \mathbb{R}^2 \mid x_3 \leq 1 \} \] (see Figure 2 right). Coordinate \( P \) corresponds to \( \vec{\sigma} \), and \( F(\text{pay}_i(\vec{\sigma}))(x_{1,2,3}) \subseteq F(H_2) \) and \( F(\text{pay}_i(\vec{\sigma}))(x_{1,2,3}) \not\subseteq F(H_3) \). Thus, \( \vec{\sigma} \) is not in the core. Now, suppose we modify the game such that \( (w_i(s))_{s \in N} = (1, 1, 1) \); we obtain \( \text{PO}(G^N) = \{(2, 1, 0), (0, 2, 1), (1, 0, 2), (1, 1, 1)\} \). Let \( \vec{\sigma}' \) be a strategy profile that stays in \( S \) forever (corresponding to \( S \) in Figure 2 right); \( \vec{\sigma}' \) is in the core of the modified game, and \( F(\text{pay}_i(\vec{\sigma}'))(x_{1,2,3}) \subseteq F(H_3) \). Indeed for all \( C \subseteq N \) there exists such a half-space. Now if we take the intersection of such half-spaces and the set \( \text{val}(G^N) = \downarrow \text{PO}(G^N) \), we obtain a non-empty set namely \( \{(1, 1, 1)\} \) which corresponds to a member of the core \( \vec{\sigma}' \).

From Example 11, we observe that a member of the core can be found in the intersection of some set of half-spaces and the set of values enforceable by the grand coalition. We formalise this observation in Theorem 12, which provides a necessary and sufficient condition for the non-emptiness of the core.

**Theorem 12.** The core of a game \( G \) is non-empty if and only if there exists a set of half-spaces \( I \) such that

1. For all coalitions \( C \subseteq N \) and for all polyhedra \( P^C_j \in \text{PS}(G^C), I \cap \mathcal{H}^C_j \neq \emptyset \), and
2. There exists a polyhedron \( P_N^C \in \text{PS}(G^N) \) such that \( R = \bigcap_{H \subseteq I} F(H) \cap P_N^C \neq \emptyset \).

**Proof.** From left to right. Suppose that \( \text{Core}(G) \neq \emptyset \), then there is a strategy profile \( \vec{\sigma} \in \text{Core}(G) \). It follows from Lemma 9 that for each coalition \( C \subseteq N \) and for each polyhedron \( P^C_j \in \text{PS}(G^C) \), there exists a half-space \( H \in \mathcal{H}^C_j \) such that \( F(\text{pay}_i(\vec{\sigma}))(x_{1,2,3}) \subseteq F(H) \). Since this holds for each coalition \( C \subseteq N \) and for each polyhedron \( P^C_j \in \text{PS}(G^C) \), then it is the case that there exists a set of half-spaces \( I \) such that for all coalitions \( C \subseteq N \) and for all polyhedra \( P^C_j \in \text{PS}(G^C) \) there is a half-space \( H \in I \cap \mathcal{H}^C_j \), and \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \in \bigcap_{H \subseteq I} F(H) \). Furthermore, for each coalition \( C \subseteq N \), it is the case that \( F(\text{pay}_i(\vec{\sigma}))(x_{1,2,3}) \subseteq F(\text{val}(G^C)) \) and by Lemma 10, we have \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \in \text{proj}_C(\text{val}(G^N)) \). Thus, it is also the case that there exists a polyhedron \( P^C_N \in \text{PS}(G^N) \), such that \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \in P^C_N \). Thus, it follows that \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \in \bigcap_{H \subseteq I} F(H) \cap P^C_N \) and consequently \( \bigcap_{H \subseteq I} F(H) \cap P^C_N \neq \emptyset \).

From right to left. Suppose \( R \neq \emptyset \). Take a vector \( \vec{x} \in R \). Hence \( \vec{x} \in \bigcap_{H \subseteq I} F(H) \), then for all coalitions \( C \subseteq N \) there is a player \( i \in C \) where \( x_i \geq x'_i \) for some vector \( \vec{x}' \in \text{PO}(G^C) \). Thus, by the definition of \( C \)-Pareto optimality, there exists a player \( i \in C \) that cannot strictly improve its payoff without making other player \( j \in C, j \neq i \), worse off. Thus, for each coalition \( C \subseteq N \) there is no (partial) strategy profile \( \vec{\sigma}_C \) such that for all counter-strategy profiles \( \vec{\sigma}_{-C} \) we have \( \text{pay}_i((\vec{\sigma}_C, \vec{\sigma}_{-C})) > x_i \) for every player \( i \in C \). In other words, for each coalition \( C \) and (partial) strategy profile \( \vec{\sigma}_C \), there is a counter-strategy profile \( \vec{\sigma}_{-C} \) that ensures \( \text{pay}_i((\vec{\sigma}_C, \vec{\sigma}_{-C})) \leq x_i \). This means that there is no beneficial deviation by the coalition \( C \). Moreover, since \( \vec{x} \in P^C_N \), then we have \( \vec{x} \in \text{val}(G^N) \). As such, there exists a strategy profile \( \vec{\sigma} \in \Sigma_N \) with \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \geq \vec{x} \) and \( \vec{\sigma} \in \text{Core}(G) \).

Using the characterisation of the core from Theorem 12 above, it follows that if the core is non-empty, then the set \( R \) is a polyhedron \( \text{poly}(\lambda) \) for some system of inequations \( \lambda \). As such, there exists a vector \( \vec{x} \in R \) whose representation is polynomial in \( n \) and \( \max \{|(\vec{a}, b)| \mid (\vec{a}, b) \in \lambda\} \) [13, Theorem 2]. By Lemma 6, it is also the case that \( \max \{|(\vec{a}, b)| \mid (\vec{a}, b) \in \lambda\} \) is polynomial in the size of the game. Therefore, we obtain the following.

**Theorem 13.** Given a game \( G \), if the core is non-empty, then there is \( \vec{\sigma} \in \text{Core}(G) \) such that \( \text{pay}_i(\vec{\sigma})(x_{1,2,3}) \) can be represented polynomially in the size of \( G \).

Theorem 13 plays a crucial role in our approach to solving Non-Emptiness and E-Core problems discussed in the next section. It guarantees the existence of a polynomial witness if the core is non-empty, allowing it to be guessed and verified in polynomial time.
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We build a corresponding game that builds and returns sequentialisation of \( \mathcal{G} \) with

Theorem 14. \textsc{dominated} is \( \Sigma^p_2 \)-complete.

Proof. Observe that an instance \((\mathcal{G}, s, \vec{x}) \in \textsc{dominated}\) has a witness vector \((x'_i)_{i \in C}\) that lies in the intersection of a polyhedron \( P^C \subseteq \text{PS}(G, s) \) and the set \( \{ \vec{y} \in \mathbb{R}^C \mid \forall i \in C : y_i \geq x_i \} \). Such an intersection forms a polyhedron \( \text{poly}(\lambda) \), definable by a system of linear inequalities \( \lambda \). By Lemma 6, each \((\vec{a}, b) \in \lambda\) has polynomial representation in the size of \( G^C \). Therefore, \((x'_i)_{i \in C}\) has a representation that is polynomial in the size of \( \mathcal{G} \). To solve \textsc{dominated}, we provide Algorithm 1. The correctness follows directly from the definition of \textsc{dominated}. For the upper bound: since \((x'_i)_{i \in C}\) is of polynomial size, line 1 can be done in \( \text{NP} \). In line 2, we have subprocedure \textsc{sequentialise} that builds and returns sequentialisation of \( \mathcal{G} \) w.r.t. coalition \( C \); this can be done in polynomial time. Finally, line 3 is in \( \text{coNP} \) [60, Theorem 3, Lemma 6]. Therefore, the algorithm runs in \( \text{NP}^{\text{coNP}} = \Sigma^p_2 \).

For the lower bound, we reduce from \( \text{QSAT}_2(3\text{DNF}) \) (satisfiability of quantified Boolean formulae with 2 alternations and 3DNF clauses). The complete hardness proof can be found in the appendix.

To illustrate the reduction, consider the formula

\[
\Phi = \exists x_1 \exists x_2 \forall y_1 \forall y_2 (x_1 \land x_2 \land y_1) \lor (x_1 \land \neg x_2 \land \neg y_2) \lor (x_1 \land x_2 \land \neg y_1).
\]

We build a corresponding game \( \mathcal{G}^\Phi \) such that \((\mathcal{G}^\Phi, s_{\text{init}}, (-1, -1, -1, -1, -1, 0)) = \chi \in \text{dominated} \) if and only if \( \Phi \) is satisfiable. To this end, we construct the game \( \mathcal{G}^\Phi \) in Figure 3 with \( N = \{1, 2, 3, 4, E, A\} \) and the weight function given as vectors, such that for a given vector \((w_1, \ldots, w_6)\) in state \( s, w_i(s) = w_i, i \in \{1, 2, 3, 4\} \) and \( w_E(s) = w_5, w_A(s) = w_6 \). The \( s_{\text{sink}} \) (not shown) only has transition to itself and its weights is given by the vector \((-1, -1, -1, -1, -1, 0)\). The intuition is that if \( \Phi \) is satisfiable, then there is a joint strategy \( \vec{\sigma}_C \) by \( C = N \setminus \{A\} \) that guarantees a payoff of 0 for each \( i \in C \). If \( \Phi \) is not satisfiable, then \( A \) has a strategy that visits some state \( y_k \) (resp. \( \neg y_k \)) infinitely often and player \( 2k - 1 \) (resp. \( 2k \)) gets payoff \( < -1 \). Since \( y_k \) (resp. \( \neg y_k \)) is controlled by \( 2k - 1 \) (resp. \( 2k \)), then the player will deviate to \( s_{\text{sink}} \), and \( \chi \notin \text{dominated} \). On the other hand, if \( \chi \in \text{dominated} \), then
there exists a strategy \( \bar{\sigma}_C \) which guarantees that the play: (a) ends up in some state \( x_k \) or \( \neg x_k \), or (b) visits both \( y_k \) and \( \neg y_k \) infinitely often. For the former, it means that there is a clause with only \( x \)-literals, and the latter implies that for all (valid) assignments of \( y \)-literals, there is an assignment for \( x \)-literal that makes at least one clause evaluate to true. Both cases show that \( \Phi \) is satisfiable. Now, notice that the formula \( \Phi \) is satisfiable: take the assignment that set \( x_1 \) and \( x_2 \) to be both true. Indeed, \( \chi \in \text{Dominated} \): the coalition \( \{1, 2, 3, 4, E\} \) have a strategy that results in payoff vector \( \vec{0} \), e.g., take a strategy profile that corresponds to the cycle \((C_1, y_1, C_3, \neg y_1)\)\( \omega \).

Our next problem \( \exists \text{-Ben-Dev} \) simply asks if a given game has a beneficial deviation from a provided strategy profile:

\[ \text{Given: Game } \mathcal{G}, \text{ strategy profile } \bar{\sigma}. \]

\( \exists \text{-Ben-Dev} \): Does there exist some coalition \( C \subseteq N \) such that \( C \) has a beneficial deviation from \( \bar{\sigma} \)?

Notice that \( \exists \text{-Ben-Dev} \) is closely related to \text{Dominated}. Firstly, we fix \( s \) to be the initial state. Secondly, instead of a vector, we are given a strategy profile. If we can compute the payoff induced by the strategy profile, then we can immediately reduce \( \exists \text{-Ben-Dev} \) to \text{Dominated}. [57] studies this problem in the memoryless setting, but the approach presented there (i.e., by “running” the strategy profile and calculating the payoff vectors) does not generalise to finite-state strategies \( \bar{\sigma} \) as the lasso \( \pi(\bar{\sigma}) \) may be of exponential size. To illustrate this, consider a profile \( \bar{\sigma} \) that acts like a binary counter. We have \(|\bar{\sigma}| \) that is of polynomial size, but when we run the profile, we obtain an exponential number of step before we encounter the same configuration of game and strategies states. However, in order to compute the payoff vector of a finite-state strategy profile \( \bar{\sigma} \), we only need polynomial space. First, we recall that for deterministic, finite-state strategies, the path \( \pi(\bar{\sigma}) \) is ultimately periodic (i.e., a lasso-path). As such, there exist \( (s^k, a\bar{\sigma}^k) \) and \( (s^l, a\bar{\sigma}^l) \) with \( l > k \) and \( \text{cfg}(\bar{\sigma}, k) = \text{cfg}(\bar{\sigma}, l) \). With this observation, computing the payoff vector can be done by Algorithm 2.

Line 1 can be done non-deterministically in polynomial space. In line 2, we have \text{ComputeIndex} subprocedure that computes and returns \( k, l \). This procedure is also in polynomial space: we run the profile \( \bar{\sigma} \) from \( s_{init} \) and in each step only store the current configuration; for the first time we have \( \text{cfg}(\bar{\sigma}, t) = (s^l, q_1^l, \ldots, q_n^l) \), assign \( k = t \), and the
Algorithm 2 Algorithm for computing payoff.

\begin{algorithm}
\caption{Algorithm for computing payoff.}
\begin{algorithmic}[1]
\State \textbf{input:} $G, \vec{\sigma}$
\State 1: guess $s^i$ and a vector $(q_{i1}^j, \ldots, q_{in}^j) \in \prod_{i \in N} Q_i$
\State 2: $k, l \leftarrow \text{Compu}te\text{Index}(G, \vec{\sigma}, (s^i, q_{i1}^j, \ldots, q_{in}^j))$
\State 3: return $\left( \sum_{t \in \mathbb{N}} \sum_{i \in N} w_i(\pi(\vec{\sigma})[t]) \right)_{i \in N}$
\end{algorithmic}
\end{algorithm}

second time $cfg(\vec{\sigma}, t') = (s^i, q_{i1}^j, \ldots, q_{in}^j)$, assign $l = t'$, and we are done. Note that this subprocedure returns the smallest pair of $k, l$. Line 3 is in polynomial time. So, overall we have a function problem that can be solved in $\text{NPSPACE}$, and by Savitch's theorem we obtain the following.

\begin{lemma}
For a given $G$ and $\vec{\sigma}$, the payoff vector $(\text{pay}_i(\vec{\sigma}))_{i \in N}$ can be computed in $\text{PSPACE}$.
\end{lemma}

This puts us in position to determine the complexity of $3\text{-Ben-Dev}$ as follows.

\begin{theorem}
$3\text{-Ben-Dev}$ is $\text{PSPACE}$-complete.
\end{theorem}

\begin{proof}
To solve $3\text{-Ben-Dev}$, we reduce it to $\text{Dominated}$ as follows. First, using Algorithm 2 we compute $(\text{pay}_i(\vec{\sigma}))_{i \in N}$ in $\text{PSPACE}$ (Lemma 15). Then, using Algorithm 1 we can check whether $(G, s_{\text{init}}, (\text{pay}_i(\vec{\sigma}))_{i \in N}) \in \text{Dominated}$. Since $\Sigma^P_2 \subseteq \text{PSPACE}$, $3\text{-Ben-Dev}$ can be solved in $\text{PSPACE}$. For the lower bound, we reduce from the non-emptiness problem of intersection of automata that is known to be $\text{PSPACE}$-hard [44]. The full proof is provided in the appendix.

\end{proof}

Another decision problem that is naturally related to the core is asking whether a given strategy profile $\vec{\sigma}$ is in the core of a given game. The problem is formally stated as follows.

\begin{given}
Given: Game $G$ and strategy profile $\vec{\sigma}$.
\end{given}

\begin{membership}
Membership: Is it the case that $\vec{\sigma} \in \text{Core}(G)$?
\end{membership}

Observe that we can immediately see the connection between $3\text{-Ben-Dev}$ and Membership: they are essentially dual to each other. Therefore, we immediately obtain the following lemma.

\begin{lemma}
For a given game $G$ and strategy profile $\vec{\sigma}$, it holds that $\vec{\sigma} \in \text{Core}(G)$ if and only if $(G, \vec{\sigma}) \notin 3\text{-Ben-Dev}$.
\end{lemma}

Using Lemma 17 and the fact that co-$\text{PSPACE} = \text{PSPACE}$, we obtain the following theorem.

\begin{theorem}
Membership is $\text{PSPACE}$-complete.
\end{theorem}

In rational verification, we check which temporal logic properties are satisfied by a game’s stable outcomes. Two key decision problems are formally defined as follows.

\begin{given}
Given: Game $G$, formula $\phi$.
\end{given}

\begin{ecore}
E-Core: Is it the case that there exists some $\vec{\sigma} \in \text{Core}(G)$ such that $\vec{\sigma} \models \phi$?
\end{ecore}

\begin{acore}
A-Core: Is it the case that for all $\vec{\sigma} \in \text{Core}(G)$, we have $\vec{\sigma} \models \phi$?
\end{acore}
Algorithm 3 Algorithm for Non-Emptiness.

input: $\mathcal{G}$

1: $G^N \leftarrow \text{Sequentialise}(\mathcal{G}, N)$
2: guess a vector $\vec{x} \in \mathbb{Q}^n$ s.t. $\vec{x} \in \text{val}(G^N)$
3: if $(G, s_{\text{init}}, \vec{x}) \in \text{Dominated} \ (\text{Alg. 1})$ then
   4: return NO
5: return YES

Figure 4 Illustration for solving E-Core.

Algorithm 4 Algorithm for E-Core.

input: $G, \varphi$

1: $G^N \leftarrow \text{Sequentialise}(G, N)$
2: guess a vector $\vec{x} \in \mathbb{Q}^n$ s.t. $\vec{x} \in \text{val}(G^N)$ and set of states $S \subseteq \text{St}$
3: if there is no $s \in S$ s.t. $(G, s, \vec{x}) \in \text{Dominated} \ (\text{Algorithm 1})$ then
   4: $G[S] \leftarrow \text{UpdateArena}(G, S)$
5: if $\pi \models \psi$ for some $\pi \in G[S]$ then
   6: return YES
7: return NO

To illustrate the decision problems above, let us revisit Example 1. Consider a query of A-Core for Example 1 with property $\varphi = \text{GF}l \land \text{GF}r$. Such a query will return a positive answer, i.e., every strategy profile that lies in the core satisfies $\varphi$.

Another key decision problem in rational verification is determining whether a given game has any stable outcomes. This involves checking if the game has a non-empty core.

Given: Game $G$.
Non-Emptiness: Is it the case that $\text{Core}(G) \neq \emptyset$?

As demonstrated in Example 11, there exist mean-payoff games with an empty core – this is in stark contrast to the dichotomous preferences setting (cf. [40, 36]). As such, Non-Emptiness problem is non-trivial in mean-payoff games.

To solve Non-Emptiness, it is important to recall the following two results. Firstly, if a game $G$ has a non-empty core, then there is a payoff vector $\vec{x}$ resulting from $\vec{\sigma} \in \text{Core}(G)$ whose representation is polynomial (Theorem 13). Secondly, if $\vec{x}$ is a witness for the core, then $(G, s_{\text{init}}, \vec{x}) \notin \text{Dominated}$. With these observations, solving Non-Emptiness can be done by Algorithm 3. The subprocedure in line 1 is polynomial. Line 2 is in $\text{NP}$ (Theorem 13) and we call $\Sigma_3^p$ oracle for line 3. Thus, Algorithm 3 runs in $\Sigma_3^p$. For hardness, we reduce from $\text{QSAT}_3(3\text{CNF})$ (satisfiability of quantified Boolean formulae with 3 alternations and 3CNF clauses). The reduction has a similar flavour to the one used in Theorem 14, albeit a bit more involved. The complete hardness proof is included in the appendix.

Theorem 19. Non-Emptiness is $\Sigma_3^p$-complete.

Now we turn our attention to E-Core. Observe that for a game $G$ and an LTL specification $\varphi$, a witness to E-Core would be a path $\pi$ such that $(\text{pay}_i(\pi))_{i \in \mathbb{N}} \geq (\text{pay}_i(\vec{\sigma}))$ for some $\vec{\sigma} \in \text{Core}(G)$, and $\pi \models \varphi$. Furthermore, a (satisfiable) LTL formula $\varphi$ has an ultimately
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Algorithm 4. An intuitive illustration is provided in Figure 4. We begin by guessing a vector \( \vec{\sigma}, \vec{x} \)

We could use any other "easy" fragment of LTL to avoid this bottleneck, as we will discuss later.

For every player \( \pi \), \( \phi \) formula occurs. This can be avoided by considering classes of properties with easier model checking problem. In this section, we address E/A-Core with GR(1) specifications\(^5\). The approach is similar to that in [58, Theorem 18]. The main idea is to define a linear program \( \mathcal{L} \) such that it has a feasible solution if and only if the condition in line 5 of Algorithm 4 is met.

To this end, first recall that a GR(1) formula \( \varphi \) has the following form

\[
\varphi = \bigwedge_{i=1}^{m} \text{GF}\psi_{i} \rightarrow \bigwedge_{r=1}^{n} \text{GF}\theta_{r},
\]

and let \( V(\psi_{i}) \) and \( V(\theta_{r}) \) be the subset of states in \( G \) that satisfy the Boolean combinations \( \psi_{i} \) and \( \theta_{r} \), respectively. Observe that property \( \varphi \) is satisfied over a path \( \pi \) if, and only if, \( \pi \) visits \( V(\psi_{i}) \) infinitely many times or visits some of the \( V(\psi_{i}) \) only a finite number of times. To check the satisfaction of \( \bigwedge_{i=1}^{m} \text{GF}\psi_{i} \) we define linear programs \( \mathcal{L}(\psi_{i}) \) such that it admits a solution if and only if there is a path \( \pi \) in \( G[S] \) such that \( \text{pay}_{i}(\pi) \geq x_{i} \) for every player \( i \) and visits \( V(\psi_{i}) \) only finitely many times. Similarly, for \( \bigwedge_{r=1}^{n} \text{GF}\theta_{r} \), define a linear program \( \mathcal{L}(\theta_{1}, \ldots, \theta_{n}) \) that admits a solution if and only if there exists a path \( \pi \) in \( G[S] \) such that \( \text{pay}_{i}(\pi) \geq x_{i} \) for every player \( i \) and visits \( V(\theta_{r}) \) infinitely many times. Both linear programs are polynomial in the size of \( G \) and \( \varphi \), and at least one of them admits a solution if and only if \( \varphi \) is satisfied in some path in \( G[S] \). Therefore, given \( G[S] \) and GR(1) formula \( \varphi \) it is possible to check in polynomial time whether \( \varphi \) is satisfied by a suitable path \( \pi \) in \( G[S] \). The detailed construction is provided in the full version [41].

\(^5\) We could use any other “easy” fragment of LTL to avoid this bottleneck, as we will discuss later.
Table 1 Summary of complexity results. The NE column shows complexity results for the corresponding decision problems with NE. Complexity results for decision problems related to the core in the memoryless setting can be found in [57], whereas for NE in [58, 43].

<table>
<thead>
<tr>
<th>Problem</th>
<th>Finite Memory</th>
<th>Memoryless</th>
<th>NE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dominated</td>
<td>$\Sigma_2^p$-c (Thm. 14)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\exists$-Ben-Dev</td>
<td>PSPACE-c (Thm. 16)</td>
<td></td>
<td>NP-c</td>
</tr>
<tr>
<td>Membership</td>
<td>PSPACE-c (Thm. 18)</td>
<td>coNP-c</td>
<td></td>
</tr>
<tr>
<td>Non-Emptiness</td>
<td>$\Sigma_3^p$-c (Thm. 19)</td>
<td>$\Sigma_2^p$</td>
<td>NP-c</td>
</tr>
<tr>
<td>E-Core with LTL spec.</td>
<td>PSPACE-c (Thm. 20)</td>
<td>PSPACE-c</td>
<td></td>
</tr>
<tr>
<td>A-Core with LTL spec.</td>
<td>PSPACE-c (Thm. 20)</td>
<td>PSPACE-c</td>
<td></td>
</tr>
<tr>
<td>E-Core with GR(1) spec.</td>
<td>$\Sigma_3^p$-c (Thm. 21)</td>
<td>$\Sigma_2^p$</td>
<td>NP-c</td>
</tr>
<tr>
<td>A-Core with GR(1) spec.</td>
<td>$\Pi_3^p$-c (Thm. 21)</td>
<td>$\Pi_2^p$</td>
<td>coNP-c</td>
</tr>
</tbody>
</table>

Therefore, to solve E-Core with GR(1) specifications, we can use Algorithm 4 with polynomial time check for line 5. Thus, it follows that E-Core with GR(1) specifications can be solved in $\Sigma_3^p$. The lower bound follows directly from hardness result of Non-Emptiness by setting $\varphi = \top$. Moreover, since A-Core is the dual of E-Core, we obtain the following theorem.

▶ Theorem 21. The E-Core and A-Core problems with GR(1) specifications are $\Sigma_3^p$-complete and $\Pi_3^p$-complete, respectively.

E/A-Core with Other Specifications. We conclude this section by noting that the approach presented here for solving E/A-Core problem is easily generalisable to different types of specification languages without incurring additional computational costs. For instance, the approach for GR(1) is directly applicable to the $\omega$-regular specifications considered in [57]. Furthermore, Algorithm 4 can also be easily adapted for LTL fragments whose witnesses are of polynomial size w.r.t. $G$ and $\varphi$ [28, 48]. This can be done by (1) guessing a witness $\pi$ in line 2 and (2) checking whether $\pi \models \varphi$ and $\text{pay}_i(\pi) \geq x_i$ for all $i \in N$ in line 5, resulting in the same complexity classes as stated in Theorem 21.

5 Concluding remarks

In this paper, we present a novel characterisation of the core of cooperative concurrent mean-payoff games using discrete geometry techniques which differs from previous methods that relied on logical characterisation and punishment/security values [57, 39]. We have also determined the exact complexity of several related decision problems in rational verification. Our results and other related results from previous work are summarised in Table 1.

It is interesting to note that Non-Emptiness of the core is two rungs higher up the polynomial hierarchy from its NE counterpart. This seems to be induced by the fact that for a given deviation, the punishment/counter-strategy is not static as in NE. It is also worth mentioning that generalising to finite-memory strategies (second column) results in an increase in complexity classes compared to the memoryless setting (third column). In particular, $\exists$-Ben-Dev and Membership jump significantly from NP-complete and coNP-complete, respectively, to PSPACE-complete. Furthermore, and rather surprisingly, in the finite memory setting, $\exists$-Ben-Dev and Membership are harder than Non-Emptiness, which sharply contrasts with the memoryless setting. This seems to be caused by the
following: Algorithm 3 for NON-EMPTINESS is “non-constructive”, in the sense that we only care about the existence of a strategy profile that lies in the core without having to explicitly construct one. On the other hand, with MEMBERSHIP, we have to calculate the payoff from a compact representation of a given strategy profile, which requires us to “unpack” the profile.

Our characterisation of the non-emptiness of the core (Theorem 12) provides a way to ensure that the core always has a polynomially representable witness. However, it would be interesting to establish the sufficient and necessary conditions in a broader sense. Previous work has addressed the sufficient and necessary conditions for the non-emptiness of the core in non-transferable utility (NTU) games. For example, [55] showed that the core of an NTU game is non-empty when the players have continuous and quasi-concave utility functions. [59] relaxed the continuity assumption (which aligns more closely with the setting in this paper) and achieved a result similar to [55]. However, their game models differ from ours, and the results do not directly apply to our setting. We conjecture that a similar condition, namely the quasi-concavity of utility functions, plays a vital role in the non-emptiness of the core in concurrent multi-player mean-payoff games. Nevertheless, this still needs to be formally proven and would make for interesting future work.

As previously mentioned, a key difference between the core of concurrent multi-player mean-payoff games and games with dichotomous preferences is that the former may have an empty core. This raises the question: what can we do when the core is empty? One might want to introduce stability, thereby making the core non-empty. One approach, which relates to the above conjecture, involves modifying the utility functions, for instance, through subsidies or rewards [42, 2]. Another approach is to introduce norms [51]. This is an area for future exploration.

It would also be interesting to generalise the current work to decidable classes of imperfect information mean-payoff games [27]. Another potential avenue is to relax the concurrency, for instance, by making agents loosely coupled. A different but intriguing direction would be to investigate the possibility of using our construction and characterisation here to extend ATL* with mean-payoff semantics.

References


The Core in Concurrent Multi-Player Mean-Payoff Games

A Appendix: Proofs

A.1 Proof of Theorem 16

Theorem 16. $\exists$-BEN-DEV is PSPACE-complete.

Proof. To solve $\exists$-BEN-DEV, we reduce it to DOMINATED as follows. First we compute \((\text{pay}_i(\bar{\sigma}))_{i \in \mathbb{N}}\) in PSPACE (Lemma 15). Then we can query whether \((G, s_{init}, (\text{pay}_i(\bar{\sigma}))_{i \in \mathbb{N}}) \in \text{DOMINATED}. Since \(\Sigma_2^P \subseteq \text{PSPACE}, \exists$-BEN-DEV can be solved in PSPACE.

For the lower bound, we reduce from the non-emptiness problem of the intersection of deterministic finite automata (DFA) that is known to be PSPACE-hard [44]. Let \(A_1, \ldots, A_n\) be a set of deterministic finite automata (DFAs), and let \(F_i = \{q_i^*\}\) be the set of accepting state of \(A_i\). Note that we can always assume that \(F_i\) only has one state; otherwise, we can simply introduce a new symbol in the alphabet (call it \(a\)), a new state \(f_i\) for \(A_i\), and define the final state of \(A_i\) to be \(f_i\), as well as defining \(\Delta_i(q, a) := f_i\), for each \(q \in F_i\), where \(\Delta_i\) is the transition function of \(F_i\). We construct from each \(A_i = (Q_i, \Sigma_i, \delta_i, q_i^0, F_i)\) a strategy \(\sigma_i = (Q_i, q_i^0, \delta_i, \tau_i)\) where \(\tau_i(q_i) = q_i^*\). We build a game with \(N = \{1, \ldots, n\}\) and arena with 3 states \(\mathcal{S} = \{s_0, s_1, s_2\}\). For each \(i \in N, \mathcal{A}_i = Q_i \cup \{d_i\}\), where \(d_i\) is a fresh variable. The transition function is defined by Figure 5 left, and the weight function by Figure 5 right.

![Diagram](image_url)

**Figure 5** Left: The game arena where \(q_i' \neq q_i^*\). Right: The weight function of the game.

Given \((G, \bar{\sigma})\) where \(\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)\), we claim that \((G, \bar{\sigma}) \notin \exists$-BEN-DEV if and only if the intersection of \(A_1, \ldots, A_n\) has non-empty language. From left to right: it is easy to see that in order for \(\bar{\sigma}\) to admit no beneficial deviation, the game has to eventually enter \(s_2\), because otherwise the grand coalition can deviate to \(s_1\) and obtain better payoffs. The only possible way to enter \(s_2\) is when each of \(A_i\) arrives at the accepting state, and thus the intersection has non-empty language. From right to left, we argue in a similar way.
A.2 Proof of Theorem 14

\textbf{Theorem 14.} \textit{Dominated} is $\Sigma^p_2$-complete.

\textbf{Proof.} For the lower bound, we reduce from QSAT$_2$(3DNF) (satisfiability of quantified Boolean formulae with 2 alternations and 3DNF clauses), which is known to be $\Sigma^p_2$-hard [50]. Consider a formula of the form $\Phi = \exists x_1 \cdots \exists x_p \forall y_1 \cdots \forall y_q C_1 \lor \cdots \lor C_r$ where each $C_i$ is the conjunction of three literals $C_i = l_{i,1} \land l_{i,2} \land l_{i,3}$, and the literals are of the form $x_k, \neg x_k, y_k, \text{or} \neg y_k$. For clauses $C$ and $C'$, we say that they are not clashing if there is no literal $x_k$ appears in $C$ and $\neg x_k$ in $C'$.

For a given formula $\Phi$ we build a corresponding game $G^\Phi$ such that $(G^\Phi, s_{\text{init}}, (-1, \ldots, -1, 0)) \in \text{Dominated}$ if and only if $\Phi$ is satisfiable, as follows.

- $N = \{1, \ldots, 2q, E, A\}$;
- $\text{St} = \{s_{\text{init}}, C_1, \ldots, C_r, l_{1,1}, \ldots, l_{r,3}, s_{\text{sink}}\}$ where
  - the states $s_{\text{init}}$ and $K$-literal states are controlled by player $E$;
  - each state $l_{i,j}$ of the from $y_k$ (resp. $\neg y_k$) is controlled by player $2k$ (resp. $2k - 1$), and
  - $\{C_1, \ldots, C_r\}$ (i.e., the clause states) by player $A$;
- the transition function is given as:
  - from $s_{\text{init}}$, player $E$ can decide to which state in $\{C_1, \ldots, C_r\}$ the play will proceed – she picks the clause;
  - from each state $C_i$, player $A$ can decide to which state in $\{l_{i,1}, \ldots, l_{i,3}\}$ the play will proceed – he picks the literal;
  - from each $l_{i,j}$, there is a self-loop transition,
  - from each $l_{i,j}$ of the form $y_k$ (resp. $\neg y_k$), the transitions are controlled by player $2k$ (resp. $2k - 1$), and defined as follows:
    * there is a transition from $l_{i,j}$ to every $C_h, i \neq h$, where $y_k$ or $\neg y_k$ occurs in $C_h$, and $C_i, C_h$ are not clashing, and
    * there is also a transition to $s_{\text{sink}}$.
- $s_{\text{sink}}$ has only self-loop transition.
- the weight function is given as:
  - for a literal state $l_{i,j}$
    * if $l_{i,j}$ is of the form $y_k$, then $w_{2k-1}(l_{i,j}) = 2q$ and $w_{2k}(l_{i,j}) = -2q$, and for each $a \in N \setminus \{2k - 1, 2k\}$, $w_a(l_{i,j}) = 0$;
    * if $l_{i,j}$ is of the form $\neg y_k$, then $w_{2k-1}(l_{i,j}) = -2q$ and $w_{2k}(l_{i,j}) = 2q$, and for each $a \in N \setminus \{2k - 1, 2k\}$, $w_a(l_{i,j}) = 0$;
    * if $l_{i,j}$ is of the form $x_k$ or $\neg x_k$, $(w_a(l_{i,j}))_{a \in N} = \vec{0}$.
  - for each non-literal state $s \in \{s_{\text{init}}, C_1, \ldots, C_r\}$, we have $(w_i(s))_{i \in N} = \vec{0}$.
  - for each $i \in N \setminus \{A\}$, $w_i(s_{\text{sink}}) = -1$ and $w_A(s_{\text{sink}}) = 0$.

We show that $(G^\Phi, s_{\text{init}}, (-1, \ldots, -1, 0)) \in \text{Dominated}$ if and only if the formula $\Phi$ is satisfiable.

($\Leftarrow$) Assume that $\Phi$ is satisfiable, then there is a (partial) assignment $v(x_1, \ldots, x_p)$ such that the formula $\forall y_1 \cdots \forall y_q C_1 \lor \cdots \lor C_r$ is valid. Let $\sigma_K$ and $\sigma_A$ denote strategies of coalition $K = N \setminus \{A\}$ and player $A$, respectively. According to [60], it is enough to only consider memoryless strategies $\sigma_A$. The strategies correspond to some assignments of variables, that is, by choosing the literal $y_k$ or $\neg y_k$, player $A$ sets the assignment of the literal such that it evaluates to false. Similarly, by choosing the clause $C_i$, $K$ pick the correct assignments for literals $x_k$ or $\neg x_k$ in $C_i$. We distinguish between strategies that are admissible and those that are not. A non-admissible strategy is a strategy that chooses two contradictory literals $y_k$ in $C$ and $\neg y_k$ in $C'$. If $\sigma_A$ is non-admissible, then $K$ can achieve $\vec{0}$ by choosing the strategy that alternates between $C$ and $C'$, and thus we have a yes-instance of Dominated.
Now suppose that \( A \) chooses an admissible strategy \( \sigma_A \). Then it corresponds to a valid assignment \( v(y_1, \ldots, y_q) \). Since for \( v(x_1, \ldots, x_p) \) the formula \( \forall y_1 \cdots \forall y_q \exists x_1 \cdots \exists x_p C_1 \lor \cdots \lor C_r \) is valid, the (full) assignment \( v(x_1, \ldots, x_p, y_1, \ldots, y_q) \) makes the formula \( C_1 \lor \cdots \lor C_r \) evaluate to true. Thus, \( \mathcal{K} \) can pick a clause state \( C_i \) that is true under \( v(x_1, \ldots, x_p, y_1, \ldots, y_q) \) and \( A \) picks a literal state of the form \( x_k \) or \( \neg x_k \) in clause \( C_i \), and not \( y_k \) or \( \neg y_k \) since it will contradict the assumption that \( C_i \) evaluates to true. Therefore, the strategy profile \((\sigma_K, \sigma_A)\) induces the payoff \( 0 \), and we have a yes-instance of \textsc{Dominated}.

\( \Rightarrow \) Assume that the strategy profile \((\sigma_K, \sigma_A)\) induces a payoff \( \text{pay}_j((\sigma_K, \sigma_A)) > -1 \) for each \( j \in \mathcal{K} \). Let \( \mathcal{C} \) and \( \mathcal{C}' \) be the set of clauses that are chosen and not chosen in \((\sigma_K, \sigma_A)\), respectively. We define the (partial) assignment of \( v(x_1, \ldots, x_p) \) as follows:

1. for each \( C_i \in \mathcal{C} \) and for each literal \( x_k \) or \( \neg x_k \) in \( C_i \)
   a. \( v(x_k) \) is true;
   b. \( v(\neg x_k) \) is false;
2. for each \( C_h \in \mathcal{C}' \) and for each literal \( x_k \) or \( \neg x_k \) in \( C_h \), if it does not appear in \( C_i \in \mathcal{C} \), then \( v(x_k) \) or \( v(\neg x_k) \) is true.

Let \( v' \) be an (extended) arbitrary assignment of \( x_1, \ldots, x_p, y_1, \ldots, y_q \) compatible with \( v(x_1, \ldots, x_p) \). Assume towards a contradiction that \( v' \) does not make any of the clauses evaluate to true. Then in each \( C_i \in \mathcal{C} \), \( A \) can choose a literal that makes \( C_i \) false. Either (i) \( A \) chooses a literal \( y_k \) or \( \neg y_k \) and there is only a self-loop from the state \( y_k \) or \( \neg y_k \), or (ii) we visit some clauses infinitely often. We distinguish between these two cases:

(i) If the run arrives in literal \( y_k \) or \( \neg y_k \) and there is only a self-loop from the state \( y_k \) or \( \neg y_k \), then player \( 2k \) or \( 2k - 1 \) will choose to move into the sink state and the players get payoff \((-1, \ldots, -1, 0)\). This contradicts our previous assumption that \( \text{pay}_j((\sigma_K, \sigma_A)) > -1 \) for each \( j \in \mathcal{K} \);

(ii) If the play visits some clauses infinitely often, then by the construction of the game graph there exists a literal state \( y_k \) (resp. \( \neg y_k \)) visited infinitely often with \( w_{2k-1}(y_k) = -2q \) (resp. \( w_{2k}(\neg y_k) = -2q \)) and the state \( \neg y_k \) (resp. \( y_k \)) is never visited. This means that either \( \text{pay}_{2k}((\sigma_K, \sigma_A)) < -1 \) or \( \text{pay}_{2k-1}((\sigma_K, \sigma_A)) < -1 \), and player \( 2k \) or \( 2k - 1 \) will choose to go to \( s_{\text{sink}} \) and the players get \((-1, \ldots, -1, 0)\). This contradicts our previous assumption that \( \text{pay}_j((\sigma_K, \sigma_A)) > -1 \) for each \( j \in \mathcal{K} \);

This implies that assignment \( v' \) makes at least one clause evaluate to true. Furthermore, since this holds for any arbitrary \( v' \) compatible with \( v(x_1, \ldots, x_p) \), we conclude that \( \Phi \in \text{QSAT}_2 \).

A.3 Proof of Theorem 19

\textbf{Theorem 19.} \textsc{Non-Emptiness} is \( \Sigma_3^P \)-complete.

\textbf{Proof.} For hardness, we reduce from \( \text{QSAT}_3(3\text{CNF}) \) (satisfiability of quantified Boolean formulae with 3 alternations and 3CNF clauses). Consider a formula of the form

\[ \Psi := \exists x_1 \cdots \exists x_p \forall y_1 \cdots \forall y_q \exists z_1 \cdots \exists z_2 C_1 \land \cdots \land C_r, \]

where each \( C_i \) is the disjunction of three literals \( C_i = l_{i,1} \lor l_{i,2} \lor l_{i,3} \), and the literals are of the form \( x_k, \neg x_k, y_k, \neg y_k, z_k, \text{ or } \neg z_k \). For clauses \( C \) and \( C' \), we say that they are not \( y\text{-clashing} \) if there is no literal \( y_k \) (resp. \( \neg y_k \)) appears in \( C \) and \( \neg y_k \) (resp. \( y_k \)) in \( C' \).

For a given formula \( \Psi \) we build a corresponding game \( G^\Psi \) such that the core of \( G^\Psi \) is not empty if and only if \( \Psi \) is satisfiable, as follows.
\[ N = \{1, \ldots, 2p, 2p + 1, \ldots, 2p + 2t, E, A, P, Q, R\} \]
\[ \text{St} = \{s_{\text{init}}, s_{\text{sink}}\} \cup \{C_v, 1 \leq v \leq r\} \cup \{l_{1,1}, \ldots, l_{r,3}\}, \text{where} \]
- state \( s_{\text{init}} \) is controlled by player \( A \)
- states \( C_1, \ldots, C_r \) are controlled by player \( E \)
- each state \( l_{i,j} \) of the form \( x_k \) (resp. \( \neg x_k \)) is controlled by player \( 2k-1 \) (resp. \( 2k \))
- each state \( l_{i,j} \) of the form \( z_k \) (resp. \( \neg z_k \)) is controlled by player \( 2(p + k) - 1 \) (resp. \( 2(p + k) \)) and player \( A \), where player \( 2(p + k) - 1/2(p + k) \) has a “veto” power to either follow player \( A \)'s decision or, instead, unilaterally choose to go to \( s_{\text{sink}} \)
- each state \( l_{i,j} \) of the form \( y_k \) or \( \neg y_k \) is controlled by player \( A \) \( ^6 \)
- the state \( s_{\text{sink}} \) is a sink state, and implemented by a gadget that will be explained later.

the transition function is given as:
- from \( s_{\text{init}} \), player \( A \) can choose to move to a clause state \( C_v, 1 \leq v \leq r \)
- from a state \( C_v \), player \( E \) can choose to move to a literal state \( l_{v,j} \)
- from a literal state \( l_{i,j} \) of the form \( x_k \) (resp. \( \neg x_k \)), player \( 2k - 1 \) (resp. \( 2k \)) can choose to move to \( s_{\text{init}} \) or \( s_{\text{sink}} \)
- from a literal state \( l_{i,j} \) of the form \( z_k \) or \( \neg z_k \), player \( E \) can choose to stay in the current state or to move to any clause state \( C' \) that is not \( y \)-clashing with \( C_i \)
- from a literal state \( l_{i,j} \) of the form \( z_k \) (resp. \( \neg z_k \)) player \( 2(p + k) - 1 \) (resp. \( 2(p + k) \)) can overrule player \( A \)'s decision, and move to \( s_{\text{sink}} \).

the weight function is given as:
- for each literal state \( l_{i,j} \)
  * if it is of the form \( x_k \), then \( w_{2k-1}(l_{i,j}) = 3r, w_{2k}(l_{i,j}) = -3r \) and for each \( a \in N \setminus \{2k-1, 2k\}, w_a(l_{i,j}) = 0 \)
  * if it is of the form \( \neg x_k \), then \( w_{2k}(l_{i,j}) = 3r, w_{2k-1}(l_{i,j}) = -3r \) and for each \( a \in N \setminus \{2k-1, 2k\}, w_a(l_{i,j}) = 0 \)
  * if it is of the form \( z_k \), then \( w_{2(p+k)-1}(l_{i,j}) = 3r, w_{2(p+k)}(l_{i,j}) = -3r \) and for each \( a \in N \setminus \{2(p+k) - 1, 2(p+k)\}, w_a(l_{i,j}) = 0 \)
  * if it is of the form \( \neg z_k \), then \( w_{2(p+k)}(l_{i,j}) = 3r, w_{2(p+k)-1}(l_{i,j}) = -3r \) and for each \( a \in N \setminus \{2(p+k) - 1, 2(p+k)\}, w_a(l_{i,j}) = 0 \)
  * otherwise, \( w_a(l_{i,j}) = 0 \) for each \( a \in N \).
- \( w_a(s_{\text{init}}) = w_a(s_{\text{sink}}) = w_a(C_i) = 0 \) for each \( a \in N \) and \( 1 \leq i \leq r \).

Now we explain the construction of \( s_{\text{sink}} \) gadget which is a small variation of a game with an empty core provided in the proof of Proposition 5. Consider a graph arena with four states \( I, U, M, B \) in which the players \( P, Q, R \) each has two actions: \( H, T \), and only the actions of those players matter in these states (i.e., the rest of the players are dummy players.) The weight function and the transition function are given below – we only specify the transitions for the state \( I \) as the other states only have self-loops.

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\(^6 \) Note that the controller of these states is ultimately not important because, as later defined, from these states we can only go to \( s_{\text{sink}} \).
Observe that once we enter $s_{sink}$, we cannot get out. Furthermore, every strategy profile that starts at state $I$ admits beneficial deviations. If the run stays at $I$ forever, the players can beneficially deviate by moving to one of $U, M, B$. However, if the game ends up at either of those states, then there will always be a coalition (of 2 players) that can beneficially deviate.

We now show that the core of $G^\Psi$ is not empty if and only if $\Psi$ is satisfiable.

($\Rightarrow$) Suppose $\vec{\sigma} \in \text{Core}(G^\Psi)$. By the construction of the game, there are three cases:

(a) $\pi(\vec{\sigma})$ visits some literal state of the form $x_k$ (resp. $\neg x_k$) infinitely often, and $\text{pay}_{2k-1}(\vec{\sigma}) \geq 1$ (resp. $\text{pay}_{2k}(\vec{\sigma}) \geq 1$)

(b) $\pi(\vec{\sigma})$ visits some literal state of the form $z_k$ (resp. $\neg z_k$) infinitely often, and $\text{pay}_{2(p+k)-1}(\vec{\sigma}) \geq 1$ (resp. $\text{pay}_{2(p+k)}(\vec{\sigma}) \geq 1$)

(c) both (a) and (b).

The condition $\text{pay}_i(\vec{\sigma}) \geq 1$ is necessary, because otherwise player $i$ can deviate to $s_{sink}$ and gets a payoff of 1 which contradicts $\vec{\sigma}$ being in the core.

We start with (a). This implies that for each clause $C_i, 1 \leq i \leq r$, there is a strategy $\sigma_E$ for player $E$ that agrees with $\vec{\sigma}$ for choosing a literal state $l_{i,j}$ such that for a literal of the form $x_k$ (resp. $\neg x_k$) we have $w_{2k-1}(l_{i,j}) \geq 3$ (resp. $w_{2k}(l_{i,j}) \geq 3$). Moreover, if such a strategy exists, then it is a valid assignment for $x_1, \ldots, x_p$ (i.e., contains no contradictions), since otherwise player $A$ can alternate between the two contradictory choices and gets $\text{pay}_{2k}(\vec{\sigma}) = 0$ or $\text{pay}_{2k-1}(\vec{\sigma}) = 0$, which implies that there is a beneficial deviation by player $2k$ or $2k-1$—contradicting our assumption that $\vec{\sigma}$ being in the core. Since this assignment is valid and makes all clauses evaluate to true, then it is the case that $\Psi$ is satisfiable.

For case (b), the argument is similar to (a). The main difference is that from a literal state $l_{i,j}$ of the form $z_k$ or $\neg z_k$, player $A$ can choose to go to state $C'$ that is not $y$-clashing with $C_i$. This assures that player $A$ can only choose a valid assignment for $y_1, \ldots, y_q$. Moreover, since we have $\text{pay}_{2(p+k)-1}(\vec{\sigma}) \geq 1$ or $\text{pay}_{2(p+k)}(\vec{\sigma}) \geq 1$, then for each clause visited, there exists an assignment of $z_1, \ldots, z_t$ that makes the clause evaluates to true. This assignment is a satisfying assignment for $\Psi$. For case (c), we combine the arguments from (a) and (b), and obtain a similar conclusion.

($\Leftarrow$) Now, suppose that $\Psi$ is satisfiable, then we have the following cases:

(1) there exists an assignment $v(x_1, \ldots, x_p)$ such that $\Psi(v)$ is a tautology, where $\Psi(v)$ is the resulting formula after applying the assignment $v(x_1, \ldots, x_p)$.

(2) there exists an assignment $v(x_1, \ldots, x_p)$ such that for each assignment $w(y_1, \ldots, y_q)$, there is an assignment $u(z_1, \ldots, z_t)$ that makes $\Psi(v, w, u)$ evaluates to true.

For case (1), we start by turning the assignment $v(x_1, \ldots, x_p)$ into a strategy $\sigma_E$ that prescribes to which literal state $l_{i,j}$ from each clause state $C_i$ the play must proceed. For instance, if $v(x_k)$ is true and $x_k$ is a literal in $C_i$, then player $E$ will choose to go to $x_k$ from $C_i$. Notice that it may be the case that there are more than one possible ways to choose a literal according to a given assignment, in which we can just arbitrarily choose one. Observe
that by following $\sigma_E$, for all strategy of player $A$ $\sigma_A$, corresponding to the assignments of $y_1, \ldots, y_q$, and for all literal state $x_k$ (resp. $\neg x_k$) visited infinitely often in $\pi((\sigma_E, \sigma_A))$ we have $\text{pay}_{2k-1}((\sigma_E, \sigma_A)) \geq 1$ (resp. $\text{pay}_{2k}((\sigma_E, \sigma_A)) \geq 1$). This means that $(\sigma_E, \sigma_A)$ admits no beneficial deviation and thus it is in the core.

For case (2), we perform a similar strategy construction as in (1). First, observe that the resulting formula $\Psi(v)$ may contain clauses that evaluate to true. We denote this by $\chi(\Psi(v))$. Notice that if $\chi(\Psi(v)) = \{ C_i | 1 \leq v \leq r \}$, then $\Psi(v)$ is a tautology – the same as case (1), and we are done. Otherwise, there is $C_i \notin \chi(\Psi(v))$ and $C_i$ contains some $z$-literals. Now, using $u(z_1, \ldots, z_t)$ we construct a strategy $\sigma'_E$ that prescribes which $x$-literal and $z$-literal to choose from each clause $C_i$. Since $\Psi(v, w, u)$ evaluates to true, then for each $C_i$ it is the case that $C_i \in \chi(\Psi(v, w, u))$. This means that for any $C_i, C_j \notin \chi(\Psi(v))$ that are visited infinitely often in a play resulting from $(\sigma_A, \sigma'_E)$, there exist no clashing $z$-literals in $C_i, C_j$ visited infinitely often. That is, for any $C_i, C_j \notin \chi(\Psi(v))$ we have only $z_k$ (resp. $\neg z_k$) visited infinitely often, and by the weight function of the game, we have $\text{pay}_{2(p+k)-1}((\sigma_A, \sigma'_E)) \geq 1$ (resp. $\text{pay}_{2(p+k)}((\sigma_A, \sigma'_E)) \geq 1$). Thus, it is the case that $(\sigma_A, \sigma'_E) \in \text{Core}(G^\Psi)$. \hfill $\blacksquare$