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Asymptotic spatial behavior for the heat equation on noncompact regions

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Abstract

We consider the isotropic initial boundary value problem for the heat equation on open regions with noncompact boundary and construct differential inequalities for a generalized heat flow measure defined over a spherical cross section. Under suitable assumptions, integration of the differential inequality leads to spatial growth and decay rate estimates for mean-square cross-sectional measures of the time-weighted temperature spatial gradient. The estimates are then used to obtain similar results for the time-weighted temperature. In particular, when the base heat flow measure is positive, the time-weighted temperature becomes pointwise unbounded at large spatial distance.

KEYWORDS

heat equation, noncompact region, spatial growth and decay rate estimates

1 | INTRODUCTION

The initial boundary value problem is considered for the isotropic heat equation on open three-dimensional regions with noncompact boundary subject to homogeneous initial data and zero temperature on appropriate portions of the boundary. Spatial asymptotic behavior, however, is not prescribed but is to be determined. The aim is to derive spatial growth and decay rate estimates for mean-square spherical cross-sectional integral measures of the time-weighted temperature gradient. Corresponding estimates for the time-weighted temperature follow with those for growth involving pointwise unbounded behavior.

Spherical cross-sectional surfaces are selected since, unlike plane cross sections, when the half-space and exterior regions are considered the a priori specification of asymptotic behavior is not required.

Although all results are believed to be new, that for unbounded generalized temperature is regarded of particular significance.

Our treatment is motivated by the procedure originally developed by Payne and Weinberger[1] and later by others including Ref. [2]. We construct a differential inequality for the generalized heat flow across the cross section whose integration leads at each time instant to conditions for spatial growth and decay of the mean-square spatial gradient of the time-weighted temperature. The growth component, together with suitably adapted versions of the maximum principle and theorems due to Evans,[3] is then used to prove that the time-weighted temperature itself becomes pointwise unbounded at large spatial distances. The rate of increase is not established. On the other hand, the decay estimate leads to decay rates for the time-weighted cross-sectional measures of both the temperature and its spatial gradient. Although the rates of growth and decay are algebraic, for a large subset of measures, the rates can be improved to become exponential.

Simple known examples illustrate essential features of the general procedure.

A similar treatment is applied by Quintanilla[4] to general parabolic equations for exponentially time-weighted measures related to the thermal energy. Spatial growth and decay rates are derived that are either at least or at most exponential.

Section 2 extends the notion of a geometrical structure introduced by Evans.[3] This preliminary section also states the initial boundary problem for both the temperature and its time-weighted generalizations. Section 3, for ease of reference, states without proof the usual maximum principle but is mainly devoted to a crucial generalization of an upper bound derived by Evans[3] in connection with regularity of the temperature on a bounded spatial region.

Section 4 discusses known explicit solutions to the one-dimensional isotropic heat equation on the semi-infinite line to illustrate the chief characteristics of the general treatment and also to demonstrate that our analysis is not merely formal. Section 5.1 introduces generalized heat flow measures defined on spherical cross-sections and derives various bounds. These together with bounds for the radial derivative obtained in Section 5.3 lead in Section 5.4 to a differential inequality whose integration in Section 6 under appropriate conditions yields growth estimates for cross-sectional measures of the spatial gradient of the time-weighted temperature valid at each positive time instant within the interval of existence.

The growth estimates combined with an extension of Evans' theorem are employed in Section 7 to prove by contradiction that the generalized temperature itself becomes asymptotically pointwise unbounded. The growth rate, however, remains open. Section 8 derives decay rate estimates for cross-sectional mean-square measures of both the generalized temperature and its spatial gradient. The method exploits the differential inequality established in Section 5.4. Taken together, our results correspond to a Phragmén–Lindelöf principle. Section 9 is devoted to brief concluding remarks.

Vector and scalar quantities are not typographically distinguished. Subscripts, however, denote the respective Cartesian components of a vector. The comma notation indicates spatial partial differentiation, while a superposed dot denotes differentiation with respect to the time variable t. The convention of summation over repeated suffixes is adopted, with Latin suffixes ranging over 1,2,3 and Greek suffixes over 1,2.

2 | NOTATION AND OTHER PRELIMINARIES

2.1 | Basic geometry

Consider the open region $\Omega \subseteq \mathbb{R}^3$ with noncompact boundary for which the half-space and exterior region are special cases. Suppose Ω is such that at least one semi-infinite ray drawn from a point on a finite part of the boundary lies entirely within Ω. Let this point be the origin $O \in \mathbb{R}^3$ of a rectangular Cartesian system of coordinates (x_1, x_2, x_3) whose positive x_3-axis coincides with the semi-infinite ray lying within Ω. Denote the spherical cross section $\Sigma(r)$ by

$$\Sigma(r) := \Omega \cap B(O, r), \qquad r = |x| \equiv (x_i x_i)^{1/2}, \tag{1}$$

where $B(x, r)$ is the ball of radius $r > 0$ centered at the point $x \in \mathbb{R}^3$. Moreover, we set

$$\Sigma(\infty) := \lim_{r \to \infty} \Sigma(r), \tag{2}$$

and let the noncompact boundary $\partial\Omega$ of Ω, supposed Lipschitz smooth, be the union of disjoint parts $\partial\Omega_1$ and $\partial\Omega_2$ where $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, and $\partial\Omega_1$ satisfies

$$\partial\Omega_1 \subset \partial\Omega \cap B(O, r_0), \tag{3}$$

for given $r_0 > 0$. Without loss, take the origin O of the Cartesian coordinate system to be a point on $\partial\Omega_1$, and define the time-independent spatial region $\Omega(r_1, r_2)$ by

$$\Omega(r_1, r_2) := \Omega \cap [B(O, r_2) \backslash B(O, r_1)], \qquad 0 < r_1 < r_2. \tag{4}$$

2.2 | Initial boundary value problem

The heat flux vector field $q_i(x,t)$ and temperature $\theta(x,t) \geq 0$, assumed sufficiently smooth functions of position and time, are related on $\Omega(r_0, \infty)$ by

$$q_i = \kappa_{ij}\theta_{,j}, \qquad (x,t) \in \Omega(r_0, \infty) \times (0, T), \tag{5}$$

$$q_{i,i} = c\dot{\theta}, \qquad (x,t) \in \Omega(r_0, \infty) \times (0, T), \tag{6}$$

where $(0, T)$, $0 < T \leq \infty$, is the maximal time interval of existence of smooth solutions, c is a given positive bounded constant, and the components $\kappa_{ij}(x)$ of the symmetric heat conduction tensor are supposed to be (piecewise) smooth and positive-definite in the sense that for vectors ξ, there exists a positive constant κ_0 such that

$$\kappa_0 \xi_i \xi_i \leq \kappa_{ij} \xi_i \xi_j, \qquad \forall \xi_i. \tag{7}$$

We also suppose that for positive bounded constant κ_1, the heat conduction tensor satisfies

$$\kappa_1^2 = \max_{\Omega} \kappa_{ij}\kappa_{ij}. \tag{8}$$

The differential equation for the temperature, obtained from (5) and (6), is given by

$$\left(\kappa_{ij}\theta_{,j}\right)_{,i} = c\dot{\theta}, \qquad (x,t) \in \Omega(r_0,\infty) \times (0,T), \tag{9}$$

whose fundamental solution $\Phi(x,t)$ for each $t > 0$ has the property:

$$\int_{\mathbb{R}^3} \Phi(x,t)\, dx = 1. \tag{10}$$

Remark 1. Treatments of the general heat conduction equation (9) are presented, for example, in the book [5, p. 24] where derivations of the fundamental solution may also be found. Explicit expressions are not required here.

Specification of the heat conduction problem on $\Omega(r_0,\ \infty)$ is completed by assigning homogeneous initial and Dirichlet boundary conditions to be

$$\theta(x,0) = 0, \qquad x \in \Omega(r_0,\infty), \tag{11}$$

$$\theta(x,t) = 0, \qquad (x,t) \in \partial\Omega_2 \times (0,T]. \tag{12}$$

In $\Omega(0,r_0) \times [0,T)$, the heat conduction equation (9) is supplemented by prescribed heat sources, given mixed thermal boundary conditions on $\partial\Omega_1 \times [0,T)$, and homogeneous initial temperature in $\Omega(0,r_0)$. Adjustment of these data yields the heat flux and temperature on $\Sigma(r_0) \times [0,T)$ required to satisfy later assumptions.

Data are not specified on the interior of $\Sigma(r)$ for $r_0 < r \leq \infty$; in particular, spatial asymptotic behavior is to be determined.

Remark 2 (Derived solutions). Define the time-weighted integrals $\phi^{(m)}(x,t),\ m = 0,1,2,\ldots$ by

$$\phi^{(m)}(x,t) := \int_0^t (t-\eta)^m \theta(x,\eta)\, d\eta, \qquad (x,t) \in \Omega(r_0,\infty) \times [0,T), \tag{13}$$

and note that besides the initial and boundary conditions analogous to (11) and (12), $\phi^{(m)}(x,t)$ also satisfies relations corresponding to (5) and (6), with $q_i(x,t)$ replaced by $q_i^{(m)}(x,t) = \kappa_{ij}\phi^{(m)}_{,j}$.

It is immediate from (13) that

$$\begin{aligned}\frac{d\phi^{(m)}(x,t)}{dt} &= \int_0^t m(t-\eta)^{m-1}\theta(x,\eta)\, d\eta \\ &= -(t-\eta)^m \theta(x,\eta)\big|_0^t + \int_0^t (t-\eta)^m \frac{d\theta(x,\eta)}{d\eta}\, d\eta.\end{aligned}$$

Relations (9) and (11) therefore imply that

$$\left(\kappa_{ij}\phi^{(m)}_{,j}\right)_{,i} = c\dot{\phi}^{(m)}, \qquad (x,t) \in \Omega(r_0,\infty) \times (0,T). \tag{14}$$

A similar remark applies to the first- and higher order time derivatives of θ provided conditions corresponding to (6) and (9) hold at $t = 0$.

The subsequent general discussion involves the integrals $\phi^{(m)}(x,t)$ and not directly the temperature $\theta(x,t)$. For convenience, however, the temperature is retained in Section 3 devoted to extensions of Evans' theorems and the maximum principle but obviously the results are valid also for $\phi^{(m)}(x,t)$.

2.3 | Additional geometry

Various geometric structures based on those treated in Ref. [3] are now introduced. Let the space-time region $\Omega_{(t_1,t_2]}$ be defined by

$$\Omega_{(t_1,t_2]} := \Omega(r_0,\infty) \times (t_1,t_2], \tag{15}$$

where $0 \leq t_1 < t_2 \leq T$. A corresponding notation is introduced for other space-time regions.

The union (c.p., [3, p. 51]),

$$\Gamma_{(t_1,t_2)} := (\Omega(r_0,\infty) \times \{t = t_1\}) \cup ((\partial\Omega_2 \cup \Sigma(\infty)) \times (t_1,t_2)) \cup \Sigma(r_0) \times (t_1,t_2) \tag{16}$$

that explicitly excludes the region $\Omega(r_0,\infty) \times \{t = t_2\}$, is called the parabolic boundary $\Gamma_{(t_1,t_2)}$ of $\Omega_{(t_1,t_2)}$.

Introduce spherical polar coordinates (r,ϕ,ψ) whose origin is that of the Cartesian coordinate system located at a point on $\partial\Omega_1$. Here, ψ is the angle of inclination of the radius with the positive x_3-axis. For fixed r, let $z = (r,0,0)$ be the point of intersection of the x_3-axis with $\Sigma(r)$, which is not necessarily at the center of the spherical cap $\Sigma(r)$.

For given r, define $R(r)$ by

$$R(r) := dist(z,\partial\Omega_2), \qquad r = |z|, \tag{17}$$

so that, for sufficiently large r,

$$B(z,\ R(r)) \subset \Omega(r_0,\infty),$$

which holds for all time since the point z and boundary $\partial\Omega_2$ are time independent.

Let $\tau \in [R^2(r),T]$, and suppose that $R^2(r) < T$. (When $T < \infty$ and $R^2(r) > T$, it is necessary to replace $R(r)$ by

$$\bar{R}(r) = \min(T,\ R(r)),$$

and modify the following arguments appropriately.)

We now follow Ref. [3, p. 615, (xii)] and consider the closed circular cylinders

$$C(x,t;R) := \left\{(y,s) : |x-y| \leq R,\ t - R^2 \leq s \leq t\right\} \tag{18}$$

that are of radius R, height R^2, and have top center at (x,t).

Accordingly, we have

$$C(z,\tau;R(r)) \subset \Omega_{(0,T]}, \tag{19}$$

so that for given r, $C(z,\tau;R(r))$ touches $\partial\Omega_2 \times [\tau - R^2(r), \tau]$ on a "vertical" path in space-time. Moreover, we have the inclusions

$$C\left(z,\tau;\frac{1}{2}R(r)\right) \subset C\left(z,\tau;\frac{3}{4}R(r)\right) \subset C(z,\tau;R(r)) \subset \Omega_{(0,T]}. \tag{20}$$

The cylinder $C(z,\tau;R(r))$ given by (19) relates to the fixed point $z \in \partial\Sigma(r)$. We next seek additional geometric structures that enable all points of $\Sigma(r)$ to be included.

For this purpose, consider the great circle $\Delta(r,\phi)$ on $\Sigma(r)$ through the point z and lying in the plane that intercepts the x_3-axis at angle ϕ to the x_1-axis. For fixed r,ϕ, let the points $z^{(\alpha)}(r,\phi) = (r,\phi,\psi^{(\alpha)})$, $\alpha = 1,2$, lie on the great circle $\Delta(r,\phi)$. The angles $\psi^{(\alpha)}$ are chosen such that the points $z^{(\alpha)}$ are on adjacent sides of z.

Without loss, take $\psi^{(1)} > 0$ and $\psi^{(2)} < 0$, and denote by $p^{(\alpha)}(\phi) \in \partial\Sigma(r)$ the respective points of intersection of the great circle $\Delta(r,\phi)$ with the boundary $\partial\Sigma(r)$.

For given r, ϕ, define $R^{(\alpha)}(r,\phi)$ by

$$R^{(\alpha)}(r,\phi) := 2dist(z^{(\alpha)}(r,\phi), p^{(\alpha)}(\phi)), \qquad \alpha = 1,2, \tag{21}$$

and further select the points $z^{(\alpha)}$ (or equivalently choose the angles $\psi^{(\alpha)}(r,\phi)$) to ensure that

$$B\left(z,\frac{1}{2}R(r)\right) \cap B\left(z^{(\alpha)}(r,\phi),\frac{1}{2}R^{(\alpha)}(r,\phi)\right) \neq \emptyset. \tag{22}$$

It is convenient to subsequently omit the arguments of $R, z^{(\alpha)}, R^{(\alpha)}$, and to assume that

$$R > R^{(1)} \geq R^{(2)}. \tag{23}$$

The ball $B(z^{(\alpha)}, R^{(\alpha)})$ may not be contained entirely within $\Omega(r_0,\infty)$ and therefore the space-time cylinders $C(z^{(\alpha)},\tau;R^{(\alpha)})$ also may not lie entirely within $\Omega_{(0,T]}$. This observation explains the following definitions of $D^{(\alpha)}(\tau)$ and $D_1^{(\alpha)}(\tau)$. The region $D_2^{(\alpha)}(\tau)$ becomes relevant depending upon the geometry of $\partial\Omega$.

For the given r, ϕ, and for given $\tau \in ((R^{(2)}/2)^2, T] \subset (0,T]$, the space-time regions $D^{(\alpha)}(\tau)$, $D_1^{(\alpha)}(\tau)$, $D_2^{(\alpha)}(\tau)$ are defined as

$$D^{(\alpha)}(\tau) := C(z^{(\alpha)},\tau;R^{(\alpha)}) \cap \Omega_{(0,T]}, \tag{24}$$

$$D_1^{(\alpha)}(\tau) := C\left(z^{(\alpha)},\tau;\frac{3}{4}R^{(\alpha)}\right) \cap \Omega_{(0,T]}, \tag{25}$$

$$D_2^{(\alpha)}(\tau) := C\left(z^{(\alpha)},\tau;\frac{1}{2}R^{(\alpha)}\right) \cap \Omega_{(0,T]}, \tag{26}$$

and therefore satisfy the inclusions

$$D_2^{(\alpha)}(\tau) \subset D_1^{(\alpha)}(\tau) \subset D^{(\alpha)}(\tau) \subset \Omega_{(0,T]}.$$

Note that $C(z^{(\alpha)}, \tau; R^{(\alpha)}) \subset \Omega_{(0,T]}$ intersects $\partial\Omega_2 \times (0, T]$ in a closed surface for $\alpha = 1,\ 2$. By construction (see (22)) for the given r, ϕ, we have

$$\Delta(r, \phi) \subset B(z^{(1)}, R^{(1)}/2) \cup B(z, R) \cup B(z^{(2)}, R^{(2)}/2),$$

which as ϕ varies over $0 \leq \phi \leq \pi$ implies that

$$\Sigma(r) = \bigcup_{0\leq\phi\leq\pi} \Delta(r, \phi) \subset \left(\bigcup_{0\leq\phi\leq\pi} B\left(z^{(1)}, R^{(1)}/2\right) \right) \cup B(z, R) \cup \left(\bigcup_{0\leq\phi\leq\pi} B\left(z^{(2)}, R^{(2)}/2\right) \right).$$

In consequence, as the angle ϕ varies in the interval $[0, \pi]$, we have

$$\Sigma(r) \times \left(\tau - \left(\frac{R^{(2)}}{2} \right)^2, \tau \right] \subset D(\tau),$$

where the space-time region $D(\tau)$ is given by

$$D(\tau) := \left(\bigcup_{0\leq\phi\leq\pi} D_2^{(1)}(\tau) \right) \cup C\left(z, \tau; \frac{R(r)}{2}\right) \cup \left(\bigcup_{0\leq\phi\leq\pi} D_2^{(2)}(\tau) \right). \tag{27}$$

3 | MAXIMUM PRINCIPLE AND RELATED INEQUALITIES

This section derives inequalities satisfied by solutions to the heat equation (9) on noncompact regions subject to homogeneous initial and boundary conditions (11) and (12). Proofs are adaptations of those presented in [3, pp. 59, 61] for the first- and higher order temperature gradients on the cylinder $C(x, t; R)$ and therefore rely upon the version of the maximum principle applicable to the unbounded spatial regions of present concern. For convenience, we state this version of the maximum principle without proof but in the current notation. (See, for example, [6, p. 183] [3], and [7, chapter 7].)

Theorem 1 (Maximum Principle). *Suppose that $\theta(x, t)$ satisfies (9) and $\kappa_{ij}(x)$, $\kappa_{ij,j}(x)$ remain bounded. Let the components κ_{ij} be positive-definite in the sense of (7). Assume that for some positive constants A, B,*

$$|\theta(x, t)| \leq A \exp(Br^2), \qquad r = |x|, \qquad t \in [0, T]. \tag{28}$$

When $\theta(x, 0) \leq 0,\ x \in \Omega(r_0, \infty)$ and $\theta(x, t) \leq 0,\ (x, t) \in \Gamma_{(0,T)}$ then $\theta(x, t) \leq 0,\ (x, t) \in \Omega_{(0,T)}$.

In particular, $\theta(x, t)$ attains its maximum on the parabolic boundary of $\Sigma_{(0,T)}$; that is, either on $\Omega(r_0, \infty) \times \{t = 0\}$ or on $(\partial\Omega_2 \cup \Sigma(\infty)) \times (0, T)$. An interior maximum at $(x, t_0) \in \Omega_{[0,T]}$ implies that θ is constant in $\Omega_{(0,t_0]}$.

Remark 3. Subject to prescribed initial and boundary conditions, (11) and (12), and to (28), the solution to (9) is unique, and attains its maximum on either $\Sigma(r_0) \times \{t = 0\}$ or $\Sigma(\infty) \times (0, T]$, where (see (2))

$$\Sigma(\infty) := \lim_{r \to \infty} \Sigma(r).$$

The next result adapts proofs of two regularity theorems derived in Ref. [3, chapter 2, Theorems 8 and 9] for bounded spatial regions subject to (11) and (12). Estimates for the first- and higher order derivatives are also established. The generalizations involve the cylindrical regions $D^{(\alpha)}(\tau)$, $D_\beta^{(\alpha)}(\tau)$, $\beta = 1, 2$ defined by (24)–(26), which reduce to the corresponding regions considered in Ref. [3] on letting $z^{(\alpha)} \to z$ and on replacing $R^{(\alpha)}$ by R.

Theorem 2 (Bounds). *For each pair of integers $p, q = 0, 1, 2, \ldots$ there exist positive bounded constants D_{pq} such that the bounded solution $\theta(x, t)$ to (9) subject to homogeneous initial and boundary conditions (11) and (12) satisfies*

$$\max_{D_2^{(\alpha)}(\tau)} \left| \frac{\partial^{p+q}\theta}{\partial x^p \partial t^q} \right| \leq \frac{D_{pq}}{[R^{(\alpha)}]^{(p+2q+5)}} \|\theta\|_{L^1(D^{(\alpha)}(\tau))}, \qquad \alpha = 1, 2. \tag{29}$$

Proof. The argument is developed for the point $z^{(1)}$. The point $z^{(2)}$ is similarly treated.

Assume that $\theta(x, t) \subset C^\infty(\Omega_{(0,T]})$ is bounded and satisfies the heat equation (9). (When θ is not smooth but still bounded, see Ref. [3, p. 629] and Remark 5.)

Extend $\theta(x, t)$ to be any bounded smooth function in $\mathbb{R}^3 \times [0, T] \backslash \Omega_{[0,T]}$ that vanishes on $\partial\Omega_2 \times [0, T]$ and also at time $t = 0$. For given τ, let the parabolic boundary of $D^{(1)}(\tau)$ be denoted by $\Gamma^{(1)}(\tau)$.

Introduce the smooth cut-off function $\zeta(x, t)$ that satisfies

$$0 \leq \zeta(x, t) \leq 1, \qquad (x, t) \in \mathbb{R}^3 \times [0, T],$$

together with

$$\zeta(x, t) = 1, \qquad (x, t) \in D_1^{(1)}(\tau), \tag{30}$$

$$= 0, \qquad (x, t) \in \left(\mathbb{R}^3 \times (0,\ T]\right) \backslash D^{(1)}(\tau), \tag{31}$$

and, moreover, vanishes near

$$\Lambda := \Gamma^{(1)}(\tau) \backslash \left(\partial\Omega_2 \times (\tau - R^{(1)2},\ \tau]\right), \tag{32}$$

for fixed τ. Recall that θ vanishes on $\partial\Omega_2$ for all $t \in (0, T]$.

Define the function $v(x, t)$ by

$$v(x, t) := \zeta(x, t)\theta(x, t), \qquad (x, t) \in \mathbb{R}^3 \times [0, T], \tag{33}$$

where in particular

$$v(x,t) = \theta(x,t), \qquad (x,t) \in D_1^{(1)}(\tau), \tag{34}$$

$$= 0, \qquad (x,t) \in (\mathbb{R}^3 \times (0,\ T])\backslash D^{(1)}(\tau), \tag{35}$$

$$= 0, \qquad (x,t) \in \partial\Omega_2 \times (0,T], \tag{36}$$

while the homogeneous initial condition for the extended function $\theta(x,t)$ implies

$$v(x,0) = \zeta(x,0)\theta(x,0) = 0, \qquad x \in \mathbb{R}^3. \tag{37}$$

A direct computation yields

$$c\dot{v} - \left(\kappa_{ij}v_{,i}\right)_{,j} = f(x,t), \qquad (x,t) \in \mathbb{R}^3 \times (0,T],$$

where

$$f(x,t) := \left[\theta\left(c\dot{\zeta} - \left(\kappa_{ij}\zeta_{,i}\right)_{,j}\right) - 2\kappa_{ij}\zeta_{,i}\theta_{,j}\right], \qquad (x,t) \in \mathbb{R}^3 \times (0,T] \tag{38}$$

and consequently,

$$f(x,t) = 0, \qquad (x,t) \in \left[\mathbb{R}^3 \times (0,T]\right]\backslash D^{(1)}(\tau), \tag{39}$$

$$= 0, \qquad (x,t) \in D_1^{(1)}(\tau), \tag{40}$$

$$\neq 0, \qquad (x,t) \in D^{(1)}(\tau)\backslash D_1^{(1)}(\tau). \tag{41}$$

Define $\tilde{v}(x,t)$ by

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^3} \Phi(x-y,t-s)f(y,s)\,dy\,ds, \tag{42}$$

where $\Phi(x,t)$ is the fundamental solution to the heat equation (9). It follows that

$$c\dot{\tilde{v}} - \left(\kappa_{ij}\tilde{v}_{,i}\right)_{,j} = f(x,t), \qquad (x,t) \in \mathbb{R}^3 \times (0,T],$$

$$\tilde{v}(x,t) = 0, \qquad (x,t) \in \mathbb{R}^3 \times \{t = 0\}.$$

The assumed boundedness of θ implies that $|v|,\ |\tilde{v}|$ are bounded. Hence, we deduce from the maximum principle stated in Theorem 1 that $\tilde{v} = v$ and consequently

$$v(x,t) = \int_0^t \int_{\mathbb{R}^3} \Phi(x-y,t-s)f(y,s)\,dy\,ds. \tag{43}$$

On recalling (31) together with (38)–(41), and that $\zeta(x,t) = 1$ when $(x,t) \in D_2^{(1)}(\tau)$, we have

$$\theta(x,t) = \int_{D^{(1)}(\tau)} \Phi(x-y,t-s)f(y,s)\,dy\,ds \qquad (x,t) \in D_2^{(1)}(\tau),$$

$$= \int_{D^{(1)}(\tau)} \Phi(x-y,t-s)\theta(y,s)\{c\zeta_{,s}(y,s)\} - \left(\kappa_{ij}\zeta_{,y_i}(y,s)\right)_{,y_j}\} \, dy \, ds$$

$$-2\int_{D^{(1)}(\tau)} \Phi(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)\theta_{,y_j}(y,s) \, dy \, ds. \tag{44}$$

Consider the last term in (44). Integration by parts yields

$$2\int_{D^{(1)}(\tau)} \Phi(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)\theta_{,y_j}(y,s) \, dy \, ds$$

$$= 2\int_{D^{(1)}(\tau)} [\Phi(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)\theta(y,s)]_{,y_j} \, dy \, ds$$

$$- 2\int_{D^{(1)}(\tau)} [\Phi_{,y_j}(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)\theta(y,s) + \Phi(x-y,t-s)(\kappa_{ij}\zeta_{,y_i})_{,y_j}\theta(y,s)] \, dy \, ds$$

$$= 2\oint_{\partial D^{(1)}(\tau)} n_j\Phi(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)\theta(y,s) \, dS \, ds \tag{45}$$

$$-2\int_{D^{(1)}(\tau)} \left[\Phi_{,y_j}(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s) + \Phi(x-y,t-s)\left(\kappa_{ij}\zeta_{,y_i}\right)_{,y_j}\right]\theta(y,s) \, dy \, ds$$

$$= -2\int_{D^{(1)}(\tau)} \Big[\Phi_{,y_j}(x-y,t-s)\kappa_{ij}\zeta_{,y_i}(y,s)$$

$$+ \Phi(x-y,t-s)\left(\kappa_{ij}\zeta_{,y_i}\right)_{,y_j}\Big]\theta(y,s) \, dy \, ds. \tag{46}$$

Remark 4 (Comment on (45)). In the treatment of (45), n_i are components of the unit outward normal on the spatial part of $\partial D^{(1)}(\tau)$ whose element of area is represented by dS. Moreover, integration by parts is with respect to the spatial variable y and therefore is over the spatial boundary of $D^{(1)}(\tau)$ at each time $s \in [\tau - (R^{(1)}/2)^2, \tau]$. But $\theta(x,t)$ vanishes on $\partial\Omega_2 \times (0,T]$ while $\zeta(x,t)$ vanishes on and near Λ defined in (32). Furthermore, $\zeta_{,i} = 0$ on $D_1^{(1)}(\tau)$ by (30). Consequently, ζ and its derivatives vanish on

$$\partial B(z^{(1)}, R^{(1)}) \cap \left(\Omega(r_0,\infty) \times [\tau - (R^{(1)}/2)^2, \tau]\right).$$

Hence, the integral (45) vanishes identically.

Insertion of (46) into (44) gives for $(x,t) \in D_2^{(1)}(\tau)$ the expression

$$\theta(x,t) = \int_{D^{(1)}(\tau)} K(x,t;y,s)\theta(y,s) \, dy \, ds, \qquad (x,t) \in D_2^{(1)}(\tau), \tag{47}$$

where by (30) and (31) we have

$$K(x,t;y,s) := \left[\Phi(x-y,t-s)\{c\zeta_{,s} + (\kappa_{ij}\zeta_{,y_i})_{,y_j}\} + 2\kappa_{ij}\Phi_{,y_i}\zeta_{,y_j}\right] \tag{48}$$

$$= 0, \qquad (y,s) \in \left(\mathbb{R}^3 \times [0,T]\right)\backslash D^{(1)}(\tau) \tag{49}$$

$$= 0, \qquad (y,s) \in D_1^{(1)}(\tau). \tag{50}$$

Observe that $K(x,t;y,s)$ is smooth for $(y,s) \in D^{(1)}(\tau)\backslash D_1^{(1)}(\tau)$, and that $\theta(x,t)$ is smooth for $(x,t) \in D_2^{(1)}(\tau)$.

Remark 5. Inequality (29) remains valid when $\theta \in C_1^2(\Omega_{(0,T)})$ but $\theta \notin C^\infty(\Omega_{(0,T)})$. The proof applies the preceding arguments to the mollification $\eta_\epsilon \star \theta$, where η_ϵ denotes the standard mollifier; see Ref. [3, p. 629]. However, θ must still be assumed bounded.

Now fix $z^{(1)}$ and τ. Rescale to obtain $R^{(1)} = 1$ and let $(\tilde{x}, \tilde{\tau}) \in \tilde{D}_2^{(1)}(\tau)$ where

$$\tilde{D}_2^{(1)}(\tau) := C\left(z^{(1)}, \tau; \frac{1}{2}\right) \cap \Omega_{(0,T]},$$

with a corresponding definition for $\tilde{D}^{(1)}(\tau)$. Then

$$\theta(\tilde{x}, \tilde{\tau}) = \int_{\tilde{D}^{(1)}(\tau)} K(\tilde{x}, \tilde{\tau}; y, s)\theta(y,s)\, dy\, ds,$$

and the smoothness of K implies

$$\begin{aligned} \frac{\partial^{p+q}\theta}{\partial \tilde{x}^p \partial \tilde{\tau}^q} &= \int_{\tilde{D}^{(1)}(\tau)} \frac{\partial^{p+q} K(\tilde{x}, \tilde{\tau}; y, s)}{\partial \tilde{x}^p \partial \tilde{\tau}_q} \theta(y,s)\, dy\, ds \\ &\leq D_{pq} \|\theta\|_{L^1(\tilde{D}^{(1)}(\tau))}, \end{aligned} \tag{51}$$

for some constant D_{pq} that remains bounded by properties of the fundamental solution $\Phi(x,t)$.

Upon rescaling according to

$$x = R^{(1)}\tilde{x}, \qquad t = R^{(1)2}\tilde{t}, \tag{52}$$

and defining

$$\tilde{\theta}(\tilde{x}, \tilde{t}) := \theta(R^{(1)}\tilde{x}, R^{(1)2}\tilde{t}),$$

it follows that $\tilde{\theta}$ satisfies the heat equation (9) in $\tilde{D}^{(1)}(\tau)$. Then, the rescaling (52) recovers estimate (29).

Remark 6 (Generalization). As stated in Remark 2, the time-weighted integrals $\phi^{(m)}(x,t)$ defined by (13) satisfy the heat equation (14) and homogeneous initial and boundary conditions corresponding to (11) and (12). In consequence, Theorem 1 and Theorem 2 hold also for $\phi^{(m)}(x,t)$, which is exclusively used in the discussion from Section 5 onwards. It is shown, moreover, that additional results become possible that apparently are not available for $\theta(x,t)$.

4 | ILLUSTRATIVE EXAMPLES

We discuss two simple known examples that illustrate the growth and decay behavior predicted later and serve to confirm that the subsequent general discussion does not lack meaning. Both examples involve the one-dimensional isotropic initial boundary value problem on the semi-infinite real line specified by

$$u_{,xx} = \dot{u}, \qquad (x,t) \in [0,\infty) \times (0,T), \tag{53}$$

$$u(0,t) = g(t), \qquad t \in [0,T], \tag{54}$$

$$u(x,0) = 0, \qquad x \in [0,\infty), \tag{55}$$

where the prescribed function $g(t)$ satisfies $g(0) = 0$, the solution $u(x,t)$ is assumed sufficiently smooth, and $(0,T)$ is the maximal interval of existence.

4.1 | Spatial growth

An explicit solution to the initial boundary value problem (53)–(55), attributed by F. John in Ref. [7, p. 211] to Tychonov[8] (see also Hellwig[9]), exhibits unbounded spatial asymptotic growth and is given by

$$u(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}, \tag{56}$$

where $g^{(n)}(t)$ denotes the n-th derivative of $g(t)$ and $g^{(0)}(t) = g(t)$. Set

$$g(t) = \exp(-t^{-d}), \qquad t > 0, \tag{57}$$

$$= 0, \qquad t \leq 0, \tag{58}$$

for constant $d > 1$. It is shown, for example, by F. John in Ref. [7, p. 212], that the infinite series (56) with $g(t)$ given by (57) and (58) converges uniformly in x and t for bounded x and all t, and so may be termwise differentiated. Substitution confirms that (56) is a solution to (53)–(55), but is not unique. Instead, there exists $\lambda(d)$ such that the series solution (56) satisfies (see Ref. [7, p. 217])

$$|u(x,t)| \leq M \exp\left(\frac{x^2}{\lambda(d)t}\right), \tag{59}$$

for positive constant M.

However, unbounded spatial growth as $x \to \infty$ of the infinite series (56) may be established irrespective of whether $u(x,t)$ is positive or negative. The conclusion follows on first supposing that the infinite series (56) is positive and convergent for all $x \in [0,\ \infty)$. Rearrangement gives

$$u(x,t) = g(t) + x^2 S(x,t), \tag{60}$$

where

$$S(x,t) := \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2(k-1)}. \tag{61}$$

Let us further suppose that S is positive and bounded for all x. Accordingly, by (60), $u(x,t)$ becomes unbounded with x, which contradicts the convergence assumption on $u(x,t)$. Hence, assume that S remains positive but unbounded as $x \to \infty$ to give a second contradiction. The assumption that S is negative and bounded implies that u is negative for sufficiently large x contrary to the assumption of positive u. Consequently, u cannot be both positive and bounded. Suppose therefore that u is negative and bounded. A repeat of the argument leads to similar contradictions and to the conclusion that $u(x,t)$ must be unbounded as $x \to \infty$. That is,

$$\lim_{x\to\infty} |u(x,t)| = \infty, \qquad t \in [0,T], \quad T < \infty. \tag{62}$$

Example (56) also may be used to illustrate sufficient conditions later derived for spatial growth. These conditions are expressed in terms of the heat flow, which in the present case is given by

$$H(x,t) := \int_0^t u(x,s)u_{,x}(x,s)\, ds. \tag{63}$$

Substitution from (53) and use of (55) show that

$$H_{,x}(x,t) = \int_0^t u_{,x}^2(x,s)\, ds + (1/2)u^2(x,t), \tag{64}$$

so that the spatial derivative of H is nonnegative, characteristic of the general analysis for both growth and decay. In particular, the later sufficient general condition for growth requires the heat flow to be positive at some finite spatial distance, which by (64) implies the heat flow remains positive at all greater spatial distances. We sketch how this condition is satisfied for the heat flow measure (63) with $g(t)$ given by (57).

Observe that the derivatives of $g(t)$ may be expressed as the convergent infinite series

$$g^{(k)}(t) = \exp(-t^{-d}) \sum_{p=0}^{k} \frac{A_p^k}{t^{(dp+k)}}, \qquad k = 0,1,2,\dots, \tag{65}$$

where the nontensorial coefficients A_p^k are recursively obtained from

$$A_p^k = dA_{p-1}^{k-1} - (dp+k-1)A_p^{k-1}, \qquad p = 0,1,2,\dots, \tag{66}$$

with

$$A_0^0 = 1, \qquad A_0^k = 0, \tag{67}$$

$$A_p^k = 0, \qquad p > k. \tag{68}$$

Coefficients of special interest are

$$A_k^k = d^k, \tag{69}$$

$$A_1^k = (-1)^{k-1} \prod_{n=1}^{k} (d+k-n). \tag{70}$$

Now rewrite the infinite series (65) as

$$g^{(k)}(t) = t^{-k(d+1)} \exp(-t^{-d}) \sum_{p=1}^{k} t^{d(k-p)} A_p^k, \tag{71}$$

to conclude that for $k = 0, 1, 2, \ldots$ each derivative $g^{(k)}(t)$ remains positive for sufficiently small time and all x. On the other hand, for sufficiently large time, the first term in (71) becomes dominant and according to (70), $g^{(k)}(t)$ alternates in sign with k. Examination of the leading term in either (65) or (71) leads to the conclusion that for sufficiently large time there holds

$$g^{(2k-1)}(t) + g^{(2k)}(t) > 0, \tag{72}$$

despite $g^{(2k)}(t)$ being negative. Furthermore, the infinite series (56) may be expressed as the sum of two convergent series of positive and negative terms given by

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \frac{g^{(2k)}(t)}{(2k)!} x^{2k} + \frac{g^{(2k-1)}(t)}{(2(k-1))!} x^{2(k-1)} \right\}. \tag{73}$$

We appeal to (72) to obtain

$$u(x,t) > \sum_{k=1}^{\infty} \frac{g^{(2k)}(t)}{(2(k-1))!} x^{2k} \left\{ \frac{1}{2k(2k-1)} - \frac{1}{x^2} \right\}, \tag{74}$$

which is positive provided

$$x^2 < 2 \leq 2k(2k-1), \tag{75}$$

or $0 \leq x < \sqrt{2}$.

The infinite series (56) may be differentiated termwise for bounded x, and the result decomposed into two convergent infinite series of positive and negative terms, which together with (72) leads to

$$\begin{aligned} u_{,x}(x,t) &= \sum_{k=1}^{\infty} \left\{ \frac{g^{(2k)}(t)}{(4k-1)!} x^{4k-1} + \frac{g^{(2k-1)}(t)}{(4k-3)!} x^{4k-3} \right\} \\ &> \sum_{k=1}^{\infty} \frac{g^{(2k)}(t)}{(4k-3)!} x^{4k-1} \left\{ \frac{1}{2(4k-1)(2k-1)} - \frac{1}{x^2} \right\}, \end{aligned}$$

which is positive for $0 \leq x < \sqrt{6}$.

In consequence, the heat flow (63) is positive for $0 \leq x < \sqrt{2}$ consistent with the general sufficient growth condition derived later.

4.2 | Spatial decay

Decay behavior is subject to the asymptotic boundedness condition

$$\lim_{x \to \infty} |u(x,t)| < \infty, \tag{76}$$

and is illustrated by a second solution to the initial boundary value problem (53)–(55). Thus, consider (see, for example, Ref. [7, p. 222] and Ref. [3, p. 87])

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} \exp\left(-\frac{x^2}{4(t-s)}\right) g(s)\, ds. \tag{77}$$

It is immediate from (77) that $u(x,t)$ is nonnegative on its domain of definition and possesses exponential spatial decay that implies the asymptotic behavior

$$\lim_{x\to\infty} u(x,t) = 0, \qquad t \in [0,T). \tag{78}$$

The general sufficient condition for spatial decay derived later is that the heat flow is negative at some finite distance. To check that the heat flow (63) satisfies this condition, we note from (77) that

$$\begin{aligned} 2\sqrt{\pi} u_{,x}(x,t) &= \int_0^t \frac{[2(t-s)^2 - x^2]}{2(t-s)^{5/2}} \exp\left(-\frac{x^2}{4(t-s)}\right) g(s)\, ds \\ &\leq (2t^2 - x^2) \int_0^t \frac{1}{2(t-s)^{5/2}} \exp\left(-\frac{x^2}{4(t-s)}\right) g(s)\, ds, \end{aligned} \tag{79}$$

and consequently, $u_{,x}(x,t)$ is negative for $t\sqrt{2} < x_0 < x$ at each time instant $t \in [0,T)$.

The heat flow (63) is therefore negative at a single sufficiently large x_0 consistent with the general sufficient condition for spatial decay. Note that the heat flow must remain nonpositive for all $x \geq x_0$ otherwise, as explained in the first example, sufficient conditions for unbounded spatial growth become satisfied in contradiction to (76).

5 | DIFFERENTIAL INEQUALITIES FOR CROSS-SECTIONAL HEAT FLOW MEASURES

This and all later sections revert to the weighted temperature $\phi^{(m)}(x,t)$ defined in (13) and assumed to satisfy (14) on the space-time region $\Omega_{(0,T)}$ together with initial and boundary conditions corresponding to (11) and (12). The analysis, conducted mainly in terms of the cross-sectional heat flow measure $H(r,t)$, constructs differential inequalities whose integration lead to the required growth and decay rate estimates.

5.1 | Generalized heat flow

For $m = 0, 1, 2, \ldots$, define the cross-sectional heat flow measure $H(r,t)$ by

$$H(r,t) := \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma} n_i q_i^{(m)} \phi^{(m)}\, dS d\eta, \qquad t \in [0,T), \tag{80}$$

where $\gamma \geq 0$ for convenience is chosen to be nonnegative integer, dS is the element of area on $\Sigma(r)$, n_i are the Cartesian components of the unit normal on $\Sigma(r)$ in the increasing radial direction, the generalized heat flux $q_i^{(m)}(x,t)$ satisfies

$$q_i^{(m)}(x,t) = \kappa_{ij}\phi_{,j}^{(m)}(x,t), \qquad (x,t) \in \Omega_{(0,T)}, \tag{81}$$

and

$$0 \le \eta \le t < T. \tag{82}$$

In what follows, we frequently distinguish the cases $\gamma \ge 1$ and $\gamma = 0$.

5.1.1 | $\gamma \ge 0$

Standard inequalities together with (7) and (8) lead to

$$|H(r,t)| = \left|\int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma n_i \kappa_{ij} \phi_{,j}^{(m)} \phi^{(m)}\, dS d\eta\right|$$
$$\le \kappa_1^{1/2} \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma |\phi^{(m)}| \left\{\kappa_{ij}\phi_{,i}^{(m)}\phi_{,j}^{(m)}\right\}^{1/2} dS d\eta \tag{83}$$

$$\le \left(\kappa_1^{1/2}/2\right)\left[\epsilon \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \phi^{(m)2}\, dS d\eta\right.$$
$$\left. + \epsilon^{-1} \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \kappa_{ij}\phi_{,i}^{(m)}\phi_{,j}^{(m)}\, dS d\eta\right], \tag{84}$$

where ϵ is an arbitrary positive constant.

Set

$$\epsilon = \kappa_1^{1/2}/2,$$

to obtain from (84) the bound

$$|H(r,t)| \le \frac{\kappa_1}{4} \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \phi^{(m)2}\, dS d\eta + \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \kappa_{ij}\phi_{,i}^{(m)}\phi_{,j}^{(m)}\, dS d\eta. \tag{85}$$

5.1.2 | $\gamma \ge 1$

We now consider bounds derived from (84) when $\gamma \ge 1$. Take $\epsilon = 1$ and when T is infinite let $t \in [0, T_1]$ for some $T_1 < \infty$ to have the simple inequality

$$(t-\eta) \le t \le T_1 < \infty. \tag{86}$$

We conclude from (83) that

$$|H(r,t)| \leq \mu \left[\int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1} \phi^{(m)2} \, dSd\eta + \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j} \, dSd\eta \right], \tag{87}$$

where

$$\mu := \left(\kappa_1^{1/2}/2\right) \max\,(T_1, 1). \tag{88}$$

On the other hand, the choice

$$\epsilon = \frac{c\gamma}{T_1 \kappa_1^{1/2}},$$

in (84) yields the estimate

$$|H(r,t)| \leq \frac{c\gamma}{2} \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1} \phi^{(m)2} \, dSd\eta + \frac{T_1 \kappa_1}{2c\gamma} \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j} \, dSd\eta, \tag{89}$$

which, subject to the condition

$$T_1 \kappa_1 \leq 2c\gamma, \qquad \gamma \geq 1, \tag{90}$$

reduces to

$$|H(r,t)| \leq \frac{c\gamma}{2} \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1} \phi^{(m)2} \, dSd\eta + \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j} \, dSd\eta. \tag{91}$$

Condition (90) for given $\gamma \geq 1$ and $T_1 < \infty$ restricts the values of the coefficients c, κ_1 for which (91) is valid.

5.2 | Alternative expression for the heat flow

The heat flow $H(r,t)$ is next expressed in terms of space-time integrals. Different expressions are obtained for $\gamma \geq 1$ and $\gamma = 0$.

5.2.1 | $\gamma \geq 1$

When $\gamma \geq 1$ and $r_0 < r$, the divergence theorem together with a time integration yield

$$\begin{aligned} H(r,t) - H(r_0,t) &= \int_0^t \int_{\Omega(r_0,r)} (t-\eta)^{\gamma} \left\{ q^{(m)}_{i,i} \phi^{(m)} + q^{(m)}_i \phi^{(m)}_{,i} \right\} dxd\eta \\ &= \frac{c\gamma}{2} \int_0^t \int_{\Omega(r_0,r)} (t-\eta)^{\gamma-1} \phi^{(m)2} \, dxd\eta \\ &\quad + \int_0^t \int_{\Omega(r_0,r)} (t-\eta)^{\gamma} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j} \, dxd\eta, \end{aligned} \tag{92}$$

since $\phi^{(m)}(x,0) = 0$.

Let $0 < r_0 \leq r_1 < r$ and introduce the energy functionals

$$E_1(r_1, r; t) := \int_0^t \int_{\Omega(r_1,r)} (t-\eta)^{\gamma-1} \phi^{(m)2}\, dx d\eta, \tag{93}$$

$$E_2(r_1, r; t) := \int_0^t \int_{\Omega(r_1,r)} (t-\eta)^{\gamma} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dx d\eta, \tag{94}$$

$$E(r_1, r; t) := (c\gamma/2) E_1(r_1, r; t) + E_2(r_1, r; t), \tag{95}$$

in terms of which (92) may be written as

$$H(r,t) = E(r_0, r; t) + H(r_0, t). \tag{96}$$

5.2.2 | $\gamma = 0$

When $\gamma = 0$, a similar treatment shows that the identity corresponding to (92) is

$$H(r,t) = E^{(0)}(r_0, r; t) + H(r_0, t), \tag{97}$$

where $0 < r_0 \leq r_1 < r$ and

$$E^{(0)}(r_1, r; t) := (c/2) E_1^{(0)}(r_1, r; t) + E_2^{(0)}(r_1, r; t), \tag{98}$$

with

$$E_1^{(0)}(r_1, r; t) := \int_{\Omega(r_1,r)} \phi^{(m)2}\, dx, \tag{99}$$

$$E_2^{(0)}(r_1, r; t) := \int_0^t \int_{\Omega(r_1,r)} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dx d\eta. \tag{100}$$

Remark 7 ($H(r,t)$ is nondecreasing). We conclude from (96) and (97) irrespective of the sign of $H(r,t)$ and for $\gamma \geq 0$, including $\gamma \geq 1$, that at each time instant $t \in (0,T)$

$$H_{,r}(r,t) \geq 0, \qquad (x,t) \in \Omega(r_0, r) \times (0,T), \tag{101}$$

so that $H(r,t)$ is nondecreasing with respect to r. Consequently, we have for each $t \in (0,T)$

$$H(r,t) \geq H(r_0,t), \qquad r_0 \leq r \leq \infty. \tag{102}$$

Nontrivial lower bounds for the derivative of $H(r,t)$ at fixed time are next derived.

5.3 | Bounds for the radial derivative of the heat flow

As previously, the cases $\gamma \geq 1$ and $\gamma = 0$ are separately discussed.

5.3.1 | $\gamma \geq 1$

Consider the expression (92) and let $\gamma \geq 1$. Differentiation with respect to r shows that

$$H_{,r}(r,t) = \frac{c\gamma}{2}\int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1}\phi^{(m)2}\, dSd\eta + \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma}\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta \tag{103}$$

$$\geq \lambda\left[\int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1}\phi^{(m)2}\, dSd\eta + \int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma}\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta\right], \tag{104}$$

where $t \in (0,T)$ and

$$\lambda := \min(c\gamma/2, 1). \tag{105}$$

To continue, we introduce the following inequality stated without proof (see, for example, Refs. [10, 11]).

Proposition 1 (Poincaré–Wirtinger). *Let $\Psi(x)$ be a differential function defined on $\Sigma(r)$ that vanishes on the boundary $\partial\Sigma(r)$. Then Ψ satisfies the inequality*

$$\int_{\Sigma(r)} \Psi^2(x)\, dS \leq kr^2 \int_{\Sigma(r)} |\nabla_s \Psi|^2\, dS, \tag{106}$$

where ∇_s denotes the tangential derivative on $\Sigma(r)$. The positive constant k is bounded by

$$0 < k \leq [p(p+1)]^{-1}, \tag{107}$$

where p is the smallest value of n such that $P_n(\cos\beta) = 0$, $P_n(.)$ specifies the Legendre polynomial of order n, and 2β is the vertical angle of the circular cone containing $\Omega(r_0, \infty)$. In particular, $k = 1/2$ both for the half-space and exterior regions provided that on such exterior regions Ψ is normalized by

$$\int_{\partial B(0,r)} \Psi\, dS = 0.$$

We deduce from (106) that

$$\int_{\Sigma(r)} \Psi^2\, dS \leq kr^2 \int_{\Sigma(r)} \Psi_{,i}\Psi_{,i}\, dS \leq \frac{kr^2}{\kappa_0}\int_{\Sigma(r)} \kappa_{ij}\Psi_{,i}\Psi_{,j}\, dS. \tag{108}$$

Application of (108) to (103) establishes for $\gamma \geq 1$ the upper and lower bounds

$$Q\int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma}\phi^{(m)2}\, dSd\eta \leq H_{,r}(r,t) \leq \frac{QT_1kr^2}{\kappa_0}\int_0^t \int_{\Sigma(r)} (t-\eta)^{\gamma-1}\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta, \tag{109}$$

where $t \in (0, T_1]$, $T_1 < \infty$, and

$$Q := \left(\frac{c\gamma}{2T_1} + \frac{\kappa_0}{kr^2} \right). \tag{110}$$

5.3.2 | $\gamma = 0$

When $\gamma = 0$, (97) leads to

$$H_{,r}(r,t) = \frac{c}{2} \int_{\Sigma(r)} \phi^{(m)2}\, dS + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS d\eta \tag{111}$$

$$\geq \lambda_0 \left[\int_{\Sigma(r)} \phi^{(m)2}\, dS + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS d\eta \right], \tag{112}$$

where $t \in (0, T)$ and

$$\lambda_0 := \min(c/2, 1) \leq \lambda. \tag{113}$$

Moreover, (111) and (108) imply the bounds

$$\left[\frac{c}{2} \int_{\Sigma(r)} \phi^{(m)2}\, dS + \frac{\kappa_0}{kr^2} \int_0^t \int_{\Sigma(r)} \phi^{(m)2}\, dS d\eta \right] \leq H_{,r}(r,t)$$

$$\leq \left[\frac{ckr^2}{2\kappa_0} \int_{\Sigma(r)} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS \right.$$

$$\left. + \int_0^t \int_{\Sigma(r)} \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS d\eta \right]. \tag{114}$$

5.4 | Differential inequalities

A differential inequality for the heat flow $H(r,t)$ is derived for the general case, with additional results derived in Section 5.4.2 for $\gamma \geq 1$.

5.4.1 | $\gamma \geq 0$

Schwarz's inequality applied to (83) followed by the extended Poincaré–Wirtinger inequality (108) gives

$$|H(r,t)| \leq \kappa_1^{1/2} \left(\int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \phi^{(m)2}\, dS d\eta \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS d\eta \right)^{1/2}$$

$$\leq a^{-1} r \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \kappa_{ij} \phi^{(m)}_{,i} \phi^{(m)}_{,j}\, dS d\eta, \tag{115}$$

where

$$a := \left(\frac{\varkappa_0}{k\varkappa_1}\right)^{1/2}. \tag{116}$$

In consequence, (103) and (111) imply for $\gamma \geq 1$ or $\gamma = 0$ and for each $t \in (0, T)$, that we have the differential inequality

$$|H(r,t)| \leq a^{-1} r H_{,r}(r,t), \qquad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T), \tag{117}$$

which is explicitly independent of $\gamma\ (\geq 0)$.

5.4.2 | $\gamma \geq 1$

Now let $\gamma \geq 1$. A second differential inequality obtained from (91) and (103) subject to condition (90) is given by

$$|H(r,t)| \leq H_{,r}(r,t), \qquad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T_1]. \tag{118}$$

Otherwise, we conclude from (87) and (104) that

$$|H(r,t)| \leq b^{-1} H_{,r}(r,t), \qquad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T_1], \tag{119}$$

where

$$b := \lambda \mu^{-1}, \tag{120}$$

and λ and μ are defined in (105) and (88).

The respective differential inequalities (117)–(119) generate quantitatively different estimates as shown in subsequent sections.

6 | SPATIAL GROWTH ESTIMATES

6.1 | $\gamma \geq 1$, $\gamma = 0$

The heat supply and heat source distribution in $\Omega(0, r_0)$ for all $t \in (0,T)$ are supposed such that

$$H(r_0,t) > 0, \qquad t \in (0,T). \tag{121}$$

But $H(r,t)$ is nondecreasing with respect to r by inequality (102) so that

$$H(r,t) > 0, \qquad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T), \tag{122}$$

and consequently differential inequality (117) may be written as

$$H(r,t) \leq a^{-1} r H_{,r}(r,t), \qquad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T), \tag{123}$$

where the positive constant a is defined in (116). Integration leads to the algebraic increasing lower bound:

$$H(r_0,t)\left(\frac{r}{r_0}\right)^a \le H(r,t), \qquad 0 < r_0 \le r \le \infty, \quad t \in (0,T), \tag{124}$$

which combined with (115) yields

$$aH(r_0,t)\left(\frac{r^{a-1}}{r_0^a}\right) \le J_1(r,t) \tag{125}$$

$$\le T_1^\gamma J_1^{(0)}(r,t), \tag{126}$$

where

$$J_1(r,t) := \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta, \qquad \gamma \ge 0, \tag{127}$$

$$J_1^{(0)}(r,t) := \int_0^t \int_{\Sigma(r)} \kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta. \tag{128}$$

Observe that for $a \ge 1$, the quadratic forms J_1, $J_1^{(0)}$ for the mean-square cross-sectional spatial gradient of the (time-weighted) temperature remain strictly positive. They algebraicly increase with respect to r when $a > 1$, which, however, restricts the constant k appearing in the Poincaré–Wirtinger inequality (106).

6.2 | $\gamma \ge 1$

Exponential growth occurs when $\gamma \ge 1$ subject to condition (90). By virtue of (122), differential inequality (118) reduces to

$$H(r,t) \le H_{,r}(r,t),$$

which for fixed $t \in (0,T)$ integrates to give the growth rate estimate

$$H(r_0,t)\exp(r-r_0) \le H(r,t), \qquad 0 < r_0 \le r \le \infty, \quad t \in (0,T_1]. \tag{129}$$

An improved estimate, based on inequality (119), is given by

$$H(r_0,t)\exp b(r-r_0) \le H(r,t), \qquad 0 < r_0 \le r \le \infty, \quad t \in (0,T_1], \tag{130}$$

where b is defined in (120). Consequently, $H(r,t)$ exhibits at least exponential spatial growth without restriction on either λ, μ, or k but only for $\gamma \ge 1$. Furthermore, on noting (115), we may derive from (129) and (130) estimates in terms of $J_1(r,t)$ or $J_1^{(0)}$ introduced in (127) and (128). For example, we have the growth rate bound

$$aH(r_0,t)r^{-1}\exp b(r-r_0) \le J_1(r,t), \qquad \gamma \ge 1. \tag{131}$$

It is of interest to also express the growth estimates in terms of the energy functions (95) and (98). On using (96) and (97), we obtain

$$H(r_0,t)[\exp b(r-r_0)-1] \leq E(r_0,r;t), \qquad \gamma \geq 1, \tag{132}$$

$$H(r_0,t)\left[\left(\frac{r}{r_0}\right)^a - 1\right] \leq E^{(0)}(r_0,r;t), \qquad \gamma = 0, \tag{133}$$

and consequently conclude that $E(r_0,\infty;t)$ and $E^{(0)}(r_0,\infty;t)$ do not exist when $H(r,t)$ becomes positive at any distance $r \geq r_0$.

Unbounded asymptotic behavior is alternatively represented on application of l'Hôpital's theorem to (132), which for $\gamma \geq 1$ gives:

$$\lim_{r\to\infty} \exp -b(r-r_0)[(c\gamma/2)J_2(r,t)+J_1(r,t)] \geq bH(r_0,t), \qquad t \in (0,T_1], \tag{134}$$

where $0 < r_0 < \infty$, $J_1(r,t)$ is defined in (127), and

$$J_2(r,t) := \int_0^t \int_{\Sigma(r)} (t-\eta)^\gamma \phi^{(m)2}\, dS d\eta. \qquad \gamma \geq 1. \tag{135}$$

Similar conclusions hold for $\gamma = 0$ with $J_1(r,t)$ replaced by $J_1^{(0)}(r,t)$ and $J_2(r,t)$ replaced by

$$J_2^{(0)}(r,t) = \int_{\Sigma(r)} \phi^{(m)2}\, dS, \qquad \gamma = 0. \tag{136}$$

We then have

$$\lim_{r\to\infty} r^{1-a}\left[(c/2)J_2^{(0)}(r,t)+J_1^{(0)}(r,t)\right] \geq \frac{a}{r_0^a}H(r_0,t), \qquad \gamma = 0, \quad t \in (0,T). \tag{137}$$

Although, for example, (125) and (131) establish growth of the cross-sectional gradient measure $J_1(r,t)$, it is not possible to prove from these estimates or from the asymptotic behavior (134) whether various cross-sectional measures of the time-weighted temperature become asymptotically unbounded and in what sense. The analysis required to prove such behavior is described in the next section where it is shown that $\phi^{(m)}(r,t)$ subject to (121) and other appropriate conditions must become unbounded for sufficiently large r. The actual rate of growth is not determined.

7 | SPATIAL ASYMPTOTIC GROWTH OF THE TIME-WEIGHTED TEMPERATURE

The argument is developed for $\gamma \geq 1$, with reference to $\gamma = 0$ as required. Consequently, in this section we employ the spatial growth estimates (125), and (131) for the mean-square gradient $J_1(r,t)$ to prove that subject to (121) the time-weighted temperature $\phi^{(m)}(x,t)$, $m = 0,1,2\ldots$, becomes unbounded for sufficiently large r at each time $t \in (0,T)$.

We proceed to establish a contradiction and accordingly suppose that $\phi^{(m)}(x,t)$ remains bounded on $\Omega_{[0,T]}$ and that Theorem 2 applies. Recall that $\phi^{(m)}(x,t)$ achieves its maximum either on $\Sigma(r_0)\times(0,T]$ or on $\Sigma(\infty)\times(0,T]$. The maximum cannot occur at an interior point since $\phi^{(m)}$ would then be constant in contradiction to the gradient growth estimates for J_1 and $J_1^{(0)}$.

Fix τ and select $x\in\Sigma^{(0)}(r)$, where

$$\Sigma^0(r) := \Sigma(r)\cap C\left(z,\tau;\frac{1}{2}R\right),$$

and the closed circular cylinder $C(x,t;R)$ is defined in (18). Let $|\Sigma(r)|$ denote the surface area of the spherical cross-section $\Sigma(r)$. By Theorem 2 with $p=1,\ q=0$, we have

$$\begin{aligned}
\int_{\Sigma^{(0)}(r)}\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\,dS &\le \kappa_1\int_{\Sigma^{(0)}(r)}\phi^{(m)}_{,i}\phi^{(m)}_{,i}\,dS\\
&\le \kappa_1|\Sigma^{(0)}(r)|\max_{C(z,\tau;R/2)}\phi^{(m)}_{,i}\phi^{(m)}_{,i}\\
&\le \kappa_1\left|\Sigma^{(0)}(r)\right|\left(\max_{C(z,\tau;R/2)}\left|\phi^{(m)}_{,i}\right|\right)\left(\max_{C(z,\tau;R/2)}|\phi^{(m)}_{,i}|\right)\\
&\le \kappa_1\left|\Sigma^{(0)}(r)\right|\left(\frac{D_{10}}{R^6}\int_{L^1(C)}\left|\phi^{(m)}\right|\,dx\,d\eta\right)^2\\
&\le \kappa_1\left(\max_{D(\tau)}|\phi^{(m)}|\right)^2\frac{16|\Sigma^{(0)}(r)|D_{10}^2\pi^2}{9R^2},
\end{aligned}\tag{138}$$

where $D(\tau)$ is defined in (27) and

$$C := C(z,\tau;R).\tag{139}$$

We conclude for $t\in[0,T_1]$ and γ a nonnegative integer that

$$\int_0^t\int_{\Sigma^{(0)}(r)}(t-\eta)^\gamma\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\,dS\,d\eta \le \frac{T_1^{(\gamma+1)}}{(\gamma+1)}\kappa_1\max_{D(\tau)}|\phi^{(m)}|^2||\Sigma^{(0)}(r)|\left(\frac{4\pi D_{10}}{3R}\right)^2.\tag{140}$$

The same argument applied to $x\in\Sigma^{(\alpha)}(r)$ where

$$\Sigma^{(\alpha)}(r) := \Sigma(r)\cap D_2^{(\alpha)}(\tau),\qquad \alpha=1,2,$$

leads to

$$\int_0^t\int_{\Sigma^{(\alpha)}(r)}(t-\eta)^\gamma\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\,dS\,d\eta \le \frac{T_1^{(\gamma+1)}}{(\gamma+1)}\kappa_1\max_{D(\tau)}|\phi^{(m)}|^2||\Sigma^{(\alpha)}(r)|\left(\frac{4\pi\tilde{D}_{10}}{3R^{(\alpha)}}\right)^2,\tag{141}$$

for bounded constant $\tilde{D}_{10}$. On noting that

$$|\Sigma^{(0)}(r)|\le|\Sigma(r)|,\qquad |\Sigma^{(\alpha)}(r)|\le|\Sigma(r)|<4\pi r^2,\qquad \alpha=1,2,$$

and that $R>R^{(1)}\ge R^{(2)}$, we obtain by addition of (140) and (141) the upper bound

$$\int_0^t\int_{\Sigma(r)}(t-\eta)^\gamma\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\,dSd\eta \le \int_0^t\int_{\Sigma^{(0)}(r)}(t-\eta)^\gamma\kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\,dSd\eta$$

$$+\int_0^t \int_{\Sigma^{(1)}(r)} (t-\eta)^\gamma \kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta$$

$$+\int_0^t \int_{\Sigma^{(2)}(r)} (t-\eta)^\gamma \kappa_{ij}\phi^{(m)}_{,i}\phi^{(m)}_{,j}\, dSd\eta$$

$$\leq \frac{12\pi T_1^{(\gamma+1)} r^2}{(\gamma+1)}\kappa_1 \max_{D(\tau)} |\phi^{(m)}|^2 \left(\frac{4\pi\hat{D}_{10}}{3R^{(2)}}\right)^2, \tag{142}$$

for some bounded constant $\hat{D}_{10}$. The various increasing lower bounds (125) (for $\gamma \geq 0$ and $a > 3$), or (131) (for $\gamma \geq 1$ but unrestricted a), may then be combined with inequality (142) to show that

$$\max_{D(\tau)} |\phi^{(m)}|$$

becomes unbounded as $r \to \infty$, contrary to the assumption that $\phi^{(m)}(x,t)$ remains bounded. Under these conditions, $\phi^{(m)}(x,t)$ must be unbounded at each $t \in (0,T)$ as $r \to \infty$. The rate of growth, however, has not been established, and it cannot be concluded that the class of unbounded solutions satisfying the increasing upper bound (28) is necessarily empty.

8 | SPATIAL DECAY ESTIMATES

Let us suppose that the energies $E(r_0,\infty;t)$ and $E^{(0)}(r_0,\infty;t)$ are bounded and in particular that

$$\lim_{r\to\infty} [(c\gamma/2)J_2(r,t) + J_1(r,t)] < \infty, \qquad \gamma \geq 1, \tag{143}$$

$$\lim_{r\to\infty} \left[(c/2)J_2^{(0)}(r,t) + J_1^{(0)}(r,t)\right] < \infty, \qquad \gamma = 0, \quad a > 1. \tag{144}$$

When $a = 1$, the right side of (144) must be replaced by zero. The asymptotic conditions (134) and (137) are contradicted and it follows that $H(r,t) \leq 0$ for all $0 < r_0 \leq r \leq \infty$. In consequence, (117) assumes the form

$$0 \leq a^{-1} r H_{,r}(r,t) + H(r,t), \quad 0 < r_0 \leq r \leq \infty, \quad t \in (0,T). \tag{145}$$

Integration of (145) yields the decay rate estimate

$$-H(r,t) \leq -\frac{r_0^a}{r^a} H(r_0,t). \tag{146}$$

Corresponding inequalities obtained from (118) and (119) lead to exponential decay but only for $\gamma \geq 1$.

Let $H(r_0,t) \equiv 0$ so that (146) implies $H(r,t) \equiv 0$ for all $r \geq r_0$, and therefore $H_{,r}(r,t) = 0,\ r \geq r_0$. We conclude from (109) and (114) that $\phi^{(m)}(x,t) \equiv 0,\ r \geq r_0,\ t \in [0,T)$.

8.1 | $\gamma \geq 1$

Next consider $H(r_0, t) < 0$. We again separately examine the cases $\gamma \geq 1$ and $\gamma = 0$. For $\gamma \geq 1$, and on noting that $a > 0$, we obtain from inequality (146) the limiting behavior

$$\lim_{r\to\infty} H(r,t) = 0, \qquad t \in [0,T). \tag{147}$$

Insertion into (96) then gives for each $t \in [0, T)$ the algebraic decay estimate

$$E(r,\infty;t) = -H(r,t) \leq -\frac{r_0^a}{r^a} H(r_0,t) = \frac{r_0^a}{r^a} E(r_0,\infty;t). \tag{148}$$

We remark that inequality (119) leads to the exponential decay estimate

$$E(r,\infty;t) \leq \exp -b(r-r_0) E(r_0,\infty;t), \qquad t \in (0,T_1]. \tag{149}$$

From either (148) or (149), we have

$$\lim_{r\to\infty} E(r,\infty;t) = 0, \tag{150}$$

and consequently

$$\lim_{r\to\infty} J_\alpha(r,t) = 0, \qquad \alpha = 1,2, \tag{151}$$

which improves the limiting behavior postulated in (143).

8.2 | $\gamma = 0$

The treatment of the case $\gamma = 0$ varies only in obvious detail and yields the decay rate estimate

$$E^{(0)}(r,\infty;t) \leq (r_0/r)^a E^{(0)}(r_0,\infty;t), \tag{152}$$

along with the asymptotic behavior corresponding to (151), which may equivalently be expressed as

$$J_\alpha^{(0)}(r,t) = o(r^{-(a+1)}), \qquad r\to\infty, \quad \alpha = 1,2. \tag{153}$$

Remark 8 (Summary). In summary, we have proved asymptotic assumptions (143) and (144) imply that

$$\int_{\Sigma(r)} \phi^{(m)2}\, dS, \qquad \int_0^t \int_{\Sigma(r)} \phi^{(m)2}\, dSd\eta, \qquad \int_0^t \int_{\Sigma(r)} \kappa_{ij}\phi_{,i}^{(m)}\phi_{,j}^{(m)}\, dSd\eta, \tag{154}$$

vanish at most to the order $r^{-(a+1)}$ as $r \to \infty$ at each time $t \in [0, T)$. The decay becomes exponential subject to previously noted restrictions.

8.3 | Decay for mean-square cross-sectional measure of $\phi^{(m)}$

The limit (153) is sufficient to derive a decay rate estimate for the mean-square cross-sectional measure of the time-weighted temperature. The divergence theorem leads to the identity

$$r^{-2}\int_{\Sigma(r)} \phi^{(m)2}(x,t)x_i n_i\, dS = -\int_{\Omega(r,\infty)} \phi^{(m)2}(y,t)|y|^{-2}\, dy$$
$$-2\int_{\Omega(r,\infty)} \phi^{(m)}(y,t)\phi^{(m)}_{,y_i}(y,t)y_i|y|^{-2}\, dy, \tag{155}$$

where $|y|^2 = y_i y_i$. Young's inequality applied to the second term on the right then gives

$$r^{-1}\int_{\Sigma(r)} \phi^{(m)2}\, dS \le \int_{\Omega(r,\infty)} \phi^{(m)}_{,y_i}\phi^{(m)}_{,y_i}\, dy,$$

which together with (7) and (152) leads to the desired estimate:

$$\int_{\Sigma(r)} \phi^{(m)2}\, dS \le \kappa_0^{-1}\frac{r_0^a}{r^{(a-1)}}E^{(0)}(r_0,\infty;t), \qquad t\in[0,T). \tag{156}$$

9 | CONCLUDING REMARKS

Asymptotic spatial growth and decay has been established for various mean-square cross-sectional measures of the time-weighted temperature satisfying the homogeneous isotropic heat equation on an open region with noncompact boundary. Behavior at asymptotically large spatial distance is not a priori specified. The rate of growth is at least algebraic when the heat flow is positive over a spherical cross section of radius $r_0 > 0$. Restrictions on material parameters lead to a growth rate that is at least exponential. When the heat flow over the cross section is nonpositive, the decay rate becomes at most algebraic or exponential for material parameters similarly restricted. Conclusions for mean-square spatial gradient measures supplement those derived in Ref. [4], while decay rate estimates for the thermal energy improve those obtained in Ref. [12] by entirely different methods.

The analysis applies also to first- and higher order time derivatives of the temperature provided the heat equation is valid at zero time to ensure that each initial time derivative vanishes.

The results of our investigation clearly have counterparts in systems analogous to the heat equation; for example, the diffusion equation. They also relate to linear thermoelasticity and other coupled theories.

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