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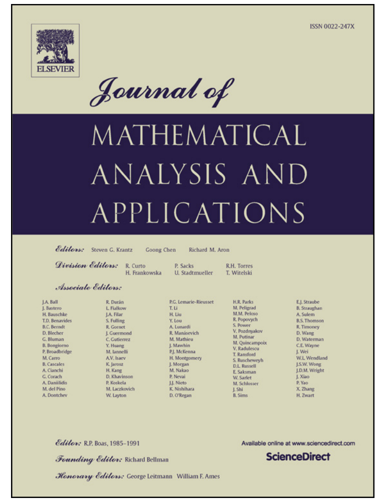


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# Generalised cosine functions, basis and regularity properties

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**Abstract**

We examine regularity and basis properties of the family of rescaled $p$-cosine functions. We find sharp estimates for their Fourier coefficients. We then determine two thresholds, $p_0 < 2$ and $p_1 > 2$, such that this family is a Schauder basis of $L_s(0,1)$ for all $s > 1$ and $p \in [p_0, p_1]$.

## 1 Introduction

The contents of this paper can be summarised as follows. Consider a continuous 2-periodic function $f : \mathbb{R} \longrightarrow \mathbb{C}$. Denote by $\mathcal{F}$ the family of rescalings $\mathcal{F} = \{f(nx)\}_{n\in\mathbb{N}}$. When does $\mathcal{F}$ form a Schauder basis of $L_s \equiv L_s(0,1)$ for all $s > 1$? This question can be traced back to a 1945 note by Arne Beurling [1]. However, quite remarkably, there are still a number of open problems associated to it. As it turns, finding a concrete answer can be extremely difficult, even for apparently simple functions $f$.

In a series of recent papers the above question has been addressed for the particular choice $f(x) = \sin_p(\pi_p x)$, the $p$-sine functions. Let $p > 1$. Let the increasing function $F_p : [0,1] \longrightarrow [0, \frac{\pi_p}{2}]$ be defined by means of the integral

$$F_p(y) := \int_0^y (1 - t^p)^{-\frac{1}{p}} \mathrm{d}t \tag{1}$$

where

$$\pi_p := 2F_p(1) = \frac{2\pi}{p\sin(\frac{\pi}{p})}.$$

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Denote the inverse of $F_p$ by $\sin_p$, which is increasing in the segment $[0, \frac{\pi_p}{2}]$. Extend to the whole of $\mathbb{R}$ by means of the rules

$$(2) \qquad \sin_p(-x) = -\sin_p(x) \qquad \text{and} \qquad \sin_p\left(\frac{\pi_p}{2} - x\right) = \sin_p\left(\frac{\pi_p}{2} + x\right),$$

which makes it $2\pi_p$-periodic and continuous in $\mathbb{R}$. The choice $p = 2$ corresponds to the standard trigonometric setting $\sin_2 \equiv \sin$, $\pi_2 = \pi$ and in this case $\mathcal{F}$ is a Schauder basis of $L_s$ for all $s > 1$ as a consequence of Fourier's Theorem.

The study of generalised trigonometric functions has a long history which dates back to the XIX century, [14] and [9, Note 4.1]. The study of the $p$-sine functions is closely related to the one-dimensional $p$-Laplacian non-linear eigenvalue problem, see the work of Elbert [10] and Ôtani [15]. Their basis properties were first examined in [2], where it was announced that the family $\{\sin_p(n\pi_p\,\cdot)\}_{n\in\mathbb{N}}$ forms a Schauder basis of $L_s$ for all $s > 1$ and $p \geq \frac{12}{11}$. Further development in this respect were settled in [5], [6] and [4]. Currently we know that this family is a Schauder basis of $L_s$ for all $s > 1$ when $p > \tilde{p}_0$, and also a Riesz basis of $L_2$ for $p \in (\hat{p}_0, \tilde{p}_0]$, where $\tilde{p}_0 \approx 1.087$ and $\hat{p}_0 \approx 1.044$ satisfy complicated identities involving hypergeometric functions [4].

Let

$$(3) \qquad \cos_p x := \frac{\mathrm{d}}{\mathrm{d}x} \sin_p x \qquad \forall x \in \mathbb{R}$$

and set $f(x) = \cos_p(\pi_p x)$, the $p$-cosine functions. From the various results established in the recent paper [7], it follows that $\mathcal{F} \cup \{1\}$ is a Schauder basis of $L_s$ for all $s > 1$ and $p \in (p_0^\dagger, 2]$ where $p_0^\dagger \approx 1.75$. In the present work we establish that this basis property in fact holds true for $p$ in a wider segment. To be precise, we show the following.

**Theorem 1.** *There exist $p_0 < \frac{3}{2}$ and $p_1 > \frac{11}{5}$, such that $\{\cos_p(n\pi_p\,\cdot)\}_{n=0}^{\infty}$ is a Schauder basis of $L_s$ for all $s > 1$ and $p \in [p_0, p_1]$.*

The constants $p_0$ and $p_1$ will be given analytically as the zeros of corresponding equations involving the parameter $p$. Their approximated values turn out to be $p_0 \approx 1.46$ and $p_1 \approx 2.43$.

The proof of Theorem 1 is naturally divided into the cases $1 < p < 2$ and $p > 2$. The different parts of the paper follow this division. In Section 2 we collect various properties of the $p$-trigonometric functions which will be useful later on. In Section 3 we establish precise upper bounds on the asymptotic behaviour of the Fourier coefficients of $\cos_p(\pi_p\cdot)$. In Section 4 we recall the framework for determining invertibility of the change of coordinates map between the families $\{\cos(n\pi\cdot)\}_{n=0}^{\infty}$ and $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$. In Section 5 we assemble the proof of Theorem 1, by combining the crucial criterion (12) of Section 4 with the estimates of Section 3. In the final Section 6 we describe the relation between the results announced here and other existing work.

## 2 The generalised trigonometric functions

We begin by recalling various elementary properties of the $p$-cosine functions. A more complete account on this matter can be found in [5, Section 2] and [9, Chapter 2].

Throughout we shall assume that $1 < p < \infty$. Note that $\pi_p$ is a decreasing function, smooth in $p > 1$, such that

$$\begin{cases} \pi_p \to \infty & p \to 1^+ \\ \pi_p = \pi & p = 2 \\ \pi_p \to 2 & p \to \infty. \end{cases}$$

Here and everywhere below we write $p' := p/(p-1)$. According to [5, (2.3)], we know that

$$p'\pi_{p'} = p\pi_p. \tag{4}$$

From (2) and (3) it immediately follows that $\cos_p$ is $2\pi_p$-periodic,

$$\cos_p(x) = \cos_p(-x) \quad \text{and} \quad \cos_p\left(x + \frac{\pi_p}{2}\right) = -\cos_p\left(x - \frac{\pi_p}{2}\right) \qquad \forall x \in \mathbb{R}.$$

Moreover, setting $y = \sin_p(x)$ for $x \in [0, \pi_p/2]$ in the formula for the derivative of the inverse function of (1), gives

$$\cos_p(x) = (1 - y^p)^{1/p} = (1 - \sin_p(x)^p)^{1/p}. \tag{5}$$

Thus, $\cos_p$ is decreasing in $(0, \pi_p/2]$, $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. In fact we have,

$$|\sin_p x|^p + |\cos_p x|^p = 1 \qquad \forall x \in \mathbb{R}.$$

See [5, (2.7)].

**Lemma 1.** *For all $x \in [0, \frac{1}{2})$,*

*a.*

$$\cos_p(\pi_p x) = \sin_{p'}\left(\pi_{p'}\left(\frac{1}{2} - x\right)\right)^{p'-1}$$

*b.*

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos_p(x) = -\sin_p(x)^{p-1}\cos_p(x)^{2-p}$$

*c.*

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos_p(x) = \sin_p(x)^{p-2}\cos_p(x)^{3-2p}[2 - p - \cos_p(x)^p].$$

*Proof.* The calculations leading to "a" and "b" can be found in the proofs of [5, Proposition 2.2] and [5, Proposition 2.1], respectively. From (5) we get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos_p(x) &= (2-p)\sin_p(x)^{2p-2}\cos_p(x)^{3-2p} - (p-1)\sin_p(x)^{p-2}\cos_p(x)^{3-p} \\ &= \sin_p(x)^{p-2}\cos_p(x)^{3-2p}\left[(2-p)\sin_p(x)^p - (p-1)\cos_p(x)^p\right],\end{aligned}$$

which is "c". □

The following inequalities will be important below.

**Lemma 2.** *Let $1 < p \le q < \infty$ and $x \in [0, \frac{1}{2}]$. Then*

*a.* $\sin_p(\pi_p x) \ge \sin_q(\pi_q x)$

*b.* $\cos_p(\pi_p x) \le \cos_q(\pi_q x)$.

*Proof.* Statement "a" is [5, Corollary 4.4-(iii)].

Let us show "b". A direct evaluation at $x = 0$ and $x = 1/2$ gives equality for all $p$ and $q$ at these points, so these two cases are immediate. Let $x \in (0, \frac{1}{2})$ be fixed. Since $p'$ is decreasing in $p > 1$, from part "a" it follows that

$$\frac{\mathrm{d}}{\mathrm{d}p}\sin_{p'}\left(\pi_{p'}\left(\frac{1}{2}-x\right)\right) \ge 0 \qquad \forall p \in (1,\infty).$$

Note that, $0 < \sin_{p'}(\pi_{p'}(\frac{1}{2}-x)) < 1$ and hence $\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2}-x))) < 0$. Substituting the identity from Lemma 1(a), yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}p}\cos_p(\pi_p x) = \frac{\mathrm{d}}{\mathrm{d}p}\left[\sin_{p'}\left(\pi_{p'}\left(\frac{1}{2}-x\right)\right)\right]^{\frac{1}{p-1}} \\ &= \left[-\frac{\ln(\sin_{p'}(\pi_{p'}(\frac{1}{2}-x)))}{(p-1)^2} + \frac{\frac{\mathrm{d}}{\mathrm{d}p}\left[\sin_{p'}(\pi_{p'}(\frac{1}{2}-x))\right]}{(p-1)\sin_{p'}(\pi_{p'}(\frac{1}{2}-x))}\right]\cos_p(\pi_p x) > 0.\end{aligned}$$

This implies "b". □

### 2.1 The case $1 < p < 2$

For $1 < p < 2$, let $\mathrm{u}_p : [0, \frac{1}{2}] \longrightarrow \mathbb{R}$ be given by

$$\mathrm{u}_p(x) := \cos_p'(\pi_p x) = -\sin_p(\pi_p x)^{p-1}\cos_p(\pi_p x)^{2-p}.$$

This function will simplify the notation when we determine estimates for the Fourier coefficients of the $p$-cosine functions in Section 3.1. Here and everywhere below we write

$$c_p := (p-1)^{\frac{p-1}{p}}(2-p)^{\frac{2-p}{p}}. \tag{6}$$

**Lemma 3.** *Let $1 < p < 2$. Then*

*a.* $\mathrm{u}_p(x) \leq 0$ *for all* $x \in [0, \frac{1}{2}]$

*b.* $\mathrm{u}_p(x) = 0$ *if and only if* $x = 0$ *or* $x = \frac{1}{2}$

*c.* $\mathrm{u}_p(x) = -c_p$ *for* $x \in [0, \frac{1}{2}]$ *if and only if* $x = m_p \in (0, \frac{1}{2})$, *where* $m_p$ *is the unique point such that* $\cos_p(\pi_p m_p)^p = 2 - p$

*d.* $\mathrm{u}_p : [0, m_p] \longrightarrow [-c_p, 0]$ *is decreasing*

*e.* $\mathrm{u}_p : [m_p, \frac{1}{2}] \longrightarrow [-c_p, 0]$ *is increasing*

*f.* $\min\limits_{x \in [0, \frac{1}{2}]} \mathrm{u}_p(x) = -c_p$.

*Proof.* Since $\sin_p(\pi_p x)$ and $\cos_p(\pi_p x)$ are non-negative over $[0, \frac{1}{2}]$, then "a" holds true. Since $\sin_p(\pi_p x)$ only vanishes at $x = 0$ and $\cos_p(\pi_p x)$ only vanishes at $x = \frac{1}{2}$ in this interval, then "b" holds true.

Lemma 1-c gives

$$\mathrm{u}_p'(x) = \pi_p \sin_p(\pi_p x)^{p-2} \cos_p(\pi_p x)^{3-2p}[2 - p - \cos_p(\pi_p x)^p].$$

Neither $\sin_p$ nor $\cos_p$ vanish in $(0, \frac{1}{2})$. On the other hand, $\cos_p(0) = 1 > 2-p$, $\cos_p(\frac{\pi_p}{2}) = 0 < 2 - p$ and $\cos_p(\pi_p x)^p$ is decreasing for $x \in (0, \frac{1}{2})$. Then the term $\cos_p(\pi_p x)^p + p - 2$ indeed vanishes at the unique point $m_p \in (0, \frac{1}{2})$ as stated in "c".

At $m_p$,

$$\begin{aligned} \mathrm{u}_p(m_p) &= -\sin_p(\pi_p m_p)^{p-1} \cos_p(\pi_p m_p)^{2-p} \\ &= -(1 - \cos_p(\pi_p m_p)^p)^{\frac{p-1}{p}} \cos_p(\pi_p m_p)^{2-p} = -c_p. \end{aligned}$$

Hence, the proof of "d" and "e", and thus of "f", is achieved as follows. Just observe that in the expression for $\mathrm{u}_p'(x)$ above, $\cos_p(\pi_p x)^p > 2 - p$ for $x \in [0, m_p)$ and $\cos_p(\pi_p x)^p < 2 - p$ for $x \in (m_p, \frac{1}{2})$, because $\cos_p(\pi_p x)$ is decreasing in $x \in (0, \frac{1}{2})$. □

According to parts "d" and "e" of Lemma 3, the function $\mathrm{u}_p$ is invertible, when restricted to the segments $[0, m_p]$ and $[m_p, \frac{1}{2}]$. We denote the inverses by $\mathrm{w}_{1,p} : [-c_p, 0] \longrightarrow [0, m_p]$ and $\mathrm{w}_{2,p} : [-c_p, 0] \longrightarrow [m_p, \frac{1}{2}]$, respectively, so that

$$\mathrm{u}_p(\mathrm{w}_{k,p}(x)) = x \qquad \forall x \in [-c_p, 0] \quad k = 1, 2.$$

### 2.2 The case $p > 2$

For $p > 2$, let $\mathrm{v}_p : (0, \frac{1}{2}] \longrightarrow [0, \infty)$ be given by

$$\mathrm{v}_p(x) := (p' - 1) \sin_{p'}(\pi_{p'} x)^{p'-2} \cos_{p'}(\pi_{p'} x).$$

Let us summarise various properties of this function, which will be employed in Section 3.2.

**Lemma 4.** *Let $p > 2$. Then*

*a.* $\mathrm{v}_p$ *is decreasing in* $(0, \frac{1}{2}]$

*b.* $\lim_{x \to 0^+} x\, \mathrm{v}_p(x) = 0$

*c.* $\lim_{x \to 0^+} \mathrm{v}_p(x) = +\infty$ *and* $\mathrm{v}_p(\frac{1}{2}) = 0$

*d.* $\lim_{x \to 0^+} \mathrm{v}'_p(x) = -\infty$ *and* $\mathrm{v}'_p(\frac{1}{2}) = 0$.

*Proof.* For $p > 2$, $p' \in (1, 2)$ and so $p' - 2 < 0$. Since, $\sin_{p'}(\pi_{p'} x)$ is increasing and $\cos_{p'}(\pi_{p'} x)$ is decreasing in $x \in (0, \frac{1}{2})$, then "a" holds true.

Let us show "b". L'Hôpital's Rule gives

$$\lim_{x \to 0^+} \frac{x}{[\sin_{p'}(\pi_{p'} x)]^{2-p'}} = \lim_{x \to 0^+} \frac{[\sin_{p'}(\pi_{p'} x)]^{p'-1}}{(2 - p')\pi_{p'} \cos_{p'}(\pi_{p'} x)} = 0.$$

Then,

$$\lim_{x \to 0^+} x\, \mathrm{v}_p(x) = \lim_{x \to 0^+} (p' - 1) \frac{x \cos_{p'}(\pi_{p'} x)}{[\sin_{p'}(\pi_{p'} x)]^{2-p'}} = 0,$$

as claimed in "b".

Both statements "c" and "d" follow directly from (5), the expression

$$\mathrm{v}'_p(x) = (p' - 1)\pi_{p'} \sin_{p'}(\pi_{p'} x)^{p'-3} \cos_{p'}(\pi_{p'} x)^{2-p'} \left[(p' - 1) \cos_{p'}(\pi_{p'} x)^{p'} - 1\right],$$

and continuity of $\sin_p$ and $\cos_p$ at $x = 0$. ∎

According to this lemma, there exists a function $\mathrm{z}_p : [0, \infty) \to (0, \frac{1}{2}]$ such that $\mathrm{z}_p$ is the inverse function of $\mathrm{v}_p$. This inverse function has the following characteristics.

a. $\mathrm{z}_p$ is decreasing in $[0, \infty)$

b. $\mathrm{z}_p(0) = \frac{1}{2}$ and $\lim_{x \to \infty} \mathrm{z}_p(x) = 0$

c. $\lim_{x \to 0^+} \mathrm{z}'_p(x) = +\infty$ and $\lim_{x \to \infty} \mathrm{z}'_p(x) = 0$.

## 3 The Fourier coefficients of the $p$-cosine functions

Let

$$a_j(p) \equiv a_j := 2\int_0^1 \sin_p(\pi_p x)\sin(j\pi x)\mathrm{d}x \qquad \forall j \in \mathbb{N}$$

be the Fourier sine coefficients of $\sin_p(\pi_p x)$. Let

$$b_j(p) \equiv b_j := 2\int_0^1 \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x \qquad \forall j \in \mathbb{N}\cup\{0\}$$

be the Fourier cosine coefficients of $\cos_p(\pi_p x)$. Since $\sin_p$ is an odd function and $\cos_p$ is an even function, $a_j = b_j = 0$ for all $j \equiv_2 0$. Here and elsewhere below we will write $j \equiv_2 k$ to denote that $j \equiv k \pmod 2$.

**Lemma 5.** *For $j \in \mathbb{N}$,*

$$b_j(p) = \frac{j\pi}{\pi_p} a_j(p).$$

*Proof.* Let $j \equiv_2 1$. Integration by parts alongside with the fact that $\cos_p(\pi_p x)$ and $\cos(j\pi x)$ are odd with respect to $\frac{1}{2}$, yield

$$\begin{aligned} b_j &= 2\int_0^1 \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x = 4\int_0^{\frac{1}{2}} \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x \\ &= \frac{4}{\pi_p}\cos(j\pi x)\sin_p(\pi_p x)\Big|_0^{\frac{1}{2}} + \frac{4j\pi}{\pi_p}\int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(j\pi x)\mathrm{d}x \\ &= \frac{j\pi}{\pi_p} a_j. \end{aligned}$$

□

We now find estimates on $|b_j(p)|$ in terms of the parameter $p > 1$.

### 3.1 The case $1 < p < 2$

**Lemma 6.** *For $1 < p < 2$, let $c_p > 0$ be given by (6). Then*

$$|b_j(p)| < \frac{8\pi_p}{j^2\pi^2} c_p \qquad \forall j \geq 1.$$

*Proof.* Integrate by parts twice to get

$$
\begin{aligned}
b_j &= 4\int_0^{\frac{1}{2}} \cos_p(\pi_p x)\cos(j\pi x)\mathrm{d}x \\
&= \frac{4}{j\pi}\cos_p(\pi_p x)\sin(j\pi x)\Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j\pi}\int_0^{\frac{1}{2}} \cos_p'(\pi_p x)\sin(j\pi x)\mathrm{d}x \\
&= -\frac{4\pi_p}{j\pi}\int_0^{\frac{1}{2}} \cos_p'(\pi_p x)\sin(j\pi x)\mathrm{d}x \\
&= \frac{4\pi_p}{j^2\pi^2}\cos_p'(\pi_p x)\cos(j\pi x)\Big|_0^{\frac{1}{2}} - \frac{4\pi_p}{j^2\pi^2}\int_0^{\frac{1}{2}} \frac{d}{\mathrm{d}x}[\cos_p'(\pi_p x)]\cos(j\pi x)\mathrm{d}x.
\end{aligned}
$$

From the identities in Lemma 3(b), it follows that the boundary term in the fourth equality always vanishes. Thus,

$$
\begin{aligned}
b_j &= -\frac{4\pi_p}{j^2\pi^2}\int_0^{\frac{1}{2}} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x \\
&= -\frac{4\pi_p}{j^2\pi^2}\Big(\int_0^{m_p} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x + \int_{m_p}^{\frac{1}{2}} \mathrm{u}_p'(x)\cos(j\pi x)\mathrm{d}x\Big) \\
&= -\frac{4\pi_p}{j^2\pi^2}\Big(\int_0^{-c_p} \cos(j\pi\, \mathrm{w}_{1,p}(s))ds + \int_{-c_p}^{0} \cos(j\pi\, \mathrm{w}_{2,p}(s))\mathrm{d}s\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|b_j| &\le \frac{4\pi_p}{j^2\pi^2}\Big[\int_{-c_p}^{0} |\cos(j\pi\, \mathrm{w}_{1,p}(s))|ds + \int_{-c_p}^{0} |\cos(j\pi\, \mathrm{w}_{2,p}(s))|\mathrm{d}s\Big] \\
&< \frac{8\pi_p}{j^2\pi^2}c_p,
\end{aligned}
$$

because the functions inside the integrals are not constants identically equal to 1. □

### 3.2 The case $p > 2$

Let $p > 2$. According to Lemma 1(a),

$$
b_j(p) = 4\int_0^{\frac{1}{2}} \sin_{p'}\left(\pi_{p'}\left(\frac{1}{2} - x\right)\right)^{\frac{1}{p-1}} \cos(j\pi x)\mathrm{d}x.
$$

Since $\cos(j\pi(\frac{1}{2} - t)) = (-1)^{\frac{j-1}{2}}\sin(j\pi t)$ for $j \equiv_2 1$, changing variables to $t = \frac{1}{2} - x$ gives

$$
b_j = (-1)^{\frac{j-1}{2}} 4\int_0^{\frac{1}{2}} \sin_{p'}(\pi_{p'}t)^{\frac{1}{p-1}}\sin(j\pi t)\mathrm{d}t.
$$

By virtue of Lemma 4 and integration by parts twice, then

$$\begin{aligned} b_j &= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \int_0^{\frac{1}{2}} \mathrm{v}_p(t)\cos(j\pi t)\mathrm{d}t \\ &= (-1)^{\frac{j-1}{2}} \frac{4\pi_{p'}}{j\pi} \left[ \frac{1}{j\pi} \mathrm{v}_p(t)\sin(j\pi t)\Big|_0^{\frac{1}{2}} - \frac{1}{j\pi}\int_0^{\frac{1}{2}} \mathrm{v}_p'(t)\sin(j\pi t)\mathrm{d}t \right] \\ &= (-1)^{\frac{j+1}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^{\frac{1}{2}} \mathrm{v}_p'(t)\sin(j\pi t)\mathrm{d}t \\ &= (-1)^{\frac{j+3}{2}} \frac{4\pi_{p'}}{j^2\pi^2} \int_0^{\infty} \sin\left(j\pi\, \mathrm{z}_p(y)\right)\mathrm{d}y. \qquad (7) \end{aligned}$$

**Lemma 7.** *Let $p > 2$. Then*

$$|b_j(p)| < \frac{2\pi_{p'}}{\pi^2(p-1)} \left[ 2 + \frac{\pi^2}{2}(p-2) \right] j^{-p'} \qquad \forall j \geq 3.$$

*Proof.* Since $p > 2$, then $1 < p' < 2$. Let $r = p' - 1$. In view of Lemma 2, we have

$$\mathrm{v}_p(t) \leq r\left[\sin_{p'}(\pi_{p'} t)\right]^{r-1} \leq r\left[\sin(\pi t)\right]^{r-1}$$

and so

$$\mathrm{z}_p(y) \leq \frac{1}{\pi} \arcsin\left[ \left(\frac{y}{r}\right)^{\frac{1}{r-1}} \right] =: \mathrm{r}_p(y) \qquad \forall y \in [r, \infty). \qquad (8)$$

Set

$$\eta(j) := r\sin\left(\frac{\pi}{2j}\right)^{r-1}.$$

Then,

$$\mathrm{r}_p(\eta(j)) = \frac{1}{2j} < \frac{1}{2}.$$

Here we use the requirement $j \geq 3$, in order to make sure that the arc-sine does not change branches.

Set

$$\mathrm{J}_1 = \int_0^{\eta(j)} \mathrm{d}x = \eta(j)$$

and

$$\mathrm{J}_2 = \int_{\eta(j)}^{\infty} \sin\left(j\pi\, \mathrm{r}_p(y)\right)\mathrm{d}y.$$

Then, (7) yields

$$|b_j| \leq \frac{4\pi_{p'}}{j^2\pi^2}(\mathrm{J}_1 + \mathrm{J}_2).$$

Here $\mathrm{J}_2$ is guaranteed to be on the right hand side, because

$$0 < j\pi\, \mathrm{z}_p(y) \leq j\pi\, \mathrm{z}_p(\eta(j)) \leq j\pi\, \mathrm{r}_p(\eta(j)) = \frac{\pi}{2},$$

so that $0 < \sin(j\pi\,\mathrm{z}_p(y)) \leq \sin(j\pi\,\mathrm{r}_p(y))$ for $y \in [\eta(j), \infty)$.

Let us estimate an upper bound for $\mathrm{J}_2$. Changing variables to

$$t = j\pi\,\mathrm{r}_p(y) \Longleftrightarrow y = r\sin\left(\frac{t}{j}\right)^{r-1}$$

gives

$$\begin{aligned}
\mathrm{J}_2 &= \int_0^{\frac{\pi}{2}} \frac{r(1-r)}{j}\sin\left(\frac{t}{j}\right)^{r-2}\cos\left(\frac{t}{j}\right)\sin(t)\mathrm{d}t \\
&= r(1-r)\int_0^{\frac{\pi}{2}} \sin\left(\frac{t}{j}\right)^{r-1}\left[\frac{\frac{t}{j}}{\sin\left(\frac{t}{j}\right)}\right]\left(\frac{\sin t}{t}\right)\cos\left(\frac{t}{j}\right)\mathrm{d}t.
\end{aligned}$$

Note that,

$$\max_{0<\theta\leq\frac{\pi}{2}} \frac{\theta}{\sin\theta} = \frac{\pi}{2}, \qquad \max_{0<\theta\leq\frac{\pi}{2}} \frac{\sin\theta}{\theta} = 1 \tag{9}$$

and

$$0 < t < j\pi\,\mathrm{r}_p(\eta(j)) = \frac{\pi}{2}.$$

Here we are using once again the fact that $j \geq 3$. Then

$$\mathrm{J}_2 < \frac{\pi}{2}r(1-r)\int_0^{\frac{\pi}{2}} \sin\left(\frac{t}{j}\right)^{r-1}\cos\left(\frac{t}{j}\right)\mathrm{d}t.$$

Changing variables to

$$\tau = \sin\left(\frac{t}{j}\right),$$

yields

$$\mathrm{J}_2 < \frac{j\pi}{2}r(1-r)\int_0^{\sin\frac{\pi}{2j}} \tau^{r-1}\mathrm{d}\tau = \frac{j\pi}{2}(1-r)\sin\left(\frac{\pi}{2j}\right)^r.$$

Then

$$|b_j| < \frac{2\pi_{p'}}{j^2\pi^2}\left[2 + \frac{j\pi(1-r)}{r}\sin\left(\frac{\pi}{2j}\right)\right]\eta(j).$$

According to (9), we get

$$\eta(j) \leq rj^{1-r}$$

and

$$|b_j| < \frac{2\pi_{p'}r}{j^2\pi^2}\left[2 + \frac{j\pi(1-r)}{r}\frac{\pi}{2j}\right]j^{1-r}. \tag{10}$$

Simplifying the expression on the right hand side, ensures the validity of the lemma. $\square$

## 4 The change of coordinates map

We now derive various properties of the change of coordinates maps that take the 2-cosine functions into the $p$-cosine functions. Most of the material in this section can also be found in [2], [5], [7] and [4]. We keep a self-contained presentation here by including details of the main arguments.

Given any $g \in L_s$, denote the even extension of $g$ with respect to 1 by

$$\tilde{g}(x) = \begin{cases} g(x) & x \in [0,1] \\ g(2-x) & x \in (1,2]. \end{cases}$$

A 2−periodic extension of $g$ to the whole of $\mathbb{R}$ is then written as

$$g^*(x) = \tilde{g}(x - 2\left\lfloor \frac{x}{2} \right\rfloor).$$

The floor function $\lfloor y \rfloor \in \mathbb{Z}$ is the unique integer such that $y - \lfloor y \rfloor \in [0,1)$. For any $n \in \mathbb{N}$, let

$$M_n g(x) := g^*(nx).$$

**Lemma 8.** *The operators $M_n : L_s \longrightarrow L_s$ are linear isometries.*

*Proof.* Indeed,

$$\begin{aligned}
\|M_n g\|_{L_s}^s &= \int_0^1 |M_n g(x)|^s \mathrm{d}x = \int_0^1 |g^*(nx)|^s \mathrm{d}x = \int_0^1 |\tilde{g}(nx - 2\left\lfloor \frac{nx}{2} \right\rfloor)|^s \mathrm{d}x \\
&= \frac{1}{n}\int_0^n |\tilde{g}(y - 2\left\lfloor \frac{y}{2} \right\rfloor)|^s \mathrm{d}y = \frac{1}{n}\sum_{l=0}^{n-1}\int_l^{l+1} |\tilde{g}(y - 2\left\lfloor \frac{y}{2} \right\rfloor)|^s \mathrm{d}y \\
&= \frac{1}{n}\left[\sum_{\substack{l=0 \\ l\equiv_2 0}}^{n-1}\int_l^{l+1} |\tilde{g}(y - 2\left\lfloor \frac{y}{2} \right\rfloor)|^s \mathrm{d}y + \sum_{\substack{l=1 \\ l\equiv_2 1}}^{n-1}\int_l^{l+1} |\tilde{g}(y - 2\left\lfloor \frac{y}{2} \right\rfloor)|^s \mathrm{d}y\right].
\end{aligned}$$

Changing variables to $w = y - l$ for $l \equiv_2 0$ and $z = y - (l-1)$ for $l \equiv_2 1$, gives

$$\left\lfloor \frac{y}{2} \right\rfloor = \begin{cases} \frac{l}{2} & \text{whenever } l \equiv_2 0 \\ \frac{l-1}{2} & \text{whenever } l \equiv_2 1. \end{cases}$$

Hence,

$$\|M_n g\|_{L_s}^s = \frac{1}{n}\left[\sum_{\substack{l=0 \\ l\equiv_2 0}}^{n-1}\int_0^1 |g(w)|^s \mathrm{d}w + \sum_{\substack{l=1 \\ l\equiv_2 1}}^{n-1}\int_1^2 |\tilde{g}(z)|^s \mathrm{d}z\right].$$

Another change of variables $z = 2 - w$, then yields

$$\|M_n g\|_{L_s}^s = \frac{1}{n}\left[n\int_0^1 |g(w)|^s \mathrm{d}w\right] = \|g\|_{L_s}^s$$

as claimed. □

Let $e_n(x) := \cos(n\pi x)$. If

$$g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_j \in L_s$$

where

$$\widehat{g}(k) := 2 \int_0^1 g(x) e_k(x) \mathrm{d}x \qquad \forall k \in \mathbb{N} \cup \{0\}$$

are the corresponding cosine Fourier coefficients, then

$$M_n g = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) M_n e_j = \frac{\widehat{g}(0)}{2} e_0 + \sum_{j=1}^{\infty} \widehat{g}(j) e_{nj} \in L_s.$$

Now, let $f_n(x) := \cos_p(n\pi_p x)$. Note that $e_0(x) = f_0(x) = 1$ for all $x \in \mathbb{R}$. Suitable linear extensions of the map $A : e_n \mapsto f_n$ are the changes of coordinates between $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$. Our next goal is to find a canonical decomposition for $A$ in terms of $M_n$ and the Fourier coefficients $b_n(p)$. After that, we show that these are bounded operators of the Banach spaces $L_s$ for all $s > 1$.

**Proposition 1.** *For all $p > 1$,*

$$\sum_{j=1}^{\infty} |b_j(p)| < \infty.$$

*Proof.* This is a direct consequence of lemmas 6 and 7. See (14) and (23) below. □

In the notation of Section 3, we have $\widehat{f_1}(k) = b_k(p)$ for all $k \in \mathbb{N} \cup \{0\}$. Recall that $b_k = 0$ for $k \equiv_2 0$. Since any of the functions $f_n(x)$ is continuous, then they all have a Fourier cosine expansion

$$f_n(x) = \frac{1}{2} \widehat{f_n}(0) e_0(x) + \sum_{k=1}^{\infty} \widehat{f_n}(k) e_k(x)$$

which is both pointwise convergent for all $x \in [0, 1]$ and also convergent in the norm of $L_s$ for all $s > 1$. Then, for all $n > 1$,

$$\begin{aligned} \widehat{f_n}(k) &= 2 \int_0^1 f_1(nx) \cos(k\pi x) \mathrm{d}x \\ &= 2 \int_0^1 \Big( \sum_{m=1}^{\infty} \widehat{f_1}(m) \cos(m\pi n x) \Big) \cos(k\pi x) \mathrm{d}x \\ &= 2 \sum_{m=1}^{\infty} \widehat{f_1}(m) \int_0^1 \cos(mn\pi x) \cos(k\pi x) \mathrm{d}x \\ &= \begin{cases} b_m(p) & \text{for } mn = k,\ m \equiv_2 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we can exchange the infinite summation with the integral sign, due to the pointwise convergence of the series, Proposition 1 and the Dominated Convergence theorem.

Let

$$A := \sum_{j=1}^{\infty} b_j(p) M_j. \tag{11}$$

By virtue of Proposition 1, Lemma 8 and the triangle inequality, it follows that the expression (11) is convergent in the operator norm of $L_s$ and that $A : L_s \longrightarrow L_s$ is a bounded linear operator such that

$$\|A\|_{L_s \longrightarrow L_s} \leq \sum_{j=1}^{\infty} |b_j| \|M_j\|_{L_s \longrightarrow L_s} = \sum_{j=1}^{\infty} |b_j|.$$

Moreover,

$$Ae_0 = \sum_{j=1}^{\infty} b_j M_j e_0 = \sum_{j=1}^{\infty} b_j e_0 = \sum_{j=1}^{\infty} b_j e_j(0) = \cos_p(\pi_p 0) = 1 = f_0$$

and

$$Ae_n = \sum_{j=1}^{\infty} b_j M_j e_n = \sum_{j=1}^{\infty} \widehat{f}_1(j) e_{nj} = \sum_{k=1}^{\infty} \widehat{f}_n(k) e_k = f_n \qquad \forall n \in \mathbb{N}.$$

These are the change of basis maps between $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$.

The operator $A$ is an homeomorphism of $L_s$ if and only if the family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ is a Schauder basis of $L_s$, cf. [12] or [16]. Then we have the following criterion, which is a consequence of [13, Theorem IV-1.16],

$$\sum_{\substack{j=3 \\ j \equiv_2 1}}^{\infty} |b_j(p)| < |b_1(p)| \quad \Rightarrow \quad \begin{cases} \{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty} \text{ is a Schauder} \\ \text{basis of } L_s \text{ for all } s > 1. \end{cases} \tag{12}$$

We employ this criterion below in order to determine the basis thresholds for the family $\{\cos_p(n\pi_p \cdot)\}_{n=0}^{\infty}$ claimed in Theorem 1.

## 5 Proof of Theorem 1

The proof is separated into two cases.

### 5.1 The case $1 < p < 2$

Recall the expression for $c_p$ given in (6) and consider the identity

$$\pi_p^2 c_p = \frac{\pi^3}{\pi^2 - 8}. \tag{13}$$

**Lemma 9.** *There exists* $1 < p_0 < 2$ *such that* (13) *holds true for* $p = p_0$. *Moreover,*

$$\pi_p^2 c_p < \frac{\pi^3}{\pi^2 - 8} \qquad \forall p \in (p_0, 2).$$

*Proof.* It will be enough to prove that $\pi_p^2 c_p$ is a convex function of the parameter $p$ for all $1 < p < 2$. Indeed, since

$$\lim_{p \to 1^+} \pi_p^2 c_p = \infty \qquad \text{and} \qquad \lim_{p \to 2^-} \pi_p^2 c_p = \pi^2 < \frac{\pi^3}{\pi^2 - 8},$$

both statements will immediately follow from this property.

Firstly note that

$$\frac{\mathrm{d}}{\mathrm{d}p} \ln(p-1)^{\frac{p-1}{p}} = \frac{1}{p^2} \ln(p-1) + \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} \ln(p-1)^{\frac{p-1}{p}} = \frac{2-p}{p^2(p-1)} - 2\frac{\ln(p-1)}{p^3} > 0.$$

Then $\ln(p-1)^{\frac{p-1}{p}}$ is convex for $1 < p < 2$.

Similarly, we have

$$\frac{\mathrm{d}}{\mathrm{d}p} \ln(2-p)^{\frac{2-p}{p}} = \frac{-2}{p^2} \ln(2-p) - \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} \ln(2-p)^{\frac{2-p}{p}} = \frac{4-p}{p^2(2-p)} + 4\frac{\ln(2-p)}{p^3} > 0.$$

Then, also $\ln(2-p)^{\frac{2-p}{p}}$ is convex for $1 < p < 2$.

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}p}[\ln \pi_p] = \frac{\pi \cot(\frac{\pi}{p})}{p^2} - \frac{1}{p}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2} \ln \pi_p = \frac{(p^2 + \pi^2)}{p^4} - \frac{2\pi}{p^3} \cot\left(\frac{\pi}{p}\right) + \frac{\pi^2}{p^4} \cot^2\left(\frac{\pi}{p}\right) > 0.$$

The latter is a consequence of the fact that $\cos \frac{\pi}{p} < 0$ and $\sin \frac{\pi}{p} > 0$. Hence, also $\ln \pi_p^2$ is convex for $1 < p < 2$.

The convexity of the logarithm of each one of the multiplying terms in the expression for $\pi_p^2 c_p$, implies that $\ln \pi_p^2 c_p$ is convex for $1 < p < 2$. This ensures that indeed $\pi_p^2 c_p$ is convex in the same segment and the validity of the statement is ensured. $\square$

**Corollary 1.** *Let $1 < p_0 < 2$ be such that (13) holds true for $p = p_0$. The family $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ is a Schauder basis of $L_s$ for all $s > 1$ and $p_0 \le p \le 2$.*

*Proof.* According to Lemma 6,

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j(p)| < \frac{8\pi_p c_p}{\pi^2} \sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} \frac{1}{j^2} = \frac{\pi_p^2 c_p(\pi^2-8)}{\pi^2 \pi_p}. \tag{14}$$

On the other hand, in view of Lemma 5 and Lemma 2(a), we have

$$\begin{aligned} b_1(p) &= \frac{\pi}{\pi_p} a_1 = \frac{4\pi}{\pi_p} \int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x \\ &\ge \frac{4\pi}{\pi_p} \int_0^{\frac{1}{2}} \sin(\pi x)^2 \mathrm{d}x = \frac{\pi}{\pi_p}. \end{aligned}$$

Then, Lemma 9 yields

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j(p)| < b_1(p)$$

for all $p \in [p_0, 2)$. By virtue of (12) the claimed conclusion follows. $\square$

Since

$$\pi_{\frac{4}{3}}^2 c_{\frac{4}{3}} = \frac{\pi^2 3^{\frac{5}{4}}\sqrt{2}}{2} > \frac{\pi^3}{\pi^2-8}$$

and

$$\pi_{\frac{3}{2}}^2 c_{\frac{3}{2}} = \frac{64\pi^2}{27\sqrt[3]{4}} < \frac{\pi^3}{\pi^2-8},$$

then $\frac{4}{3} < p_0 < \frac{3}{2}$. This settles the proof of Theorem 1 for $1 < p < 2$.

**Remark 1.** *An implementation of the Newton method gives $p_0 \approx 1.458801$ as an approximated solution of (13) with all digits correct.*

### 5.2 Case $p > 2$

Recall the following identities involving the Riemann Zeta function [11, 3.411, 9.522 & 9.524],

$$\zeta(q) = \frac{1}{\Gamma(q)} \int_0^{\infty} \frac{t^{q-1}}{e^t - 1}\mathrm{d}t \qquad \mathrm{Re}(q) > 1, \tag{15}$$

$$\sum_{\substack{j=1\\ j\not\equiv_2 0}}^{\infty} \frac{1}{j^q} = \left(1 - \frac{1}{2^q}\right)\zeta(q) \tag{16}$$

and

$$\frac{\zeta'(q)}{\zeta(q)} = -\sum_{k=1}^{\infty} \frac{\Delta(k)}{k^q} \tag{17}$$

where

$$\Delta(k) = \begin{cases} \ln(r) & \text{if } k = r^m \text{ for some } r \text{ prime and } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 10.** *Let*

$$t_0 = \frac{2(e^2 - 3e + 1)}{(e^2 - 2e - 1)}.$$

*Then*

$$\zeta\left(\frac{3}{2}\right) < \frac{2}{\sqrt{\pi}}\left(2\sqrt{2}\arctan\frac{1}{\sqrt{2}} + \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0-1)^2}{2(e-1)^2} - \frac{t_0(e-2)+1}{e-1}\right). \tag{18}$$

*Proof.* Since $\Gamma(1+\frac{1}{2}) = \frac{\sqrt{\pi}}{2}1!! = \frac{\sqrt{\pi}}{2}$, the representation (15) gives

$$\begin{aligned} \zeta\left(\frac{3}{2}\right) &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} \frac{t^{1/2}}{e^t - 1}\mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}}\left(\int_0^1 + \int_1^{\infty} \frac{t^{1/2}}{e^t-1}\mathrm{d}t\right) = \frac{2}{\sqrt{\pi}}(\mathrm{J}_1 + \mathrm{J}_2). \end{aligned}$$

We estimate separately upper bounds for $\mathrm{J}_1$ and $\mathrm{J}_2$.

The change of variables $t = u^2$, yields

$$\begin{aligned} \mathrm{J}_1 &= \int_0^1 \frac{t^{1/2}}{e^t-1}\mathrm{d}t < \int_0^1 \frac{t^{1/2}}{t + \frac{t^2}{2}}\mathrm{d}t \\ &= \int_0^1 \frac{2u^2}{u^2 + \frac{u^4}{2}}\mathrm{d}u = 2\sqrt{2}\arctan\frac{1}{\sqrt{2}}. \end{aligned}$$

On the other hand, we know that $\zeta(2) = \int_0^{\infty} \frac{t}{e^t-1}\mathrm{d}t = \frac{\pi^2}{6}$, so

$$\mathrm{J}_2 \le \int_1^{\infty} \frac{t}{e^t-1}\mathrm{d}t = \frac{\pi^2}{6} - \int_0^1 \frac{t}{e^t-1}\mathrm{d}t.$$

We find lower bound for the integral on the right hand side, by interpolating the curve $c(t) = \frac{t}{e^t-1}$ at two points, $t = 0$ and $t = 1$. Firstly observe that $c(t) \to 1$ as $t \to 0$, $c(t)$ is decreasing and $c''(t) \ge 0$ for $t \in [0,1]$. Let $t_0$ be as in the hypothesis and let

$$\tilde{c}(t) = \begin{cases} 1 - \frac{1}{2}t & 0 \le t \le t_0 \\ \frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1} & t_0 \le t \le 1 \end{cases}$$

be the piecewise linear interpolant of $c(t)$ in the two segments $[0, t_0]$ and $[t_0, 1]$, which is continuous at $t_0$. Note that $\tilde{c}(t)$ and $c(t)$ are tangent at $t = 0$ and $t = 1$. Then

$$c(t) \geq \tilde{c}(t) \qquad \forall t \in [0, 1].$$

Hence

$$\begin{aligned}\int_0^1 c(t)\mathrm{d}t &\geq \int_0^{t_0} \left(1 - \frac{1}{2}t\right)\mathrm{d}t + \int_{t_0}^1 \left(\frac{1}{(e-1)^2}(1-t) + \frac{1}{e-1}\right)\mathrm{d}t \\ &= -\frac{t_0^2}{4} + \frac{(t_0-1)^2}{2(e-1)^2} + \frac{t_0(e-2)+1}{e-1}.\end{aligned}$$

Thus

$$\mathrm{J}_2 \leq \frac{\pi^2}{6} + \frac{t_0^2}{4} - \frac{(t_0-1)^2}{2(e-1)^2} - \frac{t_0(e-2)+1}{e-1}.$$

Alongside with the upper bound above for $\mathrm{J}_1$, this ensures the validity of the claimed statement. □

Now, consider the equation

$$\frac{2\pi_{p'}}{\pi^2(p-1)}\left[2 + \frac{\pi^2}{2}(p-2)\right]\left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] = \frac{8}{\pi\pi_p}. \tag{19}$$

**Lemma 11.** *There exists $p_1 \in (\frac{11}{5}, 3)$ such that* (19) *holds true for $p = p_1$. Moreover,*

$$\frac{2\pi_{p'}}{\pi^2(p-1)}\left[2 + \frac{\pi^2}{2}(p-2)\right]\left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] < \frac{8}{\pi\pi_p} \qquad \forall p \in [2, p_1).$$

*Proof.* From (4) it follows that the identity (19) reduces to

$$\frac{\pi}{p^2\sin(\frac{\pi}{p})^2}\left(2 + \frac{\pi^2}{2}(p-2)\right)\left[\left(1 - \frac{1}{2^{p'}}\right)\zeta(p') - 1\right] = 1. \tag{20}$$

Denote by $h(p)$ the left hand side of (20). Then $h : (1, \infty) \longrightarrow \mathbb{R}$ is continuous and

$$h(2) = \frac{\pi}{2}\left(\frac{\pi^2}{8} - 1\right) < 1.$$

Since

$$\zeta\left(\frac{3}{2}\right) > 1 + \frac{\sqrt{2}}{4} + \sqrt{3}\sum_{k=3}^{\infty}\frac{1}{k^2} = \frac{4+\sqrt{2}}{4} + \sqrt{3}\left(\frac{\pi^2}{6} - \frac{5}{4}\right),$$

we get

$$\begin{aligned} h(3) &= \frac{\pi}{9\sin(\frac{\pi}{3})^2}\left[2 + \frac{\pi^2}{2}\right]\left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right)\zeta\left(\frac{3}{2}\right) - 1\right] \\ &> \frac{\pi}{9\sin(\frac{\pi}{3})^2}\left[2 + \frac{\pi^2}{2}\right]\left[\left(1 - \frac{1}{2^{\frac{3}{2}}}\right)\left(\frac{4+\sqrt{2}}{4} + \sqrt{3}\left(\frac{\pi^2}{6} - \frac{5}{4}\right)\right) - 1\right] \\ &> 1.\end{aligned}$$

Hence, there exists $p_1 \in (2,3)$ such that $h(p_1) = 1$.

The derivative

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\left(1-\frac{1}{2^q}\right)\zeta(q)\right] = \frac{\ln(2)}{2^q}\zeta(q) + \left(1-\frac{1}{2^q}\right)\zeta'(q)$$

is negative for any $q \in (1,2)$. Indeed the identity (17) gives

$$\begin{aligned}\frac{\zeta'(q)}{\zeta(q)} &< -\frac{\ln(2)}{2^q} - \frac{\ln(3)}{3^q} - \frac{\ln(2)}{4^q} \\ &< -\ln(2)\left[\frac{1}{2^q}+\frac{1}{3^q}+\frac{1}{4^q}\right] < \frac{\ln(2)}{1-2^q},\end{aligned}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}q}\left[\left(1-\frac{1}{2^q}\right)\zeta(q)\right] = \zeta(q)\left[\frac{\ln(2)}{2^q} + \frac{2^q-1}{2^q}\frac{\zeta'(q)}{\zeta(q)}\right] < 0.$$

Since $p'$ and $\sin\left(\frac{\pi}{p}\right)$ are decreasing functions of $p > 2$, then

$$\frac{\pi}{\sin(\frac{\pi}{p})^2}\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p') - 1\right]$$

is an increasing function of $p > 2$.

As

$$\frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{p^2}\Big(2+\frac{\pi^2}{2}(p-2)\Big)\right] = \frac{1}{p^3}\left(-\frac{\pi^2}{2}p + 2\pi^2 - 4\right) > 0 \qquad \forall p \in [2,3],$$

then $h(p)$ is increasing for $p \in [2,3]$ and so indeed

$$h(p) < h(p_1) = 1 \qquad \forall p \in [2, p_1).$$

Let us now show that $p_1 > \frac{11}{5}$. Let $c_1$ denote the right hand side of the estimate (18) in Lemma 10. Since $\zeta(q)$ is convex in the segment $[\frac{3}{2}, 2]$, then

$$\zeta(q) \leq \left(\frac{\pi^2}{3} - 2c_1\right)(q-2) + \frac{\pi^2}{6}.$$

That is, the straight line joining the points $(\frac{3}{2}, c_1)$ and $(2, \frac{\pi^2}{6})$ is above the curve $\zeta(q)$ for all $q \in [\frac{3}{2}, 2]$. Then

$$\zeta\left(\frac{11}{6}\right) \leq \frac{\pi^2}{9} + \frac{c_1}{3}. \tag{21}$$

Note that for $p = \frac{11}{5}$, $p' = \frac{11}{6}$. Now, $\sin(\pi y)$ is concave for $y \in [\frac{5}{12}, \frac{1}{2}]$. Then it is above the straight line joining the points $(\frac{5}{12}, \sin\frac{5\pi}{12})$ and $(\frac{1}{2}, 1)$. That is

$$\sin(\pi y) \geq \left(12 - 12\sin\frac{5\pi}{12}\right)\left(y - \frac{1}{2}\right) + 1 \qquad \forall y \in \left[\frac{5}{12}, \frac{1}{2}\right].$$

Then

$$\sin\frac{5\pi}{11} > \frac{\sqrt{6}}{22}\left(\sqrt{3}+3\right) + \frac{5}{11}. \tag{22}$$

Denote by $c_2$ the right hand side of the latter inequality. From (21) and (22), it follows that

$$\begin{aligned} h\left(\frac{11}{5}\right) &= \frac{\pi}{(\frac{11}{5})^2 \sin(\frac{5\pi}{11})^2}\left[2+\frac{\pi^2}{2}\left(\frac{11}{5}-2\right)\right]\left[\left(1-\frac{1}{2^{11/6}}\right)\zeta\left(\frac{11}{6}\right)-1\right] \\ &< \frac{\pi}{\frac{121}{25}c_2^2}\left(2+\frac{\pi^2}{10}\right)\left[\left(1-\frac{1}{2^{11/6}}\right)\left(\frac{\pi^2}{9}+\frac{c_1}{3}\right)-1\right] < 1. \end{aligned}$$

As $h(p)$ is increasing, then indeed $p_1 > \frac{11}{5}$. □

**Corollary 2.** *Let $p_1 > 2$ be such that* (19) *holds true for $p = p_1$. The family $\{\cos_p(n\pi_p\cdot\}_{n=0}^{\infty}$ forms a Schauder basis of $L_s$ for all $s > 1$ and $2 \le p \le p_1$.*

*Proof.* From Lemma 7 and (16), we have

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j| < \frac{2\pi_{p'}}{\pi^2(p-1)}\left[2+\frac{\pi^2}{2}(p-2)\right]\left[\left(1-\frac{1}{2^{p'}}\right)\zeta(p')-1\right]. \tag{23}$$

According to part "b" of Lemma 1, $\sin_p(\pi_p x)$ is strictly concave on $(0, \frac{1}{2})$. Then

$$\begin{aligned} a_1 &= 2\int_0^1 \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x = 4\int_0^{\frac{1}{2}} \sin_p(\pi_p x)\sin(\pi x)\mathrm{d}x \\ &> 4\int_0^{\frac{1}{2}} (2x)\sin(\pi x)\mathrm{d}x = \frac{8}{\pi^2}. \end{aligned}$$

Hence, in view of Lemma 5, we get

$$b_1 = \frac{\pi}{\pi_p}a_1 > \frac{8}{\pi\pi_p}. \tag{24}$$

From Lemma 11, it then follows that

$$\sum_{\substack{j=3\\ j\equiv_2 1}}^{\infty} |b_j(p)| < b_1(p) \qquad \forall p \in [2, p_1].$$

By virtue of (12) this implies the claimed conclusion. □

**Remark 2.** *An approximation of the solution of* (19) *via the Newton Method gives $p_1 \approx 2.42865$ with all digits correct.*

# 6 Connections with other work

In this final section we describe various connections between the statements established above and those reported in the literature.

### The $p$-exponential functions

Let

$$\exp_p(iy) = \cos_p(y) + i\sin_p(y) \qquad \forall y \in \mathbb{R}.$$

By combining Theorem 1 with [2, Theorem 1] or [5, Theorem 4.5], it immediately follows that the family $\tilde{\mathcal{F}} = \{\exp_p(in\pi_p\cdot)\}_{n=-\infty}^{\infty}$ is a Schauder basis of the Banach space $L_s(-1,1)$ for all $p \in [p_0, p_1]$.

Indeed, recall that every $f \in L^s(-1,1)$ decomposes as $f = f_\mathrm{e} + f_\mathrm{o}$ for

$$f_\mathrm{e}(x) = \frac{f(x)+f(-x)}{2} \qquad \text{and} \qquad f_\mathrm{o}(x) = \frac{f(x)-f(-x)}{2},$$

the even and odd parts of $f$, respectively. The family $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ comprises only even functions, the family $\{\sin_p(n\pi_p\cdot)\}_{n=1}^{\infty}$ comprises only odd functions and they are Schauder bases of the corresponding subspaces of $L_s(-1,1)$ for $p \in [p_0, p_1]$. This implies that there exist two unique scalar sequences $(\alpha_k)_{k=0}^{\infty}$ and $(\beta_k)_{k=1}^{\infty}$, such that

$$f(\cdot) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos_p(k\pi_p\cdot) + i\beta_k \sin_p(k\pi_p\cdot)$$

in $L_s(-1,1)$. In order to see this, one expands $f_\mathrm{e}$ in $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ and $f_\mathrm{o}$ in $\{\sin_p(n\pi_p\cdot)\}_{n=1}^{\infty}$, in the corresponding even and odd subspaces.

By letting $c_0 = \alpha_0$,

$$c_k = \frac{\alpha_k + \beta_k}{2} \qquad \text{and} \qquad c_{-k} = \frac{\alpha_k - \beta_k}{2} \qquad \forall k \in \mathbb{N},$$

we get

$$f(\cdot) = \sum_{k=-\infty}^{\infty} c_k \exp_p(ik\pi_p\cdot)$$

in $L_s(-1,1)$. Since there is a 1:1 correspondence between the scalar sequences via

$$\alpha_k = c_k + c_{-k} \qquad \text{and} \qquad \beta_k = c_k - c_{-k},$$

then in fact $(c_k)_{k=-\infty}^{\infty}$ is unique for the given $f$. Thus, $\tilde{\mathcal{F}}$ satisfies the definition of a Schauder basis for the Banach space $L_s(-1,1)$.

**The regularity of the $p$-sine functions**

Let $r > 0$ and denote by $H^r \equiv H^r(0,1)$ the (Hilbert) Sobolev space of order $r$. Let $1 < p < 2$. According to the formula [5, (4.4)], it follows that the Fourier coefficients of the $p$-sine function are such that

$$|a_j(p)| \leq \frac{16\pi_p^2 c_p}{\pi^3} j^{-3} \qquad \forall j \in \mathbb{N}.$$

Then, $\sin_p(\pi_p \cdot) \in H^\rho$ for all $\rho < \frac{5}{2}$.

Numerical estimates for the Sobolev regularity of $\sin_p(\pi_p \cdot)$ for $2 < p < 100$ were reported in [3, Figure 2]. From that picture, one may conjecture that for $p > 3$, $\sin_p(\pi_p \cdot) \notin H^2$. Moreover, the regularity appears to drop asymptotically to $\frac{3}{2}$ for $p$ large. By contrast, it appears that $\sin_p(\pi_p \cdot) \in H^2$ for $2 < p < 3$. The following statement, which is a consequence of Lemma 7, settles this conjecture.

**Corollary 3.** *For $p > 2$ set $r(p) = p' + \frac{1}{2}$. Then $\sin_p(\pi_p \cdot) \in H^\rho$ for all $0 \leq \rho < r(p)$.*

*Proof.* According to Lemma 5,

$$|a_j(p)| = \frac{\pi_p}{j\pi} |b_j(p)|.$$

Then, by virtue of Lemma 7,

$$|a_j(p)| \leq \frac{2\pi_p \pi_{p'}}{\pi^3 (p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] j^{-(p'+1)} \qquad \forall j \geq 3.$$

Let $\langle j \rangle^2 = 1 + j^2$. For $\rho < p' + \frac{1}{2}$,

$$\sum_{j=1}^{\infty} \langle j \rangle^{2\rho} |a_j(p)|^2 \leq 2^\rho a_1(p)^2 + c(p) \sum_{\substack{j=3 \\ j \equiv_2 1}}^{\infty} \frac{1}{j^{1+\epsilon(p)}} < \infty$$

where

$$c(p) = \frac{2\pi_p \pi_{p'}}{\pi^3 (p-1)} \left[2 + \frac{\pi^2}{2}(p-2)\right] \quad \text{and} \quad \epsilon(p) = 1 - 2\rho + 2p' > 0.$$

Hence $\sin_p(\pi_p \cdot) \in H^\rho$ as claimed. $\square$

The recent paper [8] includes various intriguing results connected to Corollary 3.

**The paper [7]**

The recent paper [7] seems to be the only one in the existing literature which conducts an analysis of the basis properties of the $p$-cosine functions. In the notation of [7] we fix $\alpha = 1$ and $p = q > 1$. The Fourier coefficients of the $p$-cosine functions are

$$\tau_j(p,p,1) = b_j(p) \qquad \forall j \in \mathbb{N} \cup \{0\}.$$

The condition [7, (2.2)] as well as the criterion for determining whether $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ is a Schauder basis of $L^s$ are exactly the same as (12). Let us compare some of the results of [7] with those of the present work.

In [7, Proposition 2.5], the estimate [7, (2.20)] is equivalent to the following. There exists $p_0^* = \frac{72(\pi-2)-2\pi^3}{96(\pi-2)-3\pi^3}$, such that

$$\tau_1(p,p,1) \geq \begin{cases} \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2} & 1 < p < p_0^* \\ \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} & p_0^* < p < \infty. \end{cases} \tag{25}$$

Here $p_0^*$ satisfies the identity

$$\frac{4p-3}{3p-2} = \frac{\pi^3}{24(\pi-2)}.$$

Note that $p_0^* \approx 1.22$.

Let us consider firstly the regime $1 < p < 2$. From [7, Proposition 2.2] it follows that

$$\sum_{k=1}^{\infty} |\tau_{2k+1}(p,p,1)| \leq \frac{\pi_p(\pi^2-8)}{\pi^2} \qquad \forall p \in (1,2). \tag{26}$$

As $c_p < 1$ whenever $1 < p < 2$ in (6), then (14) is sharper than (26) in this regime.

If $1 < p < p_0^*$, then

$$\frac{\pi_p(\pi^2-8)}{\pi^2} > \frac{\pi(p-1)}{2p-1} - \frac{(\pi-2)(p-1)}{3p-2},$$

and no conclusion about the validity of (12) can be derived in this case from (25) and (26). For $p_0^* < p < 2$, on the other hand,

$$\frac{\pi_p(\pi^2-8)}{\pi^3} < \frac{p-1}{2p-1} - \frac{\pi^2(p-1)}{24(4p-3)} \qquad \Longleftrightarrow \qquad p \in (p_0^\dagger, 2),$$

where $p_0^\dagger \approx 1.75$. In order to see this, note that $\pi_p$ is decreasing and $\lim_{p\to1^+} \pi_p = \infty$, while the right hand side of this identity is increasing for $1 < p < 2$. Thus, a combination of [7, Proposition 2.2] and [7, Proposition 2.5], only guarantees that $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ is a Schauder basis of $L^s$ for $p \in [p_0^{\dagger}, 2)$ where $p_0^{\dagger} > \frac{3}{2} > p_0$.

As it turns, it is not possible to deduce from the results of [7] any basis property of the family $\{\cos_p(n\pi_p\cdot)\}_{n=0}^{\infty}$ in the complementary regime $p > 2$. Here is how the different estimates on the Fourier coefficients compare in this case.

From [7, Proposition 2.4], we gather that

$$\sum_{k=1}^{\infty} |\tau_{2k+1}(p,p,1)| \leq \frac{2\pi_{p'}}{\pi^2(p-1)} [4 + \pi(p-1)] \left[ \left(1 - \frac{1}{2^{p'}}\right) \zeta(p') - 1 \right]. \tag{27}$$

Since

$$4 + \pi(p-1) \geq 2 + \frac{\pi^2}{2}(p-2) \qquad \forall p \leq \frac{4 + 2\pi^2 - 2\pi}{\pi^2 - 2\pi},$$

the upper bound (23) is sharper than (27) for $2 \leq p \leq 3$. The latter is the relevant regime in the proof of Theorem 1.

Since $\pi_p < \pi$ for $p > 2$, the lower bound (24) is sharper than [7, (2.19)]. Moreover,

$$\frac{8}{\pi\pi_p} > \frac{\pi(p-1)}{2p-1} - \frac{\pi^3(p-1)}{24(4p-3)} \qquad \forall p > 2.$$

Hence the estimate (25), which is [7, (2.20)], is also superseded by (24) for $p > 2$.

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