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Transient Thermal Behaviour in a Model of Linear Friction Welding

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Abstract

We derive a non-local model for the evolution of temperature in workpieces being joined by linear friction welding. The non-locality arises through the velocity being fixed by the temperature gradient at the weld. Short- and long-time behaviours are considered.

Keywords Friction welding. Thermal modelling. Lubrication theory. Non-local problem.

1 Introduction

We consider how a model of linear friction welding predicts the way temperature approaches equilibrium. In linear friction welding, two components, usually both metal or metal alloy, are rubbed together to heat their surfaces of contact, and then forced together while maintaining a linear oscillatory motion; similar procedures are carried out in the related methods of rotary and stir friction welding. (See [1] - [9].) A thin softened layer forms between the components (the layer eventually makes up the weld after the process has ceased).

If the oscillations and squeezing are maintained long enough, a steady state can be reached, with periodic behaviour in the thin, softened, welding region, and, to leading order, constant temperature elsewhere. The unpublished Study Group report [10] and paper [11] looked at models for the temperature and mechanical deformation in such steady states. To be more precise, in [10]

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the velocity towards the weld and the temperature were constant while the velocity parallel to the weld was periodic (with, to leading order, constant magnitude). In [11], the situation was either as in [10], or the velocity and temperature could vary periodically in the thin welding region.
A key part of the model in both [10] and [11] is that virtually all the deformation and heat dissipation is confined to the central, relatively warm, soft layer, where the material can be modelled as a non-linear liquid and lubrication theory can be applied.
In the paper [12] a transient model for the thermo-mechanical behaviour of the thin central layer, where the welding occurs, was considered. In particular, for a "soft material", for which the temperature dependency in the stress-strain relation can be taken to be exponential in the thin layer, it was shown that the non-local PDE for temperature which applied in the thin layer had a global solution, and that this solution tended to a steady state. Numerical solutions for the corresponding non-local PDEs for "hard material", for which the temperature dependence was locally a power, indicated the same good behaviour. This variation with respect to time was very fast, over a short time scale associated with thermal diffusion on the scale of the width of the soft layer. The present paper looks at much slower changes, associated with thermal diffusion on the scale of the components, referred to as "workpieces", being welded.

We start by outlining some key parts of the modelling of the lubricating layer as done in [10] and [11], as this is needed to build our model for the time-dependent thermal problem. These are then used to derive a key relation between the velocity of the rigid workpieces, away from the softened layer, and the matching temperature gradient between the rigid and softened regions. Because of the appearance of a function of an end value of temperature gradient in the convective term in the heat equation which applies in the rigid region, this PDE is non-local. The qualitative behaviour of solutions of these non-local PDEs for cases of both hard and soft materials are briefly discussed, and one set of numerical solutions is presented.

2 The Model of the Lubricating Layer for a Hard Material

The physical situation is indicated by Fig. 1, showing, for simplicity, a two-dimensional symmetric case.
The workpieces have sides $x = \pm L$ which are taken to be effectively insulated. Away from the thin softened layer, of width of order h, where $h \ll L$, the workpieces move rigidly and linearly so that the velocity $\boldsymbol{v} = (u, v)$ is given by

$$\boldsymbol{v} \sim (U(t), 0) \ \text{ for } \ y \gg h\,.$$

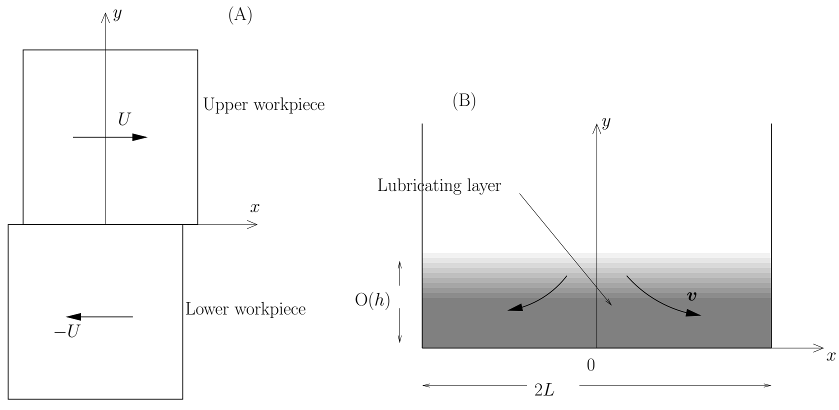

Figure 1: (A) Upper workpiece sliding in x direction with velocity $U(t)$ and lower one with velocity $-U(t)$. (B) Upper workpiece with its thin, lubricating, softened, welding layer, of width $\mathrm{O}(h)$, $h \ll L$, with L the half-length of each workpiece.

Following [10] and most of [11], the simple case of

$$U(t) = U_e \mathrm{sign}\,(\sin \omega t)$$

is discussed in the present paper.
Focusing attention on the upper workpiece and times when $U > 0$ (to avoid writing modulus signs), the basic model for the weld in $0 < x < L$, $y > 0$ is here taken as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.1}$$

the usual continuity equation (the flow can be regarded as incompressible);

$$\sigma = \kappa(T) \left(\frac{\partial u}{\partial y} \right)^{1/4}, \tag{2.2}$$

the constitutive law for shear stress σ in terms of temperature T and strain rate $\partial u / \partial y$, with u being the component of velocity parallel to the weld (in the x direction);

$$\kappa(T) = \kappa_m \left(1 - \frac{T}{T_m} \right) \exp\left(\frac{T_a}{T_m} \left(\frac{T_m}{T} - 1 \right) \right), \tag{2.3}$$

the form of the temperature dependence in the constitutive law – here κ_m is a material constant, T_a is an activation temperature and T_m is the melting temperature;

$$\frac{\partial p}{\partial y} = 0\,, \quad \frac{\partial \sigma}{\partial y} = \frac{\partial p}{\partial x}\,, \tag{2.4}$$

the usual momentum-balance equations for two-dimensional lubrication theory;

$$k \frac{\partial^2 T}{\partial y^2} + \sigma \frac{\partial u}{\partial x} = 0\,, \tag{2.5}$$

the heat equation for T, neglecting any x variation of temperature, presently neglecting the time derivative of T and the convective term, and accounting for viscous dissipation.
Note that in (2.3), the presence of the $(1 - T/T_m)$ factor models the stress becoming negligible as temperature approaches melting point. The density ρ, specific heat c_p and thermal conductivity k are all regarded as constant.

For a symmetric problem, the conditions

$$\frac{\partial p}{\partial x} = 0 \text{ on } x = 0\,, \quad \frac{\partial v}{\partial y} = 0 \text{ on } y = 0\,, \quad \frac{\partial T}{\partial y} = 0 \text{ on } y = 0\,, \tag{2.6}$$

are all taken to hold.
At the edge of the weld, pressure is atmospheric, so

$$p = 0 \text{ at } x = L$$

is imposed.
For distances large compared with the weld thickness, the velocity component parallel to the weld is specified,

$$u \to U_e \text{ as } y/h \to \infty\,,$$

while the normal component tends to some other value,

$$v \to -V \text{ as } y/h \to \infty\,,$$

with the approach speed V to be determined.
The workpieces are forced together with a specified average pressure $P > 0$ so that

$$\frac{1}{L} \int_0^L p \,\mathrm{d}x = P\,.$$

The temperature should match with that in an outer region, where the workpiece simply moves as a rigid body (and where there is no dissipation term in the heat equation) so we shall require

$$\frac{\partial T}{\partial y} \sim -G \text{ for } y/h \gg 1\,, \tag{2.7}$$

with $G > 0$ to be found later.

Following [10] (and [11]), the basic approach in solving the problem is to decouple the velocity component u and stress σ into their “sliding” and “squeezing” parts,

$$u \sim \tilde{u} + \bar{u} \text{ and } \sigma \sim \tilde{\sigma} + \bar{\sigma}\,, \qquad \text{where } \bar{u} \ll \tilde{u} \text{ and } \bar{\sigma} \ll \tilde{\sigma}\,.$$

The relative sizes of the sliding and squeezing parts can be justified, [11], in the limit of

$$\epsilon^2 = \frac{P}{\rho c_p (T_m - T_e)} \to 0$$

for T_e the ambient temperature, say temperature at $y = l > 0$, l the length of the workpiece. Then, following [11], asymptotic approximations

$$u \sim u_0 + \epsilon u_1 + \dots\,, \qquad \sigma \sim \sigma_0 + \epsilon\sigma_1 + \dots\,, \qquad T \sim T_0 + \epsilon T_1 + \dots$$

can be sought. The leading terms u_0 and σ_0 can be identified with $\tilde{u}$ and $\tilde{\sigma}$ respectively, ϵu_1 with $\bar{u}$ and $\epsilon\sigma_1$ with $\bar{\sigma}$. Note that using these asymptotic expansions gives somewhat different numerical values for quantities such as approach speed V from those found by the simpler decoupling method. However, the orders of magnitude of these quantities, and the way they depend on constants (such as κ_m or k) or controllable physical parameters (such as U_e or P) are the same for the two approaches, [11]. Hence, for simplicity, only the decoupling method is briefly described here. The analysis covered in [11] is partially repeated as the outcomes there are not given in the form required for later use in the present paper.

We first simplify the temperature dependency. For a “hard material”, to get appreciable softening, the temperature in the thin welding layer must be very close to melting point and the temperature function (2.3) can then be approximated by

$$\kappa(T) \sim \kappa_m (1 - T/T_m)\,. \tag{2.8}$$

Turning to the decoupling approach, the second momentum balance equation (2.4) separates as

$$\frac{\partial \tilde{\sigma}}{\partial y} = 0 \text{ (sliding motion)} \qquad \text{and} \qquad \frac{\partial \bar{\sigma}}{\partial y} = \frac{\mathrm{d}p}{\mathrm{d}x} \text{ (squeezing motion)}, \tag{2.9}$$

the constitutive law (2.2), on using (2.8), as

$$\tilde{\sigma} = \kappa_m (1 - T/T_m) \left(\frac{\partial \tilde{u}}{\partial y}\right)^{1/4} \qquad \text{and} \qquad \bar{\sigma} = \frac{1}{4}\tilde{\sigma}^{-3}\kappa_m^4 (1 - T/T_m)^4 \frac{\partial \bar{u}}{\partial y}\,, \tag{2.10}$$

and the continuity equation (2.1) as

$$\frac{\partial \tilde{u}}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial \bar{u}}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.11}$$

while the thermal equation (2.5) is approximately

$$k\frac{\mathrm{d}^2T}{\mathrm{d}y^2}+\tilde{\sigma}\frac{\partial\tilde{u}}{\partial y}=0\,. \tag{2.12}$$

The matching conditions with the outer region are now

$$\tilde{u}\to U_e\,,\quad \bar{u}\to 0\,,\quad v\to -V\quad\text{and}\quad \frac{\partial T}{\partial y}\to -G\quad\text{as}\quad \frac{y}{h}\to\infty\,. \tag{2.13}$$

The symmetry conditions to be imposed are

$$\tilde{u}=0\,,\quad v=0\,,\quad \frac{\partial\bar{u}}{\partial y}=0\,,\ \text{and}\ \frac{\partial T}{\partial y}=0\ \text{on}\ y=0 \tag{2.14}$$

and

$$\bar{u}=0\ \text{and}\ \frac{\mathrm{d}p}{\mathrm{d}x}=0\ \text{on}\ x=0\,. \tag{2.15}$$

Finally,

$$p=0\ \text{on}\ x=L\quad\text{and}\quad \frac{1}{L}\int_0^L p\,\mathrm{d}x=P\,. \tag{2.16}$$

A solution for $\bar{u}$ is sought in the form $\bar{u}=xw(y)$, while, because of the first of (2.9), of (2.10) and of (2.11), and because T is independent of x, $\tilde{\sigma}$ is constant so that $w(y)$ and $p(x)$ must satisfy

$$\frac{\mathrm{d}v}{\mathrm{d}y}=-w\,,\qquad \frac{\mathrm{d}p}{\mathrm{d}x}=\frac{\partial}{\partial y}\left(\frac{\kappa_m^4}{4\sigma^3}\left(1-\frac{T}{T_m}\right)^4\frac{\partial\bar{u}}{\partial y}\right)=\frac{\kappa_m^4}{4\sigma^3}\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(1-\frac{T}{T_m}\right)^4\frac{\mathrm{d}w}{\mathrm{d}y}\right)x\,,$$

where, for convenience, $\tilde{\sigma}$ has been replaced by σ (they agree to leading order). Thus

$$p=\frac{3P(L^2-x^2)}{2L^2}\,,$$

and

$$\frac{\kappa_m^4}{4\sigma^3}\frac{\mathrm{d}}{\mathrm{d}y}\left(\left(1-\frac{T}{T_m}\right)^4\frac{\mathrm{d}w}{\mathrm{d}y}\right)+\frac{3P}{L^2}=0\,,$$

with

$$w\to 0\ \text{as}\ \frac{y}{h}\to\infty\ \text{and}\ \frac{\mathrm{d}w}{\mathrm{d}y}=0\ \text{at}\ y=0$$

from the second of (2.13) and the third of (2.14). Then

$$w=\frac{12P\sigma^3}{L^2\kappa_m^4}\int_y^\infty\frac{y\,\mathrm{d}y}{(1-T/T_m)^4}\ \text{and}\ V=\frac{12P\sigma^3}{L^2\kappa_m^4}\int_0^\infty\frac{y^2\,\mathrm{d}y}{(1-T/T_m)^4}\,. \tag{2.17}$$

Meanwhile, from the first of (2.10), of (2.13) and of (2.14),

$$U_e = \frac{\sigma^4}{\kappa_m^4} \int_0^\infty \frac{\mathrm{d}y}{(1 - T/T_m)^4}\,, \tag{2.18}$$

while the heat equation (2.12) becomes

$$k\frac{\mathrm{d}^2 T}{\mathrm{d}y^2} + \frac{\sigma^5}{\kappa_m^4 (1 - T/T_m)^4} = 0\,, \tag{2.19}$$

which, with (2.18) and the last of (2.13) and of (2.14), yields

$$G = \frac{\sigma^5}{k\kappa_m^4} \int_0^\infty \frac{\mathrm{d}y}{(1 - T/T_m)^4}$$

$$\text{so} \qquad G = \frac{\sigma U_e}{k}\,. \tag{2.20}$$

Note that all these integrations to infinity, in (2.17) to (2.20), mean taking $y/h \to \infty$ with the inner approximations valid (in the soft layer). It is appropriate to scale distance with h, $y = h\eta$, and also carry out other scalings, $\sigma = Ss$, $T = T_m - \delta\theta$, $V = V_E V^*$, $G = G_E G^*$, where $h =$ typical width of lubricating layer, $S =$ size of stress, $\delta =$ size of temperature variation in layer, $V_E =$ size of approach velocity and the scaling temperature gradient G_E is that taken at equilibrium in a "large" workpiece. The non-dimensional problem (2.45) - (2.46) of [11] almost results:

$$V^* = 12 s^3 \int_0^\infty \eta^2 \theta^{-4}\, \mathrm{d}\eta \ \text{ and} \tag{2.21}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} = s^5 \theta^{-4} \ \text{ for } 0 < \eta < \infty\,, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\eta} = 0 \ \text{ at } \ \eta = 0\,, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\eta} \to G^* \ \text{ as } \ \eta \to \infty\,, \tag{2.22}$$

where T_e is the ambient temperature of the workpiece, taken at sufficiently large distances from the weld. (The second part of (2.45) of [11], $s = V^*$, is not required here, as that was derived from matching with a steady temperature in the far field.) The scaling constants satisfy

$$\frac{k\delta}{h^2} = \frac{T_m^4 S^5}{\kappa_m^4 \delta^4}\,, \quad \frac{\delta}{h} = G_E\,, \quad G_E = \frac{S U_e}{k}\,,$$

$$V_E = \frac{P T_m^4 S^3 h^3}{L^2 \kappa_m^4 \delta^4} \quad \text{and} \quad G_E = \frac{V_E \rho c_p (T_m - T_e)}{k}\,.$$

(The first, third and fourth of these are got by having balances in (2.19), (2.20) and (2.17) respectively, while the second is obtained from the matching condition for temperature gradient, (2.13). The last comes from having

T satisfy a steady convection-diffusion equation, with velocity $-V_E$, in the outer, rigid region, with $T \to T_e$ as $y \to \infty$.) After some manipulation,

$$V_E = (kT_m/\kappa_m)^{4/3}(P^{1/2}/L)(\rho c_p(T_m - T_e))^{-1/2}U_e^{-2/3}\,,$$
$$S = (kT_m/\kappa_m)^{4/3}(P^{1/2}/L)(\rho c_p(T_m - T_e))^{1/2}U_e^{-5/3}\,,$$
$$h = (kT_m/\kappa_m)^{4/3}U_e^{-5/3},$$
$$\delta = k^{5/3}(T_m/\kappa_m)^{8/3}(\rho c_p(T_m - T_e))^{1/2}(P^{1/2}L^{-1})\,U_e^{-7/3}, \tag{2.23}$$
$$\text{and } G_E = k^{1/3}(T_m/\kappa_m)^{4/3}(P^{1/2}/L)(\rho c_p(T_m - T_e))^{1/2}U_e^{-2/3}\,.$$

Eqn. (2.20) scales to

$$G^* = s$$

and then the change of variable $\theta = s\varphi$ alters (2.22) to

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\eta^2} = \varphi^{-4} \text{ for } 0 < \eta < \infty\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\eta} = 0 \text{ at } \eta = 0\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\eta} \to 1 \text{ as } \eta \to \infty \tag{2.24}$$

and (2.21) to

$$G^*V^* = sV^* = 12\int_0^\infty \eta^2\varphi^{-4}\,\mathrm{d}\eta = N\,, \tag{2.25}$$

where N is a numerical constant. From this decoupling approach of [10] and [11], $N \approx 8.123$. Returning to dimensional quantities,

$$VG = Nk^{5/3}(T_m/\kappa_m)^{8/3}(P/L^2)U_e^{-4/3}\,. \tag{2.26}$$

Note that the factor $\rho c_p(T_m - T_e)$, which comes from considering long-time behaviour and might suggest influence at a distance, has dropped out.
Note also that a different value of N would be obtained from using a formal asymptotic approach, as in Section 4 of [11].

3 The Non-Local Model for a Hard Material

In the outer region, away from the welding layer, the workpiece is simply modelled with regard to its thermal behaviour, regarding it as a rigid body moving with speed $V(t)$ towards the weld (the oscillatory motion in the x direction can be disregarded):

$$\rho c_p\left(\frac{\partial T}{\partial t} - V\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} \quad \text{for} \quad 0 < y < l\,,\, t > 0\,. \tag{3.1}$$

We assume the side $x = L$ to be thermally insulated so that $T = T(y, t)$. The temperature is taken to be ambient at the far end of the workpiece and initially:

$$T = T_e \quad \text{for} \quad y = l\,,\, t \geq 0\,; \quad T = T_e \quad \text{for} \quad 0 < y \leq l\,,\, t = 0\,. \tag{3.2}$$

As we are considering the case of a hard material, the weld should be at melting temperature, so, to leading order,

$$T = T_m \quad \text{for} \quad y = 0\,,\, t > 0\,. \tag{3.3}$$

The problem is completed by fixing $V(t)$ through the non-local condition (2.26):

$$V(t) = -M \Bigg/ \frac{\partial T}{\partial y}\bigg|_{y=0}\,, \tag{3.4}$$

with

$$M = N k^{5/3} (T_m/\kappa_m)^{8/3} (P/L^2) U_e^{-4/3}\,. \tag{3.5}$$

It is possible to non-dimensionalise the problem (as for the case of soft materials, Sec. 4) but as no great simplification of the problem results in this case, this is not done here.

Long-time solution. We expect T to tend towards the steady-state, which satisfies

$$k\frac{\mathrm{d}^2T}{\mathrm{d}y^2} + \rho c_p V_\infty \frac{\mathrm{d}T}{\mathrm{d}y} = 0 \text{ for } 0 < y < l\,, \quad T = T_m \text{ at } y = 0\,, \quad T = T_e \text{ at } y = l\,,$$

$$\text{with } V_\infty = -M \Bigg/ \frac{\mathrm{d}T}{\mathrm{d}y} \text{ at } y = 0\,.$$

With large enough l (so that $\exp(-\rho c_p V_\infty l/k) \ll 1$),
$T \sim T_e + (T_m - T_e)\exp(-\rho c_p V_\infty y/k)$, $\mathrm{d}T/\mathrm{d}y \sim -\rho c_p V_\infty (T_m - T_e)/k$ at $y = 0$, and

$$V_\infty \sim \sqrt{\frac{Mk}{\rho c_p (T_m - T_e)}}$$

is the steady velocity of approach.

Short-time solution. For a sufficiently short time, the "hard-material approximation", $\kappa(T) \sim \kappa_m(1 - T/T_m)$, would not be valid and in particularly extreme cases we would have $T \sim T_c(t)$, with T_c not close to T_m, in the softened layer. Such cases will be included in the discussions of short-time regimes for soft materials in Sec. 4. For the present, we take time to be small compared with the time scale of (3.1) (*i.e.* $t \ll l^2 \rho c_p/k$, or $t \ll l_\infty^2 \rho c_p/k = (T_m - T_e)/M$ if the length scale for the steady state, $l_\infty = k/(\rho c_p V_\infty) = \sqrt{k(T_m - T_e)/(\rho c_p M)}$, is small compared with the length of l of the workpiece), but still with $T \sim T_m$ near $y = 0$ and $\kappa(T) \sim \kappa_m(1 - T/T_m)$ in the welding layer, so that (3.1) - (3.4) apply.
For short times, high temperature gradients are expected near $y = 0$ so V should be small and (3.1) reduces to the heat equation:

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial y^2} \quad \text{for} \quad 0 < y < l\,,\, t > 0\,. \tag{3.6}$$

Combined with the boundary condition (3.3) and initial condition $T = T_e$, this leads to the standard local similarity solution

$$T \sim T_e + (T_m - T_e)\mathrm{erfc}\left(\frac{x}{2}\sqrt{\frac{\rho c_p}{kt}}\right) = T_e + \frac{2}{\sqrt{\pi}}(T_m - T_e)\int_{\frac{x}{2}\sqrt{\frac{\rho c_p}{kt}}}^{\infty} \mathrm{e}^{-z^2}\,\mathrm{d}z$$

(see, for example, [13] or [14]). In particular,

$$-\left.\frac{\mathrm{d}T}{\mathrm{d}y}\right|_{y=0} \sim (T_m - T_e)\sqrt{\frac{\rho c_p}{\pi k t}} \quad \text{so} \quad V \sim \frac{M}{(T_m - T_e)}\sqrt{\frac{\pi k t}{\rho c_p}}.$$

Numerical solution. The set of simulations carried out here used a backward-time, central-space, implicit approximation. Fig. 2 shows results for $k/(\rho c_p) = 4.52\times10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $M = 28.024\,\mathrm{K}\,\mathrm{s}^{-1}$, $l = 0.018\,\mathrm{m}$, $T_e = 300\,\mathrm{K}$, $T_m = 1350\,\mathrm{K}$ (so $l_\infty \approx 0.013\,\mathrm{m} \approx l$). Note the high initial slope near $y = 0$ and approach, with increasing t, towards the steady state.

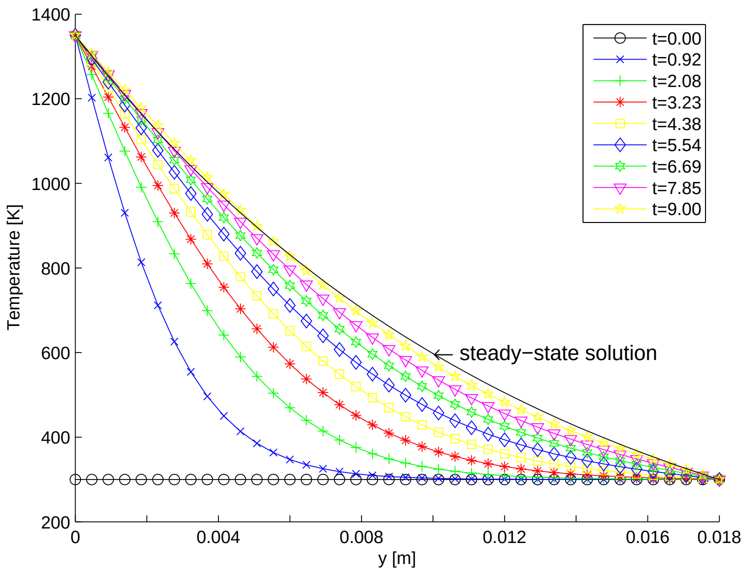

Figure 2: Plot of temperature, T, for different times, taking the case of a hard material, so that approximation (2.8) applies.

4 Soft Materials

If the material is "soft" (or is hard but the temperature is well below melting in the thin lubricating layer), the temperature dependency function in the thin layer is now approximated by

$$\kappa(T) \sim K_c \exp\left(\frac{T_a^2}{T_c}(T_c - T)\right) \tag{4.1}$$

where

$$K_c = \kappa_m \left(1 - \frac{T_c}{T_m}\right) \exp\left(\frac{T_a}{T_c} - \frac{T_a}{T_m}\right) , \tag{4.2}$$

with $T_c = T|_{y=0}$ generally varying with time.

Following the calculations of [11], which means, for the simple decoupling approach, replacing the negative fourth power of temperature in the heat equation and integrals by an exponential, so that (2.19) is replaced by

$$k\frac{\mathrm{d}^2T}{\mathrm{d}y^2} + \frac{\sigma^5}{K_c^4}\mathrm{e}^{-4T_a(T_c-T)/T_c^2} = 0 \tag{4.3}$$

(see (3.6) in [11] where the dimensionless temperature is now $\theta = T_a(T_c - T)/T_c^2)$ and (2.17) by

$$V = \frac{12P\sigma^3}{L^2K_c^4}\int_0^\infty y^2 \exp\left(\frac{-4T_a}{T_c^2}(T_c - T)\right) \mathrm{d}y\,. \tag{4.4}$$

The solution of (4.3) subject to (2.7) and $\partial T/\partial y = 0$ on $y = 0$, where $T = T_c$, is

$$T = T_c - \frac{T_c^2}{2T_a}\ln\left(\cosh by\right) , \tag{4.5}$$

$$\text{with } b = 2T_aG/T_c^2 \text{ and } b^2 = 2\sigma^5T_a/(kK_c^4T_c^2)\,. \tag{4.6}$$

Using (4.5) and the first of (4.6) in (4.4) gives

$$V = \frac{\pi^2T_c^6P\sigma^3}{8K_c^4T_a^3L^2G^3}$$

(see [11]) while (4.6) combined with (2.20) (which holds for any approximation of $\kappa(T)$) leads to

$$T_c^2G^3/K_c^4 = 2T_aU_e^5/k^4\,, \tag{4.7}$$

and hence (2.20) gives

$$V = \frac{\pi^2k^3T_c^6P}{8K_c^4T_a^3L^2U_e^3} = \frac{\pi^2PU_e^2T_c^4}{4L^2kT_a^2G^3}\,. \tag{4.8}$$

As K_c is given in terms of T_c by (4.2), (4.7) can be written as

$$\kappa_m^{-4}\left(1 - \frac{T_c}{T_m}\right)^{-4} G^3T_c^2 \exp\left(\frac{4T_a}{T_m} - \frac{4T_a}{T_c}\right) = \frac{2T_aU_e^5}{k^4}\,. \tag{4.9}$$

Eqn. (4.9), applied as a matching condition for the outer problem, gives a boundary condition at $y = 0$, while (4.8) now gives a non-local effect through having a velocity for the outer problem of the form

$$V = \frac{NT_c^4 PU_e^2}{kT_a L^2 G^3} \quad \text{with} \quad G = -\left.\frac{\partial T}{\partial y}\right|_{y=0} . \tag{4.10}$$

The value of N according to the simple-minded decoupling approach of [10] and [11] outlined in Sec. 2 is $\pi^2/4$, but a more complete asymptotic solution (see [11]) leads to $N = \frac{3}{4}(\frac{5}{2} - \frac{\pi^2}{12})$.

The outer problem, representing just heat conduction and convection, is again (3.1) with (3.2),

$$\rho c_p \left(\frac{\partial T}{\partial t} - V \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} \quad \text{for} \quad 0 < y < l\,,\, t > 0\,, \tag{4.11}$$

$$T = T_e \quad \text{for} \quad y = l\,,\, t \geq 0\,; \quad T = T_e \quad \text{for} \quad 0 < y \leq l\,,\, t = 0\,, \tag{4.12}$$

but now with the second boundary condition on $y = 0$ based on (4.9),

$$\frac{T^2}{\kappa_m^4 (1 - T/T_m)^4} \left(-\frac{\partial T}{\partial y} \right)^3 \exp \left(\frac{4T_a}{T_m} - \frac{4T_a}{T} \right) = \frac{2T_a U_e^5}{k^4} \tag{4.13}$$

and convective velocity given by

$$V = M(T) \Bigg/ \left(-\left.\frac{\partial T}{\partial y}\right|_{y=0} \right)^3 , \tag{4.14}$$

$$\text{for} \quad M(T) = \frac{NPU_e^2}{kT_a L^2} \left(T|_{y=0} \right)^4 . \tag{4.15}$$

The steady problem is

$$k \frac{\mathrm{d}^2 T}{\mathrm{d}y^2} + \rho c_p V \frac{\mathrm{d}T}{\mathrm{d}y} = 0 \ \text{ for } \ 0 < y < l\,, \quad \text{with } T = T_e \ \text{ on } \ y = l\,,$$

$$\frac{T^2}{\kappa_m^4 (1 - T/T_m)^4} \left(-\frac{\mathrm{d}T}{\mathrm{d}y} \right)^3 \exp \left(\frac{4T_a}{T_m} - \frac{4T_a}{T} \right) = \frac{2T_a U_e^5}{k^4} \ \text{ on } \ y = 0\,, \tag{4.16}$$

and

$$V = M \Bigg/ \left(-\left.\frac{\partial T}{\partial y}\right|_{y=0} \right)^3 . \tag{4.17}$$

This fixes a limiting central temperature $T_\infty = T(0)$, velocity V_∞, length scale $l_\infty = k/(\rho c_p V)$ (which is significant if it is less than l) and hence a

corresponding time scale for the problem (4.11) - (4.14), $t_\infty = l_\infty/V_\infty = l_\infty^2 \rho c_p/k$.
The dimensionless version of (4.11) - (4.14), given by scaling:

$$y = l_\infty z\,, \quad t = t_\infty s\,, \quad T = T_e + (T_\infty - T_e)\phi\,, \quad V = V_\infty V^*\,;$$

is then, assuming, for simplicity, that $l^* = l/l_\infty \gg 1$:

$$\frac{\partial \phi}{\partial s} - V^* \frac{\partial \phi}{\partial z} = \frac{\partial^2 \phi}{\partial z^2} \quad \text{for} \quad z > 0\,,\, s > 0\,;$$

$$\phi = 0 \quad \text{for} \quad s = 0\,,\, z \geq 0\,;$$

$$\begin{aligned} &\left(\frac{T_e + (T_\infty - T_e)\phi}{T_\infty}\right)^2 \left(\frac{1 - T_\infty/T_m}{1 - (T_e + (T_\infty - T_e)\phi)/T_m}\right)^4 \left(-\frac{\partial \phi}{\partial z}\right)^3 \\ &\quad = \exp\left(\frac{4T_a}{T_e + (T_\infty - T_e)\phi} - \frac{4T_a}{T_\infty}\right) \quad \text{at } z = 0\,; \end{aligned} \tag{4.18}$$

$$\text{and} \qquad V^* = \left(\frac{T_e + (T_\infty - T_e)\phi|_{y=0}}{T_\infty}\right)^4 \Bigg/ \left(-\left.\frac{\partial \phi}{\partial z}\right|_{z=0}\right)^3 .$$

Taking T_∞ not close to T_e with $T_\infty = \mathrm{O}(T_e)$, activation temperature to be large, $T_a \gg T_\infty$, and assuming that $|\partial\phi/\partial z|$ is not (exponentially) large, (4.18) reduces to

$$\phi = 1 \;\text{ at }\; z = 0\,;$$

the (dimensional) temperature near the centre of the weld varies by only $\mathrm{O}(T_\infty^2/T_a) \ll T_\infty$. With the assumptions on how T_e and T_∞ compare, this means that the central temperature varies much less than $T_\infty - T_e$. The same will hold true for l^* not large.

Long-time solution. Again we simply expect an approach to the equilibrium, as determined above.

Short-time solution. Consideration of the model (4.11) - (4.14) indicates a number of different regimes. In practice, because of the high activation temperature T_a, these would last exponentially short times, quite possibly comparable with, or smaller than, the period of oscillation. In such extreme situations the model would not be valid.

(i) Initial stage: $T_c(t) = T|_{y=0}$ very close to T_e. From (4.13),

$$\begin{aligned} G \sim G_e = &\left(2T_a \kappa_m^4 k^{-4} U_e^5 (1 - T_e/T_m)^4 T_e^{-2} \exp\left(\frac{4T_a}{T_e} - \frac{4T_a}{T_m}\right)\right)^{1/3} \\ = &\; G_0 \exp\left(\frac{4}{3}\left(\frac{T_a}{T_e} - \frac{T_a}{T_m}\right)\right)\,, \end{aligned}$$

for

$$G_0 = \left(2T_a\kappa_m^4 k^{-4}U_e^5(1-T_e/T_m)^4 T_e^{-2}\right)^{1/3}, \tag{4.19}$$

so G is expected to be exponentially large for cases of interest, V is neglected and T is given by

$$\rho c_p \frac{\partial T}{\partial t} \sim k\frac{\partial^2 T}{\partial y^2}, \text{ with } -\frac{\partial T}{\partial y} \sim G_e \text{ at } y = 0.$$

Hence, near $y = 0$, T is given by a similarity solution,

$$T \sim T_e + at^{1/2}f(\eta) \text{ with } \eta = \left(\frac{\rho c_p}{kt}\right)^{1/2} y, \quad a = \left(\frac{k}{\rho c_p}\right)^{1/2} G_e$$

and where f satisfies

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2} + \frac{\eta}{2}\frac{\mathrm{d}f}{\mathrm{d}\eta} - \frac{1}{2}f = 0 \text{ for } \eta > 0, \quad \text{with } \frac{\mathrm{d}f}{\mathrm{d}\eta} = -1 \text{ at } \eta = 0.$$

(For completeness, $f(\eta) = \frac{2}{\sqrt{\pi}}\left(\mathrm{e}^{-\eta^2/4} - \frac{\eta}{2}\int_\eta^\infty \mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi\right)$.)
This approximation for T remains valid only as long as G can be approximated by G_e, which requires that $|T - T_e| \ll T_e^2/T_a$. Hence this initial stage applies for

$$t \ll \frac{\rho c_p}{k} G_0^{-2}\frac{T_e^4}{T_a^2}\exp\left(\frac{8}{3}\left(\frac{T_a}{T_m} - \frac{T_a}{T_e}\right)\right).$$

(ii) First intermediate stage: $T_c(t) - T_e$ of order T_e^2/T_a. On writing $T = T_e + (T_e^2/T_a)\theta$,

$$G \sim G_0 \exp\left(\frac{4}{3}\left(\frac{T_a}{T_e} - \frac{T_a}{T_m}\right)\right)\mathrm{e}^{-\frac{4}{3}\theta} = G_e \mathrm{e}^{-\frac{4}{3}\theta}$$

so that V^* (which is of size G^{-3}) is still exponentially small and therefore negligible,

$$\rho c_p \frac{\partial \theta}{\partial t} \sim k\frac{\partial^2 \theta}{\partial y^2}, \text{ with } -\frac{\partial \theta}{\partial y} \sim \frac{G_e T_a}{T_e^2}\mathrm{e}^{-\frac{4}{3}\theta} \text{ at } y = 0$$

and $\theta = 0$ at $t = 0$.
Rescaling time, distance and temperature,

$$\begin{aligned} t &= \frac{9}{16}\frac{\rho c_p}{k}\frac{T_e^4}{T_a^2}\frac{1}{G_0^2}\exp\left(\frac{16}{9}\left(\frac{T_a}{T_m} - \frac{T_a}{T_e}\right)\right)\tau, \\ y &= \frac{3}{4}\frac{T_e^2}{T_a}\frac{1}{G_0}\exp\left(\frac{4}{3}\left(\frac{T_a}{T_m} - \frac{T_a}{T_e}\right)\right)Y, \quad \theta = \frac{3}{4}\varphi, \end{aligned} \tag{4.20}$$

leads to

$$\frac{\partial\varphi}{\partial\tau} \sim \frac{\partial^2\varphi}{\partial Y^2}\,, \text{ with } -\frac{\partial\varphi}{\partial Y} \sim \mathrm{e}^{-\varphi} \text{ at } Y=0\,, \tag{4.21}$$

and $\varphi = 0$ at $\tau = 0$. The form of the boundary condition in (4.21) suggests that there is no (exact) similarity solution in this regime. (The decaying exponential in the boundary condition distinguishes this phase of the problem from the very early stage, when a constant condition applied – the constant value being needed for the similarity solution.)
This regime will be left when θ and φ are large which, from (4.21), clearly corresponds to $\tau \gg 1$, *i.e.*

$$t \gg \frac{\rho c_p}{k}\frac{T_e^4}{T_a^2}\frac{1}{G_0^2}\exp\left(\frac{16}{9}\left(\frac{T_a}{T_m}-\frac{T_a}{T_e}\right)\right).$$

(iii) Second intermediate stage: $T_c(t) - T_e \gg T_e^2/T_a$ but $T_\infty - T_c(t) \gg T_\infty^2/T_a$. ($T_\infty$ being the equilibrium temperature at the centre of the weld.) The second condition states that we do not yet have T_c close to T_∞. The velocity V is still small so that the PDE can be approximated by the heat equation but no further simplifications are apparent; the condition at $y = 0$ is

$$-\frac{\partial T}{\partial y} \sim \left(\frac{2\kappa_m^4 T_a U_e^5(1-T/T_m)}{k^4T^2}\right)^{1/3}\exp\left(\frac{4}{3}\left(\frac{T_a}{T_e}-\frac{T_a}{T_m}\right)\right).$$

Note that each of these early stages could also be applicable to a material regarded as hard – hard in that the steady maximum temperature has to be so close to the melting temperature that the temperature-dependency function $\kappa(T)$ is, locally, approximately linear. The second intermediate stage would cease for $T_m - T_c$ of order T_m^2/T_a in such cases. (There would then be time regimes, before $T_m - T_c$ falls to $\mathrm{O}(\delta) = \mathrm{O}(k^{5/3}(T_m/\kappa_m)^{8/3}(\rho c_p(T_m - T_e))^{1/2}(P^{1/2}L^{-1})U_e^{-7/3})$ (see (2.23)) where $T_m - T_c$ and the matching $-\partial T/\partial y$ are so large that $\kappa(T)$ cannot be approximated by the linear function, and $V(t)$ is not given by (3.4). However, during such later intermediate times, $V(t)$ can still be regarded as negligible and, for the outer problem, $T = T_m$, to leading order, at $y = 0$.

Numerical solution. Avoiding the short-time regimes, with associated exponentially short time scales (for large T_a), the numerical method used for hard materials, Sec. 3, can be employed again. Unsurprisingly, the results are qualitatively very similar and are therefore not presented here.

5 Conclusions

The steady-temperature modelling of [10] and [11] has been extended to consider the transient conditions which can occur when linear friction welding is only run for finite times. In both limiting cases of soft and hard materials, in which the stress-strain-temperature constitutive law can be approximated in different ways, the key problem to be solved here is an outer problem for temperature which contains a non-local term. The non-locality arises in the convective term in the heat equation, through the velocity being fixed (thanks to the behaviour of the inner problem) by the temperature gradient in the vicinity of the weld.

It can be observed that for an early enough stage in the process, so that what might be regarded as a "hard material" (as its equilibrium temperature gets so close to melting) has everywhere relatively low temperatures, the approximate theory is the same as that used for soft materials. However, whether the workpieces are hard or soft, these early stages of the process, with the central temperature significantly lower than its equilibrium level, happen at rates exponentially quickly compared with the later stages (see, for example, (4.20)), assuming that the activation temperature is large, $T_a \gg T_c(t)$. (For $T_a = \mathrm{O}(T_c(t))$, the exponential approximation (4.1) used in analysing the thin welding layer becomes invalid.) For T_c too small, the predicted changes will occur over time scales comparable with, or even shorter than, the period of oscillation of the workpieces, and again modelling assumptions needed for the analysis of the thin layer would be violated. It should be emphasised that, in practice, a preliminary stage of linear friction welding entails the workpieces being rubbed against each other without them being forced together. In this stage, simple surface friction results in localised heating at the contact interfaces, without material deformation. This present paper only considers what happens subsequently: when, after such initial preheating the workpieces are pressed together, so that material deformation starts, while continuing with the oscillatory motion.

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