Non-triviality of some one-relator products of three groups

NON-TRIVIALITY OF SOME ONE-RELATOR PRODUCTS OF THREE GROUPS

I. CHINYERE AND J. HOWIE

Abstract. In this paper we study a group G which is the quotient of a free product of three non-trivial groups by the normal closure of a single element. In particular we show that if the relator has length at most eight, then G is non-trivial. In the case where the factors are cyclic, we prove the stronger result that at least one of the factors embeds in G.

1. Introduction

A one-relator product of groups is the quotient of a free product by the normal closure of a single element, called the *relator*. In [10] and [13] (see also [18]) the following conjecture was proposed.

Conjecture 1.1. *A one-relator product on three non-trivial groups is non-trivial.*

Conjecture 1.1 is an extension of the Scott-Wiegold conjecture (see Problem 5.53 in [23]). The latter problem was solved by the second author [18], who also conjectured that a free product of $(2n-1)$ groups is not the normal closure of n elements.

Our aim in this paper is to prove Conjecture 1.1 under certain conditions. First we assume that the factors are all finite cyclic groups. Under this condition the conjecture is already known [18], but here we prove a stronger result.

Theorem 1.2. *Let G_a, G_b and G_c be non-trivial cyclic groups with generators a, b and c respectively. For any word $w \in G_a * G_b * G_c$ whose exponent sum in each of the generators is non-zero modulo the order of that generator, each of the factors G_a, G_b and G_c embed in*

$$G = \frac{G_a * G_b * G_c}{N(w)}$$

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via the natural maps.

An immediate consequence (Corollary 4.2) is that in any one-relator product G of three cyclic groups, at least one of the factors embeds in G; this is Conjecture 9.4 of [13] for cyclic groups.
Also we can place a restriction on the length of the relator.

Theorem 1.3. *The one-relator product on three non-trivial groups is non-trivial when the relator has length at most eight.*

Placing an upper bound on the relator allows us to apply techniques in combinatorial group theory such as pictures.

The rest of the paper is organised as follows. Section 2 discusses pictures. As mentioned in the introduction, pictures combined with curvature arguments (discussed in Section 3) are used in the proof of Theorem 1.3. In Sections 4 and 5, we give proofs of Theorems 1.2 and 1.3 respectively.

Throughout, we shall use the following notations. Normal closure, $N(.)$; length, $\ell(.)$; real part, $\Re(.)$; imaginary part, $\Im(.)$; homeomorphic, $\approx$; conjugate, $\simeq$; isomorphic, $\cong$; union $\cup$, and disjoint union, $\sqcup$.

2. Pictures

Pictures are one of the most powerful tools available in combinatorial group theory. Essentially, pictures are the duals of van Kampen diagrams [27]. We will describe pictures briefly in the context of groups with presentations of the form $G = \langle X_1, X_2 \mid R_1, R_2, R \rangle$, where $G_1 = \langle X_1 \mid R_1 \rangle$, $G_1 = \langle X_2 \mid R_2 \rangle$, and R is a word with free product length at least two.

Groups of the form G above are called *one-relator products of groups G_1 and G_2*. In the next section we shall discuss such groups in more detail. Pictures were first introduced by Rourke [25] and adapted to work for such groups (as G) by Short [26]. Since then they have been used extensively and successfully by various authors in a variety of different ways (see [[5], [6], [7], [12], [16], [17], [20]]). We describe below the basic idea, following closely the account in [19]. A more detailed description can be found in [15] and also [[4], [1], [21], [9], [24]].

Let G be as above, a picture Γ over G on an oriented surface $\mathcal{S}$ (usually D^2) consists of the following:

(1) A collection of disjoint closed discs in the interior of $\mathcal{S}$ called vertices;
(2) A finite number of disjoint arcs, each of which is either:

(a) a simple closed curve in the interior of $\mathcal{S}$ that meets no vertex,
(b) an arc joining two vertices (or one vertex to itself),
(c) an arc joining a vertex to the boundary $\partial\mathcal{S}$ of $\mathcal{S}$, or
(d) an arc joining $\partial\mathcal{S}$ to $\partial\mathcal{S}$;

(3) A collection of labels, one at each corner of Γ. (A *corner* of Γ is a connected component of the complement, in ∂v for some vertex v or in $\partial\mathcal{S}$, of the arcs of Γ.) Reading the labels round a vertex in the clockwise direction yields $R^{\pm 1}$ (up to cyclic permutation), as a cyclically reduced word in $G_1 * G_2$.

A *region* of Γ is a connected component of the complement in $\mathcal{S}$ of the arcs and vertices of Γ. A region is a *boundary region* if it meets $\partial\mathcal{S}$, and an interior region otherwise. If $\mathcal{S} \approx S^2$ or if $\mathcal{S} \approx D^2$ and no arcs of Γ meet the boundary of D^2, then Γ is called *spherical.* In the latter case ∂D^2 is one of the boundary components of a non-simply connected region (provided, of course, that Γ contains at least one vertex or arc), which is called the *exceptional region.* All other regions are interior. The labels of any region $\triangle$ of Γ are required all to belong to either G_1 or G_2. Hence we can refer to regions as G_1-regions and G_2-regions accordingly. Similarly a corner is called a G_i-corner or more specially a g_i-corner if it is labelled by the element $g_i \in G_i$. Each arc is required to separate a G_1-region from a G_2-region. Observe that this is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from G_1 and G_2. A region bounded by arcs that are closed curves will have no labels; nevertheless the above convention requires that it be designated a G_1- or G_2-region. An important rule for pictures is that the labels within any G_1-region (respectively G_2-region) allow the solution of a quadratic equation in G_1 (respectively G_2). The labels around any given boundary component of the region are formed into a single word read anti-clockwise. The resulting collection of elements of G_1 or G_2 is required to have genus no greater than that of the region (in the sense of [7]). This technical general requirement is much simpler in the commonest case of a simply connected region - it means merely that the resulting word represents the identity element in G_1 or G_2.

Two distinct vertices of a picture are said to *cancel* along an arc e if they are joined by e and if their labels, read from the endpoints of e, are mutually inverse words in $G_1 * G_2$. Such vertices can be removed from a picture via a sequence of *bridge moves* (see Figure 1 below and [7] for more details), followed by deletion of a *dipole* without changing the boundary label. A *dipole* in Γ is a connected spherical sub-picture of

Γ containing precisely two vertices, which does not meet $\partial\mathcal{S}$, such that none of its interior regions contain other components of Γ. Removal of a dipole gives an alternative picture with the same boundary label and two fewer vertices.

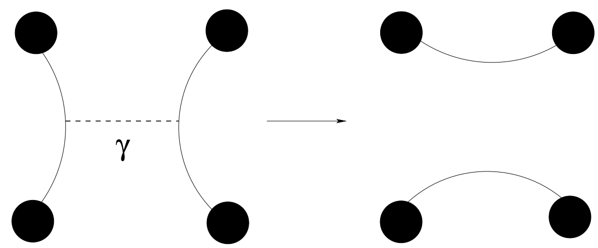

FIGURE 1. Diagram showing bridge-move.

A *bridge-move* on a picture $\Gamma \subset \mathcal{S}$ is a surgery of its arcs along an embedded path γ in $\mathcal{S}$ that is disjoint from the vertices of Γ and meets the arcs of Γ precisely in $\partial\gamma$ (see Figure 1), provided that the result of the surgery is another picture Γ' (satisfying the conditions set out above). In particular, if γ divides a simply-connected region of Γ into two regions of Γ', then the boundary labels of these two new regions are required be trivial in G_1 or G_2.
We say that a picture Γ is *reduced* if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. If $\mathcal{W}$ is a set of words, then a picture is $\mathcal{W}$-*minimal* if it is non-empty and has the minimum number of vertices amongst all pictures over G with boundary label in $\mathcal{W}$. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of G occurs as the boundary label of some reduced picture on D^2. A picture is *connected* if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed arc or joins two points of $\partial\mathcal{S}$, unless the picture consists only of that arc.

Two arcs of Γ are said to be *parallel* if they are the only two arcs in the boundary of some simply-connected region $\triangle$ of Γ. We will also use the term *parallel* to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a *class of* ω *parallel arcs* or ω-*zone*. Given a ω-*zone* joining vertices u and v of Γ, consider the $\omega - 1$ two-sided regions separating these arcs. Each such region has a corner label x_u at u and a corner label x_v at v, and the picture axioms imply that $x_u x_v = 1$ in G_1 or G_2. The $\omega - 1$ corner labels at v spell a cyclic subword s of length $\omega - 1$ of the label of v. Similarly the corner labels at u spell out a cyclic subword t of length $\omega - 1$. Moreover, $s = t^{-1}$. If we assume that Γ is reduced, then u and v do not cancel. Hence the cyclic permutations of the labels at v and u of which s and t are initial segments respectively are not equal. Hence t and s are pieces, in the sense of small cancellation theory.

3. Combinatorial curvature

For any compact orientable surface (with or without boundary) $\mathcal{S}$ with a cellular subdivision, we assign real numbers β to the corners of the faces in $\mathcal{S}$. We will think of these numbers as interior angles. A vertex which is on $\partial\mathcal{S}$ is called a boundary vertex, and otherwise interior. The *curvature* of an interior vertex v in $\mathcal{S}$ is defined as

$$\kappa(v) = \left[2 - \sum_i \beta(v)_i\right]\pi,$$

where i ranges from 1 to the index $d(v)$ of v, and $\beta(v)_i$ denotes the angle of the i-th corner at v. If v is a boundary vertex then we define

$$\kappa(v) = \left[1 - \sum_i \beta(v)_i\right]\pi.$$

The curvature of a face Δ is defined as

$$\kappa(\Delta) = \left[2 - d(\Delta) + \sum_i \beta(\Delta)_i\right]\pi$$

where i ranges from 1 to the number $d(\Delta)$ of corners of Δ, and $\beta(\Delta)_i$ denotes the angle of the i-th corner of Δ. The combinatorial version of the Gauss-Bonnet theorem states that the total curvature is the multiple of Euler characteristic of the surface by 2π:

$$\kappa(\mathcal{S}) = \left[\sum_v \kappa(v) + \sum_\Delta \kappa(\Delta)\right]\pi = 2\pi\chi(\mathcal{S}).$$

We use curvature to prove results by showing that this value cannot be realised. We assign to each corner of a region of degree k an angle $(k-2)/k$. This will mean that regions are flat in the sense that they have zero curvature (alternatively we can make vertices flat instead). This will be the standard assignment for this work. In other words, wherever curvature is mentioned with no specified assignments, it is implicitly assumed that we are using the one described above.

In some cases, it may be needful to redistribute curvature (see [8]). This involves locating positively-curved vertices (or regions), and using their excess curvature to compensate their negatively curved neighbours. Hence the total curvature is preserved. We shall describe how to do this in Section 5 (see the proof of Theorem 5.11).

4. One-relator products of cyclics

In this section we give a proof of Theorem 1.2. Recall that $G_a = \langle a \mid a^p \rangle$, $G_b = \langle b \mid b^q \rangle$, $G_c = \langle c \mid c^r \rangle$, and w is a word in the free product $G_a * G_b * G_c$ with non-zero exponent sum in each of the generators a, b and c. The proof we present follows closely the one in [18].

Lemma 4.1. *Suppose that p, q and r are prime powers. Then each of G_a, G_b, G_c embeds via the natural map into*

$$G = \frac{(G_a * G_b * G_c)}{N(w)}.$$

Proof. We know the result holds when p, q, r are primes by [18, Theorem 4.1]. Here we assume that at least one of p, q, r is a prime power but not prime.

Suppose that n is the exponent sum of a in w. The assumption that n is not divisible by p implies that $n = tp + s$ with $0 < s < p$. By replacing w with wa^{-tp} which changes n to s (leaving G unchanged), we can always assume that $0 < n < p$. If m is co-prime to p then $a \mapsto a^m$ induces an automorphism of G_a. Thus, replacing a by a^m in w gives a new word $w' \in G_a * G_b * G_c$ such that the resulting group

$$G' = \frac{(G_a * G_b * G_c)}{N(w')}$$

is isomorphic to G (and such that G_a embeds in G' if and only if it embeds in G). Moreover, the exponent sum of a in w' is mn. By Bezout's Lemma we may choose m such that $mn \equiv gcd(n, p) \bmod p$. Thus without loss of generality we may assume that n divides p (and similarly the exponent sums of b, c in w divide q, r respectively). So in particular if p is prime, then $n = 1$.

Now suppose that p and n (the exponent sum of a in w) are powers of a prime τ – say $p = \tau^t$ and $n = \tau^s$ where $0 \leq s < t$. If τ is an odd prime, define

$$\theta_p = \frac{(\tau^{t-s} - 1)\pi}{2\tau^t}.$$

If $\tau = 2$, define

$$\theta_p = \frac{(2^{t-s-1}-1)\pi}{2^t}$$

unless $s = t-1$, in which case define $\theta_p = \frac{\pi}{2}$. Recall that an element $\cos(\theta) + \sin(\theta)\mathbf{v} \in S^3$ represents an element of order p in $SO(3) \approx S^3/\{\pm 1\}$ if and only if θ is a multiple of π/p but not of π/τ^{t-1}, for any vector $\mathbf{v} \in S^2$. Hence for any $\mathbf{v} \in S^2$, the map

$$\alpha_{\mathbf{v}} : G_a \to \mathbb{H},\ a \mapsto \cos(\theta_p) + \sin(\theta_p)\mathbf{v},$$

induces a faithful representation $G_a \to SO(3)$ (where $\mathbb{H}$ denotes the quaternions). Moreover, $\Re(\alpha_{\mathbf{v}}(a^n)) = \cos(\psi_p)$ where $\psi_p := n\theta_p$ and $\frac{\pi}{4} \le \psi_p \le \frac{\pi}{2}$.

Similarly, we can define maps $\beta_{\mathbf{v}} : G_b \to \mathbb{H}$ and $\gamma_{\mathbf{v}} : G_c \to \mathbb{H}$ that induce faithful representations $G_b, G_c \to SO(3)$, and such that, if e, f denote the exponent-sums of b, c in w, then $\Re(\beta_{\mathbf{v}}(b^e)) = \cos(\psi_q)$ and $\Re(\gamma_{\mathbf{v}}(c^f)) = \cos(\psi_r)$ where $\psi_q, \psi_r \in [\frac{\pi}{4}, \frac{\pi}{2}]$.

The numbers ψ_p, ψ_q, and ψ_r satisfy a triangle inequality. In other words none is greater than the sum of the other two. Hence, for example, the triple $(\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}, \gamma_{\text{-}\mathbf{i}})$ induces a homomorphism

$$\delta : G_a * G_b * G_c \to S^3$$

that sends w to $\cos(\theta) + \mathbf{i}\sin(\theta)$ with $0 \le \theta \le \frac{3\pi}{4}$. If $\theta = 0$ then δ induces a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c and we are done. So assume that $\theta > 0$. In other words $\Im(\delta(w)) > 0$.

Similar remarks apply to the triples $(\alpha_{\mathbf{i}}, \beta_{\text{-}\mathbf{i}}, \gamma_{\mathbf{i}})$ and $(\alpha_{\text{-}\mathbf{i}}, \beta_{\mathbf{i}}, \gamma_{\mathbf{i}})$. Hence also the triple $(\alpha_{\mathbf{i}}, \beta_{\text{-}\mathbf{i}}, \gamma_{\text{-}\mathbf{i}})$ induces a representation δ with $\Im(\delta(w)) < 0$. The map $S^2 \to S^3$, $\mathbf{v} \mapsto (\alpha_{\mathbf{i}}, \beta_{\text{-}\mathbf{i}}, \gamma_{\mathbf{v}})(w)$ is an S^1-equivariant map under the conjugation action. It follows that it either sends some $\mathbf{v}$ to $\pm 1 \in S^3$ (in which case $(\alpha_{\mathbf{i}}, \beta_{\text{-}\mathbf{i}}, \gamma_{\mathbf{v}})$ gives a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c), or by [18, Corollary 2.2], it represents $+1 \in H_2(S^3 - \{\pm 1\}) \cong \mathbb{Z}$.

Similarly, the map $\mathbf{v} \mapsto (\alpha_{\mathbf{i}}, \beta_{\mathbf{i}}, \gamma_{\mathbf{v}})(w)$ either maps some $\mathbf{v} \in S^2$ to $\pm 1 \in S^3$ and so gives a representation $G \to SO(3)$ that is faithful on each of G_a, G_b, G_c, or represents one of $0, -1 \in H_2(S^3 - \{\pm 1\}) \cong \mathbb{Z}$. Now any path $P : [0,1] \to S^2$ from $-\mathbf{i}$ to $\mathbf{i}$ gives rise to a homotopy

$$t \mapsto \big(\mathbf{v} \mapsto (\alpha_i, \beta_{P(t)}, \gamma_v)(w)\big)$$

between the above two maps. If $(\alpha_{\mathbf{i}}, \beta_{P(t)}, \gamma_{\mathbf{v}})(w) \neq \pm 1$ for all t and for all $\mathbf{v}$, then we can regard this as a homotopy of maps $S^2 \to S^3 \setminus \{\pm 1\}$. This is a contradiction since the two maps belong to different homology

classes in $H_2(S^3 \setminus \{\pm 1\})$. Hence for some t and some $\mathbf{v} \in S^2$, the map $(\alpha_{\mathbf{i}}, \beta_{P(t)}, \gamma_{\mathbf{v}})$ sends w to ± 1 and so induces a representation $G \to SO(3)$ that is faithful on each of G_a, G_b and G_c.
It follows that each of the natural maps from G_a, G_b and G_c to G is injective, as required. □

The general case of the Freiheitssatz for G follows from the special case of prime powers together with the Chinese Remainder Theorem by an easy induction.

Proof of Theorem 1.2. For the inductive step, suppose that $p = mn$ with $gcd(m, n) = 1$. Since the exponent sum of the generator a in w is non-zero modulo p, we can assume that it is non-zero modulo m. Now factor out a^m and apply the inductive hypothesis. This shows that the maps $G_b \to \mathbb{H}$ and $G_c \to \mathbb{H}$ are injective. It also shows that the kernel K of $G_a \to \mathbb{H}$ is contained in the subgroup $\langle m \rangle$.

If the exponent sum of a in w is also non-zero modulo n, then by interchanging the roles of m and n in the above we see that K is contained in $\langle n \rangle$. However, if the exponent-sum of a in w is divisible by n, then the same is automatically true: K is contained in $\langle n \rangle$. Finally, we know that K is contained in the intersection of $\langle m \rangle$ and $\langle n \rangle$. But this intersection is trivial by the Chinese Remainder Theorem, so we deduce that $G_a \to \mathbb{H}$ is injective. □

In a one-relator product of groups $G = (*_\lambda G_\lambda)/N(R)$, we say that a factor group G_λ is a *Freiheitssatz factor* if the natural map $G_\lambda \to G$ is injective. It is clear that any G_λ such that the product of the G_λ-letters in R is trivial is a Freiheitssatz factor. Combining this remark with Theorem 4.1 we obtain:

Corollary 4.2. *Any one-relator product of three cyclic groups contains a Freiheitssatz factor.*

In [3], Chiodo used the result of [18] to show that the free product G of three cyclic groups of distinct prime orders is *finitely annihilated*. In other words, for every non-trivial element $g \in G$, there exist a finite index normal subgroup N of G such that g is trivial in G/N. The proof uses nothing more than the fact that finitely generated subgroups of $SO(3)$ are residually finite. Hence our result extends this to the case where the cyclic groups are arbitrary.

Corollary 4.3. *Any free product of three cyclic groups is finitely annihilated.*

5. One-relator product with short relator

In this section we give a proof of Theorem 1.3. As mentioned in the introduction the proof uses pictures, as well as Bass-Serre theory and Nielsen transformations, and is broken down into a number of lemmas. Also we shall need the following results (Theorems 5.1 and 5.3 below).

Theorem 5.1. *Suppose that A and B are non-cyclic two-generator groups with generators $\{a,c\}$ and $\{b,d\}$ respectively. If A and B have irreducible faithful representations in $PSL_2(\mathbb{C})$, then $G=(A*B)/N(abcd)$ satisfies the Freiheitssatz: the natural maps $A\to G$ and $B\to G$ are injective.*

Proof. Let X,Y and Z be variable matrices in $SL_2(\mathbb{C})$. The aim of the proof is to show that one can choose X, Y and Z such that $a\mapsto X$, $c\mapsto Z$ gives a faithful representation $A\mapsto PSL_2(\mathbb{C})$ and $b\mapsto Y$, $d\mapsto (XYZ)^{-1}$ gives a faithful representation $B\mapsto PSL_2(\mathbb{C})$.
Such a triple of matrices is a representation of the free group F_3 of rank 3. Recall [11] that the character variety of representations $F_3\to PSL(2,\mathbb{C})$ is given by the seven parameters $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XY)$, $\mathrm{Tr}(XZ)$, $\mathrm{Tr}(YZ)$ and $\mathrm{Tr}(XYZ)$ subject to a single polynomial equation

(5.2)
$$\begin{aligned}&\mathrm{Tr}(X)^2+\mathrm{Tr}(Y)^2+\mathrm{Tr}(Z)^2+\mathrm{Tr}(XY)^2+\mathrm{Tr}(XZ)^2+\mathrm{Tr}(YZ)^2\\&+\mathrm{Tr}(XYZ)^2+\mathrm{Tr}(XY)\mathrm{Tr}(XZ)\mathrm{Tr}(YZ)\\&-\mathrm{Tr}(X)\mathrm{Tr}(Y)\mathrm{Tr}(XY)-\mathrm{Tr}(X)\mathrm{Tr}(Z)\mathrm{Tr}(XZ)-\mathrm{Tr}(Y)\mathrm{Tr}(Z)\mathrm{Tr}(YZ)\\&+\mathrm{Tr}(X)\mathrm{Tr}(Y)\mathrm{Tr}(Z)\mathrm{Tr}(XYZ)-\mathrm{Tr}(X)\mathrm{Tr}(YZ)\mathrm{Tr}(XYZ)\\&-\mathrm{Tr}(Y)\mathrm{Tr}(XZ)\mathrm{Tr}(XYZ)-\mathrm{Tr}(Z)\mathrm{Tr}(XY)\mathrm{Tr}(XYZ)=4\end{aligned}$$

By hypothesis, irreducible faithful representations of A and B in $PSL_2(\mathbb{C})$ exist. By a result of Fricke [11], irreducible representations $\rho:A\to PSL_2(\mathbb{C})$ are parametrised up to conjugacy by complex numbers $\alpha:=\mathrm{Tr}(\rho(a))$, $\gamma:=\mathrm{Tr}(\rho(c))$, and $\beta:=\mathrm{Tr}(\rho(ac))$ with $\alpha^2+\beta^2+\gamma^2+\alpha\beta\gamma\neq 4$. Hence we can ensure that $a\mapsto X$, $c\mapsto Z$ gives a faithful irreducible representation $A\to PSL_2(\mathbb{C})$ by fixing suitable values for $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$ and $\mathrm{Tr}(XZ)$. Similarly, we can ensure that $b\mapsto Y$, $d\mapsto (XYZ)^{-1}$ gives a faithful irreducible representation $B\to PSL_2(\mathbb{C})$ by fixing suitable values for $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XYZ)$ and $\mathrm{Tr}(XYZY^{-1})$.
By the repeated application of the trace relation

$$\mathrm{Tr}(MN)=\mathrm{Tr}(M)\mathrm{Tr}(N)-\mathrm{Tr}(MN^{-1})$$

for arbitrary matrices M and N we can write:

$\mathrm{Tr}(XYZY^{-1})=\mathrm{Tr}(Y)\mathrm{Tr}(XYZ)-\mathrm{Tr}(XY)\mathrm{Tr}(YZ)+\mathrm{Tr}(X)\mathrm{Tr}(Z)-\mathrm{Tr}(XZ)$.

Hence if we fix suitable values for $\mathrm{Tr}(X)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(XZ)$ and $\mathrm{Tr}(XYZ)$, we have two free variables $\alpha := \mathrm{Tr}(XY)$ and $\beta := \mathrm{Tr}(YZ)$ which are required to satisfy the quadratic equation which fixes the value of

$$\mathrm{Tr}(XYZY^{-1}) = \mathrm{Tr}(Y)\mathrm{Tr}(XYZ) - \alpha\beta + \mathrm{Tr}(X)\mathrm{Tr}(Z) - \mathrm{Tr}(XZ).$$

Combining this with Equation (5.2), and fixing $\mathrm{Tr}(X)$, $\mathrm{Tr}(Z)$, $\mathrm{Tr}(XZ)$, $\mathrm{Tr}(Y)$, $\mathrm{Tr}(XYZ)$, we have a pair of quadratic equations in α, β of the form

$$\alpha\beta = c_1,$$
$$\alpha^2 + \beta^2 + c_2\alpha\beta + c_3\alpha + c_4\beta = c_5$$

for suitable constants $c_1, \dots, c_5$. It is routine to check that any such pair of equations can be solved in $\mathbb{C}$. Any solution gives a representation $\langle a, b, c, d\rangle \longrightarrow SL_2(\mathbb{C})$ that induces the given faithful representations of A and B in $PSL_2(\mathbb{C})$ up to conjugacy, mapping the word $abcd$ to the identity element. This completes the proof. □

Theorem 5.3. *Let G be a one-relator product of non-trivial groups A and B, with relator r^n for some integer n. If $2 \le \ell(r) \le 6$ and $n \ge 2$, then r has order n in G.*

Theorem 5.3 is a consequence of various results proved in Chapter 4 of [2]. We omit the proof which is straightforward but lengthy. It uses standard curvature arguments on pictures.

For the proof of Theorem 1.3, let $G = (A * B * C)/N(w)$ where $w \in A*B*C$ has free product length at most 8. The theorem holds trivially if w is in the normal closure of any two of the factors A, B, C. Hence we can assume that this does not happen. It follows that w contains at least one letter from each of A, B, C.

Next suppose that some cyclic conjugate of w has the form αW where $\alpha \in A$ and $W \in B * C$ with $\ell(W) \le 7$. Then some cyclically reduced conjugate W' of W has length at most 6.
Since we are assuming that w is not in the normal closure of $A \cup B$ or of $A \cup C$, we must also have $\ell(W') \ge 2$. Thus W' – and hence also W – has infinite order in $B * C$. If α has infinite order in A, then G is just the free product of A and $B * C$ amalgamating the infinite cyclic subgroups $\langle\alpha\rangle$ and $\langle W\rangle$ respectively. If on the other hand α has order

$n < \infty$ in A, then by Theorem 5.3, G is the free product of A and $(B * C)/N(W^n)$ amalgamated over the subgroups $\langle \alpha \rangle$ and $\langle W \rangle$ of A and $B * C$ respectively. So G is non-trivial.

By the above discussion, we can assume that $\ell(w) \geq 6$. We can also assume that up to cyclic permutation w has the form c_1Uc_2V, where $c_1, c_2 \in C\backslash\{1\}$ and $U, V \in A*B\backslash\{1\}$, with $\ell(U)+\ell(V) \leq 6$. The group G is non-trivial if $c_1c_2 = 1$ or $UV \in N(A) \cup N(B)$. Hence we assume that neither of the two conditions holds. Finally, loss of generality, we assume that $\ell(U) \leq \ell(V)$.
We next prove some preliminary results towards the final result.

Lemma 5.4. *If U or V is in the normal closure of A or B, then G is non-trivial.*

Proof. Without loss of generality, we can assume that U is in the normal closure of A. In other words, there exist an element $\alpha \in A$ and a word $\gamma \in A * B$ such that $U = \gamma^{-1}\alpha\gamma$. Hence

$$\begin{aligned} w =& c_1\gamma^{-1}\alpha\gamma c_2 V \\ \simeq& \gamma c_1 \gamma^{-1}\alpha\gamma c_2\gamma^{-1}\gamma V\gamma^{-1}. \end{aligned}$$

So G is trivial if and only if

$$G' = \frac{(A * B * \tilde{C})}{N(W)}$$

is trivial, where $\tilde{C} = \gamma C\gamma^{-1}$ and $W = \tilde{c_1}\alpha\tilde{c_2}V$. Hence it is enough to consider the case where $U = \alpha$ (i.e $\gamma = 1$).

By assumption $c_1 \neq c_2^{-1}$. Hence $c_1\alpha c_2$ has infinite order in $A * C$. It follows that the subgroup of $A * C$ generated by A and $c_1\alpha c_2$ is isomorphic to the free product $A * \mathbb{Z}$. If A and V generate a subgroup of $A * B$ which is also isomorphic to $A * \mathbb{Z}$, then

$$G = (A * B)_{\langle A,V\rangle} *_{\langle A,c_1\alpha c_2\rangle} (A * C).$$

So in this case G is non-trivial.

Suppose then that $\langle A, V\rangle$ is not isomorphic to $A * \mathbb{Z}$. We can assume that V contains at least two B letters and $\ell(V) \leq 5$. If V contains exactly two B letters, then the two letters must be inverses of each other. This implies that B is a homomorphic image of G, so G is non-trivial. Hence V contains exactly three B letters. It follows that V is conjugate in $A * B$ to a letter $\beta \in B$ with order $r < \infty$. Define H to be the group

$$\begin{aligned} H :=& \frac{A * C}{N((c_1\alpha c_2)^r)} \\ =& A \underset{\langle\alpha\rangle}{*} T \underset{\langle\gamma\rangle=\langle c_2c_1\rangle}{*} C, \end{aligned}$$

where $T = \langle \alpha, \gamma \mid \alpha^p, \gamma^q, (\alpha\gamma)^r \rangle$ and p, q are the orders of α and c_2c_1 respectively. Then $G = (A * B)_{\langle A,V\rangle} *_{\langle A,c_1\alpha c_2\rangle} H$, provided of course that $\langle A, c_1\alpha c_2\rangle$ embeds in H. We show below that this is in fact the case.

By Bass-Serre theory, H acts on a tree Γ with three orbits $H(u_A)$, $H(u_T)$ and $H(u_C)$ of vertices, where u_A, u_T, u_C have stabilisers A, T, C respectively. The vertex u_T is adjacent to each of u_A, u_C. Let e_1, e denote the edges joining u_T to u_A, u_C respectively, and define $e_2 = c_1(e)$, which joins u_C to $c_1(u_T)$ (see Figure 2). Let the vertex set of Γ be X. The edge e divides Γ into two components Γ_1 and Γ_2 with vertex sets X_1 (containing vertices u_A and u_T) and X_2 (containing vertices u_C and $c_1(u_T)$ respectively such that $X = X_1 \sqcup X_2$. The stabilisers of e, e_1, and e_2 are $C \cap T = \langle\gamma\rangle$, $A \cap T = \langle\alpha\rangle$ and $c_1\langle\gamma\rangle c_1^{-1}$ respectively.

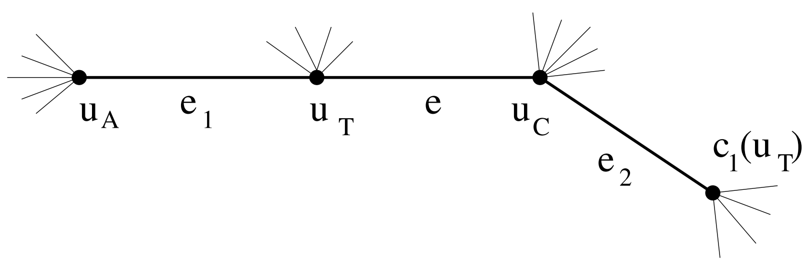

FIGURE 2. Diagram showing a section of the tree Γ on which H acts.

We aim to apply the Ping-Pong Lemma to show that the subgroup of H generated by A and $\langle c_1\alpha c_2\rangle$ is their free product. Since

$$A \cap \langle\gamma\rangle = \langle\alpha\rangle \cap \langle\gamma\rangle = 1$$

it follows that $a(X_2) \subset X_1$ for all $a \neq 1$ in A.
In a similar way, since $\alpha\gamma$ stabilizes u_T but not e, $c_1\alpha c_2 = c_1\alpha\gamma c_1^{-1}$ stabilizes $c_1(u_T)$ but not $c_1(e) = e_2$. If we can show that $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle = 1$, then it follows that $b(X_1) \subset X_2$ for every $b \neq 1$ in $\langle c_1\alpha c_2\rangle$, and the Ping-Pong Lemma will yield the result.

But $\langle\gamma\rangle \cap \langle c_1\alpha c_2\rangle$ stabilizes e and $c_1(u_T)$, and hence also e_2, so it is contained in $c_1\langle\gamma\rangle c_1^{-1}$. Hence, $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 1$ in T implies that in $c_1Tc_1^{-1}$,

$$c_1\langle\gamma\rangle c_1^{-1} \cap c_1\langle\alpha\gamma\rangle c_1^{-1} = 1.$$

Thus $\langle \gamma \rangle \cap \langle c_1 \alpha c_2 \rangle = 1$, as required. □

It follows in particular from Lemma 5.4 that $\ell(U) \geq 2$. The rest of the arguments we present rely heavily on Nielsen transformations. We transform $\{U, V\}$ into a more suitable Nielsen equivalent set depending on the subgroup they generate.

Lemma 5.5. *Suppose $\langle U, V \rangle$ is free. Then G is non-trivial.*

Proof. First we suppose $\langle U, V \rangle$ is free of rank 1 say with generator t. Then w can be expressed in the form $w = c_1 t^r c_2 t^s$, where $V = t^s$ and $U = t^r$ for integers s, r. If $s + r = 0$, then $G \neq 1$ since $w \in N(C)$. Otherwise $w = 1$ is a non-singular equation over C. We assume that $t \notin N(A) \cup N(B)$ for otherwise $G \neq 1$ by Lemma 5.4. It follows that any cyclically reduced conjugate of t has length at least 2. So since $\ell(U) + \ell(V) \leq 6$ and $\ell(t^n) \geq 2n$ for any n,

$$|s| + |r| \leq 3.$$

Consider the group

$$H := \frac{(C * \langle t \rangle)}{N(w)}.$$

If $s, r \geq 1$, then C embeds in H by [22]. Otherwise without loss of generality $s = -1$ and $r = 2$. Again C embeds in H by [14]. If t is trivial in H, then $H = C$ and so $w \in N(A * B)$; again $G \neq 1$. If t has finite order $m > 1$ in H, then

$$G = \frac{(A * B)}{N(t^m)} *_{\langle t \rangle} H.$$

By the previous comment it follows that $\ell(t) = 2$, say $t = \alpha\beta$. Hence

$$(A * B)/N(t^m) \cong A *_{\langle \alpha \rangle} T *_{\langle \beta \rangle} B,$$

where $T = \langle \alpha, \beta | \alpha^{|\alpha|} = \beta^{|\beta|} = (\alpha\beta)^m = 1 \rangle$ is a triangle group, and so t has order m as required.

Finally if t has infinite order in H, then

$$G = (A * B) *_{\langle t \rangle} H.$$

Hence G is non-trivial.

Now suppose $\langle U, V \rangle$ is free of rank 2. Let

$$H := (C * \langle U, V \rangle)/N(w) = C * \langle U \rangle.$$

Note that the subgroup of H generated by $\{U, c_1 U c_2\}$ is a free group of rank 2. It follows that G is the free product of H and $A * B$ amalgamated over the subgroups $\langle U, c_1 U c_2 \rangle$ and $\langle U, V \rangle$. It follows that G is non-trivial. □

Lemma 5.6. *Suppose $\langle U,V\rangle$ is isomorphic to $C_p * C_q$ or $C_p * \mathbb{Z}$, where C_p and C_q are finite cyclic groups. Suppose further that $2 \le \ell(U) \le \ell(V) \le 3$. Then G is non-trivial.*

Proof. By assumption $\langle U,V\rangle$ has an element of finite order. So Nielsen transformations can be applied to $\{U,V\}$ to get a new set $\{u,v\}$ with u or v having finite order. Note that UV is not in the normal closure of A, since w is not in the normal closure of $A \cup C$. Similarly UV is not in the normal closure of B.

We divide the proof into three cases, depending on the lengths of U and V

Case 1. $\ell(U) = 2 = \ell(V)$. Without loss of generality, we may write $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1$ with $\alpha, \alpha_1 \in A$ and $\beta, \beta_1 \in B$.
First suppose $U = (\alpha\beta)^{-1}$. Since UV is not in the normal closure of A or B, neither $\alpha = \alpha_1$ nor $\beta = \beta_1$ holds. Hence $\langle U,V\rangle$ is free of rank 2. The result follows from Lemma 5.5.

Suppose then that $U = \alpha\beta$. If $\alpha \ne \alpha_1$ and $\beta \ne \beta_1$, then $\{U,V\}$ is Nielsen reduced, so $\langle U,V\rangle$ is free of rank 2 and the result follows from Lemma 5.5. Hence we may assume without loss of generality that $\alpha = \alpha_1$. If $\beta = \beta_1$, then $\langle U,V\rangle$ is isomorphic to $\mathbb{Z}$. In this case the result follows from Lemma 5.5.

If $c_1 = c_2 = c$ then we can replace w with $\hat{U}\beta\hat{U}\beta_1$, where $\hat{U} = c\alpha$. In this case $\langle\hat{U}\rangle$ is free of rank 1 and the result follows from Lemma 5.5. Hence we may assume that $\beta \ne \beta_1$ and $c_1 \ne c_2$, so $\langle \beta c_2, \beta_1 c_1\rangle$ is free of rank 2. Again we apply Lemma 5.5 to show that G is non-trivial.

Case 2. $\ell(U) = 2$; $\ell(V) = 3$. Without loss of generality, we may write $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1\alpha_2$ with $\alpha, \alpha_1, \alpha_2 \in A$ and $\beta, \beta_1 \in B$.
Suppose $U = \alpha\beta$. Note that $\beta \ne \beta_1^{-1}$ as UV is not in the normal closure of A. We may assume that $\alpha_1 \ne \alpha_2^{-1}$ by Lemma 5.4. Since $\alpha_1\alpha_2 \ne 1$, either $\alpha = \alpha_1$ and $\beta = \beta_1$ or $\alpha = \alpha_2^{-1}$ and $\beta = \beta_1^{-1}$, as otherwise $\langle U,V\rangle$ is free. Since $\beta \ne \beta_1^{-1}$ we assume $\beta = \beta_1$ and $\alpha = \alpha_1$. If $c_1 = c_2 = c$, then G surjects onto $(B * C)/N((\beta c)^2)$, so is non-trivial by Theorem 5.3. Otherwise take $U' = c_2\alpha$ and $V' = \alpha_2 c_1\alpha$, and so $\langle U', V'\rangle$ is free. Hence G is non-trivial by Lemma 5.5.
The proof for the case where $U = (\alpha\beta)^{-1}$ is similar by symmetry.

Case 3. $\ell(U) = 3 = \ell(V)$. Since U and V both have odd length and $\{U,V\}$ is not Nielsen reduced, the initial and final letters of U, V must all come from the same factor – say A. Without loss of generality, we

may write $U = \alpha\beta\alpha_1$ and $V = \alpha_2\beta_1\alpha_3$, with $\alpha, \alpha_1, \alpha_2, \alpha_3 \in A$ and $\beta, \beta_1 \in B$.
As in Case 2, we have $\beta \neq \beta_1^{-1}$. Without loss of generality, there are two possibilities to consider. Either $\alpha = \alpha_2$ and $\beta = \beta_1$ or $\alpha = \alpha_2$ and $\alpha_1 = \alpha_3$. (Note that we can not have $\alpha = \alpha_3^{-1}$ and $\alpha_1 = \alpha_2^{-1}$, for otherwise w is contained in the normal closure of $B * C$).

In the first case, we take take $U' = \alpha_1 c_2 \alpha$ and $V' = \alpha_3 c_1 \alpha$. By Lemma 5.5, we can assume that $\langle U', V' \rangle$ is not free. Hence either $c_1 = c_2 = c$ or $\alpha_1 = \alpha_3$. In either case G maps onto

$$\frac{(B * C)}{N((c\beta)^2} \quad \text{or} \quad \frac{(A * B)}{N((\alpha\beta\alpha_1)^2)}$$

respectively. Hence G is non-trivial.

In the second case where $\alpha = \alpha_2$ and $\alpha_1 = \alpha_3$, we can replace B with its conjugate by α, and w by $W = c_1\beta\tilde{\alpha}c_2\beta_1\tilde{\alpha}$, where $\tilde{\alpha} = \alpha\alpha_1$. Since G is isomorphic to

$$G' = \frac{(A * \alpha^{-1}B\alpha * C)}{N(W)},$$

the result follows from Case 1. □

To complete the proof of Theorem 1.3, we need to consider the case where $\ell(U) = 2$ and $\ell(V) = 4$. Without loss of generality $U = (\alpha\beta)^{\pm 1}$ and $V = \alpha_1\beta_1\alpha_2\beta_2$ with $\alpha, \alpha_1, \alpha_2 \in A$ and $\beta, \beta_1, \beta_2 \in B$.
In this case we can prove the stronger result that each of $A * B, C$ embeds in G (the Freiheitssatz). To this end, standard arguments allow us to make the additional assumption that each of A, B, C is generated by the letters occurring in w.
Since $\langle U, V \rangle = C_p * K$, without loss of generality we have by Nielsen transformations that either

(1) $U = \alpha\beta$ and $V \in \{\alpha\beta\alpha\beta_2, \alpha\beta\alpha_2\beta\}$ with $\beta \neq \beta_2$, or
(2) $U = \beta^{-1}\alpha^{-1}$ and $V \in \{\alpha\beta\alpha\beta_2, \alpha\beta\alpha_2\beta\}$ with $\alpha \neq \alpha_2$.

Remark 5.7. In (1) and (2) above we gave two forms of V. If $V = \alpha\beta\alpha_2\beta$ we can replace U and V by $\beta U \beta^{-1}$ and $\beta V \beta^{-1}$ respectively (or equivalently replace C by $\beta^{-1}C\beta$) and interchange A and B to get the first form, $\alpha\beta\alpha\beta_2$.

In what follows we regard G as a one-relator product of $A * B$ and C. For convenience we let $U_1 = \alpha\beta$ and $U_2 = \beta^{-1}\alpha^{-1}$, so $U_2 = U_1^{-1}$. We use R to denote a relator in G which is a cyclically reduced word in $\{U, V\}$ and $\ell(R)$ denotes its length also as a word in $\{U, V\}$.

Definition 5.8. The *index* of R is the number of cyclic sub-words of the form $(UU)^{\pm 1}$, $(VV)^{\pm 1}$, $(VU^{-1})^{\pm 1}$ or $(U^{-1}V)^{\pm 1}$.

Definition 5.8 generalizes the notion of sign-index. Recall that the sign-index of R is 0 if R or R^{-1} is a positive word, and $n > 0$ (necessarily even) if a cyclic permutation of R has the form

$$W_1 W_2^{-1} W_3 \dots W_{n-1} W_n^{-1},$$

with each W_i a positive word in $\{U, V\}$. In particular the index of R is bounded below by its sign-index, and above by $\ell(R)$.

By Remark 5.7,

$$\{U, V\} = \{(\alpha\beta)^{\pm 1}, \alpha\beta\alpha\beta_2\} \xrightarrow{\text{Nielsen transformation}} \{\alpha\beta, \beta^{-1}\beta_2\}.$$

There are two possibilities to consider. If $\{\alpha\beta, \beta^{-1}\beta_2\}$ is Nielsen reduced, then $\langle U, V\rangle$ is free (if $\beta^{-1}\beta_2$ has infinite order), or $\mathbb{Z} * \mathbb{Z}_m$ (if $\beta^{-1}\beta_2$ has order m). A second possibility is that $\{\alpha\beta, \beta^{-1}\beta_2\}$ is not Nielsen reduced. In which case β is a power of $\beta^{-1}\beta_2$, so B is cyclic (generated by $\beta^{-1}\beta_2$). In particular it follows that $\beta^{-1}\beta_2$ must have order at least 3 (since by assumption $\beta \neq \beta_2$).

Proposition 5.9. *Suppose R is a non-empty, cyclically reduced word in $\{U, V\}$ of index k. Suppose also that $(\beta^{-1}\beta_2)^2 \neq 1$. If R is trivial in $A * B$, then $2k + \ell(R) \geq 12$.*

Proof. Note that $\langle U\rangle \cap \langle V\rangle = 1$ so no word of the form $U^m V^n$ is a relator. In particular $\ell(R) \geq 4$. Hence we can assume that $k \leq 3$, and so in particular we only need to consider words with sign-index 0 or 2.

First suppose that R has sign-index 0 – say R is positive. Since $\beta \neq \beta_2$, $\{U_1, V\}$ generates a free sub-semigroup of $A * B$ of rank 2. Hence we assume that $U = U_2$. If R is a counter-example then $\ell(R) < 12 - 2k$ where $k \leq 3$ is the number of cyclic subwords of the form U^2 or V^2. This leaves us with a short list of words that can be checked directly to show that none is trivial.

Suppose that R has sign-index 2. If $U = U_1$, we get an equality between two positive words in a free sub-semigroup. Since this can not happen, we assume that $U = U_2$. We can assume that R has at most a single occurrence of the subword $U^{\pm 2}$ or $V^{\pm 2}$. If there is such a subword then $\ell(R) < 6$ is odd so we may assume that $\ell(R) = 5$. Otherwise $\ell(R) < 8$ is even, so $\ell(R) \in \{4, 6\}$. Again there is a short list of candidate words R which can checked directly to show that none is trivial. ∎

Remark 5.10. Note that if $(\beta^{-1}\beta_2)^2 = 1$, then B can not be cyclic since that will imply that $\beta = \beta_2$. In other words $\{U, \beta^{-1}\beta_2\}$ is Nielsen reduced.

Theorem 5.11. *If $G = (A * B * C)/N(c_1Uc_2V)$ where $U = (\alpha\beta)^{\pm 1}$, $V = \alpha\beta\alpha\beta_2$, $\alpha \in A$, $\beta, \beta_1 \in B$, $c_1, c_2 \in C$ and $\{U, \beta^{-1}\beta_2\}$ is not Nielsen reduced in $A * B$, then each of $A * B, C$ embeds in G via the natural map.*

Proof. If C_0 is the subgroup of C generated by c_1 and c_2, then it suffices to show that $\langle U, V\rangle$ and C_0 embed into $G_0 := (\langle U, V\rangle * C_0)/N(c_1Uc_2V)$ via the natural map.
Note that $\langle U, V\rangle = \langle\alpha\rangle * \langle\beta^{-1}\beta_2\rangle$ is a free product of two non-trivial cyclic groups, so admits an irreducible faithful representation into $PSL_2(\mathbb{C})$. If c_1 and c_2 both have order 2, then (since $c_1c_2 \neq 1$) C_0 is a non-cyclic dihedral group, and so it also admits an irreducible faithful representation into $PSL_2(\mathbb{C})$. In this case the result follows from Theorem 5.1.
Thus we may assume that at most one of c_1, c_2 has order 2. Without loss of generality we assume that c_2 has order greater than 2.
Suppose by contradiction that the result fails, then there is a non-trivial $\mathcal{W}$-minimal spherical picture M over G_0, where $\mathcal{W}$ is the set of non-trivial elements of $\langle U, V\rangle \cup C_0$.

Assign angles to corners of a picture over G_0 as follows. Every c_1-corner gets angle 0, every c_2-corner gets angle $\pi/3$, and every U- and V-corner gets angle $5\pi/6$. This ensures that vertices have curvature 0, and C_0-regions have non-positive curvature. However, $\langle U, V\rangle$-regions can have positive curvature. We overcome this by redistributing any such positive curvature to neighbouring negatively curved C_0-regions, as follows.

Let Δ be an interior $\langle U, V\rangle$-region of M. We transfer $\pi/3$ of curvature across each of the edges of Δ joining a c_1-corner to c_2-corner of an adjacent C_0-region.

Now any interior $\langle U, V\rangle$-region whose label is of the form $(U_2U_2V)^n$ has label of index n and curvature at most $\pi/2$. However, $n > 2$ by Remark 5.10. Hence it has transferred at least π of curvature to neighbours, so becomes negatively curved. Similarly, it follows from Proposition 5.9 that any interior $(A * B)$-region whose label is not of the form $(U_2U_2V)^n$ has curvature at most π and that it has transferred at least π of curvature to neighbours, so it becomes non-positively curved as well.

A C-region Δ receives $\pi/3$ of positive curvature across each edge separating a c_1-corner from a c_2-corner. Suppose that in Δ there are p c_1-corners, q c_2-corners, and r edges separating a c_1-corner from a c_2-corner. The curvature of Δ after transfer is at most

$$2\pi - (p+q)\pi + \frac{q\pi}{3} + \frac{r\pi}{3} \leq \frac{(6-2p-q)\pi}{3}.$$

If $2p+q \geq 6$, then Δ still has non-positive curvature after transfer. Suppose Δ is an interior C-region and $2p+q \leq 5$. Then either $p=0$ or $q=0$ (since $p,q \neq 1$). Hence also $r=0$: so there is no transfer of curvature into Δ and it remains non-positively curved.

Since the curvature of the exceptional region is less than 4π, we get a contradiction that curvature of M is 4π. Hence each of $A*B, C$ embeds in G via the natural map. □

Theorem 5.12. *If $G = (A*B*C)/N(c_1Uc_2V)$ where $U = (\alpha\beta)^{\pm 1}$, $V = \alpha\beta\alpha\beta_2$, $\alpha \in A$, $\beta, \beta_1 \in B$, $c_1, c_2 \in C$ and $\{U, \beta^{-1}\beta_2\}$ is Nielsen reduced in $A*B$, then each of $A*B, C$ embeds in G via the natural map.*

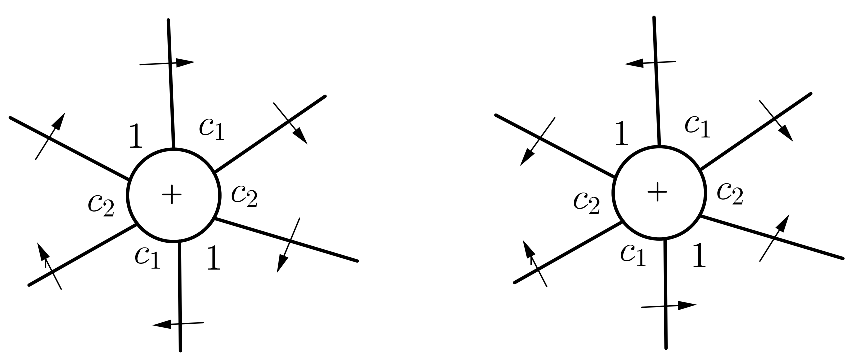

FIGURE 3. Positively oriented vertices of Γ when $n=2$. The figure on the left corresponds to a vertex of Γ when $r = (t^2c_1tc_2)^n$ and the other is when $r = (t^{-2}c_1tc_2)^n$.

Proof. By assumption $\langle U, V\rangle = \langle U\rangle * \langle \beta^{-1}\beta_2\rangle$, and is isomorphic to $\mathbb{Z} * \mathbb{Z}_n$, where $n > 1$ is the order of $\beta^{-1}\beta_2$. Consider the relative presentation

$$H = \langle C, t \mid r = (t^{\pm 2}c_1tc_2)^n\rangle.$$

The aim is to show that G is the free product of H and $A*B$ amalgamated over the subgroups $\langle U, V\rangle$ and $\langle t, c_1tc_2\rangle$. To this end we must show that the latter is also isomorphic to $\mathbb{Z}*\mathbb{Z}_n$. In other words, any relation in H which is a word in $\{t, c_1tc_2\}$ is a consequence of $(t^{\pm 2}c_1tc_2)^n$.

If this is not so, we obtain a $\mathcal{W}$-minimal non-trivial picture Γ over H on D^2 where $\mathcal{W}$ is the set of non-trivial words in $\{t, c_1tc_2\}$. Figure 3 shows typical vertices with positive orientation in the case of $n = 2$. Note that there are no 2-zones with corners labelled by $1^{\pm 1}$, for in such case, either the two vertices cancel, or we can combine the vertex with the boundary. In both cases we get a smaller picture, thereby contradicting minimality of Γ.

Make regions of Γ *flat* by assigning angle $(d(\Delta) - 2)\pi/d(\Delta)$ to each of the corners of a degree $d(\Delta)$ region Δ. We claim that interior vertices Γ are non-positively curved. The proof is in two stages depending on whether $r = (t^2c_1tc_2)^n$ or $r = (t^{-2}c_1tc_2)^n$.

Suppose that $r = (t^2c_1tc_2)^n$. Since r is a positive word, Γ is bipartite. More precisely, only vertices of opposite orientations are adjacent in Γ. In particular this implies that regions have even degrees. Every interior vertex v bounds at least two regions with a corner labelled 1. By minimality of Γ, every such region has degree at least 4. Also every 2-zone gives the relation $c_1 = c_2$, and so each of the two regions on both sides of the 2-zone has a corner labelled 1, hence is at least a 4-gon. Hence v bounds at least four regions of degree at least 4 and so is non-positively curved.

The case of $(t^{-2}c_1tc_2)^n$ is slightly different as regions can have odd degree. Note that any corner is either a *source* (the two arrows point outwards), *sink* (the two arrows point inwards), or *saddle* (one arrow points inwards and the other points outwards) depending on whether it is a c_1-, c_2- or 1-corner (see Figure 3). So in particular any 2-zone gives the relation $c_1^2 = 1$ or $c_2^2 = 1$. If an interior vertex v does not bound a 2-zone, then v satisfies $C(3n)$. Suppose it does. As before any region adjacent to a 2-zone has a 1-corner (a saddle). But any region must have an even number of saddles, no two of which are adjacent (for otherwise a cancellation would be possible). It follows that such a region has degree at least 4. There are at least two regions with 1-corner at v. If v bounds only one 2-zone, then it has degree at least 5 and bounds at least three regions of degree 4. Otherwise v bounds at least four 4-gons. In all cases v has non-positive curvature.

It follows that there exists a boundary vertex of degree at most 3. This is clearly impossible if $n > 2$ (since we will get a 2-zone with corners labelled 1). So we assume that $n = 2$. An argument similar to the ones given above shows that such a vertex must connect to ∂D^2 by an

ω-zone, with $\omega \geq 3$. It follows that either one of c_1 or c_2 is trivial or we can combine such a vertex with ∂D^2 to get a smaller picture. Both possibilities lead to a contradiction which completes the proof. $\square$

Proof of Theorem 1.3. Suppose that $G = (A*B*C)/N(w)$ with $\ell(w) \leq 9$

By earlier comments we can assume that w has the form $w = c_1 U c_2 V$ (up to cyclic permutation) where $U, V \in A * B$ with $2 \leq \ell(U) \leq \ell(V)$ and $\ell(U) + \ell(V) \leq 6$; and $c_1, c_2 \in C$ with $c_1 c_2 \neq 1$. It follows from Grushko's theorem that the subgroup of $A * B$ generated by U and V is isomorphic to one of the following:

(1) Conjugate to a subgroup of A (or B).
(2) Free group of rank one.
(3) Free group of rank two.
(4) Free product of two finite cyclic groups.
(5) Free product of finite and infinite cyclic groups.

In Case (1) the result is immediate, as w is belongs to the normal closure of $A \cup C$ or $B \cup C$.
In Cases (2) and (3) the result follows from Lemma 5.5.
Finally, Cases (4) and (5) follow from Lemma 5.6 together with Theorems 5.11 and 5.12. $\square$

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