Corrigendum - Analytic representation theory of Lie groups: general theory and analytic globalizations of Harish-Chandra modules

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CORRIGENDUM: "ANALYTIC REPRESENTATION THEORY OF LIE GROUPS: GENERAL THEORY AND ANALYTIC GLOBALIZATIONS OF HARISH–CHANDRA MODULES" [COMPOS. MATH. 147 (2011), 1581–1607]

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For a representation π of a connected Lie group G on a topological vector space E we defined in [1] a vector subspace E^ω of E of analytic vectors. Further we equipped E^ω with an inductive limit topology. We called a representation (π, E) *analytic* if $E = E^\omega$ as topological vector spaces.

Some mistakes in the paper have been pointed out by Helge Glöckner (see [2]). For a representation (π, E) and a closed G-invariant subspace F of E we asserted in Lemma 3.6 (i) that $F^\omega = E^\omega \cap F$ as a topological space. Based on that we further asserted in Lemma 3.6 (ii) that the inclusion $E^\omega/F^\omega \to (E/F)^\omega$ is continuous and in Lemma 3.11 that if (π, E) is analytic then so is the restriction to F. However, there is a gap in the proof of the first assertion, and presently it is not clear to us whether the above statements are then true in this generality (for unitary representations (π, E) they are straightforward). Our proof does give the following weaker version of the two lemmas:

Lemma 1. *Let (π, E) be a representation, and let $F \subset E$ be a closed invariant subspace. Then*

(i) *$F^\omega = E^\omega \cap F$ as vector spaces and with continuous inclusion $F^\omega \to E^\omega$.*
(ii) *$E^\omega/E^\omega \cap F \subset (E/F)^\omega$ continuously.*
(iii) *If (π, E) is an analytic representation, then π induces an analytic representation on E/F.*

Indeed, for (iii) note that if E is analytic, $E/F = E^\omega/E^\omega \cap F \subset (E/F)^\omega$ continuously by (ii), and $(E/F)^\omega \subset E/F$ continuously.

Further we asserted in Proposition 3.7 a general completeness property of the functor which associates E^ω to E. However, there is a gap in the proof, which asserts that $v_i \to v$ in the topology of E^ω. As statements in this generality are not needed for the main result, we can leave out the proposition (together with Remark 3.8).

Attached to G we introduced a certain analytic convolution algebra $\mathcal{A}(G)$. A central theme of the paper is the relation of analytic representations of G to algebra representations of $\mathcal{A}(G)$ on E: $\mathcal{A}(G) \times E \to E$. In Proposition 4.2 (ii) we claimed that the bilinear map $\mathcal{A}(G) \times \mathcal{A}(G) \to \mathcal{A}(G)$ is continuous. However, the proof shows only separate continuity. For a similar reason we need to weaken Proposition 4.6 to:

Proposition 2. *Let (π, E) be an F-representation. The assignment*

$$(f, v) \mapsto \Pi(f)v := \int_G f(g)\pi(g)v \, dg$$

defines a continuous bilinear map

$$\mathcal{A}_n(G) \times E \to E_n$$

for every $n \in \mathbb{N}$*, and a separately continuous map*

$$\mathcal{A}(G) \times E \to E^\omega$$

(with convergence of the defining integral in E^ω*). Moreover, if* (π, E) *is a Banach representation, then the latter bilinear map is continuous.*

Proof. The first statement is proved in the article, and thus only the statement for π a Banach representation remains to be proved. We repeat the first part of the proof, now with p denoting the fixed norm of E. The constants c, C such that

$$p(\pi(g)v) \le Ce^{cd(g)}p(v) \qquad (g \in G, v \in E)$$

and N, C_1 such that

$$C_1 := \int_G e^{(c-N)d(g)}\, dg < \infty,$$

are then all fixed, and so is $\epsilon = 1/(CC_1)$.

Let $n \in \mathbb{N}$ and an open 0-neighborhood $W_n \subset E_n$ be given. We may assume

$$W_n = \{v \in E_n \mid p(\pi(K_n)v) < \epsilon_n\}$$

with $K_n \subset GV_n$ compact and $\epsilon_n > 0$. Let

$$O_n := \{f \in \mathcal{O}(V_nG) \mid \sup_{z \in K_n, g \in G} |f(z^{-1}g)|e^{Nd(g)} < \epsilon\epsilon_n\} \subset \mathcal{A}_n(G).$$

The computation in the given proof shows that if $f \in O_n$ and $p(v) < 1$ then $\Pi(f)v \in W_n$. The asserted bi-continuity of $\mathcal{A}(G) \times E \to E^\omega$ follows. □

As a consequence we obtain as in Example 4.10(a), but only for Banach representations (π, E), that E^ω is $\mathcal{A}(G)$-tempered. In particular $\mathcal{A}(G)$ need not itself be $\mathcal{A}(G)$-tempered, and we need to replace Lemma 5.1(i) by the following weaker version:

Lemma 3. $V^{\min}$ *is an analytic globalization of* V *and it carries an algebra action*

$$(f, v) \mapsto \Pi(f)v, \qquad \mathcal{A}(G) \times V^{\min} \to V^{\min},$$

of $\mathcal{A}(G)$*, which is separately continuous.*

The main result of the paper Theorem 5.7, has two statements concerning a Harish-Chandra module V with a globalization E:

(1) *If* E *is analytic* $\mathcal{A}(G)$*-tempered then* $E = V^{\min}$

(2) *If E is an F-globalization then $E^\omega = V^{\min}$.*

The proof, which relied on Lemma 3.11 and Proposition 4.6 respectively, needs to be corrected. The proof of (1) if V is irreducible needs no modification. For the general case it can be adjusted as follows.

Like in the paper, it suffices to consider an exact sequence of Harish–Chandra modules $0 \to V_1 \to V \to V_2 \to 0$, where both V_1 and V_2 have unique analytic $\mathcal{A}(G)$-tempered globalizations. We show that the same holds for V.

Let E_1 be the closure of V_1 in E and $E_2 = E/E_1$. By Lemma 1(iii), E_2 is an analytic $\mathcal{A}(G)$-tempered globalization of V_2, so that by assumption $E_2 = V_2^{\min} = \mathcal{A}(G)V_2$ as topological vector spaces.

In a first step we prove that $E_1 = V_1^{\min} = \mathcal{A}(G)V_1$ as vector spaces. For that we note first that E_1 is $\mathcal{A}(G)$-tempered and that $V_1^{\min} \subset E_1$ continuously. Next, by Proposition 5.3 (which holds for any $\mathcal{A}(G)$-tempered representation), we may embed $E_1 \subset F_1$ continuously into a Banach globalization of F_1 of V_1. Moreover, the proof shows that the embedding is compatible with the action by $\mathcal{A}(G)$. It follows that $E_1^\omega \subset F_1^\omega$ continuously and as $\mathcal{A}(G)$-modules. Further note that since E is analytic, from Lemma 1(i) we also obtain $E_1^\omega = E^\omega \cap E_1 = E_1$ as vector spaces. Hence $V_1^{\min} \subset E_1 \subset F_1^\omega$. By assumption V_1 has unique $\mathcal{A}(G)$-tempered globalization, hence $F_1^\omega \simeq V_1^{\min}$. Therefore $V_1^{\min} \subset E_1 \subset F_1^\omega \simeq V_1^{\min}$. As these maps respect the structure as $\mathcal{A}(G)$–module, the inclusion is also surjective: $V_1^{\min} = E_1$.

Being an inductive limit, $E_1 = F_1^\omega$ is an ultrabornological space, and $V_1^{\min}$ is webbed (see the reference in the proof of Proposition 4.6). We conclude from the open mapping theorem that $V_1^{\min} = E_1$ also as topological vector spaces.

With Lemma 5.2 we now have a diagram of topological vector spaces

$$\begin{array}{ccccccccc} 0 & \to & V_1^{\min} & \to & V^{\min} & \to & V_2^{\min} & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & E_1 & \to & E & \to & E_2 & \to & 0 \end{array}$$

where the vertical arrow in the middle signifies the continuous inclusion $V^{\min} = \mathcal{A}(G)V \subset E$, and where the rows are exact. The five–lemma implies $V^{\min} = E$ as a vector space, and as in the article we conclude from [DS79] that this is then a topological identity.

Finally, for (2) we recall from Corollary 3.5 that $(E^\infty)^\omega = E^\omega$. The Casselman-Wallach smooth globalization theorem asserts the existence of a Banach globalization F of V such that $F^\infty = E^\infty$ and therefore $F^\omega = E^\omega$. In particular, E^ω is $\mathcal{A}(G)$-tempered by Proposition 2. Now (1) applies.

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