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The algebraic structure of the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy

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ABSTRACT

We prove that each member of the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy is a Fredholm Grassmannian flow, and for the given linear dispersion relation and corresponding equivalencing group of Fredholm transformations, is unique in the class of odd-polynomial partial differential fields. Thus each member is linearisable and integrable in the sense that time-evolving solutions can be generated by solving a linear Fredholm Marchenko equation, with the scattering data solving the corresponding linear dispersion equation. At each order, each member matches the corresponding non-commutative Lax hierarchy field which thus represent odd-polynomial partial differential fields. We also show that the cubic form for the non-commutative sine–Gordon equation corresponds to the first negative order case in the hierarchy, and establish the rest of the negative order non-commutative hierarchy. To achieve this, we construct an abstract combinatorial algebra, the Pöppe skew-algebra, that underlies the hierarchy. This algebra is the non-commutative polynomial algebra over the real line generated by compositions, endowed with the Pöppe product—the product rule for Hankel operators pioneered by Ch. Pöppe for classical integrable systems. Establishing the hierarchy members at non-negative orders, involves proving the existence of a 'Pöppe polynomial' expansion for basic compositions in terms of 'linear signature expansions' representing the derivatives of the underlying non-commutative field. The problem boils down to solving a linear algebraic equation for the polynomial expansion coefficients, at each order.

1. Introduction

For the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy, we prove that each member is both, a Fredholm Grassmannian flow, and for the given linear dispersion relation and corresponding equivalencing group of Fredholm transformations, is unique in the class of odd-polynomial partial differential fields. That each member represents a Fredholm Grassmannian flow means they are linearisable in the sense that solutions can be generated by solving a linear Fredholm Marchenko equation whose scattering data is the solution to the corresponding linear dispersion relation. We also show that each member of the Lax hierarchy generates an odd-polynomial partial differential field, and so by uniqueness, at each non-negative order, the Fredholm Grassmannian flow and Lax hierarchy member are one and the same. We also show that the cubic form for the non-commutative sine–Gordon equation corresponds to the first negative order case in the Lax hierarchy, and establish the rest of the negative order non-commutative hierarchy. Our approach is inspired by the pioneering work of Ch. Pöppe. In a sequence of papers, Pöppe [1–3], Pöppe and Sattinger [4], and Bauhardt and Pöppe [5] developed the fundamental product rule for additive Hankel operators and semi-additive operators, in order to establish the integrability and specific solution forms for classical integrable systems. These included for example, the scalar sine–Gordon equation and Kadomtsev–Petviashvili hierarchy. Closely related to this is also the work of Nijhoff et al. [6,7]. Pöppe's approach has recently been substantially developed and extended. In particular, Doikou, Malham and Stylianidis [8] streamlined and extended Pöppe's approach to the non-commutative Korteweg–de Vries and nonlinear Schrödinger equations, and Malham [9] extended the approach to the quartic-order, quintic-degree non-commutative nonlinear Schrödinger equation. Further, Malham [10] developed a simpler form of the Pöppe algebra constructed herein to prove that each member of the non-commutative potential Korteweg–de Vries hierarchy is unique in the class of polynomial partial differential fields and represents a Fredholm Grassmannian flow. Blower and Newsham [11] developed a systems perspective

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to Pöppe's approach, constructing tau-functions and families of solutions to the Kadomtsev–Petviashvili equation, while Blower and Doust [12] extend this approach to the sinh–Gordon equation.

Let us now briefly outline our approach herein. Consider a non-commutative nonlinear dispersive partial differential equation for $g = g(x;t)$ of order n of the form,

$$\partial_t g = -(\mathrm{i}\mathcal{I})^{n-1}\pi_n\big(g, \partial g, \partial^2\, g, \dots, \partial^{n-2} g, \partial^n g\big),$$

where $\partial = \partial_x$ and $x \in \mathbb{R}$. Here $\pi_n = \pi_n(\cdot)$ is a polynomial of its arguments—a polynomial partial differential field. It is linear in $\partial^n g$. The diagonal matrix $\mathcal{I}$ simply has top left block '$-$id' and bottom right block is 'id'. As we see presently, $g = [\![G]\!]$ is the square-integrable kernel of a Hilbert–Schmidt operator G. Herein we use the notation $[\![G]\!]$ to denote the kernel of an operator G. Suppose P is a Hilbert–Schmidt Hankel operator on the negative real axis satisfying the linear dispersive equation,

$$\partial_t P = -(\mathrm{i}\mathcal{I})^{n-1}\partial^n P.$$

Note if there are any linear terms in π_n, we should augment this equation for P with such linear terms on the right. In particular we suppose the square-integrable kernel of P has the form $p = p(y+z+x;t)$ for $y, z \in (-\infty, 0]$, while $x \in \mathbb{R}$ represents an additive parameter. The matrix-valued kernel p satisfies the same linear dispersive partial differential equation as that above for P; it represents the scattering data. The Marchenko equation at the operator level, here has the Fredholm form,

$$2\mathrm{i}P = G\,(\mathrm{id} + P^2),$$

for the unknown operator G. Provided $U := (\mathrm{id} + P^2)^{-1}$ exists as a Fredholm operator, then the solution Hilbert–Schmidt operator $G = 2\mathrm{i}PU$ to this Marchenko equation, parametrises a Fredholm Grassmannian flow of subspaces spanned by linear dispersive solutions p. See, for example, Doikou et al. [13]. Pöppe's insight was to recognise the crucial role the Hankel properties of P played in making the connection between the solution $G = \mathrm{i}PU$ to the Marchenko equation, and that its kernel $[\![G]\!]$ satisfies a specific nonlinear dispersive partial differential equation of the form shown above. The connection is made via the Pöppe kernel product rule,

$$[\![F\partial_x(HH')F']\!](y,z;x,t) = [\![FH]\!](y,0;x,t)[\![H'F']\!](0,y;x,t),$$

where H and H' are Hankel operators as described above, and F and F' are any two Hilbert–Schmidt operators. This rule indicates at the fundamental level, that there is a connection between the matrix products of kernels of operators of the form $G = 2\mathrm{i}PU$ and/or their derivatives (on the right), and kernels of monomials involving operator compositions of similar objects, but with one order higher derivative (on the left). Using that $\mathrm{id} + P^2 = (\mathrm{id} - \mathrm{i}P)(\mathrm{id} + \mathrm{i}P)$ and setting $V := (\mathrm{id} - \mathrm{i}P)^{-1}$, we see that,

$$G = 2V(\mathrm{i}P)V^\dagger \equiv V - V^\dagger.$$

Further, we observe that $\partial V = V\partial(\mathrm{i}P)V$, and if we use sub-indices to denote partial derivatives '∂', we find, $G_1 = V(\mathrm{i}P)_1V - V^\dagger(\mathrm{i}P)_1^\dagger V^\dagger$. In particular, if for any Hilbert–Schmidt operator F we set $[F] := [\![F - F^\dagger]\!]$, then we observe that the kernels in both these cases are given by,

$$[\![G]\!] = [V] \qquad \text{and} \qquad [\![G]\!]_1 = [V(\mathrm{i}P)_1V].$$

It is now easy to imagine that the nth partial derivative of $[\![G]\!]$ has the form,

$$[\![G]\!]_n = \sum \chi\big(a_1 \cdots a_n\big)\,\big[V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V\big],$$

where the sum is over all compositions $a_1a_2\cdots a_k$ of n. Naturally $\partial_t[\![G]\!]$ is given by $[V\partial_t(\mathrm{i}P)V]$ where $\partial_t(\mathrm{i}P)$ can be expressed in terms $\partial^n(\mathrm{i}P)$ using the linear dispersion equation for P. Hence our goal is to express $\partial_t[\![G]\!]$ in terms of a polynomial $\pi_n = \pi_n\big([\![G]\!], [\![G]\!]_1, \dots, [\![G]\!]_{n-2}, [\![G]\!]_n\big)$, linear in $[\![G]\!]_n$. The monomials in the polynomial π_n consist of factors of the form $[\![G]\!]$, $[\![G]\!]_1, \dots, [\![G]\!]_{n-2}$, each expressible as a linear combination of basis elements $[V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V]$ parametrised by compositions as shown above, with the product involved being the Pöppe kernel product. If we extend the basis elements to include basis elements of the form $[V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V]$ where any of the V factors shown my be replaced by $V^\dagger$, then the operator partial fractions formulae $V = \mathrm{id} + (\mathrm{i}P)V = \mathrm{id} + (\mathrm{i}P)V$ imply that the Pöppe product generates a closed algebra on such basis elements (see Lemma 6). The playing field is thus set. It is the algebra spanned by such basis elements equipped with the Pöppe product. In fact we use an abstract version of this algebra by stripping the basis elements of their P and V labels and focusing on the compositions $a_1a_2\cdots a_k$ and a binary encoding, $\mathbf{0}$ and $\mathbf{0}^\dagger$, of the intervening V or $V^\dagger$ factors. The Pöppe product essentially only acts on these components and thus transports our *playing field* to the closed algebra of basis elements $[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}\cdots\mathbf{0}a_k\mathbf{0}]$, where any of the $\mathbf{0}$'s may be replaced by $\mathbf{0}^\dagger$. The $[\![G]\!]_n = [V]_n$ are linear combinations of such abstract basis elements (as shown above) and we label such specific linear combinations by $[\boldsymbol{n}]$. The quantity $\partial_t[\![G]\!] = \partial_t[V]$ can ultimately be expressed in terms of $[\mathbf{0}n\mathbf{0}]$ or $[\mathbf{0}n\mathbf{0}^\dagger]$, respectively depending on whether n is odd or even. Indeed, set $\mathfrak{e}_n([\mathbf{0}]) := [\mathbf{0}n\mathbf{0}]$ when n is odd, and $\mathfrak{e}_n([\mathbf{0}]) := [\mathbf{0}n\mathbf{0}^\dagger]$ when n is even. The *game* is to determine if $\mathfrak{e}_n([\mathbf{0}])$ can be expressed in terms of a polynomial $\pi_n = \pi_n([\mathbf{0}], [\mathbf{1}], \dots, [\boldsymbol{n-2}], [\boldsymbol{n}])$, linear in $[\boldsymbol{n}]$. We express π_n as a linear combination of monomials with factors chosen from $[\mathbf{0}]$, $[\mathbf{1}], \dots, [\boldsymbol{n-2}]$ and a linear term $[\boldsymbol{n}]$. The monomials present in π_n are restricted to those that can generate $[\mathbf{0}n\mathbf{0}]$ or $[\mathbf{0}n\mathbf{0}^\dagger]$ or basis elements involving compositions $a_1a_2\cdots a_k$ of n via the Pöppe product. An unknown complex coefficient is associated with each monomial in the linear combination. We use the linear expansions for $[\mathbf{0}]$, $[\mathbf{1}], \dots, [\boldsymbol{n-2}]$ and linear term $[\boldsymbol{n}]$ in terms of the basis elements, and compute the Pöppe products of all the expansion basis elements in each of the factors of the monomial. The result is a large linear combination of basis elements, each with a factor which is a linear combination of the unknown coefficients. We equate this to $\mathfrak{e}_n([\mathbf{0}])$, and then equate all the coefficients of the basis elements present. This generates a large linear algebraic system of equations for the unknown coefficients. Though over-determined, it can be solved for a unique set of coefficients. Indeed our Main result is as follows.

Theorem 1 (*Main Result: Existence and Uniqueness*)**.** *For every $n \in \mathbb{N}\cup\{0\}$, there exists a unique Pöppe polynomial $\pi_n = \pi_n([\mathbf{0}], [\mathbf{1}], \dots, [\boldsymbol{n}])$ in $\mathbb{C}[\mathbb{Z}_\mathbf{0}]$ such that $\mathfrak{e}_n([\mathbf{0}]) = \pi_n$.*

Further, the non-commutative Lax hierarchy in our context, can be iteratively generated, to obtain the next equation in the hierarchy from the previous one by applying a specific operator to π_n. Given the existence of a match up to π_n, we show that applying this specific iterative operator to π_n generates $[\mathbf{0}(n+1)\mathbf{0}^\dagger]$ when n is odd, and generates $[\mathbf{0}(n+1)\mathbf{0}]$ when n is even. Since, from our main Theorem 1, we know there is a unique polynomial expansion for $[\mathbf{0}n\mathbf{0}]$ when n is odd or $[\mathbf{0}n\mathbf{0}^\dagger]$ when n is even for any order, there is a unique polynomial expansion for $[\mathbf{0}(n+1)\mathbf{0}^\dagger]$ or $[\mathbf{0}(n+1)\mathbf{0}]$. The uniqueness property means that the polynomials π_n we establish at each order, must match the Lax hierarchy members. Thus we have the following corollary of Theorem 1.

Corollary 1 (*The Non-Commutative Lax Hierarchy Is Unique*). *For any integer $n \geqslant 0$, the Lax hierarchy generated by the iteration indicated in Corollary 2 (see Section 5) is unique. The non-commutative nonlinear equation at order n in the Lax hierarchy is simply that represented by $\mathfrak{c}_n([\mathbf{0}]) = \pi_n$ in Theorem 1.*

We also investigate pursuing the specific operator in the opposite direction, to generate non-commutative Lax hierarchy members at all negative orders—see Tracy and Widom [14] for the scalar case. In particular we show that the first negative order case $n = -1$, corresponds to the cubic form of the non-commutative sine–Gordon equation.

The solution to any of the non-commutative hierarchy equations is generated by solving the linear dispersion equation for the matrix kernel p or equivalently the Hilbert–Schmidt Hankel operator P, and then solving the linear Fredholm Marchenko equation $2\mathrm{i}P = G\,(\mathrm{id} + P^2)$ for G. The hierarchy member solution is the kernel $[\![G]\!]$. Each member of the hierarchy is thus linearisable as the solution is generated via solving a linear dispersion equation and a linear Fredholm equation. However this procedure also identifies the graph of G as a Fredholm Grassmannian flow, represented in a specific coordinate patch parametrised by Hilbert–Schmidt operators. Such flows are explored in detail in Doikou et al. [13]. We can think of the Fredholm Grassmannian as all collections of graphs of compatible linear Hilbert–Schmidt maps. Indeed briefly, suppose we set $\mathbb{V} := L^2((-\infty, 0]; \mathbb{C}^m)$ for some $m \in \mathbb{N}$, for example. Consider the column pair of operators,

$$\begin{pmatrix} \mathrm{id} - Q \\ 2\mathrm{i}P \end{pmatrix},$$

both on $\mathbb{V}$. Here we suppose $\mathrm{id} - Q$ is a Fredholm operator on $\mathbb{V}$, with Q a Hilbert–Schmidt operator, and P is a Hilbert–Schmidt operator on $\mathbb{V}$. Assuming that the regularised determinant $\det_2(\mathrm{id} - Q) \neq 0$, this pair of operators defines a Fredholm Grassmannian flow in a given coordinate patch as follows. We can think of this pair of operators as spanning a subspace of $\mathbb{H} := \mathbb{V}\times\mathbb{V}$ that is isomorphic to $\mathbb{V}$. The transformation $(\mathrm{id} - Q)^{-1}$ of this subspace generates,

$$\begin{pmatrix} \mathrm{id} \\ G \end{pmatrix},$$

where $G = 2\mathrm{i}P(\mathrm{id} - Q)^{-1}$. We can think of the Hilbert–Schmidt operator G as parametrising all such subspaces of $\mathbb{H}$, that can be projected onto the canonical subspace represented by the column pair of operators $(\mathrm{id}; O)$. This is one coordinate patch of the Fredholm Grassmannian of such subspaces of $\mathbb{H}$. Note that if we set $Q = -P^2$, then G represents the solution to the Marchenko equation. Recall, in our application we assume P satisfies the dispersion equation $\partial_t P = -(\mathrm{i}\mathcal{I})^{n-1}\partial^n P$. Further we suppose that P is a Hankel operator. This property is a natural far-field symmetry for the dispersive field in the sense that it is a natural symmetry arising as the result of the scattering, of an incident wave from one far field, into the opposite far field. See for example the construction of the Marchenko equation in Drazin and Johnson [15] or Appendix B in Doikou et al. [13]. That the Marchenko equation solution G parametrises a class of subspaces of $\mathbb{H}$ characterised by solutions of a dispersive field P, generates the following perspective. We can think of the Fredholm Grassmannian, in the coordinate patch represented by the particular pair $(\mathrm{id}; G)$, as parametrising the time-evolving *envelope* of dispersive field solutions, i.e. the time-evolving subspace represented by the pair $(\mathrm{id}; G)$. In principle we could consider $[\![G]\!] = [\![G]\!](y, z; x, t)$ or in particular $[\![G]\!](0, 0; x, t)$ as an observable.

It was Sato [16,17] and Segal and Wilson [18] who pioneered the connection between Fredholm Grassmannians and integrable systems, and more recent research on this includes, Mulase [19], Dupré et al. [20–22], Kasman [23,24], Hamanaka and Toda [25], Cafasso [26], Cafasso and Wu [27] and Arthamonov et al. [28]. Related to this is the well-studied connection between the Korteweg–de Vries hierarchy, the intersection theory of Deligne–Mumford moduli space and the string equation in two-dimensional gravity; see for example Witten [29,30] and Cafasso and Wu [27]. There has been recent interest in non-commutative systems and their solutions, see Treves [31,32], Hamanaka and Toda [25], Degasperis and Lombardo [33], Dimakis and Müller–Hoissen [34], Carillo and Schiebold [35–37], Gürses and Pekcan [38], Ma [39], Fu [40] and Fu and Nijhoff [41]. The role of Hankel operators in integrable systems has also recently re-emerged as an active and fruitful research direction, see Blower and Newsham [11], Blower and Doust [12], Grudsky and Rybkin [42–44], Grellier and Gerard [45] and Gerard and Pushnitski [46].

Our paper is organised as follows. In Section 2 we introduce the Pöppe product for Hankel operators and motivate the solution form we propose. We introduce the Pöppe algebra in Section 3 with the Pöppe product, and in particular its isomorphic abstract form as well as the skew-Pöppe subalgebra we use for the proofs of our main results. We present some examples in Section 4 illustrating the use of the abstract Pöppe algebra. In Section 5 we establish the non-commutative Lax hierarchy using the Pöppe algebra. Herein, we also generate the cubic form of the non-commutative sine–Gordon equation as the first negative order case. In Section 6 we prove our main results. Finally, in Section 7 we present some further conclusions and applications.

2. Hankel operators, the Pöppe product and the Marchenko equation

In this section we introduce the concepts and results that underlie our formulation. Herein, we: introduce the necessary Hilbert–Schmidt and Hankel operators we use and define the Pöppe product; motivate the solution ansatz we use throughout; formulate the base linear dispersion equation; elucidate the well-posedness results for the Marchenko equation we require and establish the connection between the Pöppe product and finite rank operators.

2.1. Hankel operators and the Pöppe product

To begin, let us fix some notation. Let $\mathbb{V}$ be the Hilbert space of square-integrable, complex matrix-valued functions on $(-\infty,0]$, i.e. $\mathbb{V} := L^2((-\infty,0];\mathbb{C}^m)$ for some $m\in\mathbb{N}$. Further, we denote by $\mathfrak{J}_2(\mathbb{V})$ the space of Hilbert–Schmidt operators on $\mathbb{V}$, i.e. bounded operators whose sum of the squares of their singular values is finite. For any given operator $F = F(x,t)\in\mathfrak{J}_2(\mathbb{V})$ there exists a unique square-integrable kernel $f = f(y,z;x,t)$ with $f\in L^2((-\infty,0]^{\times 2};\mathbb{C}^{m\times m})$ such that for any $\phi\in\mathbb{V}$ we have

$$(F\phi)(y;x,t) = \int_{-\infty}^{0} f(y,z;x,t)\phi(z)\,\mathrm{d}z.$$

Conversely, any such function $f\in L^2((-\infty,0]^{\times 2};\mathbb{C}^{m\times m})$ defines an operator $F = F(x,t)$ in $\mathfrak{J}_2(\mathbb{V})$ with (for each x,t): $\|F\|_{\mathfrak{J}_2(\mathbb{V})} = \|f\|_{L^2((-\infty,0]^{\times 2};\mathbb{C}^{m\times m})}$. See for example Simon [47, p. 23].

Definition 1 (*Kernel Bracket*)**.** For any Hilbert–Schmidt operator $F = F(x,t)$, which depends on the parameters $x\in\mathbb{R}$ and $t\geqslant 0$, we use the *kernel bracket* notation $[\![F]\!]$ to refer to the kernel $f = f(y,z;x,t)$ of F:

$$[\![F]\!](y,z;x,t) := f(y,z;x,t).$$

In general, since f is square-integrable, it only exists almost everywhere on $(-\infty,0]^{\times 2}$. However below, the operators we consider have continuous kernels and so f makes sense pointwise. In such cases, we can in particular set $y=z=0$, for which we use the notation $[\![F]\!]_{0,0}(x,t) := f(0,0;x,t)$.

Recall, the *trace* of any trace-class operator F on $(-\infty,0]$ is, $\mathrm{tr}\,F := \int_{-\infty}^{0} f(z,z)\,\mathrm{d}z$.

By a Hankel operator, which may depend on a parameter x, we mean the following.

Definition 2 (*Hankel Operator with Parameters*)**.** We say a Hilbert–Schmidt operator $H\in\mathfrak{J}_2(\mathbb{V})$ with corresponding square-integrable kernel h is *Hankel* or *additive* with parameter $x\in\mathbb{R}$ if its operation, for any square-integrable function $\phi\in\mathbb{V}$, is given by (here $y\in(-\infty,0]$),

$$(H\phi)(y;x) := \int_{-\infty}^{0} h(y+z+x)\phi(z)\,\mathrm{d}z.$$

Pöppe [1,2] recognised the fundamental role played by such Hankel operators in classical integrable systems. The kernel of the derivative with respect to the additive parameter x of the operator product of an arbitrary pair of Hankel operators can be expressed as the matrix product of their respective kernels as follows.

Lemma 1 (*Pöppe Product*)**.** *Assume H and H' are Hankel Hilbert–Schmidt operators with parameter x and F and F' are Hilbert–Schmidt operators. Further assume the kernels of F and F' are continuous, whilst the kernels of H and H' are continuously differentiable. Then the following Pöppe product rule holds,*

$$\big[\!\big[F\partial_x(HH')F'\big]\!\big](y,z;x) = [\![FH]\!](y,0;x)[\![H'F']\!](0,z;x).$$

The term on the left is given by a triple integral of the kernels of the four operators shown. The Hankel property of the middle pair means that the x-derivative can be transferred to a derivative with respect to the middle integral variable. Using the Fundamental Theorem of Calculus then establishes the result. See Pöppe [1,2] and Malham [10].

Remark 1. We implicitly interpret kernel products written in the form $[\![\,\cdot\,]\!][\![\,\cdot\,]\!]\cdots[\![\,\cdot\,]\!][\![\,\cdot\,]\!]$ as $[\![\,\cdot\,]\!](y,0;x)[\![\,\cdot\,]\!](0,0;x)\cdots[\![\,\cdot\,]\!](0,0;x)$ $[\![\,\cdot\,]\!](0,z;x)$.

2.2. Solution ansatz motivation

Let us formally motivate the solution form we use for the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy we study herein. This comes from the sine–Gordon equation. With $x\in\mathbb{R}$ and $t\geqslant 0$, we assume the sine–Gordon equation has the form, $\partial_t\partial u = \sin u$, where $u = u(x,t)$ and $\partial := \partial_x$. In the scalar case, when $u\in\mathbb{R}$, it is well-known that there exists a solution of the form,

$$u = 2\mathrm{i}\,\mathrm{tr}\,\log\left(\frac{\mathrm{id}-\mathrm{i}P}{\mathrm{id}+\mathrm{i}P}\right),$$

or the equivalent form $u = -4\,\mathrm{tr}\arctan P$. Here $P = P(x,t)$ is a Hankel Hilbert–Schmidt operator with an integral kernel $p = p(x,t)$ which satisfies the linearised form of the sine–Gordon equation (see for example Pöppe [1, Cor. 3.2]), $\partial_t\partial p = p$. For scalar valued kernels, we have the following. Suppose that $H = H(x)$ is a Hankel operator dependent on the parameter $x\in\mathbb{R}$. Then for any $n\in\mathbb{N}$, we have,

$$\partial\,\mathrm{tr}\,H^n = \frac{n}{2}[\![H^n]\!]_{0,0}.$$

Further, suppose that $\Theta = \Theta(H)$ is a power series function of the Hankel operator H, with scalar-valued coefficients c_n, of the form $\Theta(H) = \sum_{n\geqslant 1} c_n H^n$. Then we have,

$$\partial\,\mathrm{tr}\,\Theta(H) = \tfrac{1}{2}[\![H\,D\Theta(H)]\!]_{0,0},$$

where $D\Theta = D\Theta(H)$ is the series $D\Theta(H) = \sum_{n\geqslant 1} nc_n H^{n-1}$.

Remark 2. This is equivalent to the result embodied in equation (3.26) in Pöppe [1]. We give a proof in Proposition 2 in Section 2.5 below. Also see Blower and Doust [12].

Example 1 (*Logarithm of the Cayley Transform*). The solution ansatz for the scalar sine–Gordon equation above involves the logarithm of Cayley transform, i.e. the form,

$$\Theta(P) = \log\left(\frac{\mathrm{id} - \mathrm{i}P}{\mathrm{id} + \mathrm{i}P}\right).$$

By direct computation we observe that,

$$D\Theta(P) = -\frac{\mathrm{i}\cdot\mathrm{id}}{\mathrm{id} - \mathrm{i}P} - \frac{\mathrm{i}\cdot\mathrm{id}}{\mathrm{id} + \mathrm{i}P} \quad \Rightarrow \quad P\, D\Theta(P) = -\frac{2\mathrm{i}P}{(\mathrm{id} - \mathrm{i}P)(\mathrm{id} + \mathrm{i}P)}.$$

We deduce, $\partial\,\mathrm{tr}\,\Theta(P) = -[\![(\mathrm{id} - \mathrm{i}P)^{-1}(\mathrm{i}P)(\mathrm{id} + \mathrm{i}P)^{-1}]\!]_{0,0}$, though the order of the factors shown on the right is not important.

Recall the solution form to the scalar sine–Gordon equation given above, $u = 2\mathrm{i}\,\mathrm{tr}\,\Theta(P)$, where $\Theta = \Theta(P)$ is the logarithm of the Cayley transform given in Example 1. Let $\partial^{-1} := \partial_x^{-1}$ denote the operator, $\big(\partial^{-1}\phi\big)(x) := \int_{-\infty}^{x} \phi(\xi)\,\mathrm{d}\xi$. Then, given the final result in Example 1, the solution to the scalar sine–Gordon equation has the form,

$$u = -2\mathrm{i}\,\partial^{-1}\big[\!\big[(\mathrm{id} - \mathrm{i}P)^{-1}(\mathrm{i}P)(\mathrm{id} + \mathrm{i}P)^{-1}\big]\!\big]_{0,0}.$$

This form of the solution for the scalar sine–Gordon equation motivates the solution form we seek for the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy, which we utilise in the following sections. The sine–Gordon equation is just a special case, the order '−1' case, in that hierarchy.

Example 2 (*Non-commutative sine–Gordon Cubic-form Equation*). If u satisfies the scalar sine–Gordon equation $\partial_t\partial u = \sin u$, and $u = -2\mathrm{i}\partial^{-1}g$, then $g = g(x,t)$ satisfies the following sine–Gordon cubic-form equation,

$$\partial_t\partial g = g + g\,\partial^{-1}(\partial_t g^2) + \partial^{-1}(\partial_t g^2)\,g.$$

To see this, define the operator Γ by, $\big(\Gamma\phi\big)(x) := \int_{-\infty}^{x} \gamma(\xi)\phi(\xi)\,\mathrm{d}\xi$, where $\gamma = -2\mathrm{i}g$. Using that for any $n \in \mathbb{N}$ we have, $(\Gamma\circ 1)^n \equiv n!\,(\Gamma^n\circ 1)$, then we observe that in fact, $\sin u = (\mathrm{id} + \Gamma^2)^{-1}\circ\Gamma\circ 1$. In other words γ satisfies the integral equation $(\mathrm{id} + \Gamma^2)\circ\partial_t\gamma = \Gamma\circ 1$ or equivalently satisfies $\partial_t\gamma + \partial^{-1}\big(\gamma\partial^{-1}(\gamma\partial_t\gamma)\big) = \partial^{-1}\gamma$. Noting that $g = \gamma/(-2\mathrm{i})$ and symmetrically splitting the nonlinear term, generates the sine–Gordon cubic-form equation above. The cubic-form equation above is often interpreted to be the non-commutative sine–Gordon equation in, for example, Schiebold [48, Prop. 6.2].

2.3. The linear dispersion equation

Consider the following coupled linear system of equations for the Hilbert–Schmidt operators P_α, P_β, G_α and G_β,

$$\begin{aligned} \partial_t P_\alpha &= \mu_n\partial^n P_\alpha, \\ \mathrm{i}P_\alpha &= G_\alpha(\mathrm{id} + P_\beta P_\alpha), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \partial_t P_\beta &= (-1)^{n-1}\mu_n\partial^n P_\beta, \\ \mathrm{i}P_\beta &= G_\beta(\mathrm{id} + P_\alpha P_\beta). \end{aligned}$$

for some order $n \in \mathbb{Z}$, where the parameter $\mu_n \in \mathbb{C}$. In order for the partial differential equations for P_α and P_β shown to be dispersive, we necessarily require that μ_n is pure imaginary when n is even and real when n is odd. We suppose the matrix-valued kernel of P_β has the same shape as the transpose of the matrix-valued kernel of P_α. The matrix-valued kernels of G_α and G_β have the same size and shape as those of P_α and P_β, respectively. If we set,

$$P := \begin{pmatrix} O & P_\beta \\ P_\alpha & O \end{pmatrix}, \qquad G := \begin{pmatrix} O & G_\beta \\ G_\alpha & O \end{pmatrix}, \qquad \text{and} \qquad \mathcal{I} := \begin{pmatrix} -\mathrm{id} & O \\ O & \mathrm{id} \end{pmatrix},$$

then the system of linear equations above can be expressed in the form,

$$\begin{aligned} \partial_t P &= -\mu_n(\mathrm{i}\mathcal{I})^{n-1}\partial^n P, \\ \mathrm{i}P &= G(\mathrm{id} + P^2), \end{aligned}$$

where now the parameter $\mu_n \in \mathbb{R}$. This form is given, eg., in Schiebold [48, p.679–80]. Now consider the following second order cubic nonlinear equation for the kernel $[\![G]\!]$,

$$\partial_t[\![G]\!](y,z;x,t) = -\mu_2\mathrm{i}\mathcal{I}\big(\partial_x^2[\![G]\!](y,z;x,t) - 2\,[\![G]\!](y,0;x,t)[\![G]\!](0,0;x,t)[\![G]\!](0,z;x,t)\big).$$

Written in terms of the kernels $[\![G_\alpha]\!]$ and $[\![G_\beta]\!]$ with $y = z = 0$, we observe,

$$\begin{aligned} \mathrm{i}\partial_t[\![G_\alpha]\!] &= \mu_2\partial_x^2[\![G_\alpha]\!] - 2\mu_2\,[\![G_\alpha]\!]\,[\![G_\beta]\!]\,[\![G_\alpha]\!], \\ \mathrm{i}\partial_t[\![G_\beta]\!] &= -\mu_2\partial_x^2[\![G_\beta]\!] + 2\mu_2\,[\![G_\beta]\!]\,[\![G_\alpha]\!]\,[\![G_\beta]\!]. \end{aligned}$$

There are several different consistent choices we can make for P_α and P_β. For example, if we set $P_\beta = P_\alpha^\dagger$, the adjoint operator to P_α with respect to the L^2 inner product. Then if $G = \mathrm{i}PU$ with $U := (\mathrm{id} + P^2)^{-1}$, as defined above, at the block level it transpires $G_\beta = G_\alpha^\dagger$. In this case the kernel $[\![G_\beta]\!](0,0;x,t)$ is the complex conjugate transpose of the kernel $[\![G_\alpha]\!](0,0;x,t)$. And thus, assuming the kernel $[\![G]\!]$ generated from $G = PU$ satisfies the equation above, the equation for the block $[\![G_\alpha]\!](0,0;x,t)$ collapses to the non-commutative nonlinear Schrödinger equation. Further, for the choice $P_\beta = P_\alpha^\dagger$, the operator P is Hermitian with respect to the L^2 inner product, i.e. $P^\dagger = P$.

2.4. The Marchenko equation

Consider an operator $P \in \mathfrak{J}_N(\mathbb{V})$, where $N = 1$ or $N = 2$. For the moment P is not necessarily a Hankel operator, and $\mathbb{V}$ is an arbitrary separable Hilbert space. The space $\mathfrak{J}_1(\mathbb{V})$ denotes the set of trace-class (nuclear) operators. Crucial to the Pöppe algebra we introduce in Section 3 are both, the Marchenko equation,

$$P = G\,(\mathrm{id} - Q),$$

for the operator G, and the Pöppe product in Lemma 1. In our application, we set $Q := -P^2$. The following abstract result is proved in Doikou et al. [13, Lemma 1].

Lemma 2 (*Existence and Uniqueness; Doikou et al. [13]*). *Assume $Q_0 \in \mathfrak{J}_2$ and for some $T > 0$ we know that $Q \in C^\infty\big([0,T];\mathfrak{J}_2\big)$ with $Q(0) = Q_0$ and $P \in C^\infty\big([0,T];\mathfrak{J}_N\big)$, where N is 1 or 2. Further assume, $\det_2(\mathrm{id} - Q_0) \neq 0$. Then there exists a $T' > 0$ with $T' \leqslant T$ such that for $t \in [0,T']$ we have $\det_2(\mathrm{id} - Q(t)) \neq 0$ and there exists a unique solution $G \in C^\infty\big([0,T'];\mathfrak{J}_N\big)$ to the linear equation $P = G(\mathrm{id} - Q)$.*

Now suppose $\mathbb{V} := L^2((-\infty,0];\mathbb{C}^m)$ for some $m \in \mathbb{N}$. For any function $w \geqslant 0$, we denote the weighted L^2-norm of any complex matrix-valued function f on $(-\infty,0]$ by,

$$\|f\|^2_{L^2_w} := \int_{-\infty}^0 \mathrm{tr}\,\big(f^\dagger(x)f(x)\big)\,w(x)\,\mathrm{d}x.$$

Doikou et al. [13, Lemma 3] also establish, if $p(\cdot;t) \in L^2_w((-\infty,0])$ with $w\colon y \mapsto (1-y)^2$, the Hankel operator $P = P(t)$ generated by p is such that $P(t) \in \mathfrak{J}_2(\mathbb{V})$. Hence we *assume* the solutions $p = p(y;t)$ to the linear dispersive system $\partial_t p = -\mu_n (\mathrm{i}\mathcal{I})^{n-1}\partial_y^n p$ lie in $L^2_w((-\infty,0])$. We then take $P = P(x,t)$ to be the Hankel operator with kernel $p = p(y+z+x;t)$, where $y,x \in (-\infty,0]^{\times 2}$, with parameter $x \in \mathbb{R}$. Statements for $p = p(\cdot;t)$ on $(-\infty,0]$ translate, for each $x \in \mathbb{R}$, to statements for $p = p(\cdot + x;t)$ on $(-\infty,x]$. This is important, as we wish to include natural solutions $p = p(y;t)$ to the linear dispersion equation that are unbounded as $y \to \infty$. Examples of such solutions are exponential-form solutions that generate soliton solutions to the corresponding non-commutative integrable nonlinear partial differential equation. Explicitly, the Marchenko equation we consider herein takes the form,

$$p(y+z+x;t) = g(y,z;x,t) - \int_{-\infty}^0 g(y,\xi;x,t)q(\xi,z;x,t)\,\mathrm{d}\xi,$$

where q is the kernel of $Q := -P^2$. With this in hand, we have the following result, adapted from Doikou et al. [13, Lemma 6].

Lemma 3 (*Existence and Uniqueness: Marchenko Equation*). *Assume the smooth initial data $p_0 = p_0(\cdot)$ for $p = p(\cdot;t)$ is such that $\det_2(\mathrm{id} - Q_0) \neq 0$, where $Q_0 := -P_0^2$ and P_0 is the Hankel operator generated by p_0. Further assume there exists a $T > 0$ such that there is a solution,*

$$p \in C^\infty\big([0,T]; L^2_w((-\infty,0];\mathbb{C}^{m\times m}) \cap C^\infty((-\infty,0];\mathbb{C}^{m\times m})\big),$$

to the linear dispersion equation $\partial_t p = -\mu_n(\mathrm{i}\mathcal{I})^{n-1}\partial_y^n p$, where $w\colon y \mapsto (1-y)^2$. Then there exists a $T' > 0$ with $T' \leqslant T$ such that for $t \in [0,T']$ we know:

(i) The Hankel operator $P = P(x,t)$ with parameter $x \in \mathbb{R}$ generated by p is Hilbert–Schmidt valued on $\mathbb{V}$;
(ii) The determinant $\det_2(\mathrm{id} - Q(x,t)) \neq 0$ where $Q(x,t) := -P^2(x,t)$, and hence
(iii) There is a unique Hilbert–Schmidt valued solution $G = G(x,t)$ with $G \in C^\infty([0,T'];\mathfrak{J}_2(\mathbb{V}))$ to the linear Fredholm equation $P = G(\mathrm{id} - Q)$.

2.5. Trace formulae and finite-rank operators

We now establish that at the core of the Pöppe product is in fact a finite rank operator. For this section only, for convenience, we assume the domain of support of the functions under consideration is $[0,\infty)$ as opposed to $(-\infty,0]$. A reflection transformation translates between the two. For a bounded integral operator $K: L^2(0,\infty) \to L^2(0,\infty)$ with a continuous kernel $k = k(y,z)$, we write $[\![K]\!]_{0,0} = k(0,0)$ for the kernel bracket.

Definition 3 (*Shift Operator*). We define the shift operator $S_\eta : L^2(0,\infty) \to L^2(0,\infty)$ by $S_\eta f(x) = f(x-\eta)\,\mathrm{ind}_{(0,\infty)}(x-\eta)$, where $\mathrm{ind}_{(0,\infty)}$ is the indicator function on $(0,\infty)$.

It is well-known that S_η is a linear isometry and $(S_\eta)_{\eta>0}$ is a strongly continuous contraction semigroup. Further, the adjoint $S_\eta^\dagger$ is a linear contraction, and $(S_\eta^\dagger)_{\eta>0}$ is a strongly continuous contraction semigroup on L^2.

Proposition 1. *Set $\sigma_\eta(K) := S_\eta^\dagger K S_\eta$ for $K \in \mathfrak{J}(L^2(0,\infty))$. Then we have:*

(i) $\sigma_\eta(K) \in \mathfrak{J}(L^2(0,\infty))$ for all $K \in \mathfrak{J}(L^2(0,\infty))$ with $\|\sigma_\eta(K)\| \leqslant \|K\|$, $K \mapsto \sigma_\eta(K)$ is linear and $\sigma_{\eta+\xi} = \sigma_\eta(\sigma_\xi(K))$;
(ii) $K = K^\dagger$ implies $(\sigma_\eta(K))^\dagger = \sigma_\eta(K)$ and $K \geqslant 0$ implies $\sigma_\eta(K) \geqslant 0$;
(iii) $(S_\eta)_{\eta>0}$ gives a strongly continuous contraction semigroup on the von Neumann–Schatten ideal $\mathfrak{J}_p$ for $1 \leqslant p < \infty$ and on the space of compact operators. Also $\sigma_\eta(K) \to K$ as $\eta \to 0$ for such K;
(iv) Let δ be the generator of the semigroup in (iii), so $\sigma_\eta = \exp(t\,\delta)$. Then for continuously differentiable kernels $k = k(y,z)$ we have, $\delta k(y,z) = (\partial_y + \partial_z)k(y,z)$;
(v) Suppose that K has a continuous kernel $k = k(y,z)$, and that K is self-adjoint, non-negative and trace class. Then, $\det(\mathrm{id} + \sigma_\eta(K)) = \det(\mathrm{id} + K\mathrm{Pr}_{(\eta,\infty)})$ and,

$$[\![\sigma_\eta(K)]\!]_{0,0} = k(\eta,\eta) = -\frac{\mathrm{d}}{\mathrm{d}\eta}\mathrm{tr}\,\sigma_\eta(K);$$

(vi) Pöppe's bracket operation satisfies, $\mathrm{tr}\,\delta K = -[\![K]\!]_{0,0}$.

Proof. (i) Follows since (S_η) is a contraction semigroup, while (ii) is straightforward. (iii) The Schatten class gives an operator ideal, so we have $\|S_\eta^\dagger K S_\eta\|_{\mathfrak{J}_p} \leqslant \|K\|_{\mathfrak{J}_p}$ since $\|S_\eta\|_{\mathfrak{J}} = 1$. In view of this, we only need to check continuity in the relevant norm. For the Hilbert–Schmidt operators $\mathfrak{J}_2$, we let $\sigma_\eta(K)(y,z)$ be the kernel of $\sigma_\eta(K)$ as an integral operator. Then we have $\sigma_\eta(K)(y,z) = K(y+\eta, z+\eta)$, so by the Hilbert–Schmidt theorem,

$$\|\sigma_\eta(K) - K\|_{\mathfrak{J}_2}^2 = \int_0^\infty \int_0^\infty \|k(y+\eta, z+\eta) - k(y,z)\|^2 \,\mathrm{d}y\mathrm{d}z,$$

which converges to 0 as $\eta \to 0^+$. For the trace class operators $\mathfrak{J}_1$, we observe that the space of trace class operators may be identified with the projective tensor product $\mathfrak{J}_1 = L^2 \hat{\otimes} L^2$, so we have a nuclear expansion,

$$k(y,z) = \sum_{j=1}^\infty f_j(y) g_j(z),$$

where $\sum_{j=1}^\infty \|f_j\|_{L^2}\|g_j\|_{L^2} = \|K\|_{\mathfrak{J}_1}$. Then we have

$$\sigma_\eta(K)(y,z) - k(y,z) = \sum_{j=1}^\infty \big(f_j(y+\eta) - f_j(y)\big) g_j(z+\eta) + \sum_{j=1}^\infty f_j(y)\big(g_j(z+\eta) - g_j(y)\big),$$

and so,

$$\|\sigma_\eta(K) - K\|_{\mathfrak{J}_1} \leqslant \sum_{j=1}^\infty \|S_\eta^\dagger(f_j) - f_j\|_{L^2}\|g_j\|_{L^2} + \sum_{j=1}^\infty \|f_j\|_{L^2}\|S_\eta(g_j) - g_j\|_{L^2},$$

where the right-hand side converges to 0 as $\eta \to 0^+$ by dominated convergence.

For $1 < p < \infty$, we observe that the finite-rank operators give a dense linear subspace of $\mathfrak{J}_p$, so we can argue as with the trace class operators. Likewise, the finite-rank operators give a dense linear subspace of the space of compact operators. Hence $(\sigma_\eta)_{\eta>0}$ gives as strongly continuous contraction semigroup on these spaces. Now $S_\eta^\dagger f \to 0$ as $\eta \to \infty$ for all $f \in L^2(0,\infty)$, so for all finite rank operators F, we have $\sigma_\eta(F) \to 0$ as $\eta \to \infty$. Then for $K \in \mathfrak{J}_p$ and $\varepsilon > 0$ there exists a finite rank F such that $\|K - F\|_{\mathfrak{J}_p} < \varepsilon$, so $\|\sigma_\eta(K)\|_{\mathfrak{J}_p} \leqslant \|K - F\|_{\mathfrak{J}_p} + \|\sigma_\eta(F)\|_{\mathfrak{J}_p}$ is less than 2ε for all sufficiently large η. (Note, we do not assert that $(\sigma_\eta)_{\eta>0}$ is strongly continuous on $\mathfrak{J}$ itself.)

(iv) From the definition of generator, we have,

$$\delta k(y,z) = \frac{\mathrm{d}}{\mathrm{d}\eta}\bigg|_{\eta=0} \sigma_\eta(K)(y,z) = \frac{\mathrm{d}}{\mathrm{d}\eta}\bigg|_{\eta=0} k(y+\eta, z+\eta) = \big(\partial_y + \partial_z\big) k(y,z).$$

(v) We have, $\det(\mathrm{id} + \sigma_\eta(K)) = \det(\mathrm{id} + S_\eta^\dagger K S_\eta) = \det(\mathrm{id} + K S_\eta S_\eta^\dagger) = \det(\mathrm{id} + K\mathrm{Pr}_{(\eta,\infty)})$. By Mercer's formula we have, $\mathrm{tr}\,\sigma_\eta(K) = \int_0^\infty K(y+\eta, y+\eta)\,\mathrm{d}y$, and we can differentiate this formula using the fundamental theorem of calculus.

(vi) The result (v) may be formulated in terms of the generator without explicit mention of the semigroup. Let $\mathcal{D}^1 := \{\phi \in L^2((0,\infty);\mathbb{C}) : \phi' \in L^2((0,\infty);\mathbb{C})\}$, and recall from Hille and Phillips [49, p. 535] that $\mathcal{D}^1$ is the domain of the generator of $(S_\eta^\dagger)_{\eta>0}$. By Plancherel's theorem, we have $\mathcal{D}^1 \subset L^\infty$, so $\mathcal{D}^1$ is an algebra under pointwise multiplication of functions; hence there is a map $\mu : \mathcal{D}^1 \otimes \mathcal{D}^1 \to \mathcal{D}^1$ given by $\phi(y)\psi(z) \mapsto \phi(y)\psi(y)$. There is also a natural inclusion $\mathcal{D}^1 \otimes \mathcal{D}^1 \to L^2 \otimes L^2 = \mathfrak{J}_1(L^2)$, and the trace satisfies $\mathrm{tr}\,(K) = \int_0^\infty \mu(K)(x)\,\mathrm{d}x$. We have $\delta : \mathcal{D}^1 \otimes \mathcal{D}^1 \to L^2 \otimes L^2 : \phi \otimes \psi \mapsto \phi' \otimes \psi + \phi \otimes \psi'$; hence for $k = \sum_{j=1}^\infty \phi_j(y)\psi_j(z)$ we have,

$$\mathrm{tr}\,\delta K = \sum_{j=1}^\infty \int_0^\infty \big(\phi_j(y)\psi_j(y) + \phi_j(y)\psi_j'(y)\big)\,\mathrm{d}y = -\sum_{j=1}^\infty \phi_j(0)\psi_j(0),$$

so we have the required expression for Pöppe's bracket operation. □

Suppose that $\phi \in L^2((0,\infty);\mathbb{M}_{m\times m}(\mathbb{C}))$. Then we introduce $\phi_{(x)}(\eta) := \phi(2x+\eta)$ and the Hankel operator $\Gamma_{\phi_{(x)}} : L^2((0,\infty);\mathbb{C}^{m\times 1}) \to L^2((0,\infty);\mathbb{C}^{m\times 1})$ by,

$$\Gamma_{\phi_{(x)}} h(y) = \int_0^\infty \phi(y + z + 2x) f(z)\,\mathrm{d}z$$

for $f \in L^2((0,\infty);\mathbb{C}^{m\times 1})$. Suppose, $\int_0^\infty y\|\phi(y)\|^2\,\mathrm{d}y < \infty$ and $\int_0^\infty y\|\psi(y)\|^2\,\mathrm{d}y < \infty$. Then Γ_ϕ is a Hilbert–Schmidt operator, and $\Gamma_\phi\Gamma_\psi$ is trace class with,

$$\mathrm{tr}\,\big(\Gamma_\phi\Gamma_\psi\big) = \int_0^\infty \int_0^\infty \phi(y+z)\psi(y+z)\,\mathrm{d}y\mathrm{d}z = \int_0^\infty y\phi(y)\psi(y)\,\mathrm{d}y.$$

A bounded linear operator Γ on $L^2(0,\infty)$ is Hankel if and only if $S_\eta^\dagger \Gamma = \Gamma S_\eta$ for all $\eta > 0$. This may be interpreted as $\partial_x \Gamma = -\Gamma\partial_y$ when we consider operators on $C_c^\infty(0,\infty)$. In the context of Hankel products, this leads to the following.

Proposition 2 (*Bracket Identities for Hankel Operators*). *We have the following:*

(i) Let Γ_ϕ be the Hankel operator with kernel $\phi(y+z)$. Then $\sigma_\eta(\Gamma_\phi)$ has kernel $\phi(y+z+2\eta)$, so $\sigma_\eta(\Gamma_\phi) = \Gamma_{\phi_{(\eta)}}$;

(ii) Let $\phi,\psi \in \mathbb{M}_{m\times m}(C_c^\infty(0,\infty))$ be functions as above. Then $\sigma_{2\eta}(\Gamma_\phi\Gamma_\psi) = \Gamma_{\phi_{(\eta)}}\Gamma_{\psi_{(\eta)}}$, and,

$$\partial_\eta\big(\Gamma_{\phi_{(\eta)}}\Gamma_{\psi_{(\eta)}}\big),$$

is a bounded linear operator of finite rank with rank less than or equal to m^2;

(iii) Suppose $m = 1$, and $\phi,\psi \in C_c^\infty(0,\infty)$. Then $\delta(\Gamma_\phi\Gamma_\psi)$ has rank one;

(iv) Conversely, suppose that K is in the domain of δ and $\delta(K)$ has finite rank. Then $K = \Gamma_\Phi^\top\Gamma_\Psi$;

(v) Let $\Gamma_x = \Gamma_{\phi_{(x)}}$ and $\Gamma_x' = \partial_x\Gamma_x$. Let Θ be holomorphic on an open neighbourhood of the spectrum of Γ_x for all real x. Then we have,

$$[\![\Gamma_x\Theta'(\Gamma_x)]\!]_{0,0} = -\partial_x\mathrm{tr}\,\big(\Theta(\Gamma_x)\big).$$

Proof. We observe the following. (i) This is straightforward, and explains the notation. (ii) The Hankel operators $\Gamma_{\phi_{(\eta)}}$ and $\Gamma_{\psi_{(\eta)}}$ are Hilbert–Schmidt, so their product is trace class. Then we differentiate the kernel and obtain,

$$\partial_\eta \int_0^\infty \phi(y+\xi+2\eta)\psi(\xi+z+2\eta)\,\mathrm{d}\xi = -2\phi(y+2\eta)\psi(z+2\eta),$$

which gives an element of the vector space $\mathbb{M}_{m\times m}(\mathbb{C})$ of dimension m^2.

(iii) We have for $m=1$, $\delta(\Gamma_\phi\Gamma_\psi)(y,z) = -\phi(y)\psi(z)$.

(iv) By hypothesis, there exist $\phi_j,\psi_j \in L^2(0,\infty)$ for $j=1,\dots,n$ such that $\delta K(y,z) = -\sum_{j=1}^n \phi_j(y)\psi_j(z)$. Then we introduce the vector functions $\Phi = (\phi_1,\dots,\phi_n)^{\mathrm{T}}$ and $\Psi = (\psi_1,\dots,\psi_n)^{\mathrm{T}}$ so that $\Gamma_\Phi^\top\Gamma_\Psi = \sum_{j=1}^n \Gamma_{\phi_j}\Gamma_{\psi_j}$; and also $\delta(\Gamma_\Phi^\top\Gamma_\Psi) = -\sum_{j=1}^n \phi_j(y)\psi_j(z)$. We consider $W = K - \Gamma_\Phi^\top\Gamma_\Psi$ which belongs to the domain of δ with $\delta(W)=0$, hence, $\sigma_\eta(W) = W + \int_0^\eta \sigma_\xi(\delta W)\,\mathrm{d}\xi = W$, where $\sigma_\eta(W)\to 0$ as $t\to\infty$; thus $W=0$ and $K = \Gamma_\Phi^\top\Gamma_\Psi$ is a Hankel product.

(v) For even powers we have,

$$2\partial_x \mathrm{tr}\,\Gamma_x^{2k} = 2\sum_{j=0}^{k-1} \mathrm{tr}\left(\Gamma_x^{2j}\partial_x(\Gamma_x^2)\Gamma_x^{2(k-j-1)}\right) = 2k\,\mathrm{tr}\left(\Gamma_x^{2k-2}\partial_x\Gamma_x^2\right),$$

where the final operator has finite rank. For odd powers, since,

$$2\partial_x \mathrm{tr}\left(\Gamma_x^{2k+1}\right) = \mathrm{tr}\left(\Gamma_x'\Gamma_2^{2k} + \Gamma_x\Gamma_x'\Gamma_x^{2k-1} + \cdots + \Gamma_x^{2k}\Gamma_x'\right) + \mathrm{tr}\left(\Gamma_x'\Gamma_2^{2k} + \Gamma_x\Gamma_x'\Gamma_x^{2k-1} + \cdots + \Gamma_x^{2k}\Gamma_x'\right),$$

and then we move the terms in the second list one step to the left, except for the first, which we move to the end. Thus we obtain,

$$\mathrm{tr}\left((\Gamma_x'\Gamma_x + \Gamma_x\Gamma_x')\Gamma_x^{2k-1} + \Gamma_x(\Gamma_x'\Gamma_x + \Gamma_x\Gamma_x')\Gamma_x^{2k-2} + \cdots + \Gamma_x^{2k-1}(\Gamma_x'\Gamma_x + \Gamma_x\Gamma_x')\right) = (2k+1)\,\mathrm{tr}\left(\Gamma_x^{2k-1}\partial_x\Gamma_x^2\right),$$

where again the final operator has finite rank.

Suppose that the spectrum of Γ_x is contained in $D(0,r)$ for some $r>0$. Then for all s such that $|s|>r$, we have a convergent power series $(s-\gamma)^{-1} = \sum_{j=0}^\infty \gamma^j/s^{j+1}$ for all γ in the spectrum of Γ_x and,

$$2\,\partial_x\mathrm{tr}\left((s\cdot\mathrm{id}-\Gamma_x)^{-1}\right) = \sum_{j=1}^\infty \frac{2}{s^{j+1}}\partial_x\mathrm{tr}\left(\Gamma_x^j\right).$$

Then for the first term in the series, we have,

$$2\,\mathrm{tr}\,(\Gamma_x') = 2\,\partial_x\int_0^\infty \phi(2y+2x)\,\mathrm{d}y = -2\phi(2x) = -2[\![\Gamma_x]\!]_{0,0},$$

and for the remaining terms in the series, $\mathrm{tr}\,(\Gamma_x^{j-2}\partial_x\Gamma_x^2)$ equals,

$$-2\int_0^\infty\cdots\int_0^\infty \phi_{(x)}(y_0+y_1)\phi_{(x)}(y_1+y_2)\dots\phi_{(x)}(y_{j-1}+y_0)dy_0\dots dy_{j-1},$$

which equals $-2[\![\Gamma_x^j]\!]_{0,0}$. The result $-\partial_x\mathrm{tr}\left((s\cdot\mathrm{id}-\Gamma_x)^{-1}\right) = [\![\Gamma_x(s\cdot\mathrm{id}-\Gamma_x)^{-2}]\!]_{0,0}$ holds for all s such that $|s|>r$ follows when we multiply through by $-1/2$ and sum over j. By analytic continuation, we have the same identity for all s in the unbounded component of the complement of the spectrum of Γ_x in the complex plane.

Now let Θ be holomorphic on an open neighbourhood of the spectrum of Γ_x for all $x>0$. Note that $\|\Gamma_x\|\to 0$ as $x\to\infty$, so this uniformity is a mild restriction. Then there exists a contour C that winds round the spectrum of Γ_x once in the positive sense, so by Cauchy's integral formula $\Theta(\Gamma_x) = (2\pi i)^{-1}\int_C(\zeta\cdot\mathrm{id}-\Gamma_x)^{-1}\Theta(\zeta)\,\mathrm{d}\zeta$, hence,

$$-\partial_x\mathrm{tr}\left(\Theta(\Gamma_x)\right) = -\frac{1}{2\pi i}\int_C \partial_x\mathrm{tr}\left((\zeta\cdot\mathrm{id}-\Gamma_x)^{-1}\right)\Theta(\zeta)\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_C [\![\Gamma_x(\zeta\cdot\mathrm{id}-\Gamma_x)^{-2}]\!]_{0,0}\Theta(\zeta)\,\mathrm{d}\zeta,$$

which integrates to $[\![\Gamma_x\Theta'(\Gamma_x)]\!]_{0,0}$. The proof is complete. □

Example 3. For real-valued ϕ, and for $\Theta(\zeta)=\zeta^2$, the basic formulae are,

$$\|\Gamma_x\|_{\mathfrak{J}_2}^2 = \mathrm{tr}\,(\Gamma_x^2) = \int_0^\infty y\phi_{(x)}(y)^2\,\mathrm{d}y \quad\text{and}\quad [\![\Gamma_x^2]\!]_{0,0} = -\tfrac{1}{2}\partial_x\mathrm{tr}\,(\Gamma_x^2) = \int_0^\infty \phi_{(x)}(y)^2\,\mathrm{d}y.$$

3. Pöppe algebra

We prescribe the kernel algebra generated by the quantities $[\![V]\!]$ and $[\![V^\dagger]\!]$ and their derivatives, based on the Pöppe product in Lemma 1, as well as a subalgebra generated by the quantity $[\![V-V^\dagger]\!]$ and its derivatives. We also outline abstract versions of these algebras to aid computations. We nominate the abstract algebra as the *Pöppe algebra* and the corresponding subalgebra the *skew-Pöppe algebra*. We begin with some preliminary identities. Given a Hankel Hilbert–Schmidt operator $P=P(x,t)$ on $\mathbb{V}$, depending on the parameters $x\in\mathbb{R}$ and $t\geqslant 0$, we set,

$$V := (\mathrm{id}-\mathrm{i}P)^{-1}.$$

Recall that P is self adjoint, so that $P^\dagger = P$ and thus $(\mathrm{i}P)^\dagger = -\mathrm{i}P$.

Lemma 4 (*Operator Identities*)**.** *Given a Hankel operator $P=P(x,t)$ which is Hilbert–Schmidt valued, and the definition $V := (\mathrm{id}-\mathrm{i}P)^{-1}$, with $V^\dagger = (\mathrm{id}+\mathrm{i}P)^{-1}$ the adjoint operator to V, we observe that,*

$$V \equiv \mathrm{id} + (\mathrm{i}P)V \equiv \mathrm{id} + V(\mathrm{i}P) \quad\text{and}\quad V^\dagger \equiv \mathrm{id} + (\mathrm{i}P)^\dagger V^\dagger \equiv \mathrm{id} + V^\dagger(\mathrm{i}P)^\dagger,$$

and further that, $V - V^\dagger \equiv 2\,V(\mathrm{i}P)V^\dagger$.

Proof. These follow from the definitions of V and $V^\dagger$ and partial fraction identities. □

Definition 4 (*Fredholm Grassmannian Flow*)**.** Given a Hankel Hilbert–Schmidt operator $P = P(x,t)$ on $\mathbb{V}$, depending on the parameters $x \in \mathbb{R}$ and $t \geqslant 0$, we define the operator G by,

$$G := V - V^\dagger.$$

Remark 3. Using Lemma 4, we can write $V - V^\dagger = 2V(\mathrm{i}P)V^\dagger = 2(\mathrm{i}P)U$, where $U := \big(\mathrm{id} + P^2\big)^{-1}$. This is possible because V and $V^\dagger$ can be expressed as power series in P with scalar coefficients. Hence the order of the operators in $V(\mathrm{i}P)V^\dagger$ does not matter. Thus, the flow of G represents a Fredholm Grassmannian flow.

Definition 5 (*Signature Character*)**.** Let $\mathbb{N}^*$ denote the free monoid of words on $\mathbb{N}$, i.e. the set of all possible words of the form $a_1a_2\cdots a_k$ we can construct from letters $a_1, a_2, \dots, a_k \in \mathbb{N}$. Suppose $a_1a_2\cdots a_n \in \mathbb{N}^*$. The *signature character* $\chi : \mathbb{N}^* \to \mathbb{Q}$ of any such word is given by the product of Leibniz coefficients,

$$\chi : a_1a_2\cdots a_n \mapsto \prod_{k=1}^{n} \binom{a_k + \cdots + a_n}{a_k}.$$

Let $C(n)$ denote the set of all compositions of $n \in \mathbb{N}$. The following result is equivalent to that in Malham [10, Lemma 2] and Doikou et al. [13, Lemma 8], where detailed proofs can be found. For any integer k, we set $(\mathrm{i}P)_k := \partial^k(\mathrm{i}P)$, $V_k := \partial^k V$ and $V_k^\dagger := \partial^k V^\dagger$. For example, if $k = 2$, then $V_2 = \partial^2 V$, while if $k = -1$, then $V_{-1} = \partial^{-1}V$.

Lemma 5 (*Kernel Signature Expansion*)**.** *Given a Hankel operator $P = P(x,t)$ and that $V := (\mathrm{id}-\mathrm{i}P)^{-1}$ with $V^\dagger = (\mathrm{id}+\mathrm{i}P)^{-1}$, we observe that $\partial V \equiv V(\mathrm{i}P)_1 V$ and $\partial V^\dagger \equiv V^\dagger(\mathrm{i}P)_1^\dagger V^\dagger$. With the sum over all compositions $a_1\cdots a_k \in C(n)$, we have,*

$$V_n = \sum \chi\big(a_1\cdots a_n\big)\, V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V,$$

with the corresponding generalisation for $V_n^\dagger$. In particular we have,

$$V_n - V_n^\dagger = \sum \chi\big(a_1\cdots a_n\big)\, \big(V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V - V^\dagger(\mathrm{i}P)_{a_1}^\dagger V^\dagger \cdots V^\dagger(\mathrm{i}P)_{a_k}^\dagger V^\dagger\big).$$

We now construct the algebra generated by $[\![V - V^\dagger]\!]$ and its derivatives. For any Hilbert–Schmidt operator W, we set,

$$[W] := [\![W - W^\dagger]\!] \qquad \text{and} \qquad \{W\} := [\![W + W^\dagger]\!].$$

In other words the bracket '$[\,\cdot\,]$' generates the kernel of the difference, between its operator argument and corresponding adjoint. It is the kernel of the skew-symmetric part of its operator argument. It is not a commutator. Thus if $v = v(y,z;x,t)$ is the matrix-valued kernel corresponding to V which depends on the parameters x and t, then $[V] = v(y,z;x,t) - v^\dagger(z,y;x,t)$, where now $v^\dagger$ is the complex conjugate transpose of the matrix v. Analogously, $\{\,\cdot\,\}$ generates the kernel of the symmetric part of its operator argument. From Lemma 5, we observe that,

$$[V_n] = \sum \chi\big(a_1\cdots a_n\big)\, \big[V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V\big].$$

The kernel monomial algebra is generated by the monomials $\big[V(\mathrm{i}P)_{a_1}V \cdots V(\mathrm{i}P)_{a_k}V\big]$, including monomials of this form with one or more of the V's shown being replaced by $V^\dagger$. The Pöppe product from Lemma 1 generates closed-form identities for products of such monomials. In particular, we have the following.

Lemma 6 (*Pöppe Kernel Product Identities*)**.** *For arbitrary Hilbert–Schmidt operators F and F' and a Hankel Hilbert–Schmidt operator P with parameter x and a smooth kernel, we have the following for any $a, b \in \mathbb{Z}$,*

$$\begin{aligned}
[\![F(\mathrm{i}P)_aV]\!]\,[\![V(\mathrm{i}P)_bF']\!] &= [\![F(\mathrm{i}P)_{a+1}V(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)_{b+1}F']\!] + 2\,[\![F(\mathrm{i}P)_aV(\mathrm{i}P)_1V(\mathrm{i}P)_bF']\!],\\
[\![F(\mathrm{i}P_a)V^\dagger]\!]\,[\![V^\dagger(\mathrm{i}P_b)F']\!] &= [\![F(\mathrm{i}P)_{a+1}V^\dagger(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV^\dagger(\mathrm{i}P)_{b+1}F']\!] + 2\,[\![F(\mathrm{i}P)_aV^\dagger(\mathrm{i}P)_1V^\dagger(\mathrm{i}P)_bF']\!],\\
[\![F(\mathrm{i}P)_aV^\dagger]\!]\,[\![V(\mathrm{i}P)_bF']\!] &= [\![F(\mathrm{i}P)_{a+1}V(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV^\dagger(\mathrm{i}P)_{b+1}F']\!],\\
[\![F(\mathrm{i}P)_aV]\!]\,[\![V^\dagger(\mathrm{i}P)_bF']\!] &= [\![F(\mathrm{i}P)_{a+1}V^\dagger(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)_{b+1}F']\!].
\end{aligned}$$

Proof. The results stated are established straightforwardly. For example, using the identities in Lemma 4 and the basic Pöppe product rule, we observe,

$$\begin{aligned}
[\![F(\mathrm{i}P)_aV]\!]\,[\![V(\mathrm{i}P)_bF']\!] &= [\![F(\mathrm{i}P)_a + F(\mathrm{i}P)_aV(\mathrm{i}P)]\!][\![(\mathrm{i}P)_bF' + (\mathrm{i}P)V(\mathrm{i}P)_bF']\!]\\
&= [\![F(\mathrm{i}P)_{a+1}(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_a(\mathrm{i}P)_{b+1}F']\!] + [\![F(\mathrm{i}P)_{a+1}(\mathrm{i}P)V(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_a(\mathrm{i}P)_1V(\mathrm{i}P)_bF']\!]\\
&\quad + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)_1(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)(\mathrm{i}P)_{b+1}F']\!]\\
&\quad + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)_1(\mathrm{i}P)V(\mathrm{i}P)_bF']\!] + [\![F(\mathrm{i}P)_aV(\mathrm{i}P)(\mathrm{i}P)_1V(\mathrm{i}P)_bF']\!].
\end{aligned}$$

Combining terms using the identities in Lemma 4 generates the first result claimed. And so forth. □

Remark 4 (*Algebra of Kernel Monomials: Abstract Encoding*)**.** As mentioned, the set of all kernel monomials of the form $[\![V(\mathrm{i}P)_{a_1}V(\mathrm{i}P)_{a_2}V \cdots V(\mathrm{i}P)_{a_k}V]\!]$, where any of the V's shown may be replaced by $V^\dagger$, with the Pöppe kernel product defined in Lemma 6, form a closed algebra of such monomials. We assume here that all the derivatives of the P operator exist and are Hilbert–Schmidt valued. At this stage it is useful to consider an abstract encoding of this kernel monomial algebra, equipped with the Pöppe kernel product. The abstract algebra is constructed by simply stripping the '$\mathrm{i}P$' and 'V' labels from the kernel monomials, and respectively, replacing them by the composition components $a_1a_2\cdots a_k$, together with a binary encoding of whether an intervening operator is a V or $V^\dagger$, i.e. we replace,

$$[\![V(\mathrm{i}P)_{a_1}V(\mathrm{i}P)_{a_2}V\cdots V(\mathrm{i}P)_{a_k}V]\!]\to \mathbf{0}a_1\mathbf{0}a_2\mathbf{0}\cdots\mathbf{0}a_k\mathbf{0},$$

where any of the $\mathbf{0}$'s shown, corresponding to the V operator, may be replaced by $\mathbf{0}^\dagger$ in the corresponding position that a $V^\dagger$ operator is present in the monomial on the left. In essence, the Pöppe kernel product defined in Lemma 6 involves operations on these stripped down components only, i.e. operations on the forms $\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}\cdots\mathbf{0}a_k\mathbf{0}$, where again some $\mathbf{0}$'s may be replaced by $\mathbf{0}^\dagger$. Since we mirror the Pöppe product in the abstract setting in Definition 6, we know that the kernel monomial algebra and our abstract algebra encoding just below, are isomorphic.

Let us now introduce our abstract encoding for the algebra of operator kernel monomials equipped with the Pöppe product in Lemma 1, just mentioned. Given a word $w=a_1a_2\cdots a_k$ generated using letters $a_1,a_2,\ldots,a_k$ from $\mathbb{Z}$, and a word $\boldsymbol{\varphi}=\theta_1\theta_2\cdots\theta_{k+1}$ generated using the letters θ_1, $\theta_2,\ldots,\theta_{k+1}$ chosen from the binary set $\{\mathbf{0},\mathbf{0}^\dagger\}$, let $w\times\boldsymbol{\vartheta}$ denote the corresponding word,

$$w\times\boldsymbol{\varphi}=\theta_1a_1\theta_2a_2\theta_3\cdots\theta_ka_k\theta_{k+1},$$

in the free monoid $(\mathbb{Z}_\mathbf{0})^*$ where $\mathbb{Z}_\mathbf{0}:=\mathbb{Z}\cup\{\mathbf{0},\mathbf{0}^\dagger\}$. For such words, there is a single letter from the binary set $\{\mathbf{0},\mathbf{0}^\dagger\}$ sandwiched between each of the letters from $\mathbb{Z}$, as well as one at each end. Let $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ denote the non-commutative polynomial algebra over $\mathbb{C}$ generated by words from $(\mathbb{Z}_\mathbf{0})^*$, endowed with the following Pöppe product.

Definition 6 (*Pöppe Product*)**.** Consider four words from $(\mathbb{Z}_\mathbf{0})^*$ of the form $ua\mathbf{0}$, $ua\mathbf{0}^\dagger$, $\mathbf{0}bv$ and $\mathbf{0}^\dagger bv$, where u and v are any subwords from $(\mathbb{Z}_\mathbf{0})^*$ and $a,b\in\mathbb{Z}$. We define the *Pöppe product* from $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle\times\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ to $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ of these words to be,

$$\begin{aligned}(ua\mathbf{0})(\mathbf{0}bv)&=u(a+1)\mathbf{0}bv+ua\mathbf{0}(b+1)v+2\cdot ua\mathbf{0}1\mathbf{0}bv,\\(ua\mathbf{0}^\dagger)(\mathbf{0}^\dagger bv)&=u(a+1)\mathbf{0}^\dagger bv+ua\mathbf{0}^\dagger(b+1)v+2\cdot ua\mathbf{0}^\dagger1\mathbf{0}^\dagger bv,\\(ua\mathbf{0}^\dagger)(\mathbf{0}bv)&=u(a+1)\mathbf{0}bv+ua\mathbf{0}^\dagger(b+1)v\\(ua\mathbf{0})(\mathbf{0}^\dagger bv)&=u(a+1)\mathbf{0}^\dagger bv+ua\mathbf{0}(b+1)v.\end{aligned}$$

Let ν denote the empty word in $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$. Then for any word $w\times\boldsymbol{\varphi}\in\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ we have $\nu(w\times\boldsymbol{\varphi})=(w\times\boldsymbol{\varphi})\nu=w\times\boldsymbol{\varphi}$. Let $\mathcal{C}:=\cup_{n\geqslant0}\mathcal{C}(n)$ denote the set of all compositions.

Definition 7 (*Signature Expansion*)**.** For any $n\in\mathbb{N}\cup\{0\}$, we define the following linear *signature expansions* $\boldsymbol{n}\in\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$,

$$\boldsymbol{n}:=\sum_{a_1a_2\cdots a_k\in\mathcal{C}(n)}\chi(a_1a_2\cdots a_k)\cdot\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}\cdots\mathbf{0}a_k\mathbf{0},$$

where the sum is over all possible compositions $a_1a_2\cdots a_k$, with $k\geqslant1$ parts, of n.

For example, we note that $\mathbf{1}=\chi(1)\cdot\mathbf{010}$ and $\mathbf{2}=\chi(2)\cdot\mathbf{020}+\chi(11)\cdot\mathbf{01010}$. Further note, for the case $n=0$, the signature expansion simply corresponds to the letter $\mathbf{0}$ from the binary set $\{\mathbf{0},\mathbf{0}^\dagger\}$. Equivalently we can write the relation in Definition 7 for the case $n=0$ as $\mathbf{0}=\chi(0)\cdot\mathbf{0}$. Naturally by convention, we take $\chi(0)=1$. Let us also remark on the following basic identities in $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$, which follow from Lemma 4.

Lemma 7 (*Algebraic Identities*)**.** *We have the following basic relations in* $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$,

$$\mathbf{0}\equiv\nu+\mathbf{00}\equiv\nu+\mathbf{00},\quad \mathbf{0}^\dagger\equiv\nu+\mathbf{0}^\dagger0^\dagger\equiv\nu+0^\dagger\mathbf{0}^\dagger\quad\textit{and}\quad\mathbf{0}-\mathbf{0}^\dagger\equiv2\cdot\mathbf{000}^\dagger.$$

Definition 8. Given any word $w\times\boldsymbol{\varphi}\in\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$, say $w\times\boldsymbol{\varphi}=\theta_1a_1\theta_2a_2\theta_3\cdots\theta_ka_k\theta_{k+1}$, the letters $a_i^\dagger$ denote the letters '$-a_i$' from $\mathbb{Z}$, i.e. $a_i^\dagger=-a_i$. Further we set,

$$\big(\theta_1a_1\theta_2a_2\theta_3\cdots\theta_ka_k\theta_{k+1}\big)^\dagger:=\theta_1^\dagger a_1^\dagger\theta_2^\dagger a_2^\dagger\theta_3^\dagger\cdots\theta_k^\dagger a_k^\dagger\theta_{k+1}^\dagger,$$

i.e. we replace all the letters a_i in w by their counterparts $a_i^\dagger=-a_i$ and all the letters in $\boldsymbol{\theta}$ by their counterparts. In the latter instance this means we change all the $\mathbf{0}$'s to $\mathbf{0}^\dagger$, and vice-versa. Note we *do not* reverse the order of the terms in $w\times\boldsymbol{\varphi}$. This means, for example, that we can interpret $(w\times\boldsymbol{\varphi})^\dagger=w^\dagger\times\boldsymbol{\varphi}^\dagger$, and since $w^\dagger=(-1)^{|w|}w$, where $|w|$ is the length of w, then $(w\times\boldsymbol{\varphi})^\dagger\in\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$.

Consider the following skew-symmetric and symmetric forms on $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$.

Definition 9 (*Skew-symmetric and Symmetric Forms*)**.** Given any word $w\times\boldsymbol{\varphi}\in\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$, we define its skew-symmetric and symmetric forms in $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$, respectively, by

$$[w\times\boldsymbol{\varphi}]:=w\times\boldsymbol{\varphi}-(w\times\boldsymbol{\varphi})^\dagger\qquad\text{and}\qquad\{w\times\boldsymbol{\varphi}\}:=w\times\boldsymbol{\varphi}+(w\times\boldsymbol{\varphi})^\dagger.$$

Naturally we have $\big[(w\times\boldsymbol{\varphi})^\dagger\big]=-[w\times\boldsymbol{\varphi}]$ and $\big\{(w\times\boldsymbol{\varphi})^\dagger\big\}=\{w\times\boldsymbol{\varphi}\}$.

The following product rules based on the Pöppe product in Definition 6, are useful for our computations in all subsequent sections.

Lemma 8 (*Skew and Symmetric Pöppe Products*)**.** *Consider the elements* $[ua\mathbf{0}]$, $\{ua\mathbf{0}\}$, $[\mathbf{0}bv]$ *and* $\{\mathbf{0}bv\}$ *from* $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$, *where u and v are any subwords from* $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ *and* $a,b\in\mathbb{Z}$. *We have the following Pöppe products in* $\mathbb{C}\langle\mathbb{Z}_\mathbf{0}\rangle$ *between these elements,*

$$\begin{aligned}\{ua\mathbf{0}\}\,[\mathbf{0}bv]&=\big[u(a+1)[\mathbf{0}bv]\big]+\big[ua\mathbf{0}[(b+1)v]\big]+2\cdot[ua\mathbf{0}1\mathbf{0}bv],\\ [ua\mathbf{0}]\,\{\mathbf{0}bv\}&=\big[u(a+1)\{\mathbf{0}bv\}\big]+\big[ua\mathbf{0}\{(b+1)v\}\big]+2\cdot[ua\mathbf{0}1\mathbf{0}bv],\\ [ua\mathbf{0}]\,[\mathbf{0}bv]&=\big\{u(a+1)[\mathbf{0}bv]\big\}+\big\{ua\mathbf{0}[(b+1)v]\big\}+2\cdot\{ua\mathbf{0}1\mathbf{0}bv\},\end{aligned}$$

$$\{ua\mathbf{0}\}\,\{\mathbf{0}bv\} = \big\{u(a+1)\{\mathbf{0}bv\}\big\} + \big\{ua\mathbf{0}\{(b+1)v\}\big\} + 2\cdot\{ua\mathbf{010}bv\}.$$

These products also hold when $[ua\mathbf{0}] = [\mathbf{0}]$*, which case term on the right involving '*$(a+1)$*' is absent. Likewise, these products also hold when* $[\mathbf{0}bv] = [\mathbf{0}]$*, in which case the term on the right involving '*$(b+1)$*' is absent.*

Proof. The results are established straightforwardly using Definition 6 for the abstract Pöppe product. Consider for example the first product shown, we observe that,

$$\begin{aligned}\{ua\mathbf{0}\}\,[\mathbf{0}bv] &= \big(ua\mathbf{0} + (ua\mathbf{0})^\dagger\big)\big(\mathbf{0}bv - (\mathbf{0}bv)^\dagger\big)\\ &= u(a+1)\mathbf{0}bv + ua\mathbf{0}(b+1)v + 2\cdot ua\mathbf{010}bv - u(a+1)(\mathbf{0}bv)^\dagger - ua\mathbf{0}\big((b+1)v\big)^\dagger\\ &\quad + \big(u(a+1)\big)^\dagger\mathbf{0}bv + (ua\mathbf{0})^\dagger(b+1)v - \big(u(a+1)\mathbf{0}bv\big)^\dagger - \big(ua\mathbf{0}(b+1)v\big)^\dagger - 2\cdot\big(ua\mathbf{010}bv\big)^\dagger,\end{aligned}$$

which gives the first product result. The other three cases follow completely analogously. For the case, for example, when $[ua\mathbf{0}] = [\mathbf{0}]$ in the second product, we use that, since $[\mathbf{0}] = 2\cdot\mathbf{00}\mathbf{0}^\dagger$ and $\mathbf{00}\mathbf{0}^\dagger = \mathbf{0}^\dagger\mathbf{00}$, we have $[\mathbf{00}\mathbf{0}^\dagger] = [\mathbf{0}^\dagger\mathbf{00}] = \mathbf{00}\mathbf{0}^\dagger + \mathbf{0}^\dagger\mathbf{00} = [\mathbf{0}]$. Hence we observe, since $[\mathbf{0}] = [\mathbf{0}^\dagger\mathbf{00}]$, we can use the latter form in the corresponding product already established, so using the properties of the skew and symmetric forms in Definition 9 we have,

$$\begin{aligned}[\mathbf{0}]\,\{\mathbf{0}bv\} &= [\mathbf{0}^\dagger\mathbf{00}]\,\{\mathbf{0}bv\}\\ &= \big[\mathbf{0}^\dagger 1\{\mathbf{0}bv\}\big] + \big[\mathbf{0}^\dagger\mathbf{00}\{(b+1)v\}\big] + 2\cdot[\mathbf{0}^\dagger\mathbf{0010}bv]\\ &= [\mathbf{0}^\dagger\mathbf{10}bv] + [\mathbf{010}bv] + [\mathbf{0}^\dagger\mathbf{00}(b+1)v] + [\mathbf{00}\mathbf{0}^\dagger(b+1)v] + 2\cdot[\mathbf{0}^\dagger\mathbf{0010}bv]\\ &= [\mathbf{0}(b+1)v] - [\mathbf{0}^\dagger(b+1)v] + \big[(\mathbf{0}^\dagger + \mathbf{0} + 2\cdot\mathbf{0}^\dagger\mathbf{00})\mathbf{10}bv\big]\\ &= \big[\mathbf{0}\{(b+1)v\}\big] + 2\cdot[\mathbf{010}bv].\end{aligned}$$

The remaining cases follow completely analogously. □

Remark 5. We observe that, using the properties of the skew-symmetric and symmetric forms '$[\,\cdot\,]$' and '$\{\,\cdot\,\}$' recorded in Definition 9, the skew and symmetric Pöppe products quoted in Lemma 8 are sufficient to resolve the Pöppe products of all possible skew-symmetric or symmetric forms we might encounter. For example if the left factor is of the form $[ua\mathbf{0}^\dagger]$ or $\{ua\mathbf{0}^\dagger\}$, then we can use that $[ua\mathbf{0}^\dagger] = -[(ua)^\dagger\mathbf{0}]$ or $\{ua\mathbf{0}^\dagger\} = \{(ua)^\dagger\mathbf{0}\}$ and then apply the product rules shown to the latter forms. Similarly we can use that $[\mathbf{0}^\dagger bv] = -[\mathbf{0}(bv)^\dagger]$ and $\{\mathbf{0}^\dagger bv\} = \{\mathbf{0}(bv)^\dagger\}$.

Remark 6 (*Minimal Product Set*). We observe, to compute the Pöppe product of any monomials of the form $[w_1\times\boldsymbol{\varphi}_1]\,[w_2\times\boldsymbol{\varphi}_2]\,\ldots\,[w_k\times\boldsymbol{\varphi}_k]$, we really only need the rule for $[\,\cdot\,]\,[\,\cdot\,]$ and say the rule for $[\,\cdot\,]\,\{\,\cdot\,\}$ in Lemma 8. This is because we can work from right to left through the products in such a monomial. We can alternatively use $\{\,\cdot\,\}\,[\,\cdot\,]$ and work from left to right.

Remark 7 (*Basic Skew-form Properties*). The skew-form $[\mathbf{0}]$ corresponds to the general skew-form $[w\times\boldsymbol{\varphi}]$ in which the composition component/word $w = \nu$, the empty word, and $\boldsymbol{\varphi} = \mathbf{0}$, i.e. we have $[\nu\times\mathbf{0}] = [\mathbf{0}]$. Note if $\boldsymbol{\varphi} = \mathbf{0}^\dagger$, this simply corresponds to '$-[\mathbf{0}]$' or equivalently '$-[\nu\times\mathbf{0}]$'. In Pöppe products, the skew-form $[\mathbf{0}]$ has some rather special properties, as highlighted in Lemma 8. By Remark 6 and the second result in proof of Lemma 8, we have, for example,

$$\begin{aligned}[\mathbf{0}]\,[\mathbf{0}bv] &= \big\{\mathbf{0}[(b+1)v]\big\} + 2\cdot\{\mathbf{010}bv\},\\ [ua\mathbf{0}]\,[\mathbf{0}] &= \big\{u(a+1)[\mathbf{0}]\big\} + 2\cdot\{ua\mathbf{010}\},\\ [\mathbf{0}]\,\{\mathbf{0}bv\} &= \big[\mathbf{0}\{(b+1)v\}\big] + 2\cdot[\mathbf{010}bv].\end{aligned}$$

In particular, setting bv to be the empty word ν, we have $[\mathbf{0}]^2 = 2\cdot\{\mathbf{010}\}$.

Remark 8 (*Homomorphic Signature Character*). Consider a multi-factor product of signature expansions of the form,

$$[\boldsymbol{n}_1]\,[\boldsymbol{n}_2]\,\ldots\,[\boldsymbol{n}_k] = \sum\big(\chi(w_1)\chi(w_2)\cdots\chi(w_k)\big)\cdot[w_1\times\boldsymbol{\varphi}_1]\,[w_2\times\boldsymbol{\varphi}_2]\,\ldots\,[w_k\times\boldsymbol{\varphi}_k],$$

where the sum is over all words $w_1\times\boldsymbol{\varphi}_1$ with $w_1\in\mathcal{C}(n_1)$, $w_2\times\boldsymbol{\varphi}_2$ with $w_2\in\mathcal{C}(n_2)$, and so forth. Note, the form $[w_1\times\boldsymbol{\varphi}_1]\,[w_2\times\boldsymbol{\varphi}_2]\,\ldots\,[w_k\times\boldsymbol{\varphi}_k]$ generates many different words in $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$. We observe that it would be convenient to encode $\chi(w_1)\chi(w_2)\cdots\chi(w_k)$ as $\chi(w_1{\otimes}w_2{\otimes}\cdots{\otimes}w_k)$. Indeed, hereafter, we assume that χ acts *homomorphically* on any such tensor product of compositions so that indeed we have,

$$\chi(w_1{\otimes}w_2{\otimes}\cdots{\otimes}w_k) \equiv \chi(w_1)\chi(w_2)\cdots\chi(w_k).$$

Let us now outline some simple examples.

Example 4. By definition $[\mathbf{0}] := \mathbf{0} - \mathbf{0}^\dagger$. Using the notation $[\mathbf{0}]^2 = \big(\chi(0)\cdot[\mathbf{0}]\big)\,\big(\chi(0)\cdot[\mathbf{0}]\big)$ and so forth, then using the product rules in Lemma 8 we observe (also see Remark 7),

$$[\mathbf{0}]^2 = \chi(0\hat{\otimes}0)\cdot\{\mathbf{010}\}\qquad\text{and}\qquad[\mathbf{0}]^3 = [\mathbf{0}]\,[\mathbf{0}]^2 = \big(\chi(0)\cdot[\mathbf{0}]\big)\,\big(\chi(0\hat{\otimes}0)\{\mathbf{010}\}\big) = \chi(0{\otimes}0\hat{\otimes}0)\cdot\big[\mathbf{0}\{\mathbf{20}\}\big] + \chi(0\hat{\otimes}0\hat{\otimes}0)\cdot[\mathbf{01010}],$$

where the tensor notation '$\hat{\otimes}$' in the argument of $\chi = \chi(\cdot)$ indicates a tensor product '$\otimes$' together with the fact that an extra real factor of '2' should be included with the $\chi = \chi(\cdot)$ factor shown. See Remark 9 just below.

Remark 9. Hereafter, we also use the tensor notation '$\hat{\otimes}$' in the argument of $\chi = \chi(\cdot)$ to indicate when the skew or symmetric form were generated by the 'quasi' term $2\cdot[ua\mathbf{010}bv]$ in one of the Pöppe products in Lemma 8. We illustrated this in Example 4 just above. We observe therein that the result of the product $[\mathbf{0}]^2$ is '2' times the symmetric form $\{\mathbf{010}\}$. This symmetric form emerges from the 'quasi' term in the Pöppe product of $\chi(0)\cdot[\mathbf{0}]$ with $\chi(0)\cdot[\mathbf{0}]$ and a natural way to record this is the form $\chi(0\hat{\otimes}0)\cdot\{\mathbf{010}\}$. The tensor product of the zeros in the argument of $\chi=\chi(\cdot)$ indicates that the symmetric form is the result of the product of $[\mathbf{0}]$ with $[\mathbf{0}]$, while the fact that the tensor product is '$\hat{\otimes}$' indicates it was the result of the 'quasi' term in the Pöppe product, and an extra factor of 2 is implied. In this case if we evaluate the signature character we include an extra factor of '2' in its evaluation. Also consider the product $[\mathbf{0}]^3$ in Example 4. When we compute the Pöppe product $\big(\chi(0)\cdot[\mathbf{0}]\big)\big(\chi(0\hat{\otimes}0)\{\mathbf{010}\}\big)$, the first skew form generated, i.e. $\big[\mathbf{0}\{\mathbf{20}\}\big]$, has the coefficient $\chi(0\otimes0\hat{\otimes}0)$ as we might expect, using the homomorphic properties of χ. However the second term generated by the product $\big(\chi(0)\cdot[\mathbf{0}]\big)\big(\chi(0\hat{\otimes}0)\{\mathbf{010}\}\big)$, which is $[\mathbf{01010}]$, has the coefficient $\chi(0\hat{\otimes}0\hat{\otimes}0)$. This is because this second term is the result of the 'quasi' term $2\cdot[ua\mathbf{010}bv]$ in the Pöppe product; here $ua=v$ and $bv=1\mathbf{0}$. The factor '2' is absorbed/encoded by the fact that a '$\hat{\otimes}$' tensor (instead of just '$\otimes$') is used between the first 0 and the $0\hat{\otimes}0$, the respective χ-arguments for $[\mathbf{0}]$ and $\{\mathbf{010}\}$, in the coefficient $\chi(0\hat{\otimes}0\hat{\otimes}0)$ for $[\mathbf{01010}]$ in Example 4. See Example 5 for further illustrations of this notation.

Example 5. Using the Pöppe products in Lemma 8 and that $[\mathbf{1}]=\chi(1)\cdot[\mathbf{010}]$, we have,

$$[\mathbf{1}]\,[\mathbf{0}]^2=\big(\chi(1)\cdot[\mathbf{010}]\big)\big(\chi(0\hat{\otimes}0)\cdot\{\mathbf{010}\}\big)=\chi(1\otimes0\hat{\otimes}0)\cdot\big(\big[\mathbf{0}2\{\mathbf{010}\}\big]+\big[\mathbf{010}\{\mathbf{20}\}\big]\big)+\chi(1\hat{\otimes}0\hat{\otimes}0)\cdot[\mathbf{0101010}],$$
$$[\mathbf{0}]^2\,[\mathbf{1}]=\big(\chi(0\hat{\otimes}0)\cdot\{\mathbf{010}\}\big)\big(\chi(1)\cdot[\mathbf{010}]\big)=\chi(0\hat{\otimes}0\otimes1)\cdot\big(\big[\mathbf{0}2[\mathbf{010}]\big]+\big[\mathbf{010}[\mathbf{20}]\big]\big)+\chi(0\hat{\otimes}0\hat{\otimes}1)\cdot[\mathbf{0101010}].$$

Definition 10 (*Derivation Endomorphism*)**.** Given any word $w\times\boldsymbol{\varphi}\in\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ with $w\times\boldsymbol{\varphi}=\theta_1a_1\theta_2\cdots\theta_ka_k\theta_{k+1}$, we define the *derivation endomorphism* $\mathfrak{d}$ on $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ to be the linear expansion,

$$\mathfrak{d}(w\times\boldsymbol{\varphi}):=\sum_{\ell=1}^{k}\theta_1a_1\theta_2\cdots\theta_\ell(a_\ell+1)\theta_{\ell+1}\cdots\theta_ka_k\theta_{k+1}+\sum_{\ell=1}^{k+1}\theta_1a_1\theta_2\cdots\theta_{\ell-1}a_{\ell-1}(\mathfrak{d}\theta_\ell)a_\ell\theta_{\ell+1}\cdots\theta_ka_k\theta_{k+1},$$

where $\mathfrak{d}\theta_\ell$ equals $\mathbf{010}$ or $\mathbf{0}^\dagger1^\dagger\mathbf{0}^\dagger$, depending respectively on whether θ_ℓ is $\mathbf{0}$ or $\mathbf{0}^\dagger$.

Remark 10. The action of the derivation endomorphism on $\mathbf{0}$ and $\mathbf{0}^\dagger$ shown in the definition reflects the signature expansions, either at the kernel or abstract level. In this case here, we know $\partial V=V(\mathrm{i}P)_1V$ and $\partial V^\dagger=V^\dagger(\mathrm{i}P)_1^\dagger V^\dagger$ or equivalently $\mathbf{1}=\chi(1)\cdot\mathbf{010}$ and $\mathbf{1}^\dagger=\chi(1)\cdot\mathbf{0}^\dagger1^\dagger\mathbf{0}^\dagger=-\mathbf{0}^\dagger1\mathbf{0}^\dagger$. Similarly, the action of the derivation endomorphism on any signature expansion, say $\boldsymbol{n}$, is given by, $\mathfrak{d}:\boldsymbol{n}\mapsto(\boldsymbol{n}+\mathbf{1})$, and similarly for $\boldsymbol{n}^\dagger$.

Now suppose, within $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$, we restrict ourselves to the set of skew-symmetric forms $[w\times\boldsymbol{\varphi}]$. Naturally, as a vector space, $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ can be decomposed into the direct sum of the vector subspaces $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ of skew-symmetric forms, and $\mathbb{C}\{\mathbb{Z}_{\mathbf{0}}\}$ of symmetric forms:

$$\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle=\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]\oplus\mathbb{C}\{\mathbb{Z}_{\mathbf{0}}\}.$$

We observe from the Pöppe products in Lemma 8, the product $[w_1\times\boldsymbol{\varphi}_1]\,[w_2\times\boldsymbol{\varphi}_2]$ does not generate a skew-symmetric form but a symmetric one. However any triple product $[w_1\times\boldsymbol{\varphi}_1]\,[w_2\times\boldsymbol{\varphi}_2]\,[w_3\times\boldsymbol{\varphi}_3]$ does generate a skew-symmetric form. This is true for any Pöppe products involving an odd number of skew-symmetric forms. Hence we can define a subalgebra of the Pöppe algebra $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ which we denote by $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]\subseteq\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$, which is generated by skew-symmetric forms and triple products of such forms.

Definition 11 (*Skew-Pöppe Algebra*)**.** We call $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ the *skew-Pöppe algebra*.

Remark 11 (*Practical Pöppe Algebra Computations*)**.** In practice, in particular in the next two sections, we perform calculations in the "enveloping" algebra $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$, and at the end, show that the result remains closed within the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. However, the skew-Pöppe algebra and its triple product structure is crucial to the proof of our main result in Section 6.

4. Hierarchy examples

We use the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ to establish integrability for examples from the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries hierarchy. Hereafter we assume Hilbert–Schmidt operators P and G satisfy, respectively, the linear dispersive partial differential equation $\partial_tP=-\mu_n(\mathrm{i}\mathcal{I})^{n-1}\partial^nP$ and the linear Fredholm equation $\mathrm{i}P=G(\mathrm{id}+P^2)$. We also assume $P^\dagger=P$. We observe that with $V:=(\mathrm{id}-\mathrm{i}P)^{-1}$, we have,

$$G=V(\mathrm{i}P)V^\dagger.$$

Note we scale this by a factor '2' presently so that it matches the expression in Definition 4. We record the following identities that prove useful below. Also recall the identities in Lemma 4 and Remark 3.

Lemma 9. *The block operators P and V satisfy the following identities,*

$$P\mathcal{I}=-\mathcal{I}P,\qquad \mathcal{I}V=V^\dagger\mathcal{I}\qquad\text{and}\qquad V\mathcal{I}=\mathcal{I}V^\dagger.$$

Proof. The first identity follows from the block structures assumed for P and $\mathcal{I}$. The latter two identities follow using the power series expansion for $V:=(\mathrm{id}-\mathrm{i}P)^{-1}$. □

We now rescale our definition for G above by a factor '2', and set,

$$G:=V-V^\dagger.$$

Hereafter, we are thus concerned with the quantity $[V]:=[\![V-V^\dagger]\!]$. Using that $\partial_tV=V\partial_t(\mathrm{i}P)V$ and $\partial_tV^\dagger=V^\dagger\partial_t(\mathrm{i}P)^\dagger V^\dagger$, and that $\partial_t(\mathrm{i}P)=-\mu_n(\mathrm{i}\mathcal{I})^{n-1}\partial^n(\mathrm{i}P)$ and $\partial_t(\mathrm{i}P)^\dagger=-(-1)^{n-1}\mu_n\partial^n(\mathrm{i}P)^\dagger(\mathrm{i}\mathcal{I})^{n-1}$, we observe that for any $n\in\mathbb{Z}$, we have

$$\partial_t[V]=[\![V\partial_t(\mathrm{i}P)V]\!]-[\![V^\dagger\partial_t(\mathrm{i}P)^\dagger V^\dagger]\!]=-\mu_n\Big([\![V(\mathrm{i}\mathcal{I})^{n-1}\partial^n(\mathrm{i}P)V]\!]-(-1)^{n-1}[\![V^\dagger\partial^n(\mathrm{i}P)^\dagger(\mathrm{i}\mathcal{I})^{n-1}V^\dagger]\!]\Big).$$

For convenience set $\mathcal{M}_n := -\mu_n(\mathrm{i}\mathcal{I})^{n-1}$. Using the identities in Lemma 9, we have,

$$\mathcal{M}_n^{-1}\partial_t[V] = \begin{cases} [V(\mathrm{i}P)_n V], & \text{when } n \text{ is odd},\\ [V^\dagger(\mathrm{i}P)_n V], & \text{when } n \text{ is even}. \end{cases}$$

We express $\mathcal{M}_n^{-1}\partial_t[V]$ in the skew-Pöppe algebra as follows.

Definition 12 (*Time-derivation Endomorphism*)**.** Given $n \in \mathbb{Z}$, we define the *time-derivation endomorphism* $\mathfrak{e}_n : \mathbb{C}[\mathbb{Z}_{\mathbf{0}}] \to \mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ by,

$$\mathfrak{e}_n : [\mathbf{0}] \mapsto \begin{cases} [\mathbf{0}n\mathbf{0}], & \text{when } n \text{ is odd},\\ [\mathbf{0}n\mathbf{0}^\dagger], & \text{when } n \text{ is even}. \end{cases}$$

The nonlinear fields we seek are expressed in the skew-Pöppe algebra as follows.

Definition 13 (*Pöppe Polynomials*)**.** For $n \in \mathbb{N}\cup\{0\}$, let $\pi_n = \pi_n\big([\mathbf{0}],[\mathbf{1}],\ldots,[\boldsymbol{n}]\big)$ denote a polynomial consisting of a linear combination of odd-degree monomials of signature expansions in the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ of the form,

$$\pi_n := \sum_{k=1(\text{odd})}^{n} \sum_{a_1a_2\cdots a_k} c_{a_1a_2\cdots a_k}\cdot[\boldsymbol{a}_1]\,[\boldsymbol{a}_2]\,\ldots\,[\boldsymbol{a}_k].$$

The first sum is over odd values of k. The second sum is over all words $a_1a_2\cdots a_k$ we can construct from the alphabet $\{0,1,2,\ldots,n\}$ such that $a_1+a_2+\cdots+a_k = n-(k-1)$. This ensures π_n is an odd polynomial. The coefficients $c_{a_1a_2\cdots a_k}$ are scalar constants.

Our goal is to show $\mathfrak{e}_n\big([\mathbf{0}]\big)$ can be expressed in terms of a Pöppe polynomial $\pi_n = \pi_n\big([\mathbf{0}],[\mathbf{1}],\ldots,[\boldsymbol{n}]\big)$ in the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. Thus for each $n\in\mathbb{N}\cup\{0\}$, our goal is to determine the coefficients $c_{a_1a_2\cdots a_k}$ such that,

$$\mathfrak{e}_n = \pi_n.$$

The examples below correspond to the simple cases $n = 0,1,2,3,4$.

Example 6 (*Linear Ordinary Differential Equation: $n=0$*)**.** We observe $\mathfrak{e}_0\big([\mathbf{0}]\big) = [\mathbf{000}^\dagger]$. Recall that $a^\dagger = -a$ for letters from $\mathbb{Z}$ in $(\mathbb{Z}_{\mathbf{0}})^*$, including $a=0$. Hence we observe,

$$[\mathbf{000}^\dagger] = \mathbf{000}^\dagger - \mathbf{0}^\dagger 0^\dagger \mathbf{0} = \mathbf{000}^\dagger + \mathbf{0}^\dagger 0\mathbf{0} = \mathbf{0} - \mathbf{0}^\dagger = [\mathbf{0}].$$

In other words $\mathfrak{e}_0\big([\mathbf{0}]\big) = [\mathbf{0}]$ which translates to the following linear ordinary differential equation, $\partial_t g = \mathcal{M}_0\, g$, for $g = [\![G]\!]$, with $\mathcal{M}_0 = \mu_0\mathrm{i}\mathcal{I}$.

Example 7 (*Linear Wave Equation: $n=1$*)**.** We observe $\mathfrak{e}_1\big([\mathbf{0}]\big) = [\mathbf{010}]$. From Definition 7, we have $[\mathbf{1}] = [\mathbf{010}]$, since $\chi(1)=1$. From Definition 10 for the derivation endomorphism, we know $\mathfrak{d}[\mathbf{0}] = [\mathbf{1}]$. Hence we have, $\mathfrak{e}_1\big([\mathbf{0}]\big) = \mathfrak{d}[\mathbf{0}]$ in $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. This translates to the linear wave equation, $\partial_t g = \mathcal{M}_1\,\partial g$, for $g = [\![G]\!]$, with $\mathcal{M}_1 = -\mu_1\,\mathrm{id}$,

Example 8 (*Nonlinear Schrödinger Equation: $n=2$*)**.** We observe $\mathfrak{e}_2\big([\mathbf{0}]\big) = [\mathbf{020}^\dagger]$. Using the homomorphic properties of χ, the values for the signature coefficients given in Definition 5 and that each tensor product '$\hat{\otimes}$' under χ generates a real factor of 2, we have $\chi(0\otimes 0\hat{\otimes}0) = 2$ and $\chi(0\hat{\otimes}0\hat{\otimes}0) = 4$. Then from Example 4, we see that we have, $[\mathbf{0}]^3 = 2\cdot\big[\mathbf{0}\{\mathbf{20}\}\big] + 4\cdot[\mathbf{01010}] = 2\cdot\big[\mathbf{020}\big] - 2\cdot[\mathbf{020}^\dagger] + 4\cdot[\mathbf{01010}]$, using that $\big[\mathbf{0}\{\mathbf{20}\}\big] = [\mathbf{020}] - [\mathbf{020}^\dagger]$, and that $2^\dagger = -2$. The signature expansion for $[\mathbf{2}] = \mathfrak{d}^2[\mathbf{0}]$ is given by, $[\mathbf{2}] = \chi(2)\cdot[\mathbf{020}] + \chi(11)\cdot[\mathbf{01010}] = [\mathbf{020}] + 2\cdot[\mathbf{01010}]$. Hence we observe, $\mathfrak{e}_2\big([\mathbf{0}]\big) = [\mathbf{2}] - \frac{1}{2}\cdot[\mathbf{0}]^3$ in $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. This translates to the non-commutative nonlinear Schrödinger equation, $\mathcal{M}_2^{-1}\partial_t g = \partial^2 g - \frac{1}{2}g^3$, for $g = [\![G]\!]$, with $\mathcal{M}_2 = -\mu_2\,(\mathrm{i}\mathcal{I})$.

Remark 12 (*Rescaling*)**.** In all examples, rescaling g to $2\,g$ recovers the usual corresponding equations in the non-commutative nonlinear Schrödinger hierarchy. This is because we assumed $G := V - V^\dagger$ rather than $G = V(\mathrm{i}P)V^\dagger \equiv \frac{1}{2}(V-V^\dagger)$.

Example 9 (*Modified Korteweg–de Vries Equation: $n=3$*)**.** We observe $\mathfrak{e}_3\big([\mathbf{0}]\big) = [\mathbf{030}]$. Recall from Example 4 that, $[\mathbf{0}]^2 = \chi([0\otimes 0])\cdot\{\mathbf{010}\}$. Note this lies in $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ as opposed to the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. From the results of Example 5, evaluating the signature characters, we know $[\mathbf{1}]\,[\mathbf{0}]^2 = 2\cdot\big(\big[\mathbf{02}\{\mathbf{010}\}\big] + \big[\mathbf{010}\{\mathbf{20}\}\big]\big) + 4\cdot[\mathbf{0101010}]$ and $[\mathbf{0}]^2\,[\mathbf{1}] = 2\cdot\big(\big[\mathbf{02}[\mathbf{010}]\big] + \big[\mathbf{010}[\mathbf{20}]\big]\big) + 4\cdot[\mathbf{0101010}]$. Then using the properties of the skew form from Definition 9, we see that,

$$\big[\mathbf{02}\{\mathbf{010}\}\big] + \big[\mathbf{02}[\mathbf{010}]\big] = 2\cdot\big[\mathbf{02010}\big] \quad\text{and}\quad \big[\mathbf{010}\{\mathbf{20}\}\big] + \big[\mathbf{010}[\mathbf{20}]\big] = 2\cdot\big[\mathbf{01020}\big].$$

The signature expansion for $[\mathbf{3}] = \mathfrak{d}^3[\mathbf{0}]$ is given by,

$$\begin{aligned}[\mathbf{3}] &= \chi(3)\cdot[\mathbf{030}] + \chi(21)\cdot[\mathbf{02010}] + \chi(12)\cdot[\mathbf{01020}] + \chi(111)\cdot[\mathbf{0101010}]\\ &= [\mathbf{030}] + 3\cdot[\mathbf{02010}] + 3\cdot[\mathbf{01020}] + 6\cdot[\mathbf{0101010}].\end{aligned}$$

Hence, $\mathfrak{e}_3\big([\mathbf{0}]\big) = [\mathbf{3}] - \frac{3}{4}\cdot\big([\mathbf{1}]\,[\mathbf{0}]^2 + [\mathbf{0}]^2\,[\mathbf{1}]\big)$, in $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. This is the non-commutative modified Korteweg–de Vries equation, $\mathcal{M}_3^{-1}\partial_t g = \partial^3 g - \frac{3}{4}\big((\partial g)g^2 + g^2(\partial g)\big)$, for $g = [\![G]\!]$, with $\mathcal{M}_3 = \mu_3\,\mathrm{id}$. See Table 2.

Example 10 (*Fourth Order Quintic Nonlinear Schrödinger Equation: $n=4$*)**.** In this case we know $\mathfrak{e}_4\big([\mathbf{0}]\big) = [\mathbf{040}^\dagger]$. For this and higher orders, our procedure needs to be systematic. The Pöppe polynomial in this case has the form,

$$\pi_4 := c_4\cdot[\mathbf{4}] + c_{200}\cdot[\mathbf{2}]\,[\mathbf{0}]^2 + c_{020}[\mathbf{0}]\,[\mathbf{2}]\,[\mathbf{0}] + c_{002}\cdot[\mathbf{0}]^2\,[\mathbf{2}] + c_{110}\cdot[\mathbf{1}]^2\,[\mathbf{0}] + c_{101}\cdot[\mathbf{1}]\,[\mathbf{0}]\,[\mathbf{1}] + c_{011}\cdot[\mathbf{0}]\,[\mathbf{1}]^2 + c_{00000}\cdot[\mathbf{0}]^5.$$

The signature expansion for $[\mathbf{4}]$ has the form,

$$
\begin{aligned}
[\mathbf{4}] = {} & \chi(4)\cdot[\mathbf{040}] + \chi(31)\cdot[\mathbf{03010}] + \chi(22)\cdot[\mathbf{02020}] + \chi(13)\cdot[\mathbf{01030}] \\
& + \chi(211)\cdot[\mathbf{0201010}] + \chi(121)\cdot[\mathbf{0102010}] + \chi(112)\cdot[\mathbf{0101020}] + \chi(1111)\cdot[\mathbf{010101010}].
\end{aligned}
$$

Using the skew and symmetric Pöppe products in Lemma 8 we find, for example, that,

$$
\begin{aligned}
[\mathbf{2}]\,[\mathbf{0}]^2 &= \big(\chi(2)\cdot[\mathbf{020}] + \chi(11)\cdot[\mathbf{01010}]\big)\,\big(\chi(0\hat{\otimes}0)\cdot\{\mathbf{010}\}\big) \\
&= \chi(2\otimes 0\hat{\otimes}0)\cdot\big(\big[\mathbf{03}\{\mathbf{010}\}\big] + \big[\mathbf{020}\{\mathbf{20}\}\big]\big) + \chi(2\hat{\otimes}0\hat{\otimes}0)\cdot[\mathbf{0201010}] \\
&\quad + \chi(11\otimes 0\hat{\otimes}0)\cdot\big(\big[\mathbf{0102}\{\mathbf{010}\}\big] + \big[\mathbf{01010}\{\mathbf{20}\}\big]\big) + \chi(11\hat{\otimes}0\hat{\otimes}0)\cdot[\mathbf{010101010}].
\end{aligned}
$$

The other products shown in π_4 can be similarly expanded. In Table 3 we list all the basis elements and corresponding coefficients generated by all the Pöppe products present in π_4. The values of the coefficients are the χ-images of the tensored terms shown. Each row generates a linear algebraic equation for the expansion coefficients c_4, c_{200}, c_{110}, $\ldots c_{00000}$. Note that in Table 3 rows are ordered according to descent order, with a sub-order for the positions of the $\mathbf{0}^\dagger$ letters as indicated. The ordering of the columns is self-evident from the structure present in the table. We discuss this ordering in more explicitly in Section 6. Using all the rows shown, we generate an over-determined linear system of algebraic equations, $AC = B$, where B is the column vector shown in the right-hand column in Table 3, C is the vector of coefficients c_4, c_{020}, and so forth in the order shown across the top row. The matrix A consists of the χ-images of the entries shown in the table (neglecting the final column). From the augmented matrix $[A\;B]$, we observe the first two rows generate a system of equations, namely $c_4 + 2c_{020} = 0$ and $-2c_{020} = 1$. This system of equations corresponds to the following smaller augmented matrix subsystem $[A_0\;B_0]$ for c_4 and c_{020}, where

$$
A_0 = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \qquad \text{and} \qquad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

Hence we deduce $c_4 = 1$ and $c_{020} = -\frac{1}{2}$. With these values in hand, we then observe that the next two rows generate a system of equations for c_{200} and c_{011} given by $4c_4 + 2c_{020} + 2c_{200} + 2c_{011} = 0$ and $2c_{020} - 2c_{200} + 2c_{011} = 0$. This linear system of equations for the two unknowns c_{200} and c_{011} is represented by the smaller augmented matrix subsystem $[A_1\;B_1]$, where,

$$
A_1 = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix} \qquad \text{and} \qquad B_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.
$$

Solving the linear system of equations, we deduce that $c_{200} = -1$ and $c_{011} = -1/2$. The next four rows in the augmented matrix $[A\;B]$, given the coefficients we have already solved for, generate a system of equations for c_{110}, c_{101}, c_{002} and c_{00000}, generated by the smaller augment matrix $[A_2\;B_2]$, where,

$$
A_2 = \begin{pmatrix} 1 & 1 & 2 & 4 \\ -1 & 1 & 0 & -4 \\ 1 & -1 & 0 & -4 \\ -1 & -1 & 2 & 4 \end{pmatrix} \qquad \text{and} \qquad B_2 = \begin{pmatrix} -5/2 \\ -5/2 \\ -1/2 \\ -3/2 \end{pmatrix}.
$$

The solution to this system is given by $c_{110} = -1/2$, $c_{101} = -3/2$, $c_{002} = -1$ and $c_{00000} = 3/8$. It is easy to check the equations represented by the remaining rows in the big augmented matrix $[A\;B]$ above, are consistent. Thus, we deduce that $\mathfrak{c}_4\big([\mathbf{0}]\big) = \pi_4$, where the coefficients c_4, c_{020} and so forth, are given by the unique values outlined above. The fourth order non-commutative nonlinear Schrödinger equation for $g = [\![G]\!]$, with $\mathcal{M}_4 = \mu_4 \mathrm{i}\mathcal{I}$, is given by,

$$
\mathcal{M}_4^{-1}\partial_t g = \partial^4 g - (\partial^2\, g)g^2 - \tfrac{1}{2}g(\partial^2\, g)g - g^2(\partial^2\, g) - \tfrac{1}{2}(\partial g)^2 g - \tfrac{3}{2}(\partial g)g(\partial g) - \tfrac{1}{2}g(\partial g)^2 + \tfrac{3}{8}g^5.
$$

This matches the form given in Malham [9] and Nijhoff et al. [6, eq. B.4a].

Remark 13 (*Basis Elements*)**.** In Tables 2 and 3 we record the basis elements of the form $[w\times\boldsymbol{\varphi}]$ in the far left column. The composition components w are compositions of n, i.e. compositions of 3 and 4, in the respective tables. The $\mathbb{Z}_2^*$-component $\boldsymbol{\varphi}$ in the basis element is in principle any possible $(|w|+1)$-tuples that can be constructed from $\{\mathbf{0},\mathbf{0}^\dagger\}\cong\mathbb{Z}_2$. However, recall $\big[(w\times\boldsymbol{\varphi})^\dagger\big] = -[w\times\boldsymbol{\varphi}]$ and Definition 8. Using this property for $\big[(w\times\boldsymbol{\varphi})^\dagger\big]$, for any basis element $[w\times\boldsymbol{\varphi}]$, we can thus always arrange for the first component of $\boldsymbol{\varphi}$ to be '$\mathbf{0}$'; as can be observed in the tables. The basis elements are ordered according to descent order for the compositions w and natural binary ordering for the $\mathbb{Z}_2^*$-components $\boldsymbol{\varphi}$. For more details see Definition 15 in Section 6, and the subsequent discussion therein. Note that though the first component of $\boldsymbol{\varphi}$ can always be arranged to be $\mathbf{0}$, in our computations involving Pöppe products, we often utilise the symmetry $\big[(w\times\boldsymbol{\varphi})^\dagger\big] = -[w\times\boldsymbol{\varphi}]$ in order to use the Pöppe products listed in Lemma 8. Thus temporarily, the first component in some factors is sometimes $\mathbf{0}^\dagger$. However, we always use the same symmetry again to convert the final answer to the form with $\mathbf{0}$ as the first component in the basis element.

5. Non-commutative Lax hierarchy and the sine–Gordon equation

Herein we establish the non-commutative nonlinear Schrödinger and modified Korteweg–de Vries Lax hierarchy iteratively, order by order. The non-commutative modified Korteweg–de Vries hierarchy can be found for example in Carillo and Schiebold [35, eq. (9)]. Importantly, this iterative hierarchy extends to all negative orders. The first member of negative order, i.e. for which $n = -1$, corresponds to the non-commutative sine–Gordon cubic-form equation in Example 2; see, for example, Tracy and Widom [14] for the scalar case. Establishing the hierarchy for all orders $n\in\mathbb{Z}$ is particularly simple in the Pöppe algebra $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$. We need to define some natural actions on $\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ first.

Definition 14 (*Adjoint and Symmetric Algebra Products and Actions*)**.** We define the standard commutation and symmetric products, respectively, $\mathrm{ad}:\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle\times\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle\to\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ and $\mathrm{sd}:\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle\times\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle\to\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$. For example, for $[\mathbf{0}]\in\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]\subset\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ and any word $w\times\boldsymbol{\varphi}\in\mathbb{C}\langle\mathbb{Z}_{\mathbf{0}}\rangle$ we have,

$$
\mathrm{ad}_{[\mathbf{0}]}(w\times\boldsymbol{\varphi}) := [\mathbf{0}]\,(w\times\boldsymbol{\varphi}) - (w\times\boldsymbol{\varphi})\,[\mathbf{0}] \qquad \text{and} \qquad \mathrm{sd}_{[\mathbf{0}]}(w\times\boldsymbol{\varphi}) := [\mathbf{0}]\,(w\times\boldsymbol{\varphi}) + (w\times\boldsymbol{\varphi})\,[\mathbf{0}],
$$

which is the exclusive form of their action we use below. We also define the following two actions on the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$. For $[\mathbf{0}]\in\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ set,

$$A := \frac{1}{4}\mathrm{ad}_{[\mathbf{0}]}\mathfrak{d}^{-1}\mathrm{ad}_{[\mathbf{0}]} \qquad \text{and} \qquad S := \frac{1}{4}\mathrm{sd}_{[\mathbf{0}]}\mathfrak{d}^{-1}\mathrm{sd}_{[\mathbf{0}]}.$$

That the actions of A and S are closed in $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ is established as part of the proof of the following crucial lemma.

Lemma 10 (*Natural Iteration*). *For any $n\in\mathbb{Z}$ we have:*

$$(\mathfrak{d}-A)[\mathbf{0}n\mathbf{0}^\dagger]=[\mathbf{0}(n+1)\mathbf{0}] \qquad \textit{and} \qquad (\mathfrak{d}-S)[\mathbf{0}n\mathbf{0}]=[\mathbf{0}(n+1)\mathbf{0}^\dagger].$$

Proof. By direct computation using Lemma 8 we have,

$$\mathrm{ad}_{[\mathbf{0}]}\,[\mathbf{0}n\mathbf{0}^\dagger]=[\mathbf{0}][\mathbf{0}n\mathbf{0}^\dagger]-[\mathbf{0}n\mathbf{0}^\dagger][\mathbf{0}]=2\cdot\big(\{\mathbf{0}(n+1)\mathbf{0}^\dagger\}+\{\mathbf{0}\mathbf{1}\mathbf{0}n\mathbf{0}^\dagger\}+\{\mathbf{0}n\mathbf{0}^\dagger\mathbf{1}^\dagger\mathbf{0}^\dagger\}\big)=2\cdot\mathfrak{d}\,\{\mathbf{0}n\mathbf{0}^\dagger\}.$$

Hence we observe, using that $\{\mathbf{0}n\mathbf{0}^\dagger\}=-\{\mathbf{0}^\dagger n\mathbf{0}\}$ we have,

$$A\,[\mathbf{0}n\mathbf{0}^\dagger]=\tfrac{1}{2}\big([\mathbf{0}]\,\{\mathbf{0}n\mathbf{0}^\dagger\}+\{\mathbf{0}^\dagger n\mathbf{0}\}\,[\mathbf{0}]\big)=[\mathbf{0}(n+1)\mathbf{0}^\dagger]-[\mathbf{0}(n+1)\mathbf{0}]+[\mathbf{0}\mathbf{1}\mathbf{0}n\mathbf{0}^\dagger]+[\mathbf{0}^\dagger n\mathbf{0}\mathbf{1}\mathbf{0}]=\mathfrak{d}\,[\mathbf{0}n\mathbf{0}^\dagger]-[\mathbf{0}(n+1)\mathbf{0}].$$

This gives the first result. Again, by direct computation, we have,

$$\mathrm{sd}_{[\mathbf{0}]}\,[\mathbf{0}n\mathbf{0}]=[\mathbf{0}]\,[\mathbf{0}n\mathbf{0}]+[\mathbf{0}n\mathbf{0}]\,[\mathbf{0}]=2\,\big(\{\mathbf{0}(n+1)\mathbf{0}\}+\{\mathbf{0}\mathbf{1}\mathbf{0}n\mathbf{0}\}+\{\mathbf{0}n\mathbf{0}\mathbf{1}\mathbf{0}\}\big)=2\cdot\mathfrak{d}\,\{\mathbf{0}n\mathbf{0}\}.$$

Hence we observe,

$$S\,[\mathbf{0}n\mathbf{0}]=\tfrac{1}{2}\big([\mathbf{0}]\,\{\mathbf{0}n\mathbf{0}\}+\{\mathbf{0}n\mathbf{0}\}\,[\mathbf{0}]\big)=[\mathbf{0}(n+1)\mathbf{0}]+[\mathbf{0}\mathbf{1}\mathbf{0}n\mathbf{0}]+[\mathbf{0}n\mathbf{0}\mathbf{1}\mathbf{0}]-[\mathbf{0}(n+1)\mathbf{0}^\dagger]=\mathfrak{d}\,[\mathbf{0}n\mathbf{0}]-[\mathbf{0}(n+1)\mathbf{0}^\dagger].$$

This gives the second result. □

The following immediate corollary is established straightforwardly by induction, both when n is positive as well as negative. For the remainder of this section we refer to Pöppe polynomials π_n for $n\in\mathbb{Z}$, though in Definition 13, $n\in\mathbb{N}\cup\{0\}$ for which $\pi_n=\pi_n\big([\mathbf{0}],[\mathbf{1}],\ldots,[\boldsymbol{n}]\big)$. The form of π_n for the negative integer cases is given presently.

Corollary 2 (*Non-commutative Lax Hierarchy Iteration*). *Let $n\in\mathbb{Z}$ be a given integer, and consider the equation, $\mathfrak{e}_n\big([\mathbf{0}]\big)=\pi_n$, where π_n is a Pöppe polynomial and $\mathfrak{e}_n\big([\mathbf{0}]\big)$ equals $[\mathbf{0}n\mathbf{0}]$ if n is odd, and equals $[\mathbf{0}n\mathbf{0}^\dagger]$ if n is even. For $n=0,1,2,3,4$ such polynomials π_n exist, as demonstrated in Examples 6—10. Then we have,*

$$\mathfrak{e}_{n+1}\big([\mathbf{0}]\big)=\begin{cases}(\mathfrak{d}-A)\pi_n, & \textit{when } n \textit{ is even,}\\ (\mathfrak{d}-S)\pi_n, & \textit{when } n \textit{ is odd.}\end{cases}$$

Proof. This follows directly from Lemma 10 for $n\in\mathbb{Z}$ by induction.

Further, we have the following additional immediate corollary.

Corollary 3 (*Non-commutative Lax Hierarchy*). *For any $n\in\mathbb{Z}$, the $(n+1)$th order member equation of the non-commutative Lax hierarchy is given by,*

$$\mathfrak{e}_{n+1}\big([\mathbf{0}]\big)=\begin{cases}(\mathfrak{d}-A)\big((\mathfrak{d}-S)(\mathfrak{d}-A)\big)^{\frac{n}{2}}\,[\mathbf{0}], & \textit{when } n \textit{ is even,}\\ \big((\mathfrak{d}-S)(\mathfrak{d}-A)\big)^{\frac{1}{2}(n+1)}\,[\mathbf{0}], & \textit{when } n \textit{ is odd.}\end{cases}$$

The Lax hierarchy stated in Corollary 3, at each odd order n, exactly matches that quoted for the non-commutative modified Korteweg–de Vries hierarchy in Carillo and Schiebold [35, eq. (9)]. Given the main existence and uniqueness result we prove in Section 6, this is expected. The cases $n=0,1,2,3,4$ in Corollary 3 naturally match the non-commutative equation members given in Examples 6–10. However, Corollary 3 also applies for negative n. Consider the example case of order '-1'.

Example 11 (*Non-commutative sine–Gordon Cubic-form Equation: Order* '-1'). Setting $n=-2$, a case when n is even, in Corollary 3, generates the equation,

$$\mathfrak{e}_{-1}\big([\mathbf{0}]\big)=(\mathfrak{d}-A)\big((\mathfrak{d}-S)(\mathfrak{d}-A)\big)^{-1}\,[\mathbf{0}] \quad\Leftrightarrow\quad (\mathfrak{d}-S)\,\mathfrak{e}_{-1}\big([\mathbf{0}]\big)=[\mathbf{0}].$$

We can express the equation on the right as follows,

$$\mathfrak{d}\mathfrak{e}_{-1}\big([\mathbf{0}]\big)=[\mathbf{0}]+\tfrac{1}{4}\Big(\big(\mathfrak{d}^{-1}\mathfrak{e}_{-1}([\mathbf{0}]^2)\big)\,[\mathbf{0}]+[\mathbf{0}]\,\big(\mathfrak{d}^{-1}\mathfrak{e}_{-1}([\mathbf{0}]^2)\big)\Big),$$

where we have used that $\mathrm{sd}_{[\mathbf{0}]}\,\mathfrak{e}_{-1}([\mathbf{0}])=\mathfrak{e}_{-1}([\mathbf{0}]^2)$ from Lemma 11 just below. This relation in $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$ translates to the non-commutative sine–Gordon cubic-form equation given in Example 2 for $g=[\![G]\!]$, with $\mathcal{M}_{-1}=\mu_{-1}\,\mathrm{id}$, i.e.

$$\partial\partial_t g-\tfrac{1}{4}\big((\partial^{-1}\partial_t g^2)\,g+g\,(\partial^{-1}\partial_t g^2)\big)=\mu_{-1}g.$$

Invoking the factor '2' rescaling mentioned in Remark 12 gives the exact match.

Lemma 11. *For any $n\in\mathbb{Z}$ we have,*

$$\mathfrak{e}_n\big([\mathbf{0}]^2\big)=\begin{cases}\mathrm{sd}_{[\mathbf{0}]}\,\mathfrak{e}_n\big([\mathbf{0}]\big), & \textit{when } n \textit{ is odd,}\\ \mathrm{ad}_{[\mathbf{0}]}\,\mathfrak{e}_n\big([\mathbf{0}]\big), & \textit{when } n \textit{ is even.}\end{cases}$$

Proof. This result is established at the operator level. Recall $\mathcal{M}_n := -\mu_n(\mathrm{i}\mathcal{I})^{n-1}$ and the properties outlined in Lemma 9. We observe when n is even, we have,

$$\partial_t[V]^2 = \mathcal{M}_n\big[V(\mathrm{i}P)_n V^\dagger\big]\,[V] + [V]\,\mathcal{M}_n\big[V(\mathrm{i}P)_n V^\dagger\big] = \mathcal{M}_n\big[V(\mathrm{i}P)_n V^\dagger\big]\,[V] - \mathcal{M}_n[V]\,\big[V(\mathrm{i}P)_n V^\dagger\big],$$

where we have used that $[V]\,\mathcal{M}_n = [\![V - V^\dagger]\!]\,\mathcal{M}_n = \mathcal{M}_n\,[\![V^\dagger - V]\!] = -\mathcal{M}_n[V]$. When n is odd, we follow an analogous computation with $[V(\mathrm{i}P)_n V]$ replacing $[V(\mathrm{i}P)_n V^\dagger]$ just above, and that in this case there is no sign change in the second term on the right as $[V]\,\mathcal{M}_n = [\![V - V^\dagger]\!]\,\mathcal{M}_n = \mathcal{M}_n\,[\![V - V^\dagger]\!] = \mathcal{M}_n[V]$. $\square$

6. Hierarchy uniqueness

Herein we prove that at each order $n \geqslant 0$, the Pöppe polynomial signature expansion π_n such that $\mathfrak{c}_n([\mathbf{0}]) = \pi_n$, exists, and is unique. This is our main result, Theorem 1. Before proving this result, we present one further example, the $n = 5$ case, which acts a useful reference for our general argument, and also prove Corollary 1, which is a corollary to Theorem 1.

Example 12 (*Fifth Order Quintic Modified Korteweg–de Vries Equation: $n = 5$*). In this case $\mathfrak{c}_5\big([\mathbf{0}]\big) = [\mathbf{050}]$ and the Pöppe polynomial π_5, in general, has the form,

$$\begin{aligned}\pi_5 := {}& c_5\cdot[\mathbf{5}] + c_{300}\cdot[\mathbf{3}]\,[\mathbf{0}]^2 + c_{030}\cdot[\mathbf{0}]\,[\mathbf{3}]\,[\mathbf{0}] + c_{003}\cdot[\mathbf{0}]^2\,[\mathbf{3}] + c_{210}\cdot[\mathbf{2}]\,[\mathbf{1}]\,[\mathbf{0}] + c_{201}\cdot[\mathbf{2}]\,[\mathbf{0}]\,[\mathbf{1}] + c_{120}\cdot[\mathbf{1}]\,[\mathbf{2}]\,[\mathbf{0}]\\ &+ c_{102}\cdot[\mathbf{1}]\,[\mathbf{0}]\,[\mathbf{2}] + c_{021}\cdot[\mathbf{0}]\,[\mathbf{2}]\,[\mathbf{1}] + c_{012}\cdot[\mathbf{0}]\,[\mathbf{1}]\,[\mathbf{2}] + c_{111}\cdot[\mathbf{1}]\,[\mathbf{1}]\,[\mathbf{1}] + c_{10000}\cdot[\mathbf{1}]\,[\mathbf{0}]^4 + c_{01000}\cdot[\mathbf{0}]\,[\mathbf{1}]\,[\mathbf{0}]^3\\ &+ c_{00100}\cdot[\mathbf{0}]^2\,[\mathbf{1}]\,[\mathbf{0}]^2 + c_{00010}\cdot[\mathbf{0}]^3\,[\mathbf{1}]\,[\mathbf{0}] + c_{00001}\cdot[\mathbf{1}]\,[\mathbf{0}]^4.\end{aligned}$$

The signature expansion for $[\mathbf{5}]$ has the form,

$$\begin{aligned}[\mathbf{5}] = {}& \chi(5)\cdot[\mathbf{050}] + \chi(41)\cdot[\mathbf{04010}] + \chi(32)\cdot[\mathbf{03020}] + \chi(23)\cdot[\mathbf{02030}] + \chi(14)\cdot[\mathbf{01040}]\\ &+ \chi(311)\cdot[\mathbf{0301010}] + \chi(221)\cdot[\mathbf{0202010}] + \chi(212)\cdot[\mathbf{0201020}]\\ &+ \chi(131)\cdot[\mathbf{0103010}] + \chi(122)\cdot[\mathbf{0102020}] + \chi(113)\cdot[\mathbf{0101030}]\\ &+ \chi(2111)\cdot[\mathbf{020101010}] + \chi(1211)\cdot[\mathbf{010201010}] + \chi(1121)\cdot[\mathbf{010102010}]\\ &+ \chi(1112)\cdot[\mathbf{010101020}] + \chi(11111)\cdot[\mathbf{01010101010}].\end{aligned}$$

Using the skew and symmetric Pöppe products in Lemma 8 we find, for example, that,

$$\begin{aligned}[\mathbf{1}]\,[\mathbf{2}]\,[\mathbf{0}] &= \big(\chi(1)\cdot[\mathbf{010}]\big)\,\big(\chi(2)\cdot[\mathbf{020}] + \chi(11)\cdot[\mathbf{01010}]\big)\,\big(\chi(0)\cdot[\mathbf{0}]\big)\\ &= \big(\chi(1)\cdot[\mathbf{010}]\big)\,\big(\chi(2\otimes 0)\cdot\big\{\mathbf{03}[\mathbf{0}]\big\} + \chi(2\hat{\otimes}0)\cdot\{\mathbf{02010}\} + \chi(11\otimes 0)\cdot\big\{\mathbf{0102}[\mathbf{0}]\big\} + \chi(11\hat{\otimes}0)\cdot\{\mathbf{0101010}\}\big)\\ &= \chi(1\otimes 2\otimes 0)\cdot\big(\big[\mathbf{02}\{\mathbf{03}[\mathbf{0}]\}\big] + \big[\mathbf{010}\{\mathbf{4}[\mathbf{0}]\}\big]\big) + \chi(1\hat{\otimes}2\otimes 0)\cdot\big[\mathbf{010103}[\mathbf{0}]\big]\\ &\quad + \chi(1\otimes 2\hat{\otimes}0)\cdot\big(\big[\mathbf{02}\{\mathbf{02010}\}\big] + \big[\mathbf{010}\{\mathbf{3010}\}\big]\big) + \chi(1\hat{\otimes}2\hat{\otimes}0)\cdot[\mathbf{010102010}]\\ &\quad + \chi(1\otimes 11\otimes 0)\cdot\big(\big[\mathbf{02}\{\mathbf{0102}[\mathbf{0}]\}\big] + \big[\mathbf{010}\{\mathbf{102}[\mathbf{0}]\}\big]\big) + \chi(1\hat{\otimes}11\otimes 0)\cdot\big[\mathbf{01010102}[\mathbf{0}]\big]\\ &\quad + \chi(1\otimes 11\hat{\otimes}0)\cdot\big(\big[\mathbf{02}\{\mathbf{0101010}\}\big] + \big[\mathbf{010}\{\mathbf{201010}\}\big]\big) + \chi(1\hat{\otimes}11\hat{\otimes}0)\cdot\big[\mathbf{01010101010}\big].\end{aligned}$$

The other products shown in π_5 can be similarly expanded. In Tables 4 and 5 we list the essential basis elements and corresponding signature coefficients generated by all the Pöppe products present in π_5. The values of the coefficients are the χ-images of the tensored terms shown. Each row generates a linear algebraic equation for the expansion coefficients $c_5, c_{300}, c_{210}, \ldots c_{00001}$. The ordering of rows and columns in Tables 4 and 5 is self-evident. We discuss this ordering in detail presently. Using all the rows shown, we generate an over-determined linear system of algebraic equations, $AC = B$, where B is the column vector shown in the right-hand column in Table 5 and C is the vector of coefficients c_5, c_{030}, c_{300}, c_{021} and so forth in the order shown. The matrix A is populated with the χ-images of the signature coefficients shown in the tables. We can in fact solve $AC = B$ for C systematically, block by block, as follows. The first two rows corresponding to $[\mathbf{050}]$ and $[\mathbf{050}^\dagger]$ generate the pair of equations, $c_5 + 2c_{030} = 1$ and $c_{030} = 0$. This system of equations corresponds to the smaller augmented subsystem $[A_0\ B_0']$ where A_0 is the same matrix as in Example 10 for the order $n = 4$ case, and $B_0' = (1,0)^{\mathrm{T}}$. We deduce $c_5 = 1$ and $c_{030} = 0$. The next two rows corresponding to $[\mathbf{04010}]$ and $[\mathbf{040}^\dagger\mathbf{10}^\dagger]$ generate the pair of equations $5c_5 + 2c_{030} + 2c_{300} + 2c_{021} = 0$ and $2c_{030} - 2c_{300} + 2c_{021} = 0$ for c_{300} and c_{021}. This pair corresponds to the smaller augmented subsystem $[A_1\ B_1']$ where A_1 is the same matrix as in Example 10, and $B_1' = (-5,0)^{\mathrm{T}}$. Hence we deduce $c_{021} = -5/4$ and $c_{300} = -5/4$. Then the equations corresponding to the rows $[\mathbf{03020}]$, $[\mathbf{03020}^\dagger]$, $[\mathbf{030}^\dagger\mathbf{20}]$ and $[\mathbf{030}^\dagger\mathbf{20}^\dagger]$ generate the following smaller augmented matrix subsystem $[A_2\ B_2']$ for c_{210}, c_{201}, c_{012} and c_{01000}, where, A_2 is the same as the corresponding matrix in Example 10 and $B_2' = (-25/4, -5/4, 5/4, 5/4)^{\mathrm{T}}$. This system of linear equations is easily solved to reveal $c_{210} = -5/4$, $c_{201} = -5/2$, $c_{012} = -5/4$ and $c_{01000} = 0$. As we shall see, it is no coincidence that the coefficient matrices A_0, A_1 and A_2 match those for the analogous blocks of basis elements in Example 10. The next set of rows corresponding to the basis elements $[\mathbf{02030}]$, $[\mathbf{02030}^\dagger]$, $[\mathbf{020}^\dagger\mathbf{30}]$ and $[\mathbf{020}^\dagger\mathbf{30}^\dagger]$, generates exactly the same augmented matrix subsystem $[A_2\ B_2']$ as that discussed just above, but now for the unknown coefficients c_{120}, c_{102}, c_{003} and c_{00010}. This linear system reveals $c_{120} = -5/4$, $c_{102} = -5/2$, $c_{003} = -5/4$ and $c_{00010} = 0$. We do not deduce any new information from the equations corresponding to the rows $[\mathbf{01040}]$ and $[\mathbf{01040}^\dagger]$, other than that they are consistent. We consider the next block of four rows shown in Tables 4 and 5 corresponding to the rows $[\mathbf{0202010}]$, $[\mathbf{02020}^\dagger\mathbf{10}^\dagger]$, $[\mathbf{020}^\dagger\mathbf{2010}]$ and $[\mathbf{020}^\dagger\mathbf{20}^\dagger\mathbf{10}^\dagger]$. These basis elements generate the following augmented matrix subsystem $[A_3'\ B_3]$ for c_{111}, c_{10000}, c_{00100} and c_{00001}, where,

$$A_3' = \begin{pmatrix} 1 & 4 & 4 & 4\\ 1 & -4 & -4 & 4\\ 1 & -4 & 4 & -4\\ 1 & 4 & -4 & -4 \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} 5\\ -5\\ -5\\ -5 \end{pmatrix}.$$

This system of linear equations is easily solved to reveal $c_{111} = -5/2$, $c_{10000} = 5/8$, $c_{00002} = 5/8$ and $c_{00001} = 5/8$. We have determined the unique set of coefficients for which $\mathfrak{c}_5\big([\mathbf{0}]\big) = \pi_5$. In principle we can check the equations generated by the rows corresponding to the remaining basis elements are consistent. However, we explain in our proof of our main result (Step 8) why this is not necessary. Hence the fifth order non-commutative modified Korteweg–de Vries for $g = [\![G]\!]$, with $\mathcal{M}_5 = \mu_5\,\mathrm{id}$, is given by (this matches the form given in Nijhoff et al. [6, eq. B.5a]),

$$\mathcal{M}_5^{-1}\partial_t g = \partial^5 g - \frac{5}{4}\Big((\partial^3 g)g^2 + g^2(\partial^3 g) + (\partial^2 g)(\partial g)g + 2\,(\partial^2 g)g(\partial g) + (\partial g)(\partial^2 g)g + 2\,(\partial g)g(\partial^2 g) + g(\partial^2 g)(\partial g) + g(\partial g)(\partial^2 g) + 2\,(\partial g)(\partial g)(\partial g)\Big) + \frac{5}{8}\Big((\partial g)g^4 + g^2(\partial g)g^2 + g^4(\partial g)\Big).$$

Proof of Corollary 1. When n is even, from Corollary 2, $\mathfrak{e}_{n+1}([\mathbf{0}]) = (\partial - A)\pi_n$. From Theorem 1 we know $\mathfrak{e}_{n+1}([\mathbf{0}]) = \pi_{n+1}$ with $\pi_{n+1} = \pi_{n+1}([\mathbf{0}],[\mathbf{1}],\ldots,[\boldsymbol{n}+1])$ a unique polynomial in its arguments. We thus deduce the result of the corollary when n is even. An exactly analogous argument follows for the case when n is odd. □

We now prove Theorem 1 via a sequence of steps. It requires some preparation and the rest of this section is devoted to outlining the notation, strategy, ideas and intermediate results we require to carry through the proof in Steps 1–7 before giving the overall proof in Step 8. Roughly, we construct a 'table' for the coefficients of π_n, much like we did for the cases $n = 3,4,5$ in Tables 2–5. Indeed, Tables 4 and 5 act as a useful reference. We proceed systematically, considering basis elements $[w \times \boldsymbol{\varphi}]$ in descending order with respect to the composition C-component w and using a natural binary order for the $\mathbb{Z}_2^*$-component $\boldsymbol{\varphi}$. Both these orders are implicit in Tables 2–5.

Step 1: Descent and binary order. We introduce an order, the descent order, on the set of basis elements. We also define a new representation for the basis elements, the composition-binary representation, that we use hereafter. The descent order for compositions is given in Malham [10]; we present it here for completeness.

Definition 15 (*Descent Ordering of Compositions*)**.** A composition $u \in C$ precedes another composition $v \in C$ if the length of the composition u, i.e. the number of digits it contains, is strictly less than the length of v. If u and v have the same length, say k, so $u = u_1u_2\cdots u_k$ and $v = v_1v_2\cdots v_k$, then u precedes v if for some $\ell \in \{1,2,\ldots,k\}$ we have $u_1 = v_1$, $u_2 = v_2$, $\ldots$, $u_{\ell-1} = v_{\ell-1}$ and $u_\ell < v_\ell$. Otherwise, v precedes u. The resulting ordering induced on C, is the *descent ordering*.

The $\mathbb{Z}_2^*$-component $\boldsymbol{\varphi}$ in the basis element $[w\times\boldsymbol{\varphi}]$ is any $(|w|+1)$-tuple constructed from $\{\mathbf{0},\mathbf{0}^\dagger\} \cong \mathbb{Z}_2$. Recall from Remark 13, we can always arrange for the first component of $\boldsymbol{\varphi}$ to be '$\mathbf{0}$'; see Tables 2–5. Thus for any given composition $w \in C$, the component $\boldsymbol{\varphi} \in \mathbb{Z}_2^*$ in a basis element $[w\times\boldsymbol{\varphi}]$ is one of the following forms,

$$\boldsymbol{\phi}_1 := \mathbf{00}\cdots\mathbf{0000},\ \boldsymbol{\phi}_2 := \mathbf{00}\cdots\mathbf{0000}^\dagger,\ \boldsymbol{\phi}_3 := \mathbf{00}\cdots\mathbf{000}^\dagger\mathbf{0},\ \boldsymbol{\phi}_4 := \mathbf{00}\cdots\mathbf{000}^\dagger\mathbf{0}^\dagger,$$
$$\boldsymbol{\phi}_5 := \mathbf{00}\cdots\mathbf{00}^\dagger\mathbf{00},\ \boldsymbol{\phi}_6 := \mathbf{00}\cdots\mathbf{00}^\dagger\mathbf{00}^\dagger,\ \boldsymbol{\phi}_7 := \mathbf{00}\cdots\mathbf{00}^\dagger\mathbf{0}^\dagger\mathbf{0},$$

and so forth, all the way up to $\boldsymbol{\phi}_{2^{|w|}} := \mathbf{00}^\dagger\cdots\mathbf{0}^\dagger\mathbf{0}^\dagger\mathbf{0}^\dagger\mathbf{0}^\dagger$. This is the *natural binary ordering* of $\mathbb{Z}_2^*$ we refer to just above. In this and the next sections, we mainly use a modified encoding of the basis elements $[w\times\boldsymbol{\varphi}]$, as follows. We replace the free monoid $\mathbb{Z}_2^*$ of all forms $\boldsymbol{\varphi}$ that can be constructed from $\{\mathbf{0},\mathbf{0}^\dagger\} \cong \mathbb{Z}_2$ by the vector space $\mathbb{R}\langle\mathbb{B}\rangle$ representing the span over all the elements $\mathbb{B} := \{\boldsymbol{\phi}_i\}_{i\geqslant 1}$. We thus express any $\boldsymbol{\varphi}\in\mathbb{Z}_2^*$ in the form $\boldsymbol{\varphi} = \beta_1\boldsymbol{\phi}_1 + \beta_2\boldsymbol{\phi}_2 + \beta_3\boldsymbol{\phi}_3 + \cdots$, where the β_i for integer $i \geqslant 1$, represent the coefficients of the basis element components $\boldsymbol{\phi}_i$. Henceforth we represent any element $\boldsymbol{\varphi}\in\mathbb{Z}_2^*$ by a $2^{|w|}$-tuple $\boldsymbol{\beta} := (\beta_1,\beta_2,\ldots,\beta_{2^{|w|}}) \in \mathbb{R}\langle\mathbb{B}\rangle$. Thus we replace,

$$[w\times\boldsymbol{\varphi}] \rightsquigarrow [w]\times\boldsymbol{\beta}.$$

Example 13. Some examples matching the old notation with the new are as follows: $[\mathbf{0}a\mathbf{0}] = [a]\times(1,0)$, $[\mathbf{0}a\mathbf{0}^\dagger] = [a]\times(0,1)$, and then also,

$$\begin{aligned}
[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}] &= [a_1a_2]\times(1,0,0,0),\\
[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}^\dagger] &= [a_1a_2]\times(0,1,0,0),\\
[\mathbf{0}a_1\mathbf{0}^\dagger a_2\mathbf{0}] &= [a_1a_2]\times(0,0,1,0),\\
[\mathbf{0}a_1\mathbf{0}^\dagger a_2\mathbf{0}^\dagger] &= [a_1a_2]\times(0,0,0,1),\\
[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}a_3\mathbf{0}] &= [a_1a_2a_3]\times(1,0,0,0,0,0,0,0),\\
[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}a_3\mathbf{0}^\dagger] &= [a_1a_2a_3]\times(0,1,0,0,0,0,0,0),\\
&\ \vdots\\
[\mathbf{0}a_1\mathbf{0}^\dagger a_2\mathbf{0}^\dagger a_3\mathbf{0}^\dagger] &= [a_1a_2a_3]\times(0,0,0,0,0,0,0,1),
\end{aligned}$$

and so forth. Naturally, as constructed, for linear combinations, we have for example that for any real scalar constants β_1 and β_2,

$$\beta_1\cdot[\mathbf{0}a_1\mathbf{0}a_2\mathbf{0}] + \beta_2\cdot[\mathbf{0}a_1\mathbf{0}^\dagger a_2\mathbf{0}] = [a_1a_2]\times(\beta_1,0,\beta_2,0),$$

and so forth. There is a natural basis for elements of $\mathbb{R}\langle\mathbb{B}\rangle$ of a given length 2^n. Such a basis is given by the elements of length 2^n of the form $\boldsymbol{\beta}_i := (0,\ldots,0,1,0,\ldots)$, where the '1' is in the ith position.

Definition 16 (*Composition-binary Representation*)**.** We call the representation $[w]\times\boldsymbol{\beta}$ where $w\in C$ and $\boldsymbol{\beta}\in\mathbb{R}\langle\mathbb{B}\rangle$ the *composition-binary representation* of the basis elements. We call $[w]$ the composition component and $\boldsymbol{\beta}$ the $\mathbb{R}\langle\mathbb{B}\rangle$-component.

Step 2: Triple product action. All the Pöppe polynomials π_n are polynomials in the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_\mathbf{0}]$ and thus necessarily of odd degree. As such all the monomials therein can be constructed from triple products of signature expansions $[\boldsymbol{n}]$, as we have seen in Examples 8–12. In particular we characterise the following triple product action which is established straightforwardly using the Pöppe product rules in Lemma 8.

Lemma 12. *Consider the two signature expansions $[\boldsymbol{a}]$ and $[\boldsymbol{b}]$ and a basis element $[cw\times\boldsymbol{\varphi}]\in\mathbb{C}[\mathbb{Z}_\mathbf{0}]$, with $c\in\mathbb{Z}$ the first letter in the composition 'cw'. Let $\hat{\boldsymbol{\varphi}}$ denote the element of $\mathbb{Z}_2^*$ given by $\boldsymbol{\varphi}$ with its first letter '$\mathbf{0}$' removed. Then at leading order the triple product action $[\boldsymbol{a}][\boldsymbol{b}]\big([cw\times\boldsymbol{\varphi}]\big)$ is given by,*

$$\chi(a\otimes b\otimes cw)\cdot\Big([\mathbf{0}(a+1)\mathbf{0}(b+1)\mathbf{0}c(w\times\hat{\boldsymbol{\varphi}})] + [\mathbf{0}(a+1)\mathbf{0}^\dagger(b+1)\mathbf{0}c(w\times\hat{\boldsymbol{\varphi}})]$$

$$+[\mathbf{0}(a+1)\mathbf{0}(b+1)\mathbf{0}^\dagger c(w\times\hat{\boldsymbol{\varphi}})^\dagger]+[\mathbf{0}(a+1)\mathbf{0}^\dagger(b+1)\mathbf{0}^\dagger c(w\times\hat{\boldsymbol{\varphi}})^\dagger]$$
$$+[\mathbf{0}(a+1)\mathbf{0}b\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})]+[\mathbf{0}(a+1)\mathbf{0}^\dagger b\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})]$$
$$+[\mathbf{0}(a+1)\mathbf{0}b\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]+[\mathbf{0}(a+1)\mathbf{0}^\dagger b\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]$$
$$+2\cdot[\mathbf{0}a\mathbf{0}(b+2)\mathbf{0}c(w\times\hat{\boldsymbol{\varphi}})]+2\cdot[\mathbf{0}a\mathbf{0}(b+2)\mathbf{0}^\dagger c(w\times\hat{\boldsymbol{\varphi}})^\dagger]$$
$$+[\mathbf{0}a\mathbf{0}(b+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})]+[\mathbf{0}a\mathbf{0}(b+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})]$$
$$+[\mathbf{0}a\mathbf{0}(b+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]+[\mathbf{0}a\mathbf{0}(b+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]\Big)+\cdots.$$

Here by leading order, we mean, we do not retain terms generated by lower (descent) order terms in the signature expansions of $[\boldsymbol{a}]$ *and* $[\boldsymbol{b}]$*, nor do we retain terms generated with a quasi-product term—i.e. generated using any of the final terms with real factor '2' in the Pöppe products in Lemma 8. We use the notation '*$+\cdots$*' do denote these missing terms.*

We also define the following auto-tensorial action on $\mathbb{R}\langle\mathbb{B}\rangle$.

Definition 17 (*Tensorial Action*)**.** Given $\boldsymbol{\beta} := (\beta_1,\beta_2,\ldots)$ and $\boldsymbol{\gamma} := (\gamma_1,\gamma_2,\ldots)$ in $\mathbb{R}\langle\mathbb{B}\rangle$ we define the (left) *auto-tensorial action* of $\boldsymbol{\beta}$ on $\boldsymbol{\gamma}$, denoted $\boldsymbol{\beta}\triangleleft\boldsymbol{\gamma}$, to be,

$$\boldsymbol{\beta}\triangleleft\boldsymbol{\gamma} := \big(\beta_1\cdot\boldsymbol{\gamma},\beta_2\cdot\boldsymbol{\gamma},\ldots\big),$$

where for each $i=1,2,\ldots$, we note $\beta_i\cdot\boldsymbol{\gamma}=(\beta_i\gamma_1,\beta_i\gamma_2,\ldots)$.

In the new notation, with the tensorial action on $\mathbb{R}\langle\mathbb{B}\rangle$ just defined, the triple product action on $\mathbb{C}[\mathbb{Z}_\mathbf{0}]$ given in Lemma 12 can be expressed more succinctly as follows.

Corollary 4 (*Triple Product Action*)**.** *Given two signature expansions* $[\boldsymbol{a}]$ *and* $[\boldsymbol{b}]$ *and a generic basis element* $[cw]\times\boldsymbol{\beta}\in\mathbb{C}[C]\times\mathbb{R}\langle\mathbb{B}\rangle\cong\mathbb{C}[\mathbb{Z}_\mathbf{0}]$*, the triple product action* $[\boldsymbol{a}]\,[\boldsymbol{b}]\,\big([cw]\times\boldsymbol{\beta}\big)$ *on* $\mathbb{C}[C]\times\mathbb{R}\langle\mathbb{B}\rangle$ *is given at leading order by,*

$$\begin{aligned}[\boldsymbol{a}]\,[\boldsymbol{b}]\,\big([cw]\times\boldsymbol{\beta}\big)=\chi(a\otimes b\otimes cw)\cdot\Big(&[(a+1)(b+1)cw]\times\big((1,0,1,0)\triangleleft\boldsymbol{\beta}+(-1)^{|w|}\,(0,1,0,1)\triangleleft\boldsymbol{\beta}^\dagger\big)\\&+[(a+1)b(c+1)w]\times\big((1,0,0,1)\triangleleft(\boldsymbol{\beta}+(-1)^{|w|}\,\boldsymbol{\beta}^\dagger)\big)\\&+[a(b+2)cw]\times\big((2,0,0,0)\triangleleft\boldsymbol{\beta}+(-1)^{|w|}\,(0,2,0,0)\triangleleft\boldsymbol{\beta}^\dagger\big)\\&+[a(b+1)(c+1)w]\times\big((1,1,0,0)\triangleleft(\boldsymbol{\beta}+(-1)^{|w|}\,\boldsymbol{\beta}^\dagger)\big)\Big)+\cdots.\end{aligned}$$

Here, if $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3,\ldots,\beta_{2^n})$*, then* $\boldsymbol{\beta}^\dagger=(\beta_{2^n},\beta_{2^n-1},\ldots,\beta_2,\beta_1)$*.*

Proof. The result of the corollary is just a restatement of the triple product action in Lemma 12. That the adjoint of $\boldsymbol{\beta}$, denoted $\boldsymbol{\beta}^\dagger$, corresponds to reversing the elements in $\boldsymbol{\beta}$ is explained as follows. The entries in $\boldsymbol{\beta}$ correspond to the coefficients of the basis $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, …, $\boldsymbol{\phi}_{2^n}$, for some $n\in\mathbb{N}$. The triple product action in Lemma 12 involves the components $\hat{\boldsymbol{\varphi}}^\dagger$ where $\hat{\boldsymbol{\varphi}}$ corresponds to $\boldsymbol{\varphi}$ with its first letter '$\mathbf{0}$' removed. Let $\hat{\boldsymbol{\phi}}_1$, $\hat{\boldsymbol{\phi}}_2$, …, $\hat{\boldsymbol{\phi}}_{2^n}$, be the same sequence of basis elements each of which has the first letter '$\mathbf{0}$' removed. We observe, $\big\{\hat{\boldsymbol{\phi}}_1^\dagger,\hat{\boldsymbol{\phi}}_2^\dagger,\ldots,\hat{\boldsymbol{\phi}}_{2^n}^\dagger\big\}=\big\{\hat{\boldsymbol{\phi}}_{2^n},\hat{\boldsymbol{\phi}}_{2^n-1},\ldots,\hat{\boldsymbol{\phi}}_1\big\}$. □

Example 14. Consider computing the triple product $[\boldsymbol{a}]\,[\boldsymbol{b}]\,[\boldsymbol{c}]$ to leading order. In this case, to leading order $[\boldsymbol{c}]=[\mathbf{0}c\mathbf{0}]+\cdots$ and so in Corollary 4 we have $w=\nu$, the empty word, with $|w|=0$. We observe, $[\mathbf{0}c\mathbf{0}]=[c]\times\boldsymbol{\gamma}$ with $\boldsymbol{\gamma}=(1,0)\in\mathbb{R}\langle\mathbb{B}\rangle$. Hence we have,

$$(1,0,1,0)\triangleleft\boldsymbol{\gamma}=\big(1\cdot(1,0),\,0\cdot(1,0),\,1\cdot(1,0),\,0\cdot(1,0)\big)=(1,0,0,0,1,0,0,0),$$
$$(0,1,0,1)\triangleleft\boldsymbol{\gamma}^\dagger=\big(0\cdot(0,1),\,1\cdot(0,1),\,0\cdot(0,1),\,1\cdot(0,1)\big)=(0,0,0,1,0,0,0,1),$$

and so forth. Hence we observe from Corollary 4 that at leading order,

$$\begin{aligned}[\boldsymbol{a}]\,[\boldsymbol{b}]\,[\boldsymbol{c}]=\chi(a\otimes b\otimes c)\cdot\big(&[(a+1)(b+1)c]\times(1,0,0,1,1,0,0,1)+[(a+1)b(c+1)]\times(1,1,0,0,0,0,1,1)\\&+[a(b+2)c]\times(2,0,0,2,0,0,0,0)+[a(b+1)(c+1)]\times(1,1,1,1,0,0,0,0)\big)+\cdots.\end{aligned}$$

We extensively use such computations hereafter.

Step 3: Generators, a coarse-grain overview. For a given basis element $[w]\times\boldsymbol{\beta}$ and composition w of $n\in\mathbb{N}$, it is useful to identify the types of odd-degree monomials of signature expansions that might generate it.

Definition 18 (*Monomial Generator*)**.** Given a basis element $[w]\times\boldsymbol{\beta}$ with a composition component $[w]$ and an $\mathbb{R}\langle\mathbb{B}\rangle$-component $\boldsymbol{\beta}$, where w is a composition of $n\in\mathbb{N}$ and $\boldsymbol{\beta}$ has length 2^n, we call any odd-degree monomial of signature expansions of the form $[\boldsymbol{a}_1]\,[\boldsymbol{a}_2]\,\ldots\,[\boldsymbol{a}_{2m+1}]$ that produces $[w]\times\boldsymbol{\beta}$ as one of the terms in its expansion, a *monomial generator* or just *generator* of $[w]\times\boldsymbol{\beta}$.

At this stage and in this step, it is useful to give a brief coarse overview of our overall strategy, which we implement in detail in the subsequent steps below. We show in this step how, for any given composition component, we can identify, for the associated basis elements, specific collections of generators. We call the sets of basis elements and corresponding collections of generators "coefficient blocks" or simply "blocks". We show that such blocks are necessarily square. To start, consider any one-part composition w of n, so the two corresponding basis elements are $[\mathbf{0}n\mathbf{0}]\rightsquigarrow[n]\times(1,0)$ and $[\mathbf{0}n\mathbf{0}^\dagger]\rightsquigarrow[n]\times(0,1)$. The basis element $[n]\times(1,0)$ is generated by signature expansion $[\boldsymbol{n}]$, while $[n]\times(0,1)$ is not. On the other hand, both basis elements are generated by $[\mathbf{0}]\,[\boldsymbol{n-2}]\,[\mathbf{0}]$. This exhausts all the possible odd-degree monomials in π_n that could generate $[n]\times(1,0)$ and $[n]\times(0,1)$. Thus for a one-part composition, the possible odd-degree generators have the form,

$$[\star],\ \ [\mathbf{0}]\,[\star]\,([\mathbf{0}]),$$

where $[\star]$ represents the appropriate generic signature expansion, i.e. in the first instance it is $[\boldsymbol{n}]$ and in the second instance, i.e. for $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$, the middle $[\star]$ factor is $[\boldsymbol{n-2}]$. Note, we allow $[\star]=[\mathbf{0}]$. The only other possibilities are $[\star]\,[\mathbf{0}]\,[\mathbf{0}]$ and $[\mathbf{0}]\,[\mathbf{0}]\,[\star]$. However for $n\geqslant 3$, we can rule these two possibilities out as $[\mathbf{0}]\,[\mathbf{0}]=2\cdot\{\mathbf{010}\}$ and any subsequent Pöppe product of this term with $[\mathbf{0}]$ would generate a basis element $[w]\times\boldsymbol{\beta}$ where the composition w has two parts. Note of course, the triple Pöppe product $[\boldsymbol{a}]\,[\boldsymbol{b}]\,[\boldsymbol{c}]$ is naturally associative.

Now consider any two part composition $w=a_1a_2$ of n. We observe that basis elements with such a two-part composition component can in principle be generated by $[\star]$ and $[\mathbf{0}]\,[\star]\,[\mathbf{0}]$, which we have already come across just above. However such basis elements can also be generated by any of the following four generators of the form,

$$[\star]\,[\star]\,([\mathbf{0}]),\ [\star]\,[\mathbf{0}]\,([\star]),\ [\mathbf{0}]\,[\star]\,([\star]),\ [\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big).$$

We observe that each possible generator above contains only two '$[\star]$' factors, consistent with the two-part composition component of the basis elements we are aiming to generate. Further note that we can also see that the four generators above can be constructed from the previous two generators $[\star]$ and $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$ corresponding to one-part compositions, by applying one of the three actions $[\star]\,[\star]\,(\cdot)$, $[\star]\,[\mathbf{0}]\,(\cdot)$ or $[\mathbf{0}]\,[\star]\,(\cdot)$ to them. For example, the first generator above is constructed by applying the action $[\star]\,[\star]\,(\cdot)$ to $[\star]=[\mathbf{0}]$, where we must set the argument $[\star]=[\mathbf{0}]$ to preserve the two-part composition component of the basis element we wish to generate. The next two generators above are constructed by applying the actions $[\star]\,[\mathbf{0}]\,(\cdot)$ or $[\mathbf{0}]\,[\star]\,(\cdot)$ to $[\star]$. Now consider the final quintic generator above. Applying the action $[\star]\,[\star]\,(\cdot)$ to $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$ would produce a generator with too many '$[\star]$' factors, while in principle, either of the actions $[\star]\,[\mathbf{0}]\,(\cdot)$ or $[\mathbf{0}]\,[\star]\,(\cdot)$ could be applied to $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$. However the action of $[\star]\,[\mathbf{0}]\,(\cdot)$ on $[\mathbf{0}]\,[\star]\,(\cdot)$ is nilpotent. We demonstrate this below in Lemma 16. Hence the action $[\star]\,[\mathbf{0}]\,(\cdot)$ rigorously applied to $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$ produces zero. Thus the only viable action is $[\mathbf{0}]\,[\star]\,(\cdot)$ on $[\mathbf{0}]\,[\star]\,([\mathbf{0}])$ producing the quintic generator shown. From another perspective, for the quintic generator, in the case $n\geqslant 5$, any other quintic arrangement with two $[\star]$- and three $[\mathbf{0}]$-factors, would necessitate a consecutive pair '$[\mathbf{0}]\,[\mathbf{0}]$' that would result in generating a basis element whose composition component has more than two parts.

In the case of basis elements with a three-part composition component $w=a_1a_2a_3$, the possible generators are, in principle, any of the generators we have already seen, as well as, the generators of the form,

$$\begin{aligned}&[\star]\,[\star]\,([\star]),\ [\star]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big),\ [\star]\,[\mathbf{0}]\,\big([\star]\,[\star]\,([\mathbf{0}])\big),\ [\star]\,[\mathbf{0}]\,\big([\star]\,[\mathbf{0}]\,([\star])\big),\\ &\qquad[\mathbf{0}]\,[\star]\,\big([\star]\,[\star]\,([\mathbf{0}])\big),\ [\mathbf{0}]\,[\star]\,\big([\star]\,[\mathbf{0}]\,([\star])\big),\ [\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\star])\big),[\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big)\big).\end{aligned}$$

We remark that each possible generator above contains only three '$[\star]$' factors. We see that the first two generators are constructed by applying the action $[\star]\,[\star]\,(\cdot)$ to the generators for basis elements with one-part composition components. The next set of generators are constructed by applying the action $[\star]\,[\mathbf{0}]\,(\cdot)$ to the generators for basis elements with two-part composition components, taking into account the nilpotent action of $[\star]\,[\mathbf{0}]\,(\cdot)$ on $[\mathbf{0}]\,[\star]\,(\cdot)$. This accounts for the next two generators. Then the final four generators are constructed by applying the action $[\mathbf{0}]\,[\star]\,(\cdot)$ to all four of the generators for basis elements with two-part composition components.

For the case of basis elements with a four-part composition component $w=a_1a_2a_3a_4$, the possible generators are, in principle, besides any of the generators we have already seen, generators of the form,

$$\begin{aligned}&[\star]\,[\star]\,\big([\star]\,[\star]\,([\mathbf{0}])\big),\ [\star]\,[\star]\,\big([\star]\,[\mathbf{0}]\,([\star])\big),\ [\star]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\star])\big),\ [\star]\,[\mathbf{0}]\,\big([\star]\,[\star]\,([\star])\big),\\ &[\mathbf{0}]\,[\star]\,\big([\star]\,[\star]\,([\star])\big),\ [\star]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big)\big),\ [\star]\,[\mathbf{0}]\,\big([\star]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big)\big),\\ &[\star]\,[\mathbf{0}]\,\big([\star]\,[\mathbf{0}]\,\big([\star]\,[\star]\,([\mathbf{0}])\big)\big),\ [\star]\,[\mathbf{0}]\,\big([\star]\,[\mathbf{0}]\,\big([\star]\,[\mathbf{0}]\,([\star])\big)\big),\ [\mathbf{0}]\,[\star]\,\big([\star]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big)\big),\\ &[\mathbf{0}]\,[\star]\,\big([\star]\,[\mathbf{0}]\,\big([\star]\,[\star]\,([\mathbf{0}])\big)\big),\ [\mathbf{0}]\,[\star]\,\big([\star]\,[\mathbf{0}]\,\big([\star]\,[\mathbf{0}]\,([\star])\big)\big),\ [\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\star]\,[\star]\,([\mathbf{0}])\big)\big),\\ &[\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\star]\,[\mathbf{0}]\,([\star])\big)\big),\ [\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\star])\big)\big),[\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,\big([\mathbf{0}]\,[\star]\,([\mathbf{0}])\big)\big)\big).\end{aligned}$$

Again, each possible generator above contains only four '$[\star]$' factors. They are constructed by applying the action $[\star]\,[\star]\,(\cdot)$ to the generators for basis elements with two-part composition components, applying the action $[\star]\,[\mathbf{0}]\,(\cdot)$ to the generators for basis elements with three-part composition components, taking into account the nilpotent action of $[\star]\,[\mathbf{0}]\,(\cdot)$ on $[\mathbf{0}]\,[\star]\,(\cdot)$, and then also applying the action $[\mathbf{0}]\,[\star]\,(\cdot)$ to all of the generators for basis elements with three-part composition components.

We have seen that for basis elements with a composition component with $k=1,2,3$ or 4 parts, the number of generators that might produce such a basis element is 2^k. We have not shown that corresponding to a given basis element, the generators constructed in the manner indicated are unique at leading order. We demonstrate this below in Steps 7 and 8. Assuming this is the case for the moment, we have the following.

Lemma 13 (*Generator Block Size*)**.** *For a given basis element with composition component w, the number of monomial generators that can generate that basis element at leading order is $2^{|w|}$.*

Proof. As observed, the result is true for $|w|=1,2,3,4$. Assume the result is true for $|w|=1,2,\dots,k$ for some $k\in\mathbb{N}$. The set of generators for basis elements with composition components of $k+1$ parts are constructed by: (i) Applying the action $[\star]\,[\star]\,(\cdot)$ to the generators for basis elements with $(k-1)$-part composition components of which there are 2^{k-1} by assumption; (ii) Applying the action $[\star]\,[\mathbf{0}]\,(\cdot)$ to the generators for basis elements with k-part composition components, taking into account the nilpotent action of $[\star]\,[\mathbf{0}]\,(\cdot)$ on $[\mathbf{0}]\,[\star]\,(\cdot)$. Since there are 2^k generators corresponding to any basis element with a composition of k-parts, and half of these start with the factor '$[\mathbf{0}]\,[\star]$', there are 2^{k-1} generators constructed in this way; and then finally (iii) Applying the action $[\mathbf{0}]\,[\star]\,(\cdot)$ to all of the generators for basis elements with k-part composition components, of which there are 2^k. Adding these three contributions up, $2^{k-1}+2^{k-1}+2^k=2^{k+1}$, and the result follows by induction. □

We can also view this last result from another perspective as follows. For each k-part composition, when k is odd, the set of new generators are characterised as follows. First, we include the degree k monomial $[\star]\,[\star]\,\dots\,[\star]$, of which there is only one choice. We also include the degree $k+2$ monomials which contain two non-adjacent '$[\mathbf{0}]$' factors; there are $k+1$ choose 2 possible monomials of this form. Then we can also include degree $k+4$ monomials which contain four non-adjacent '$[\mathbf{0}]$' factors; there are $k+1$ choose 4 possible monomials of this form. And so forth until we reach the single degree $2k+1$ monomial of the form $[\mathbf{0}]\,[\star]\,[\mathbf{0}]\,[\star]\,[\mathbf{0}]\,\dots\,[\star]\,[\mathbf{0}]$. Here we have implicitly used that the number of ways to place r objects in non-adjacent slots whose total number is m, is given by $m-r+1$ choose r. In the examples just presented, we considered the number of possible ways of placing 2ℓ factors of the form '$[\mathbf{0}]$' in a monomial of degree $k+2\ell$, for $\ell=0,1,\dots,(k+1)/2$. Hence the total number of monomials of each degree outlined being $k+1$ choose 2ℓ. Thus with k odd, the total number of such new odd-degree monomials is given by the sum over $\ell = 0, 1, \dots, (k+1)/2$ of $k+1$ choose 2ℓ, i.e. the sum on the left shown in Lemma 14. Suppose now k is even. The lowest degree monomials that might generate the corresponding basis element are those of degree $k+1$ with a single factor '$[\mathbf{0}]$'. There are $k+1$ such monomials. We can also include degree $k+3$ monomials with three non-adjacent factors '$[\mathbf{0}]$'; there are $k+1$ choose 3 such possible monomials, and so forth. In the final highest degree monomial, of degree $2k+1$ has the single form $[\mathbf{0}]\,[\star]\,[\mathbf{0}]\,[\star]\,[\mathbf{0}]\,\dots\,[\star]\,[\mathbf{0}]$. Thus with k even, the total number of such new odd-degree monomials is given by the sum over $\ell = 0, 1, \dots, k/2$ of $k+1$ choose $2\ell+1$, i.e. the sum on the right shown in Lemma 14. In consequence we have the following important result (whose proof is straightforward).

Lemma 14. *The aforementioned sums, in the respective k is odd and then even cases, are equal to 2^k. In other words, respectively, when k is odd and then even, we have,*

$$\sum_{\ell=0}^{(k+1)/2} \binom{k+1}{2\ell} = 2^k \qquad \text{and} \qquad \sum_{\ell=0}^{k/2} \binom{k+1}{2\ell+1} = 2^k.$$

The crucial observation from the result of Lemmas 13 and 14 is the following.

Corollary 5 (*Generator-tuple Dimension Match*)**.** *For a given composition component w of a block set of basis elements $[w] \times \boldsymbol{\beta}$ parametrised by the tuples $\boldsymbol{\beta} \in \mathbb{R}\langle\mathbb{B}\rangle$, the number of new monomial generators equals the dimension of the tuple block, i.e. $2^{|w|}$.*

One of our main concerns now is to show that the resulting square block of signature coefficients has full rank. The next three steps address this issue, making the analysis of this section more precise.

Step 4: The three standard triple actions. We have seen that the triple product action in Corollary 4, in the full form given therein, as well as in the special forms $[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot)$ and $[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)$, are used to construct the generators corresponding to a given basis element. We call these three actions the standard triple actions.

Definition 19 (*Standard Triple Actions*)**.** We call the actions $[\boldsymbol{a}]\,[\boldsymbol{b}]\,(\cdot)$, $[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot)$ and $[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)$ the three *standard* triple actions.

The result of the action $[\boldsymbol{a}]\,[\boldsymbol{b}]\,(\cdot)$ is given in Corollary 4. As we use them frequently hereafter, we record the result of the standard actions $[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot)$ and $[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)$ in the following Corollary. They are just special cases which we call the *special actions*.

Corollary 6 (*Special Actions*)**.** *The two special actions $[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot)$ and $[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)$ are given at leading order by,*

$$\begin{aligned}
[\boldsymbol{a}]\,[\mathbf{0}]\,\big([cw]\times\boldsymbol{\beta}\big) &= \chi(a\otimes 0\otimes cw)\cdot[(a+1)(c+1)w]\times\big((1,-1)\triangleleft\big(\boldsymbol{\beta}+(-1)^{|w|}\boldsymbol{\beta}^\dagger\big)\big)+\cdots,\\
[\mathbf{0}]\,[\boldsymbol{b}]\,\big([cw]\times\boldsymbol{\beta}\big) &= \chi(0\otimes b\otimes cw)\cdot\Big([(b+2)cw]\times\big((2,0)\triangleleft\boldsymbol{\beta}+(-1)^{|w|}(0,2)\triangleleft\boldsymbol{\beta}^\dagger\big)\\
&\qquad+[(b+1)(c+1)w]\times\big((1,1)\triangleleft\big(\boldsymbol{\beta}+(-1)^{|w|}\boldsymbol{\beta}^\dagger\big)\big)\Big)+\cdots.
\end{aligned}$$

Proof. By direct computation using the Pöppe product rules in Lemma 8, we observe that $[\boldsymbol{a}]\,[\mathbf{0}]\,\big([cw\times\boldsymbol{\varphi}]\big)$ equals,

$$\begin{aligned}
\chi(a\otimes 0\otimes cw)\cdot\big(\;&[\mathbf{0}(a+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})]+[\mathbf{0}(a+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]\\
&-[\mathbf{0}(a+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})]-[\mathbf{0}(a+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger\big)+\cdots,
\end{aligned}$$

at leading order, giving the first result. Then, by direct computation for the other case, we observe that $[\mathbf{0}]\,[\boldsymbol{b}]\,\big([cw\times\boldsymbol{\varphi}]\big)$ equals,

$$\begin{aligned}
\chi(0\otimes b\otimes cw)\cdot\big(\;&2\cdot[\mathbf{0}(b+2)\mathbf{0}c)(w\times\hat{\boldsymbol{\varphi}})]+2\cdot[\mathbf{0}(b+2)\mathbf{0}^\dagger c)(w\times\hat{\boldsymbol{\varphi}})^\dagger]\\
&+[\mathbf{0}(b+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})]+[\mathbf{0}(b+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})]\\
&+[\mathbf{0}(b+1)\mathbf{0}(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger]+[\mathbf{0}(b+1)\mathbf{0}^\dagger(c+1)(w\times\hat{\boldsymbol{\varphi}})^\dagger\big)+\cdots,
\end{aligned}$$

at leading order, giving the second result. □

Remark 14. Comparing the results of Corollary 6 with Corollary 4 we emphasise two observations, that there is: (i) A natural contraction of the action forms due to the '$[\mathbf{0}]$' factors in the action; (ii) An apparent change of sign in the second term on the right in the first example. We can view both cases as the consequence of substituting $[v\times\mathbf{0}]$ for $[\boldsymbol{b}]$ in the first case and then $[v\times\mathbf{0}]$ for $[\boldsymbol{a}]$ in the second case. The sign change, perhaps more easily observed from the corresponding result in Lemma 12, is a consequence of the fact that to make the appropriate substitution of $[v\times\mathbf{0}]$ for $[\boldsymbol{b}]$, we should convert the two terms with $\mathbf{0}^\dagger b\mathbf{0}^\dagger$ on the right, to $-\mathbf{0}^\dagger b^\dagger\mathbf{0}^\dagger$ first.

Step 5: Generating blocks. We now show precisely how, given a block of basis elements characterised by a given composition component w and parametrised by the corresponding $2^{|w|}$ basis elements $\boldsymbol{\beta}$ of $\mathbb{R}\langle\mathbb{B}\rangle$, we can use the three standard actions to enumerate all the monomial generators that produce the basis elements of that block at leading order, and also establish the corresponding signature coefficient associated with each such basis element. Let us examine the three standard actions given in Corollaries 4 and 6 more closely. If we examine the right-hand side of $[\boldsymbol{a}]\,[\boldsymbol{b}]\,\big([cw]\times\boldsymbol{\beta}\big)$ in Corollary 4, then we observe that in terms of descent order, the first composition term '$[(a+1)(b+1)cw]$' on the right is highest, and thus we retain that term only. In Corollary 6, at leading order, the action $[\boldsymbol{a}]\,[\mathbf{0}]\,\big([cw]\times\boldsymbol{\beta}\big)$ in unique, while the right-hand side of $[\mathbf{0}]\,[\boldsymbol{b}]\,\big([cw]\times\boldsymbol{\beta}\big)$ contains two terms, the first of which is higher in terms of descent order, which is the one we retain. Thus at leading order the three standard actions on $[cw]\times\boldsymbol{\beta}$ are:

$$\begin{aligned}
[\boldsymbol{a}]\,[\boldsymbol{b}]\,(\cdot) &= \chi(cw)\cdot[(a+1)(b+1)cw]\times\big((1,0,1,0)\triangleleft\boldsymbol{\beta}+(-1)^{|w|}\,(0,1,0,1)\triangleleft\boldsymbol{\beta}^\dagger\big)+\cdots,\\
[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot) &= \chi(cw)\cdot[(a+1)(c+1)w]\times\big((1,-1)\triangleleft\boldsymbol{\beta}+(-1)^{|w|}(1,-1)\triangleleft\boldsymbol{\beta}^\dagger\big)+\cdots,
\end{aligned}$$

$$[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot) = \chi(cw)\cdot[(b+2)cw]\times\big((2,0)\lhd\boldsymbol{\beta}+(-1)^{|w|}(0,2)\lhd\boldsymbol{\beta}^{\dagger}\big)+\cdots.$$

Here we have used the homomorphic properties of χ, in particular that $\chi(a\otimes b\otimes cw) = \chi(a\otimes 0\otimes cw) = \chi(0\otimes b\otimes cw) = \chi(cw)$. Consider the following respective replacements in each of the three actions above: (i) $a \to a-1$, $b \to b-1$, $c \to v$; (ii) $a \to a-1$, $c \to c-1$ and (iii) $b \to b-2$. With these three choices, each of the actions generates the same composition acw—in the first case we relabel b as c and in the third case we relabel b as a. Recall from our coarse-grain overview in Step 3 that to enumerate the generators corresponding to basis elements with composition components with $k \geqslant 2$ parts, we apply the first action to the generators at level $k-2$, and the two special actions to the generators at level $k-1$, taking into account the nilpotent action outlined just below in Lemma 16. We note that, for any sequence $\hat{u} \in \mathbb{R}\langle\mathbb{B}\rangle$, with $|\hat{u}| = 2^{k-2}$, we have,

$$(1,0,1,0)\lhd\hat{u} = (1,1)\lhd(1,0)\lhd\hat{u} = (1,1)\lhd(\hat{u},0),$$
$$(0,1,0,1)\lhd\hat{u}^{\dagger} = (1,1)\lhd(0,1)\lhd\hat{u}^{\dagger} = (1,1)\lhd(0,\hat{u}^{\dagger}),$$

where $(\hat{u},0)$ and $(0,\hat{u}^{\dagger})$ are of length 2^{k-1}. Putting these observations together, we have thus established the following lemma.

Lemma 15 (*Actions Generating the Same Composition*). *At leading order, with the choices mentioned above, the following three standard actions generate the same composition with the respective* $\mathbb{R}\langle\mathbb{B}\rangle$ *components indicated,*

$$[\boldsymbol{a-1}]\,[\boldsymbol{c-1}]\,\big([w]\times\boldsymbol{\beta}\big) = \chi(w)\cdot[acw]\times\Big((1,1)\lhd\big((\hat{u},0)-(-1)^{|w|}(\hat{u},0)^{\dagger}\big)\Big),$$
$$[\boldsymbol{a-1}]\,[\mathbf{0}]\,\big([(c-1)w]\times\boldsymbol{\beta}\big) = \chi((c-1)w)\cdot[acw]\times\Big((1,-1)\lhd\big((\hat{a},\hat{b})+(-1)^{|w|}(\hat{a},\hat{b})^{\dagger}\big)\Big),$$
$$[\mathbf{0}]\,[\boldsymbol{a-2}]\,\big([cw]\times\boldsymbol{\beta}\big) = \chi(cw)\cdot[acw]\times\big((2,0)\lhd(\hat{a},\hat{b})+(-1)^{|w|}(0,2)\lhd(\hat{a},\hat{b})^{\dagger}\big).$$

Here, in the first case $\boldsymbol{\beta} = (\hat{u},0) \in \mathbb{R}\langle\mathbb{B}\rangle$ with $\hat{u}$ arbitrary, and in the second and third cases $\boldsymbol{\beta} = (\hat{a},\hat{b}) \in \mathbb{R}\langle\mathbb{B}\rangle$ is arbitrary. Each such $\boldsymbol{\beta}$ is of length $2^{|acw|-1}$, and $\hat{a}$ and $\hat{b}$ have the same length—matching that of $\hat{u}$.

Remark 15. Note, in the statement of Lemma 15, the case of the first action which corresponds to the action $[\boldsymbol{a}]\,[\boldsymbol{b}]\,(\cdot)$ applied to $[cw]\times\boldsymbol{\beta}$ in the discussion preceding the Lemma. In that discussion, when we set $c \to v$, we equivalently replaced cw by w. This means that we should effectively consider the length of w to be one less than would otherwise be the case. This explains why the sign in front of the term with the factor $(-1)^{|w|}$ in the first action case is negative in the statement of the Lemma.

Some further clarifications on the statement of Lemma 15 are required. Note that,

$$[\mathbf{0}] \rightsquigarrow [v]\times(1),$$

where v is the empty composition and (1) is the element of $\mathbb{R}\langle\mathbb{B}\rangle$ corresponding to compositions of zero parts. The special action $[\mathbf{0}]\,[\boldsymbol{a-2}]\,(\cdot)$ in Lemma 15 still applies when the argument $[cw]\times\boldsymbol{\beta} = [v]\times(1)$ and thus when $(\hat{a},\hat{b}) = (1)$. The result is that at leading order we have,

$$[\mathbf{0}]\,[\boldsymbol{a-2}]\,\big([v]\times(1)\big) = \chi(v)\cdot[a]\times\big((2,0)\lhd(1)-(0,2)\lhd(1)^{\dagger}\big) = [a]\times(2,-2).$$

Here, by convention, we take $\chi(v) := 1$. Since we have taken $cw \to v$, we can think of the number of parts of w to be '-1', explaining the sign in front of the $\mathbb{R}\langle\mathbb{B}\rangle$-element $(0,2)$. This is consistent with just computing $[\mathbf{0}]\,[\boldsymbol{a-2}]\,[\mathbf{0}]$. Further, the first two actions in Lemma 15 do not make sense when $cw \to v$, though if $w \to v$, the special action $[\boldsymbol{a-1}]\,[\mathbf{0}]\,(\cdot)$ applies with the appropriate adaptations. And of course we can compute $[\boldsymbol{a-1}]\,[\boldsymbol{c-1}]\,\big([v]\times(1)\big) = [\boldsymbol{a-1}]\,[\boldsymbol{c-1}]\,[\mathbf{0}]$.

Finally, we now also observe the following (aforementioned) nilpotency property.

Lemma 16 (*Nilpotent Action*). *At leading order, if we first apply the action $[\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)$ to an arbitrary $\mathbb{R}\langle\mathbb{B}\rangle$ component, then apply the action $[\boldsymbol{a}]\,[\mathbf{0}]\,(\cdot)$ to the result, this generates the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component. In other words at leading order we have,*

$$[\boldsymbol{a}]\,[\mathbf{0}]\,\big([\mathbf{0}]\,[\boldsymbol{b}]\,(\cdot)\big) = 0,$$

where the '0' on the right-hand side represents the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component.

Proof. We focus on the effect of the actions on the $\mathbb{R}\langle\mathbb{B}\rangle$ components only. The third (special) action applied to the input (a,b) generates $2\cdot(a,b,\pm b^{\dagger},\pm a^{\dagger})$. Set $A, B \in \mathbb{R}\langle\mathbb{B}\rangle$ to be the sub-components $A := (a,b)$ and $B := \pm(b^{\dagger},a^{\dagger})$. With these identifications we note that $B = \pm A^{\dagger}$. Ignoring the real factor 2, apply the second (special) action to the input (A,B). This is (note the sign of the second term of the action changes), $(1,-1)\lhd\big((A,B)\mp(A,B)^{\dagger}\big)$, which equals, $(A\mp B^{\dagger}, B\mp A^{\dagger}, -A\pm B^{\dagger}, -B\pm A^{\dagger})$. Since $B = \pm A^{\dagger}$, this result is the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component. □

We now explore, through a series of examples, how to construct the generators and coefficient blocks associated with any given composition. In particular we consider the cases of compositions with 1, 2 and 3 parts, before exploring the case of any given composition. Compositions containing a '1' need to be singled out, as explained below.

Example 15 (*One-part Compositions*). We observe that there are two basis elements corresponding to the one-part composition $w = a$, namely, $[a]\times(1,0)$ and $[a]\times(0,1)$. We assume $n = a \geqslant 2$. At leading order we know from the corresponding signature expansion $[\boldsymbol{a}] = [a]\times(1,0)+\cdots$. From our discussion succeeding Lemma 15, we know the first two actions do not make sense when $cw \to v$, while the final special action does make sense. As we saw directly, at leading order we have $[\mathbf{0}]\,[\boldsymbol{a-2}]\,[\mathbf{0}] = [a]\times(2,-2)$. We have thus enumerated the generators corresponding to $[a]\times(1,0)$ and $[a]\times(0,1)$ and that the signature coefficient matrix is,

$$A_0 = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}.$$

Example 16 (*Two-part Compositions*). Consider the basis elements with a two-part composition component a_1a_2, i.e. basis elements of the form $[a_1a_2]\times\boldsymbol{\beta}_i$, where the $\boldsymbol{\beta}_i$ are the four basis elements of length 4, which are zero apart from a '1' in the ith position. For the moment assume neither

a_1 nor a_2 equal 1; we consider each of these two special cases separately below. Using Lemma 15, noting that for each of the standard actions our goal is to obtain the composition component $[a_1a_2]$ on the right-hand side, we observe the following. For the first action, setting $w=v$, $a=a_1$ and $c=a_2$, we find that at leading order, we get,

$$[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,[\boldsymbol{0}] = [a_1a_2]\times\big((1,1)\lhd\big((1,0)-(0,1)\big)\big) = [a_1a_2]\times(1,-1,1,-1).$$

The first special action in Lemma 15, with the same identifications gives,

$$[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([a_2-1]\times(\hat a,\hat b)\big) = [a_1a_2]\times\big((1,-1)\lhd\big((\hat a,\hat b)+(\hat b,\hat a)\big)\big).$$

We saw in Example 15, the basis element $[a_2-1]\times(\hat a,\hat b)$ can be generated both by the corresponding signature expansion $[\boldsymbol{a_2-1}]=[a_2-1]\times(1,0)+\cdots$, and by the generator $[\boldsymbol{0}]\,[\boldsymbol{a-2}]\,[\boldsymbol{0}]$. We discount the latter case due to the nilpotent action property. Hence using this expression for $[\boldsymbol{a_2-1}]$ and inserting $(\hat a,\hat b)=(0,1)$ into the expression above, we deduce,

$$[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([\boldsymbol{a_2-1}]\big) = [a_1a_2]\times(1,1,-1,-1),$$

to leading order. Now consider the second special action in Lemma 15. Again with the same identifications for a, c and w, we observe that to leading order,

$$[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([a_2]\times(\hat a,\hat b)\big) = [a_1a_2]\times\big((2,0)\lhd(\hat a,\hat b)+(0,2)\lhd(\hat b,\hat a)\big).$$

We know from Example 15, the basis element $[a_2]\times(\hat a,\hat b)$ can be generated either by the signature expansion $[\boldsymbol{a_2}]=[a_2]\times(1,0)+\cdots$, or by the generator $[\boldsymbol{0}]\,[\boldsymbol{a_2-2}]\,[\boldsymbol{0}]=[a_2]\times(2,-2)+\cdots$. Respectively substituting the expressions $[a_2]\times(1,0)$ and $[a_2]\times(2,-2)$ for $[a_2]\times(\hat a,\hat b)$ in the relation just above, we observe that to leading order,

$$\begin{aligned}[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{a_2}]\big) &= [a_1a_2]\times\big((2,0)\lhd(1,0)+(0,2)\lhd(0,1)\big)\\ &= [a_1a_2]\times(2,0,0,2),\\ [\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{0}]\,[\boldsymbol{a_2-2}]\,[\boldsymbol{0}]\big) &= [a_1a_2]\times\big((2,0)\lhd(2,-2)+(0,2)\lhd(-2,2)\big)\\ &= [a_1a_2]\times(4,-4,-4,4).\end{aligned}$$

We have thus enumerated the four generators corresponding to the four basis elements $[a_1a_2]\times(1,0,0,0)$, $[a_1a_2]\times(0,1,0,0)$, $[a_1a_2]\times(0,0,1,0)$ and $[a_1a_2]\times(0,0,0,1)$. They are $[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,[\boldsymbol{0}]$, $[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,[\boldsymbol{a_2-1}]$, $[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,[\boldsymbol{a_2}]$ and the quintic generator $[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,[\boldsymbol{0}]\,[\boldsymbol{a_2-2}]\,[\boldsymbol{0}]$. The corresponding signature coefficient matrix is,

$$A_2 := \begin{pmatrix}1&1&2&4\\-1&1&0&-4\\1&-1&0&-4\\-1&-1&2&4\end{pmatrix}$$

which is the subsystem coefficient matrix A_2 in Examples 10 and 12 respectively concerning the quartic and quintic non-commutative nonlinear Schrödinger equations.

Let us now consider the case when $a_2=1$. If we substitute this value for a_2 into the generators above, we see that the first two generators coincide and are given by $[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,[\boldsymbol{0}]=[a_11]\times(1,-1,1,-1)+\cdots$ and $[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,[\boldsymbol{0}]=[a_11]\times(1,1,-1,-1)+\cdots$. Since we can add them together under the same coefficient $c_{(a_1-1)00}$, in this case we have the single generator, $[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,[\boldsymbol{0}]=[a_11]\times(2,0,0,-2)+\cdots$. The third generator above becomes, $[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,[\boldsymbol{1}]=[a_11]\times(2,0,0,2)+\cdots$. The final quintic generator cannot be a generator in this case if we insist on only including signature expansions corresponding to non-negative integers. There are thus only two independent generators. Hence in this case, the corresponding signature coefficient matrix, ignoring the middle two rows, is

$$A_1 := \begin{pmatrix}2&2\\-2&2\end{pmatrix}.$$

See Examples 10 and 12 and the equations for the coefficients $c_{(n-2)00}$ and $c_{0(n-3)1}$ in those cases for when $w=(n-1)1$, as well as with the coefficients in Tables 3–5. Note, when $a_1=a_2=1$, there is only one generator, $[\boldsymbol{0}]^3$, as we saw in Example 8. We treat the more general case when $a_1=1$ at the end of this step.

Example 17 (*Three-part Compositions*)**.** Consider basis elements with a three-part composition component $a_1a_2a_3$, i.e. basis elements of the form $[a_1a_2a_3]\times\boldsymbol{\beta}_i$, where the $\boldsymbol{\beta}_i$ for $i=1,\ldots,8$, contain '1' in the ith position and zeros in the remaining seven positions. For the moment assume neither a_1 nor a_2 nor a_3 are unity. Using Lemma 15, the standard actions, setting $a=a_1$, $c=a_2$ and $w=a_3$ give to leading order,

$$\begin{aligned}[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,\big([a_3]\times(\hat u,0)\big) &= [a_1a_2a_3]\times\big((1,1)\lhd(\hat u,\hat u^\dagger)\big),\\ [\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([(a_2-1)a_3]\times(\hat a,\hat b)\big) &= \chi((a_2-1)a_3)\cdot[a_1a_2a_3]\times\big((1,-1)\lhd\big((\hat a,\hat b)-(\hat a,\hat b)^\dagger\big)\big),\\ [\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([a_2a_3]\times(\hat a,\hat b)\big) &= \chi(a_2a_3)\cdot[a_1a_2a_3]\times(2\hat a,2\hat b,-2\hat b^\dagger,-2\hat a^\dagger).\end{aligned}$$

We observe, with these three relations, the task of finding the generators for any basis element with a three-part composition component, becomes the task of finding the generators for the basis element with the one-part composition component '$[a_3]$' in the first case, and then the generators for basis elements with the two-part components '$[(a_2-1)a_3]$' and '$[a_2a_3]$' in the second and third cases. We can construct the generators in these cases via Examples 15 and 16 just above. In the first case, from Example 15, the two generators for $[a_3]\times(1,0)$ and $[a_3]\times(0,1)$ are $[\boldsymbol{a_3}]=[a_3]\times(1,0)+\cdots$ and $[\boldsymbol{0}]\,[\boldsymbol{a_3-2}]\,[\boldsymbol{0}]=[a_3]\times(2,-2)+\cdots$. Hence if we substitute these expressions into the first case above, respectively setting $\hat u=(1,0)$ and then $\hat u=(2,-2)$, we find,

$$\begin{aligned}[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,\big([\boldsymbol{a_3}]\big) &= [a_1a_2a_3]\times\big((1,1)\lhd(1,0,0,1)\big)\\ &= [a_1a_2a_3]\times(1,0,0,1,1,0,0,1),\end{aligned}$$

$$\begin{aligned}[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,\big([\boldsymbol{0}]\,[\boldsymbol{a_3-2}]\,[\boldsymbol{0}]\big) &= [a_1a_2a_3]\times\big((1,1)\lhd(2,-2,-2,2)\big)\\ &= [a_1a_2a_3]\times(2,-2,-2,2,2,-2,-2,2).\end{aligned}$$

For the second case above with composition component '$[(a_2-1)a_3]$', we know from Example 16, there are four possible generators. However once we observe the nilpotent action property, we are left with two, namely, $[\boldsymbol{a_2-2}]\,[\boldsymbol{a_3-1}]\,[\boldsymbol{0}] = [(a_2-1)a_3]\times(1,-1,1,-1)+\cdots$ and $[\boldsymbol{a_2-2}]\,[\boldsymbol{0}]\,[\boldsymbol{a_3-1}] = [(a_2-1)a_3]\times(1,1,-1,-1)+\cdots$. Substituting these expressions into the second case above, respectively setting $(\hat{a},\hat{b})=(1,-1,1,-1)$ and then $(\hat{a},\hat{b})=(1,1,-1,-1)$, we find,

$$\begin{aligned}[\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([\boldsymbol{a_2-2}]\,[\boldsymbol{a_3-1}]\,[\boldsymbol{0}]\big) &= \chi((a_2-1)a_3)\cdot[a_1a_2a_3]\times\big((1,-1)\lhd\big((1,-1,1,-1)-(1,-1,1,-1)^\dagger\big)\big)\\ &= \chi((a_2-1)a_3)\cdot[a_1a_2a_3]\times(2,-2,2,-2,-2,2,-2,2),\\ [\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([\boldsymbol{a_2-2}]\,[\boldsymbol{0}]\,[\boldsymbol{a_3-1}]\big) &= \chi((a_2-1)a_3)\cdot[a_1a_2a_3]\times\big((1,-1)\lhd\big((1,1,-1,-1)-(1,1,-1,-1)^\dagger\big)\big)\\ &= \chi((a_2-1)a_3)\cdot[a_1a_2a_3]\times(2,2,-2,-2,-2,-2,2,2).\end{aligned}$$

For the third case above with composition component '$[a_2a_3]$', again, we know from Example 16, there are four possible generators. These are all four of the generators shown in Example 16 once we replace a_1 and a_2 therein respectively by a_2 and a_3. If we substitute the corresponding four expressions with the replacements mentioned into the third case above, respectively setting $(\hat{a},\hat{b})=(1,-1,1,-1)$, $(\hat{a},\hat{b})=(1,1,-1,-1)$, $(\hat{a},\hat{b})=(2,0,0,2)$ and then $(\hat{a},\hat{b})=(4,-4,-4,4)$, we find at leading order,

$$\begin{aligned}[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{a_2-1}]\,[\boldsymbol{a_3-1}]\,[\boldsymbol{0}]\big) &= \chi(a_2a_3)\cdot[a_1a_2a_3]\times(2,-2,2,-2,2,-2,2,-2),\\ [\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{a_2-1}]\,[\boldsymbol{0}]\,[\boldsymbol{a_3-1}]\big) &= \chi(a_2a_3)\cdot[a_1a_2a_3]\times(2,2,-2,-2,2,2,-2,-2),\\ [\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{0}]\,[\boldsymbol{a_2-2}]\,[\boldsymbol{a_3}]\big) &= \chi(a_2a_3)\cdot[a_1a_2a_3]\times(4,0,0,4,-4,0,0,-4),\end{aligned}$$

and finally,

$$[\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([\boldsymbol{0}]\,[\boldsymbol{a_2-2}]\,[\boldsymbol{0}]\,[\boldsymbol{a_3-2}]\,[\boldsymbol{0}]\big) = \chi(a_2a_3)\cdot[a_1a_2a_3]\times(8,-8,-8,8,-8,8,8,-8).$$

Hence, for the eight basis elements $[a_1a_2a_3]\times\boldsymbol{\beta}_i$, $i=1,\ldots,8$, with the columns corresponding to the generators above in descent order following by degree, the corresponding signature coefficient matrix, is the full rank matrix,

$$A_3 := \begin{pmatrix}1&2&2&2&2&2&4&8\\0&-2&-2&2&-2&2&0&-8\\0&-2&2&-2&2&-2&0&-8\\1&2&-2&-2&-2&-2&4&8\\1&2&-2&-2&2&2&-4&-8\\0&-2&2&-2&-2&2&0&8\\0&-2&-2&2&2&-2&0&8\\1&2&2&2&-2&-2&-4&-8\end{pmatrix},$$

where columns 3 and 4 should involve the factor $\chi((a_2-1)a_3)$, while columns 5 through to 8 should involve the factor $\chi(a_2a_3)$. The factors are omitted in A_3 for clarity.

Remark 16. Examples 15–17 precisely reflect the analysis we outlined in Step 3.

We can now discern the pattern. Suppose we wish to construct all the generators corresponding to a full set of 2^k basis elements associated with a given composition component $[a_1a_2\cdots a_k]$. We preclude for the moment, that any of a_1 through to a_k are equal to '1'. Using the standard actions in Lemma 15, we find,

$$\begin{aligned}[\boldsymbol{a_1-1}]\,[\boldsymbol{a_2-1}]\,\big([a_3\cdots a_k]\times(\hat{u},0)\big) &= [a_1\cdots a_k]\times\big((1,1)\lhd\big((\hat{u},0)-(-1)^{k-2}(0,\hat{u}^\dagger)\big)\big),\\ [\boldsymbol{a_1-1}]\,[\boldsymbol{0}]\,\big([(a_2-1)a_3\cdots a_k]\times(\hat{a},\hat{b})\big) &= \chi((a_2-1)a_3)\cdot[a_1\cdots a_k]\times\big((1,-1)\lhd\big((\hat{a},\hat{b})+(-1)^{k-2}(\hat{a},\hat{b})^\dagger\big)\big),\\ [\boldsymbol{0}]\,[\boldsymbol{a_1-2}]\,\big([a_2\cdots a_k]\times(\hat{a},\hat{b})\big) &= \chi(a_2a_3)\cdot[a_1\cdots a_k]\times\big((2,0)\lhd(\hat{a},\hat{b})+(-1)^{k-2}(0,2)\lhd(\hat{a},\hat{b})^\dagger\big),\end{aligned}$$

to leading order. We then act iteratively to substitute for the generators corresponding to the composition: (i) $[a_3\cdots a_k]$ with $(k-2)$-parts; (ii) $[(a_2-1)a_3\cdots a_k]$ with $(k-1)$-parts, taking into account the nilpotency action property; and (iii) $[a_2\cdots a_k]$ with $(k-1)$-parts. We know from Step 3, for a given composition $a_1\cdots a_k$, we can construct 2^k unique generators in this way. In Step 6 we demonstrate that the corresponding signature coefficient matrix generated in this way has full rank. However, at this stage, we note that we have the following straightforward result.

Lemma 17 (*Generator Sets Unique to Compositions*). *If we use the procedure above to construct the 2^k generators associated with the 2^k basis elements with a given composition component $a_1\cdots a_k$, then each such set of generators is unique to the given composition $a_1\cdots a_k$, i.e. each such set of generators corresponding to a given composition $a_1\cdots a_k$, does not appear elsewhere in generator sets for other compositions.*

Lastly, apart from the case of the composition 'a_11' in Example 16, including the case '11', our analysis above has precluded compositions $a_1\cdots a_k$ containing '1' in the composition sequence. We saw at the end of Example 16, that provided $a_1\neq 1$ then the set of generators for the basis elements with composition components a_11 reduces to two generators only, however, both generators only generate the basis elements $[a_11]\times(1,0,0,0)$ and $[a_11]\times(0,0,0,1)$. Consider the case of the composition a_1a_21, with both $a_1\neq 1$ and $a_2\neq 1$. Using arguments analogous to those for the case 'a_11' at the end of Example 16, if we examine the eight generators listed in Example 17, we observe that the second and last cannot be generators in the case when $a_3=1$, while the third and fourth generators combine, and the fifth and sixth generators combine, in much the same way as for the case of 'a_11' in Example 16. The latter two correspond to adding the third and fourth, and also the fifth and sixth, columns in the 8×8-matrix A_3 above. Thus for the case of the composition a_1a_21, the resulting coefficient matrix, ignoring the second, third, sixth and seventh rows which are zero, corresponds to the coefficient matrix A_3' in Example 12. With these last two examples in hand, we deduce that for any composition of the form $a_1a_2\cdots a_{k-1}1$, where we preclude any of a_1 through to a_{k-1} to be '1', we have a unique set of generators in the sense of Lemma 17, albeit with a signature coefficient matrix of size $2^{k-1}\times 2^{k-1}$. Now consider the case when the composition component is $[1a_2\cdots a_k]$, assume for the moment none of a_2 through to a_k equal '1'. Looking at the standard actions in Lemma 15, we observe that for basis elements with such a composition component, the two valid actions are,

$$\begin{aligned}[\mathbf{0}]\,[\boldsymbol{a_2-1}]\,\big([a_3\cdots a_k]\times(\hat{u},0)\big)&=\chi(a_3\cdots a_k)\cdot[a_1\cdots a_k]\times(\star,\star),\\ [\mathbf{0}]\,[\mathbf{0}]\,\big([(a_2-1)a_3\cdots a_k]\times(\hat{a},\hat{b})\big)&=\chi((a_2-1)a_3\cdots a_k)\cdot[a_1\cdots a_k]\times(\star,\star),\end{aligned}$$

where the two expressions $(\star,\star)$ are proxies for the appropriate $\mathbb{R}\langle\mathbb{B}\rangle$-components whose exact form is not important at this stage. However, we now observe that if we apply the final action in Lemma 15 respectively for the cases of the compositions $[(a_2+1)a_3\cdots a_k]$ and $[2(a_2-1)a_3\cdots a_k]$, we find,

$$\begin{aligned}[\mathbf{0}]\,[\boldsymbol{a_2-1}]\,\big([a_3\cdots a_k]\times(\hat{u},0)\big)&=\chi(a_3\cdots a_k)\cdot[(a_2+1)\cdots a_k]\times(\star,\star),\\ [\mathbf{0}]\,[\mathbf{0}]\,\big([(a_2-1)a_3\cdots a_k]\times(\hat{a},\hat{b})\big)&=\chi((a_2-1)a_3\cdots a_k)\cdot[2(a_2-1)a_3\cdots a_k]\times(\star,\star).\end{aligned}$$

We observe that the first two respective factors of the generators and their arguments match the two cases corresponding to the composition $[1a_2\cdots a_k]$. However the latter two cases generate basis elements with the respective composition components $[(a_2+1)a_3\cdots a_k]$ and $[2(a_2-1)a_3\cdots a_k]$, both of which occur before the composition $[1a_2\cdots a_k]$ in descent order. Thus these generators will not be new. A similar scenario could occur if one or more letters a_2 through to a_k are equal to 1. In our main proof below in Step 8, we are able to discount any compositions $a_1\cdots a_k$ in which any one of the letters a_1 through to a_{k-1} is equal to '1'.

Step 6: Full rank blocks. Our goal in this step is to show that for a given composition of length k, for which in general there are 2^k different possible $\mathbb{R}\langle\mathbb{B}\rangle$ components, there are 2^k independent generators, generated by the first action acting on generators at level $k-2$ and the second and third special actions on generators at level $k-1$. The following results establish that this is indeed the case.

Lemma 18 (*Actions and Independence*)**.** *We have the following, at leading order:*

(i) Given an independent set of input $\mathbb{R}\langle\mathbb{B}\rangle$ components of length 2^{k-1}, of the form $(u,0)$ for the first action, or of the form (a,b) for the second and third actions, each individual action produces an independent set of $\mathbb{R}\langle\mathbb{B}\rangle$ components of length 2^k;

(ii) Given any arbitrary length 2^{k-1} non-zero inputs, of the form $(u,0)$ for the first action or of the form (a,b) for the second and third actions, the set of three actions generate independent $\mathbb{R}\langle\mathbb{B}\rangle$ components of length 2^k.

Proof. We focus on the effect of the actions on the $\mathbb{R}\langle\mathbb{B}\rangle$ components only. In order, consider (i). It is sufficient to prove the result for two independent inputs as the general case follows suit. Consider the first action and suppose u and $\hat{u}$ are two non-trivial independent $\mathbb{R}\langle\mathbb{B}\rangle$ components. Consider an arbitrary linear combination, with scalar coefficients κ_1 and κ_2, of the first action applied to the input $(u,0)$ and the first action applied to the input $(\hat{u},0)$. Set the linear combination to zero. This gives, $\kappa_1\cdot(u,\pm u^\dagger,u,\pm u^\dagger)+\kappa_2\cdot(\hat{u},\pm\hat{u}^\dagger,\hat{u},\pm\hat{u}^\dagger)=0$, where the right-hand side represents the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component of the appropriate length. Pairing up, we observe the equation above is equivalent to $\kappa_1\cdot u+\kappa_2\cdot\hat{u}=0$, with the other pairings generating the same equation. Since by assumption u and $\hat{u}$ are two independent $\mathbb{R}\langle\mathbb{B}\rangle$ components, the result follows. Now consider the second action. Suppose (a,b) and $(\hat{a},\hat{b})$ are two non-trivial independent $\mathbb{R}\langle\mathbb{B}\rangle$ components. As above, we construct the arbitrary linear combination, $\kappa_1\cdot(a\pm b^\dagger,b\pm a^\dagger,-a\mp b^\dagger,-b\mp a^\dagger)+\kappa_2\cdot(\hat{a}\pm\hat{b}^\dagger,\hat{b}\pm\hat{a}^\dagger,-\hat{a}\mp\hat{b}^\dagger,-\hat{b}\mp\hat{a}^\dagger)=0$, for arbitrary scalar coefficients κ_1 and κ_2. We assume $a\neq\pm b^\dagger$ and $\hat{a}\neq\pm\hat{b}^\dagger$—we observe from our proof of Lemma 16 that the second action is trivial if and only if $a=\pm b^\dagger$. Since in the last equation the final two components generate the same equation as the first two, the last equation is equivalent to, $\kappa_1\cdot(a\pm b^\dagger,b\pm a^\dagger)+\kappa_2\cdot(\hat{a}\pm b^\dagger,\hat{b}\pm a^\dagger)=0$. This reduces to $\kappa_1\cdot(a,b)+\kappa_2\cdot(\hat{a},\hat{b})=0$. Hence by our assumption on (a,b) and $(\hat{a},\hat{b})$, the result follows. We now consider the third action. Suppose (a,b) and $(\hat{a},\hat{b})$ are two non-trivial independent $\mathbb{R}\langle\mathbb{B}\rangle$ components, As above, we construct the linear combination, $\kappa_1\cdot(a,b,\pm b^\dagger,\pm a^\dagger)+\kappa_2\cdot(\hat{a},\hat{b},\pm\hat{b}^\dagger,\pm\hat{a}^\dagger)=0$, for arbitrary scalar coefficients κ_1 and κ_2. This last equation is equivalent to $\kappa_1\cdot(a,b)+\kappa_2\cdot(\hat{a},\hat{b})=0$—the final two components generate the same equation as the first two. By our independence assumption on (a,b) and $(\hat{a},\hat{b})$, the result follows.

We now consider (ii). For arbitrary $(u,0)$ and (a,b) in $\mathbb{R}\langle\mathbb{B}\rangle$, consider the following linear combination of the three actions, set equal to the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component, namely: $\kappa_1\cdot(u,u^\dagger,u,u^\dagger)+\kappa_2\cdot(a-b^\dagger,b-a^\dagger,b^\dagger-a,a^\dagger-b)+\kappa_3\cdot(a,b,-b^\dagger,-a^\dagger)=0$, where κ_1, κ_2 and κ_3 are arbitrary scalar coefficients. Note we assume u and (a,b) are non-trivial. If $\kappa_1\neq0$ and $\kappa_2=\kappa_3=0$ then we observe that necessarily u must be the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component, which contradicts our assumptions. Similarly if $\kappa_3\neq0$ and $\kappa_1=\kappa_2=0$ then necessarily (a,b) is the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component, which again contradicts our assumptions. If $\kappa_1\neq0$, $\kappa_2\neq0$ and $\kappa_3=0$, the first and second components above reveal that necessarily $\kappa_1\cdot u+\kappa_2\cdot(a-b^\dagger)=0$ and $\kappa_1\cdot u^\dagger+\kappa_2\cdot(b-a^\dagger)=0$. Taking the adjoint of the second equation and adding the result to the first equation, implies $u=0$. The third and fourth components generate the same information. Thus we have a contradiction. Analogously, in the cases $\kappa_1\neq0$, $\kappa_3\neq0$ and $\kappa_2=0$, as well as $\kappa_2\neq0$, $\kappa_3\neq0$ and $\kappa_1=0$, it is straightforward to show that a necessary consequence is that (a,b) is the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component, and we have a contradiction. Now consider the case when all of κ_1, κ_2, κ_3 are non-zero. Pairing up the first component from the linear combination above with the adjoint of the fourth component reveals that necessarily a is the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component. Pairing the second component and the adjoint of the third component reveals that necessarily b is the zero $\mathbb{R}\langle\mathbb{B}\rangle$ component. We thus reach another contradiction. The final case we have not considered is the case $\kappa_1=\kappa_3=0$ and $\kappa_2\neq0$. In this case we necessarily deduce $a=b^\dagger$. As we have seen above, this is precisely the condition we need to rule out for the input when we apply the second action. The proof is complete. □

Putting the results of this and the previous steps together, we observe the following.

Proposition 3 (*Full Rank Linear System for All Compositions*)**.** *Suppose we are given a composition $a_1\cdots a_k\in C$ of k-parts. Assume that all of a_1 through to a_{k-1} are not equal to '1'. Then associated with the 2^k set of basis elements with composition component $a_1\cdots a_k$, are a unique set of 2^k generators, and the signature coefficient matrix has full rank. If $a_k=1$, the statement still holds but instead with 2^{k-1} basis elements and 2^{k-1} generators.*

Step 7: Composition and generator counts. In light of Proposition 3, we are interested in the following counts. Given $n\in\mathbb{N}$, what are the total numbers of: (i) Generators; (ii) Basis elements with composition components avoiding '1', i.e. compositions $a_1\cdots a_k$ for which none of the letters a_1 through to a_k are '1'; (iii) Basis elements with compositions ending in '1' with rest of the composition avoiding '1', i.e. compositions of the form $a_1 \cdots a_{k-1} 1$ for which $a_1 \cdots a_{k-1}$ avoids '1'. Such information will help us keep track of size of the linear system of equations for the unknown coefficients $\{c_\star\}$ we solve as part of the proof of Theorem 1 below in Step 8.

Let us begin with the total number of generators, i.e. item (i). Note that all monomial generators are of odd-degree. For a given $n \in \mathbb{N}$, there is one generator of degree 1, namely $[\boldsymbol{n}]$. We then need to enumerate all the possible degree 3 generators. Each Pöppe product corresponds to increasing the order of the compositions they generate by 1, and there are two Pöppe products in any degree 3 generator. Hence all the degree 3 generators consist of all the possible compositions of $(n-2)$ with 1, 2 and 3 parts and all the possible ways to assort them into three factors which can include "packing factors" of 0. So for example, the only 1-part compositions of $(n-2)$ assorted in this way are $[\boldsymbol{n-2}]\,[\boldsymbol{0}]\,[\boldsymbol{0}]$, $[\boldsymbol{0}]\,[\boldsymbol{n-2}]\,[\boldsymbol{0}]$ and $[\boldsymbol{0}]\,[\boldsymbol{0}]\,[\boldsymbol{n-2}]$. The first set of generators of this form associated with the 2-part compositions of $(n-2)$ are $[\boldsymbol{n-3}]\,[\boldsymbol{1}]\,[\boldsymbol{0}]$, $[\boldsymbol{n-3}]\,[\boldsymbol{0}]\,[\boldsymbol{1}]$ and $[\boldsymbol{0}]\,[\boldsymbol{n-3}]\,[\boldsymbol{1}]$, and so forth. These are just the *weak compositions* of $(n-2)$ into three parts. The next set of generators are those of degree 5, and all the generators of this degree would consist of the weak compositions of $(n-4)$ with 5 parts, and so forth. The number of weak compositions of m into k parts is $m+k-1$ choose $k-1$. We can also think of this as the number of ways of distributing m balls into k slots, allowing empty slots. From our discussion above, when n is odd, we see that we are interested in, for $k = 1, 3, 5, \ldots, n$, the number of ways of distributing $m = n-k+1$ balls into k slots, or in other words,

$$\sum_{k=1 (k \text{ odd})}^{n} \binom{n}{k-1} = \sum_{\ell=0}^{\frac{1}{2}(n-1)} \binom{n}{2\ell} = 2^{n-1},$$

where we use the substitution $k = 2\ell + 1$ for the second sum. That the sum total on the right equals 2^{n-1} follows, with some care, from the corresponding result in Lemma 14. Similarly, when n is even, we are interested in, for $k = 1, 3, 5, \ldots, n+1$, the number of ways of distributing $m = n-k+1$ balls into k slots, or in other words,

$$1 + \sum_{k=1 (k \text{ odd})}^{n-1} \binom{n}{k-1} = 1 + \sum_{\ell=0}^{\frac{1}{2}(n-2)} \binom{n}{2\ell} = 2^{n-1}.$$

Again we used the substitution $k = 2\ell+1$ for the second sum. Note that the initial '1' in the sum corresponds to the case $k = n+1$, i.e. corresponding to the monomial generator $[\boldsymbol{0}]^{n+1}$. That the sum total is 2^{n-1} again follows from the first result in Lemma 14. We have thus established the following.

Lemma 19 (*Total Number of Generators*). *Given $n \in \mathbb{N}$, the total number of monomial generators is equal to 2^{n-1}.*

The number of compositions of n with k-parts is $n-1$ choose $k-1$, and accumulating these coefficients from $k=1$ to $k=n$, the total number of compositions of n is 2^{n-1}. Of course associated with each composition, there are one or more basis elements. For example for a composition $a_1 \cdots a_k$ with k-parts which avoids 1, there are 2^k associated basis elements. The number of compositions of n into k parts avoiding 1 is given by,

$$\binom{n-k-1}{k-1}.$$

To see this we observe the following—see Axenovich and Ueckerdt [50, p. 24] or Beck and Robbins [51]. There is a bijection between: (a) the arrangements of n balls into k slots with each slot containing two or more balls; and (b) the arrangements of $n-k$ balls into k slots with no empty slots. For the map from (a) to (b), we simply remove one ball from each slot. For the map from (b) to (a) we just add one ball to each slot. The count for (b) is $n-k-1$ choose $k-1$, giving the result above. Thus, given that for a given composition with k parts that avoids '1' there are 2^k corresponding basis elements, the total number of basis elements with composition components that avoid '1' is given, with $\lambda = 2$, respectively when n is odd and then when n is even, by,

$$p(n;\lambda) := \sum_{k=1}^{(n-1)/2} \binom{n-k-1}{k-1} \lambda^k \qquad \text{and} \qquad p(n;\lambda) := \sum_{k=1}^{n/2} \binom{n-k-1}{k-1} \lambda^k.$$

By direct enumeration, as well as from Examples 8–12, we know $p(2;2) = p(3;2) = 2$, $p(4;2) = 6$ and $p(5;2) = 10$. In fact, in general, we have the following result.

Lemma 20 (*Weighted Compositions Avoiding '1' Count*). *Given an integer $n \geqslant 2$ and a real number $\lambda > 0$, the weighted sum $p = p(n;\lambda)$ of the total number of basis elements with composition components which avoid '1' satisfies the weighted Fibonacci sequence satisfying,*

$$p(n;\lambda) = p(n-1;\lambda) + \lambda\, p(n-2;\lambda).$$

In particular, when $\lambda = 2$, $p(2;2) = p(3;2) = 2$ and $p(n;2) = \frac{2}{3}(2^{n-1} + (-1)^n)$.

Proof. By direct computation, for n odd, we observe, $p(n-1;\lambda) + \lambda\, p(n-2;\lambda)$ equals,

$$\sum_{k=1}^{(n-1)/2} \binom{n-k-2}{k-1} \lambda^k + \sum_{k=1}^{(n-3)/2} \binom{n-k-3}{k-1} \lambda^{k+1} = \binom{n-3}{0} \lambda + \sum_{k=2}^{(n-1)/2} \left(\binom{n-k-2}{k-1} + \binom{n-k-2}{k-2} \right) \lambda^k,$$

which equals $p(n;\lambda)$ once we combine the two terms in the coefficient of λ^k shown and observe that the coefficient of the λ term is one. Note that in the first step we made the change of variables $\ell = k+1$ in the second sum, before relabelling ℓ as k. When n is even, we similarly observe that $p(n-1;\lambda) + \lambda\, p(n-2;\lambda)$ equals,

$$\sum_{k=1}^{(n-2)/2} \binom{n-k-2}{k-1} \lambda^k + \sum_{k=1}^{(n-2)/2} \binom{n-k-3}{k-1} \lambda^{k+1} = \binom{n-3}{0} \lambda + \sum_{k=2}^{(n-2)/2} \left(\binom{n-k-2}{k-1} + \binom{n-k-2}{k-2} \right) \lambda^k + \binom{n/2-2}{n/2-2} \lambda^{n/2},$$

which equals $p(n;\lambda)$ once we combine the terms in the coefficient of λ^k, and observe that the coefficients of the λ and $\lambda^{n/2}$ terms are one. We also used the same change of variables in the first step. The final statement specific to $\lambda = 2$, follows directly by solving the difference equation for $p = p(n;2)$ for the initial conditions indicated. □

The result of Lemma 20 provides an answer to item (ii), stated at the beginning of this step (Step 7). Item (iii) is now straightforward. The total number of compositions with k-parts of the form $a_1 \cdots a_{k-1} 1$ for which $a_1 \cdots a_{k-1}$ avoids '1' is simply $n-k-1$ choose $k-2$. This is because here we require $n-1$ balls to fit into $k-1$ slots with each slot containing two or more balls. Hence the total number of basis elements associated with composition components which end in '1', but avoid '1' elsewhere, when n is odd so $n-1$ is even, is given by,

$$\sum_{k=2}^{(n+1)/2} \binom{n-k-1}{k-2} \lambda^{k-1} = \sum_{k=1}^{(n-1)/2} \binom{n-k-2}{k-1} \lambda^{k},$$

which equals $p(n-1;\lambda)$. When n is even and thus $n-1$ is odd, the same count is,

$$\sum_{k=2}^{n/2} \binom{n-k-1}{k-2} \lambda^{k-1} = \sum_{k=1}^{(n-2)/2} \binom{n-k-2}{k-1} \lambda^{k},$$

which equals $p(n-1;\lambda)$. Finally we observe the following.

Lemma 21 (*Basis Element Count*)**.** *The total number of basis elements with composition components which avoid* '1'*, or end in* '1' *and avoid* '1' *elsewhere, equals* 2^{n-1}.

Proof. The count in question is $p(n;2)+p(n-1;2)$. Using the explicit solution for $p(n;2)$ given in Lemma 20, the result follows. □

Combining the results of Lemmas 19 and 21, we deduce the rather remarkable fact:

> For any given $n \in \mathbb{N}$, the total number of basis elements whose composition component avoids '1', or ends in '1' but avoids '1' elsewhere, exactly equals the total number of generators.

Naturally this result is important in our proof of Theorem 1 just below. We remark that the inclusion of basis elements whose composition components end in '1' but avoid '1' elsewhere, rather than composition components containing '1' at some other position with avoidance elsewhere, is merely a consequence of the ordering we have imposed, namely the descent order.

Step 8: Proof of Theorem 1. In this last step we provide the overall proof of our main theorem. We combine together the knowledge we gained in Steps 1–7. This final stage of the argument, though significantly adapted, is analogous to that outlined for the non-commutative potential Korteweg–de Vries hierarchy in Malham [10].

Proof of Theorem 1. To complete the proof, we essentially construct a table of signature coefficients for the arbitrary order n case, much like Tables 3–5 for the $n=4$ and $n=5$ cases. Indeed we refer to these example tables to demonstrate examples of the general procedure. We already know from Steps 1–7 that we can construct a linear algebraic equations of the form $AC=B$, where the vector C of length 2^{n-1} lists the unknown coefficients of the generator monomials in the Pöppe polynomial $\pi_n = \pi_n([\mathbf{0}],[\mathbf{1}],\ldots,[\boldsymbol{n}])$. The vector B, whose length exceeds 2^{n-1}, is a vector of zeros apart from a single non-zero value '1' in the first position when n is odd and in the second position when n is even. This is due to the ordering we impose which we outline briefly now, and in some more detail just below. The coefficients of C, are ordered according to the descent order and blocks as outlined in Steps 3 and 5. The signature coefficient matrix A has a lower triangular block form. It has 2^{n-1} columns and its total number of rows exceeds 2^{n-1}, though equals the length of B. The columns of A are parametrised by the blocks of monomial generators mentioned, or equivalently the order of the coefficients in C. The rows of A are parametrised by the basis elements, characterised by the composition components of the basis elements listed in descent order, and within individual composition w, the basis elements are listed according to the binary order of the $\mathbb{R}\langle\mathbb{B}\rangle$-components. The number of such $\mathbb{R}\langle\mathbb{B}\rangle$-components is $2^{|w|-\upsilon(w)}$, where $\upsilon(w)$ counts the number of 1's in the composition w.

Let us outline the forms of C and A in some more detail. We can be brief as much of the procedure has already been outlined in Steps 1–7. Corresponding to the pair of basis elements with a one-part composition component, namely $[n]\times(1,0)$ and $[n]\times(0,1)$, the first pair of coefficients in C are c_n and $c_{0(n-2)0}$, corresponding to the generators $[\boldsymbol{n}]$ and $[\mathbf{0}]\,[\boldsymbol{n-2}]\,[\mathbf{0}]$. The corresponding 2×2 top-left block in A is the matrix A_0. All the remaining entries in the first two rows of A to the right of this block are zero. We move onto basis elements whose composition components have $k=2$ parts. In descent order the first set of basis elements consists of $[(n-1)1]\times(1,0,0,0)$ and $[(n-1)1]\times(0,0,0,1)$. We know from Example 16 in Step 5, these are the only two relevant basis elements for corresponding to compositions ending in '1'. The corresponding coefficients in C are $c_{(n-2)00}$ and $c_{0(n-3)1}$ associated with the generators $[\boldsymbol{n-2}]\,[\mathbf{0}]\,[\mathbf{0}]$ and $[\mathbf{0}]\,[\boldsymbol{n-3}]\,[\mathbf{1}]$. The corresponding coefficient matrix is the block A_1 taking up rows and columns 3 and 4 in A, with the remaining entries in rows 3 and 4 to the right of this block being zero. This first set of blocks is important once we start addressing the issue of the uniqueness of the solution C to the linear system, or equivalently the consistency of the overall linear system. Next we consider the blocks of basis elements parametrised by two-part composition components of the form a_1a_2 in descent order, with neither a_1 nor a_2 equal to one. Associated with any such composition, there are four basis elements $[a_1a_2]\times\boldsymbol{\beta}_i$, $i=1,2,3,4$, and the corresponding coefficients in C are $c_{(a_1-1)(a_2-1)0}$, $c_{(a_1-1)0(a_2-1)}$, $c_{0(a_1-2)a_2}$ and $c_{0(a_1-2)0(a_2-2)0}$, corresponding to the monomial generators outlined in Example 16. For each block of four basis elements with such a composition component, the corresponding signature coefficient matrix is A_2. As we run through the compositions a_1a_2 which avoid '1' in descent order, the coefficient matrix A_2 occupies rows and columns $4(n-a_1-1)+i$ for $i=1,2,3,4$ in the signature coefficient matrix A. All the entries in A in the these rows to the right of these blocks are zero. We know from Step 5, that the basis elements with the two-part composition component $1(n-1)$ does not occupy any new columns in A, but instead, the two rows corresponding to the two basis elements concerned only contain non-zero entries in the columns parametrised by $4(n-a_1-1)+i$ for $i=1,2,3,4$ for $a_1\neq 1$.

We move onto blocks of basis elements corresponding to composition components with $k=3$ parts. We know from our arguments in Step 5 that we can focus on composition components of the form $a_1a_2a_3$ which avoid '1' or end in '1' and avoid '1' elsewhere. Basis elements with composition components lying in the complement of this set do not generate "new" columns, or equivalently, the non-zero entries in the rows in A corresponding to these basis elements only occupy columns we have already encountered/parametrised. Let us examine the blocks we generate as we run through the compositions $a_1a_2a_3$ which avoid '1' or end in '1' and avoid '1' elsewhere. We know from Step 5, that for each of the former compositions we generate the block matrix A_3, modified to include the column factors mentioned in Step 5, associated with the 8 corresponding basis elements and the 8 "new" monomial generator columns. For each of the latter compositions we generate a block matrix A_3' associated with the 4 corresponding basis elements and the 4 "new" monomial generator columns. The block matrices A_3 and A_3' are diagonal blocks in A, with all entries in the corresponding rows they occupy to the right of these blocks equal to zero.

Table 1
The top left block entries in the signature coefficient matrix A at any order n, depending on whether n is odd (top) or n is even (bottom). The coefficients are the χ-images of the signature entries shown. The forms shown for the case of when n is odd or even, are used to prove the consistency of the overdetermined linear system of algebraic equations for the Pöppe polynomial coefficients. The first rows are not shown.

n odd	$[\boldsymbol{n}]$	$[\boldsymbol{n-2}]\,[\boldsymbol{0}]^2$	$[\boldsymbol{0}]\,[\boldsymbol{n-3}]\,[\boldsymbol{1}]$	$\cdots$
$[\boldsymbol{0}(n-1)\boldsymbol{010}]$	$(n-1)1$	$2\cdot\big((n-2)_{\otimes}0_{\otimes}0\big)$	$2\cdot\big(0_{\otimes}(n-3)_{\otimes}1\big)$	
$[\boldsymbol{0}(n-1)\boldsymbol{0}^{\dagger}\boldsymbol{10}^{\dagger}]$		$-2\cdot\big((n-2)_{\otimes}0_{\otimes}0\big)$	$2\cdot\big(0_{\otimes}(n-3)_{\otimes}1\big)$	
$\vdots$				
n even	$[\boldsymbol{n}]$	$[\boldsymbol{0}]\,[\boldsymbol{n-2}]\,[\boldsymbol{0}]$	$[\boldsymbol{n-2}]\,[\boldsymbol{0}]^2$	$[\boldsymbol{0}]\,[\boldsymbol{n-3}]\,[\boldsymbol{1}]$
$[\boldsymbol{0}n\boldsymbol{0}]$	n	$2\cdot\big(0_{\otimes}(n-2)_{\otimes}0\big)$		
$[\boldsymbol{0}(n-1)\boldsymbol{010}]$	$(n-1)1$	$2\cdot\big(0_{\otimes}(n-2)_{\otimes}0\big)$	$2\cdot\big((n-2)_{\otimes}0_{\otimes}0\big)$	$2\cdot\big(0_{\otimes}(n-3)_{\otimes}1\big)$
$[\boldsymbol{0}(n-1)\boldsymbol{0}^{\dagger}\boldsymbol{10}^{\dagger}]$		$2\cdot\big(0_{\otimes}(n-2)_{\otimes}0\big)$	$-2\cdot\big((n-2)_{\otimes}0_{\otimes}0\big)$	$2\cdot\big(0_{\otimes}(n-3)_{\otimes}1\big)$

We know from our analysis in Step 5 and in particular Lemma 17 and the discussion immediately following this lemma, that we have the following. For every composition w of n that avoids '1', or ends in '1' but avoids '1' elsewhere, we can construct a $2^{|w|}\times 2^{|w|}$ block matrix A_w in the former case, or a $2^{|w|-1}\times 2^{|w|-1}$ block matrix A'_w in the latter case, which occupies a distinct diagonal block in A. Here by "distinct", we mean that its rows and columns do not coincide with the rows and columns of any of the analogous block matrices corresponding to basis elements with such composition components. All the entries in A in the rows occupied by these blocks and to the right of them, are zero. Thus for general n, the signature coefficient matrix A is indeed a lower block triangular matrix. Further, from our results in Step 6, we know that each of the block matrices A_w and A'_w has full rank. Even further, from our results in Step 7, we know that the total number of basis elements corresponding to such composition components and generating the rows of such diagonal blocks, exactly equals the total number of monomial generators generating the columns of such diagonal blocks. This means that if we ignore the basis elements with composition components in the complementary set for the moment, then we can proceed block by block, starting with A_0, and solve for the corresponding set of Pöppe polynomial coefficients in C, until we precisely exhaust the blocks and uniquely recover C.

The rest of the proof is now concerned with demonstrating the consistency of the remaining rows/linear equations for the coefficients in C associated with those basis elements with composition components that contain a '1', not including the instances of compositions ending in '1', but avoiding it elsewhere. We heavily rely on the fact that the linear system of algebraic equations for C is almost homogeneous apart from the single unit entry in the first position in B when n is odd and in the second position when n is even. The first phase of this section of the proof focuses on the blocks associated with basis elements with 1 and 2-part composition components. Assume for the moment that n is odd. In this instance the first two linear equations for the coefficients c_n and $c_{0(n-2)0}$ are $c_n+2\cdot c_{0(n-2)0}=1$ and $-2\cdot c_{0(n-2)0}=0$. Thus when n is odd, we always have $c_{0(n-2)0}=0$. This means that all the entries in the second column of the signature coefficient matrix A are not relevant to the linear system of equations for C and we thus eliminate this column in A when n is odd. Implementing this, and ignoring the first row in A corresponding to the basis element $[\boldsymbol{0}n\boldsymbol{0}]$, the top left corner of A has the form shown in the top part of Table 1. Now assume that n is even. The first two linear equations for the coefficients c_n and $c_{0(n-2)0}$ in this instance are $c_n+2\cdot c_{0(n-2)0}=0$ and $-2\cdot c_{0(n-2)0}=1$. When n is even we swap over the first two rows in the signature coefficient matrix A which is equivalent to swapping the order of the first two linear equations shown. If we ignore the *new* first row of A corresponding to the nonhomogeneous equation $-2\cdot c_{0(n-2)0}=1$, then the top left corner of A has the form shown in the bottom part of Table 1. All the remaining rows and columns in the signature coefficient matrix A, whether n is odd or even, remain the same. However, with the first rows ignored, the system of linear equations that remains is homogeneous. And, of course, the top left corners in the case that n is odd or even have the forms shown in Table 1. In both cases in Table 1 we trace a diagonal starting from the top left non-zero coefficient, which is $\chi\big((n-1)1\big)=n$ when n is odd, and $\chi(n)=1$ when n is even. When n is even the next two entries along this diagonal are $\chi\big(2\cdot(0_{\otimes}(n-2)_{\otimes}0)\big)=2$ and $\chi\big(-2\cdot((n-2)_{\otimes}0_{\otimes}0)\big)=-2$. When n is odd the next diagonal entry is $\chi\big(-2\cdot((n-2)_{\otimes}0_{\otimes}0)\big)=-2$. Thereafter the diagonal entries for the cases of n being even or odd are the same. In Table 3, when n is even, after swapping the first two rows, though not ignoring the new first row yet, we can view the diagonal we have identified as the diagonal just below the leading diagonal. Similarly in Tables 4 and 5, when n is odd, after eliminating the second column but still retaining the first row, we can again view the diagonal we have identified as that just below the leading diagonal in those Tables. In either Table 3 or in Tables 4 and 5, let us call the diagonal just below the leading diagonal the 'sub-diagonal'. Consider for example Tables 4 and 5. If we follow the sub-diagonal with the view of retaining non-zero terms along it, we observe we meet an obstruction in the first 4×4 block with matrix A_2 characterised by the composition component 32. The problem is that while the sub-diagonal of A_2 has non-zero entries, the next term along the diagonal that lies in the last column of that A_2 block, but beneath the entire block, and in fact the row corresponding to the basis element $[23]\times(1,0,0,0)$ is zero. However there is a quick fix to this obstruction. That is to simply swap the two columns in the signature coefficient matrix A corresponding to the final two columns of the A_2 block characterised by the composition component 23. Such a swap simply corresponds to changing the order of the monomial generators. We can see from Tables 4 and 5 that this column-swap procedure would guarantee that the next entry in the sub-diagonal would be non-zero. We can then continue to consider the sub-diagonal in the A_2 block corresponding to the basis elements $[23]\times\boldsymbol{\beta}_i$ for $i=1,2,3,4$. However we observe a similar obstruction necessitating an analogous swap of the columns of A corresponding to the final two columns of this second A_4 block. This is again enacted to ensure that the term in the final column immediately below this A_4 block is non-zero. The term in question corresponds to the signature coefficient $\chi(0\hat{\otimes}0_{\otimes}3)$ in the row corresponding to $[14]\times(1,0,0,0)$. We have thus established, for all the basis elements with 1 and 2-part composition components, a complete diagonal with all entries non-zero, and which can act as pivots. Since all the linear equations corresponding to the rows we are considering (we are ignoring the top row) are homogeneous we can use Gaussian elimination to render all the entries in the columns below the sub-diagonal to be zero.

The procedure for the case of general n proceeds in exactly the same manner as the $n=5$ case we have just outlined, except that now we need to establish that when we swap the columns over as just outlined, we are guaranteed a non-zero entry in the corresponding sub-diagonal entry. We also need to guarantee that the entry immediately below the final column of the 2×2 block A_1 corresponding to the rows $[(n-1)1]\times(1,0,0,0)$ and $[(n-1)1]\times(0,0,0,1)$ is also non-zero. This case corresponds to the column given by $[\boldsymbol{0}]\,[\boldsymbol{n-3}]\,[\boldsymbol{1}]$. In the other cases of the A_4 blocks corresponding to the basis elements $[(n-m)m]\times\boldsymbol{\beta}_i$ for $i=1,2,3,4$, the column in question is the third column in the A_4 block corresponding to the column given by $[\mathbf{0}]\,[\boldsymbol{n-m-2}]\,[\boldsymbol{m}]$. In particular this means we can treat the $m=1$ and $m=2,\dots,n-2$ cases simultaneously. Indeed, using Corollary 6 in Step 4, we observe that at leading order we have,

$$[\mathbf{0}]\,[\boldsymbol{n-m-2}]\,[\boldsymbol{m}] = [(n-m)m]\times(2,0,0,2) + [(n-m-1)(m+1)]\times(1,1,1,1)+\cdots,$$

where we have used that at leading order $[\boldsymbol{m}]=[m]\times(1,0)+\cdots$. The first term on the right is the term we expect at leading order for this generator, while the second term on the right is the column corresponding to the rows in the next block down—we see that $(n-m-1)(m+1)$ is obtained from $(n-m)m$ by replacing m by $m+1$, which is one composition further down in descent order. Thus indeed we are guaranteed that the next entry in the sub-diagonal is non-zero when we enact the column swap. Further we observe that in the case of the final A_4 block corresponding to the composition $2(n-2)$ for which $m=n-2$, the corresponding generator is $[\mathbf{0}]\,[\mathbf{0}]\,[\boldsymbol{n-2}]$ while the corresponding row containing the sub-diagonal entry of interest is $[1(n-1)]\times(1,0,0,0)$. At leading order we have,

$$[\mathbf{0}]\,[\mathbf{0}]\,[\boldsymbol{n-2}] = [2(n-2)]\times(2,0,0,2) + [1(n-1)]\times(2,2,0,0)+\cdots,$$

which thus guarantees a final sub-diagonal non-zero entry. We are thus in the exact same situation as described for the $n=5$ case just above, and we can use the sub-diagonal entries as pivots to render all the entries in A, in all the sub-diagonal columns, below the sub-diagonal to be zero.

The second phase of this section of the proof now focuses on all the blocks associated with basis elements with composition components with k-parts with $k\geqslant 3$. This phase is more straightforward. Let us focus on the 3-part composition cases to begin with. The first 3-part composition in descent order is $(n-2)11$, and we know from Step 5 that there are no "new" generators associated with any such composition that lies in the set of compositions complementary to those avoiding '1' or ending in '1', but avoiding it elsewhere. Hence we can use Gaussian elimination, using the pivots from the sub-diagonal outlined for the 1 and 2-part composition cases just outlined, to render the entries in for the two rows/basis elements concerned here, $[(n-2)11]\times\boldsymbol{\beta}_1$ and $[(n-2)11]\times\boldsymbol{\beta}_8$, equal to zero. Next we consider the block of rows/basis elements corresponding to the composition $(n-3)21$. As outlined in Step 5 this block is associated with 4 generators and the block matrix A_3'. We know this has full rank and we can thus use the leading diagonal as pivots to render all entries in the corresponding columns of A below this diagonal to be zero. The next block is associated with the composition $(n-3)12$, which with a '1' in the middle is not associated with any new generators, and from our Gaussian elimination processes thus far has all row entries rendered zero. The next blocks are associated with the compositions $(n-4)31$, $(n-4)22$, $(n-4)14$. The first two of these compositions are associated with separate copies of the 8×8 matrix A_3 (with the columns mentioned in Step 5 suitably scaled) and a total of 16 generators (one set of 8 each). The leading diagonals of both copies of A_3 can again be used as pivots to render all the entries, below this diagonal in the columns of A associated with these two copies, equal to zero. The entries in the rows associated with the third composition $(n-4)13$ will have been rendered zero in the Gaussian elimination process just outlined for the other two compositions. And so forth, we can see that we can proceed in descent order through the blocks associated with 3-part compositions, either in the case of compositions that avoid '1' or end in '1' but avoid it elsewhere, using the diagonals of the blocks associated with A_3 or A_3', to render the corresponding entries in A below these diagonals to be zero, or recognising for the blocks associated with the complementary set of compositions, the entries in the rows of those block will already be rendered zero. The procedure for all further blocks associated with compositions of 4 or more parts proceeds exactly analogously. Naturally that the corresponding diagonals with non-zero entries exist for all blocks associated with compositions that avoid '1' or end in '1' but avoid it elsewhere, is guaranteed by the results in Step 6, in particular Proposition 3.

Hence we have thus rendered all the entries in all the rows corresponding to basis elements with composition components which lie in the set complementary to those that avoid '1' or end in '1' but avoid it elsewhere, equal to zero. Briefly returning to the rows/blocks associated with the 1 and 2 part compositions, still ignoring the top row as indicated in Table 1. A quick count reveals that when n is even, we have $3+4(n-1)$ rows, i.e. homogeneous linear equations, in $4+4(n-1)$ unknowns, while when n is odd, we have $2+4(n-1)$ homogeneous linear equations, in $3+4(n-1)$ unknowns. In either case when n is even or odd, proceeding through all the other blocks associated with compositions of three or more parts, the remaining number of homogeneous linear equations equals the remaining number of unknowns (as outlined in the first section of this proof). Hence, in either case when n is even or odd, we can solve the entire system of linear homogeneous equations, with one less equation then the total number of unknowns, to find expressions for all the unknowns in terms of only one of them. In the case that n is odd, we solve for all of them in terms of c_n. In the case that n is even, we solve for all of them in terms of $c_{0(n-2)0}$. We now re-introduce the very first row we ignored at the beginning of this second, "consistency", section of the proof. When n is odd, that first equation is $c_n+2\cdot c_{0(n-2)0}=1$. Since we have an expression for $c_{0(n-2)0}$ in terms of c_n from the homogeneous set of linear equations, we can substitute that expression into this non-homogeneous linear equation and determine c_n. When n is even, the first equation is $-2\cdot c_{0(n-2)0}=1$ or equivalently $c_{0(n-2)0}=-1/2$. Since in this case we have expressions for all the other unknowns in terms of $c_{0(n-2)0}$, this fixes the values of all the other unknowns. In either case, whether n is odd or even, we have established a unique solution C, and the proof is complete. □

Remark 17. As mentioned, the overall proof in Step 8 just above is analogous to that for the non-commutative potential Korteweg–de Vries hierarchy in Malham [10]. Therein we proceed by considering compositions with $k=1$, $k=2$, and so forth, parts. In that case there are no blocks as there are no $\mathbb{R}\langle\mathbb{B}\rangle$ components. The basis elements are just compositions of n, there are no skew forms. Also, as it is the potential equation, the generators are just monomials of the signature expansions $\boldsymbol{n}$, with $n\in\mathbb{N}$—in particular there are no generators corresponding to $[\mathbf{0}]$. Further, since there are no skew forms, the generators can be of even or odd degree. See in particular Section 6 in Malham [10]. We observed at the end of Step 7 above that the total number of generators equalled the total number of basis elements with composition components that avoided '1' together with those that ended in '1' but avoided it elsewhere. It would be natural to wonder whether a similar situation occurs for the case of the non-commutative potential Korteweg–de Vries hierarchy, and indeed retrospectively, we can establish the exact same result for that hierarchy. In fact we can show, since there are no blocks and no generators akin to '$[\mathbf{0}]$', that for each set of compositions with k parts, the number of monomial generators with k factors equals the number of compositions (the basis elements here) that avoid '1' together with those that ended in '1' but avoided it elsewhere. Again, that we single out those compositions ending in '1' is just an artefact of the descent order we impose. To see this fact, we observe from Section 6 in Malham [10], that the number of generators with k factors, say of the form $(\boldsymbol{n}_1)(\boldsymbol{n}_2)\cdots(\boldsymbol{n}_k)$ with the Pöppe product, is given by $n-k$ choose $k-1$. This is because each Pöppe product adds a '1' to one of the composition parts in the eventual expansion in compositions. The complete set of such monomial generators is exhausted by those with $k=1,2,\dots,\frac{1}{2}(n+1)$ parts. We already know from Step 7 above, that the number of compositions of n with k parts that avoid '1' equals $n-k-1$ choose $k-1$ for $k=1,2,\dots,\frac{1}{2}(n-1)$, and the number of compositions that

Table 2
Non-zero signature coefficients appearing in the expansion of the *Pöppe polynomial* π_3 in Example 9. The coefficients are the χ-images of the signature entries shown. Each column shows the factor contributions to the real coefficients of the basis elements shown in the very left column, for each of the monomials in π_3 shown across the top row. The final column represents the coefficient on the right-hand side of the equation $\pi_3 = [\mathbf{030}]$.

Basis	[3]	$[\mathbf{0}]\,[\mathbf{1}]\,[\mathbf{0}]$	$[\mathbf{1}]\,[\mathbf{0}]^2$	$[\mathbf{0}]^2\,[\mathbf{1}]$	B
$[\mathbf{030}]$	3	$2\cdot(0\otimes 1\otimes 0)$			1
$[\mathbf{030}^\dagger]$		$-2\cdot(0\otimes 1\otimes 0)$			
$[\mathbf{02010}]$	21	$2\cdot(0\otimes 1\otimes 0)$	$2\cdot(1\otimes 0\otimes 0)$	$2\cdot(0\otimes 0\otimes 1)$	
$[\mathbf{020}^\dagger\,\mathbf{10}^\dagger]$		$2\cdot(0\otimes 1\otimes 0)$	$-2\cdot(1\otimes 0\otimes 0)$	$2\cdot(0\otimes 0\otimes 1)$	
$[\mathbf{01020}]$	12	$2\cdot(0\otimes 1\otimes 0)$	$2\cdot(1\otimes 0\otimes 0)$	$2\cdot(0\otimes 0\otimes 1)$	
$[\mathbf{01020}^\dagger]$		$-2\cdot(0\otimes 1\otimes 0)$	$-2\cdot(1\otimes 0\otimes 0)$	$2\cdot(0\otimes 0\otimes 1)$	
$[\mathbf{0101010}]$	111	$4\cdot(0\otimes 1\otimes 0)$	$4\cdot(1\otimes 0\otimes 0)$	$4\cdot(0\otimes 0\otimes 1)$	

Table 3
Non-zero signature coefficients appearing in the expansion of the *Pöppe polynomial* π_4 in Example 10. The coefficients are the χ-images of the signature entries shown. Each column shows the factor contributions to the real coefficients of the basis elements shown in the very left column, for each of the monomials in π_4 shown across the top row. The final column represents the coefficient on the right-hand side of the equation $\pi_4 = [\mathbf{040}^\dagger]$.

Basis	[4]	$[\mathbf{0}]\,[\mathbf{2}]\,[\mathbf{0}]$	$[\mathbf{2}]\,[\mathbf{0}]^2$	$[\mathbf{0}]\,[\mathbf{1}]^2$	$[\mathbf{1}]^2\,[\mathbf{0}]$	$[\mathbf{1}]\,[\mathbf{0}]\,[\mathbf{1}]$	$[\mathbf{0}]^2\,[\mathbf{2}]$	$[\mathbf{0}]^5$	B
$[\mathbf{040}]$	4	$2\cdot(0\otimes 2\otimes 0)$							
$[\mathbf{040}^\dagger]$		$-2\cdot(0\otimes 2\otimes 0)$							1
$[\mathbf{03010}]$	31	$0\otimes 2\hat{\otimes}0$	$2\cdot(2\otimes 0\otimes 0)$	$2\cdot(0\otimes 1\otimes 1)$					
$[\mathbf{030}^\dagger\,\mathbf{10}^\dagger]$		$0\otimes 2\hat{\otimes}0$	$-2\cdot(2\otimes 0\otimes 0)$	$2\cdot(0\otimes 1\otimes 1)$					
$[\mathbf{02020}]$	22	$0\otimes 11\otimes 0$	$2\cdot(2\otimes 0\otimes 0)$	$0\otimes 1\otimes 1$	$1\otimes 1\otimes 0$	$1\otimes 0\otimes 1$	$2\cdot(0\otimes 0\otimes 2)$	$4\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{02020}^\dagger]$		$-0\otimes 11\otimes 0$	$-2\cdot(2\otimes 0\otimes 0)$	$0\otimes 1\otimes 1$	$-1\otimes 1\otimes 0$	$1\otimes 0\otimes 1$		$-4\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{020}^\dagger\,\mathbf{20}]$		$-0\otimes 11\otimes 0$		$0\otimes 1\otimes 1$	$1\otimes 1\otimes 0$	$-1\otimes 0\otimes 1$		$-4\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{020}^\dagger\,\mathbf{20}^\dagger]$		$0\otimes 11\otimes 0$		$0\otimes 1\otimes 1$	$-1\otimes 1\otimes 0$	$-1\otimes 0\otimes 1$	$2\cdot(0\otimes 0\otimes 2)$	$4\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{01030}]$	13	$0\hat{\otimes}2\otimes 0$			$2\cdot(1\otimes 1\otimes 0)$		$2\cdot(0\otimes 0\otimes 2)$		
$[\mathbf{01030}^\dagger]$		$-0\hat{\otimes}2\otimes 0$			$-2\cdot(1\otimes 1\otimes 0)$		$2\cdot(0\otimes 0\otimes 2)$		
$[\mathbf{0201010}]$	211	$0\otimes 11\hat{\otimes}0$	$2\cdot(2\hat{\otimes}0\otimes 0)$	$0\otimes 1\hat{\otimes}1$	$1\otimes 1\hat{\otimes}0$	$1\otimes 0\hat{\otimes}1$	$2\cdot(0\otimes 0\otimes 11)$	$8\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{020}^\dagger\,\mathbf{10}^\dagger\,\mathbf{10}^\dagger]$		$-0\otimes 11\hat{\otimes}0$		$-0\otimes 1\hat{\otimes}1$	$1\otimes 1\hat{\otimes}0$	$1\otimes 0\hat{\otimes}1$	$-2\cdot(0\otimes 0\otimes 11)$	$-8(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{0102010}]$	121	$0\hat{\otimes}2\hat{\otimes}0$	$2\cdot(11\otimes 0\otimes 0)$	$0\hat{\otimes}1\otimes 1$	$1\otimes 1\hat{\otimes}0$	$1\otimes 0\hat{\otimes}1$	$2\cdot(0\otimes 0\otimes 11)$	$8\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{01020}^\dagger\,\mathbf{10}^\dagger]$			$-2\cdot(11\otimes 0\otimes 0)$	$0\hat{\otimes}1\otimes 1$	$1\otimes 1\hat{\otimes}0$	$1\otimes 0\hat{\otimes}1$	$-2\cdot(0\otimes 0\otimes 11)$	$-8\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{0101020}]$	112	$0\hat{\otimes}11\otimes 0$	$2\cdot(11\otimes 0\otimes 0)$	$0\hat{\otimes}1\otimes 1$	$1\hat{\otimes}1\otimes 0$	$1\hat{\otimes}0\otimes 1$	$2\cdot(0\otimes 0\hat{\otimes}2)$	$8\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{0101020}^\dagger]$		$-0\hat{\otimes}11\otimes 0$	$-2\cdot(11\otimes 0\otimes 0)$	$0\hat{\otimes}1\otimes 1$	$-1\hat{\otimes}1\otimes 0$	$1\hat{\otimes}0\otimes 1$		$-8\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	
$[\mathbf{010101010}]$	1111	$0\hat{\otimes}11\hat{\otimes}0$	$2\cdot(11\hat{\otimes}0\otimes 0)$	$0\hat{\otimes}1\hat{\otimes}1$	$1\hat{\otimes}1\hat{\otimes}0$	$1\hat{\otimes}0\hat{\otimes}1$	$2\cdot(0\otimes 0\hat{\otimes}11)$	$16\cdot(0\otimes 0\otimes 0\otimes 0\otimes 0)$	

end in '1' but avoid it elsewhere equals $n-k-1$ choose $k-2$ for $k=2,\ldots,\frac{1}{2}(n+1)$. If we restrict ourselves to $k=2,\ldots,\frac{1}{2}(n-1)$, we observe that the number of compositions satisfying either property is given by,

$$\binom{n-k-1}{k-1}+\binom{n-k-1}{k-2}=\binom{n-k}{k-1},$$

with equality following by simply adding the two relevant fractions on the left. The cases $k=1$ and $k=\frac{1}{2}(n+1)$, for which the number of such compositions and generators is singular, just follows by inspection.

7. Conclusion

There are many open directions of research we intend to pursue based on the combinatorial algebraic approach we introduced herein. One direction we have not directly addressed herein is that of alternative formulations of the modified Korteweg–de Vries hierarchy members of orders 3, 5 and higher. See for example Liu and Athorne [52], Olver and Sokolov [53], Oevel and Rogers [54] and Gerdjikov [55]. For example, the alternative non-commutative modified Korteweg–De Vries equation has the form,

$$\partial_t g=\partial^3 g+3\big(g(\partial^2\,g)-(\partial^2\,g)g\big)-6g(\partial g)g.$$

Note that the polynomial partial differential field includes even degree terms. Such alternative forms can be obtained from non-commutative modified Korteweg–De Vries equation via a suitable gauge transformation, as outlined in Carillo and Schiebold [56]. The combinatorial algebraic structure we have developed would seem a natural context to investigate such alternative hierarchy forms further. Closely related is the *Miura transformation*. This is particularly simple in our context. Assuming the order $n=2m+1$ with $m\in\mathbb{N}$ is odd, then since $\mathcal{I}^2=\mathrm{id}$, the base dispersion equation for P is, $\partial_t P=(-1)^{m+1}\partial^{2m+1}P$. We can assume this to be the base equation for the non-commutative potential Korteweg–de Vries hierarchy considered in Malham [10]. In that case the solution G^{pKdV} is given by $G^{\mathrm{pKdV}}=P(\mathrm{id}-P)^{-1}$. For the non-commutative modified Korteweg–de Vries hierarchy, we observe that when n is odd we can assume the solution G^{mKdV} to have the form $G^{\mathrm{mKdV}}=2P(\mathrm{id}+P)^{-1}(\mathrm{id}-P)^{-1}$, i.e. replacing the '$\mathrm{i}P$' everywhere simply by P, and all our results in Section 4 and thereafter follow through. This is because we carried through the quantity '$\mathrm{i}P$' throughout our computations in Section 4 and, in particular, into our abstract encoding. For example, our computation for $\partial_t[V]$ preceding Definition 12 carries through with this replacement with $V:=(\mathrm{id}-P)^{-1}$, $P^\dagger=-P$ and $V^\dagger=(\mathrm{id}+P)^{-1}$. For convenience we set $U^{\mathrm{pKdV}}:=(\mathrm{id}-P)^{-1}$ and $U^{\mathrm{mKdV}}:=(\mathrm{id}+P)^{-1}(\mathrm{id}-P)^{-1}$. Note that by operator partial fractions we have $U^{\mathrm{pKdV}}=\mathrm{id}+PU^{\mathrm{pKdV}}$ so $\partial G^{\mathrm{pKdV}}=\partial U^{\mathrm{pKdV}}$. Then as in Doikou et al. [8, Cor. 3.15] we observe that since $U^{\mathrm{pKdV}}=(\mathrm{id}+P)U^{\mathrm{mKdV}}$ we have,

Table 4
Non-zero signature coefficients appearing in the expansion of the *Pöppe polynomial* π_5 in Example 12. Not all the coefficients are shown. The remaining columns are shown in Table 5. The coefficients are the χ-images of the signature entries shown. Each column shows the factor contributions to the real coefficients of the basis elements shown in the very left column, for each of the monomials in π_5 shown across the top row.

Basis	$[\mathbf{5}]$	$[\mathbf{0}][\mathbf{3}][\mathbf{0}]$	$[\mathbf{3}][\mathbf{0}]^2$	$[\mathbf{0}][\mathbf{2}][\mathbf{1}]$	$[\mathbf{2}][\mathbf{1}][\mathbf{0}]$	$[\mathbf{2}][\mathbf{0}][\mathbf{1}]$	$[\mathbf{0}][\mathbf{1}][\mathbf{2}]$	$[\mathbf{0}][\mathbf{1}][\mathbf{0}]^3$
$[\mathbf{050}]$	5	$2\cdot(0\otimes 3\otimes 0)$						
$[\mathbf{050}^\dagger]$		$-2\cdot(0\otimes 3\otimes 0)$						
$[\mathbf{04010}]$	41	$0\otimes 3\hat{\otimes}0$	$3\otimes 0\hat{\otimes}0$	$2\cdot(0\otimes 2\otimes 1)$				
$[\mathbf{040}^\dagger\mathbf{10}^\dagger]$		$0\otimes 3\hat{\otimes}0$	$-3\otimes 0\hat{\otimes}0$	$2\cdot(0\otimes 2\otimes 1)$				
$[\mathbf{03020}]$	32	$0\otimes 21\otimes 0$	$3\otimes 0\hat{\otimes}0$	$0\otimes 2\otimes 1$	$2\otimes 1\otimes 0$	$2\otimes 0\otimes 1$	$2\cdot(0\otimes 1\otimes 2)$	$2\cdot(0\otimes 1\otimes 0\hat{\otimes}0\otimes 0)$
$[\mathbf{03020}^\dagger]$		$-0\otimes 21\otimes 0$	$-3\otimes 0\hat{\otimes}0$	$0\otimes 2\otimes 1$	$-2\otimes 1\otimes 0$	$2\otimes 0\otimes 1$		$-2\cdot(0\otimes 1\otimes 0\hat{\otimes}0\otimes 0)$
$[\mathbf{030}^\dagger\mathbf{20}]$		$-0\otimes 21\otimes 0$		$0\otimes 2\otimes 1$	$2\otimes 1\otimes 0$	$-2\otimes 0\otimes 1$		$-2\cdot(0\otimes 1\otimes 0\hat{\otimes}0\otimes 0)$
$[\mathbf{030}^\dagger\mathbf{20}^\dagger]$		$0\otimes 21\otimes 0$		$0\otimes 2\otimes 1$	$-2\otimes 1\otimes 0$	$-2\otimes 0\otimes 1$	$2\cdot(0\otimes 1\otimes 2)$	$2\cdot(0\otimes 1\otimes 0\hat{\otimes}0\otimes 0)$
$[\mathbf{02030}]$	23	$0\otimes 12\otimes 0$			$2\cdot(2\otimes 1\otimes 0)$		$0\otimes 1\otimes 2$	
$[\mathbf{02030}^\dagger]$		$-0\otimes 12\otimes 0$			$-2\cdot(2\otimes 1\otimes 0)$		$0\otimes 1\otimes 2$	
$[\mathbf{020}^\dagger\mathbf{30}]$		$-0\otimes 12\otimes 0$					$0\otimes 1\otimes 2$	
$[\mathbf{020}^\dagger\mathbf{30}^\dagger]$		$0\otimes 12\otimes 0$					$0\otimes 1\otimes 2$	
$[\mathbf{01040}]$	14	$0\hat{\otimes}3\otimes 0$						
$[\mathbf{01040}^\dagger]$		$-0\hat{\otimes}3\otimes 0$						
$[\mathbf{0202010}]$	221	$0\otimes 12\hat{\otimes}0$	$21\otimes 0\hat{\otimes}0$	$0\otimes 11\otimes 1$	$2\otimes 1\hat{\otimes}0$	$2\otimes 0\hat{\otimes}1$	$0\otimes 1\otimes 11$	$0\otimes 1\otimes 0\hat{\otimes}0\hat{\otimes}0$
$[\mathbf{02020}^\dagger\mathbf{10}^\dagger]$			$-21\otimes 0\hat{\otimes}0$	$0\otimes 11\otimes 1$	$2\otimes 1\hat{\otimes}0$	$2\otimes 0\hat{\otimes}1$	$-0\otimes 1\otimes 11$	$-0\otimes 1\otimes 0\hat{\otimes}0\hat{\otimes}0$
$[\mathbf{020}^\dagger\mathbf{2010}]$				$-0\otimes 11\otimes 1$			$0\otimes 1\otimes 11$	$0\otimes 1\otimes 0\hat{\otimes}0\hat{\otimes}0$
$[\mathbf{020}^\dagger\mathbf{20}^\dagger\mathbf{10}^\dagger]$		$-0\otimes 12\hat{\otimes}0$		$-0\otimes 11\otimes 1$			$-0\otimes 1\otimes 11$	$-0\otimes 1\otimes 0\hat{\otimes}0\hat{\otimes}0$
⋮								

Table 5
The remaining non-zero signature coefficients appearing in the expansion of the *Pöppe polynomial* π_5 in Example 12. The first set of columns appear in Table 4. The final column represents the coefficient on the right-hand side of the equation $\pi_5 = [\mathbf{050}]$.

Basis	$[\mathbf{1}][\mathbf{2}][\mathbf{0}]$	$[\mathbf{1}][\mathbf{0}][\mathbf{2}]$	$[\mathbf{0}]^2[\mathbf{3}]$	$[\mathbf{0}]^3[\mathbf{1}][\mathbf{0}]$	$[\mathbf{1}]^3$	$[\mathbf{1}][\mathbf{0}]^4$	$[\mathbf{0}]^2[\mathbf{1}][\mathbf{0}]^2$	$[\mathbf{0}]^4[\mathbf{1}]$	B
$[\mathbf{050}]$									1
$[\mathbf{050}^\dagger]$									
$[\mathbf{04010}]$									
$[\mathbf{040}^\dagger\mathbf{10}^\dagger]$									
$[\mathbf{03020}]$									
$[\mathbf{03020}^\dagger]$									
$[\mathbf{030}^\dagger\mathbf{20}]$									
$[\mathbf{030}^\dagger\mathbf{20}^\dagger]$									
$[\mathbf{02030}]$	$1\otimes 2\otimes 0$	$1\otimes 0\otimes 2$	$0\hat{\otimes}0\otimes 3$	$2\cdot(0\hat{\otimes}0\otimes 0\otimes 1\otimes 0)$					
$[\mathbf{02030}^\dagger]$	$-1\otimes 2\otimes 0$	$1\otimes 0\otimes 2$		$-2\cdot(0\hat{\otimes}0\otimes 0\otimes 1\otimes 0)$					
$[\mathbf{020}^\dagger\mathbf{30}]$	$1\otimes 2\otimes 0$	$-1\otimes 0\otimes 2$		$-2\cdot(0\hat{\otimes}0\otimes 0\otimes 1\otimes 0)$					
$[\mathbf{020}^\dagger\mathbf{30}^\dagger]$	$-1\otimes 2\otimes 0$	$-1\otimes 0\otimes 2$	$0\hat{\otimes}0\otimes 3$	$2\cdot(0\hat{\otimes}0\otimes 0\otimes 1\otimes 0)$					
$[\mathbf{01040}]$	$2\cdot(1\otimes 2\otimes 0)$		$0\hat{\otimes}0\otimes 3$						
$[\mathbf{01040}^\dagger]$	$-2\cdot(1\otimes 2\otimes 0)$		$0\hat{\otimes}0\otimes 3$						
$[\mathbf{0202010}]$	$1\otimes 2\hat{\otimes}0$	$1\otimes 0\otimes 11$	$0\hat{\otimes}0\otimes 21$	$0\hat{\otimes}0\otimes 0\otimes 1\hat{\otimes}0$	$1\otimes 1\otimes 1$	$1\otimes 0\hat{\otimes}0\otimes 0\hat{\otimes}0$	$0\hat{\otimes}0\otimes 1\otimes 0\hat{\otimes}0$	$0\hat{\otimes}0\otimes 0\hat{\otimes}0\otimes 1$	
$[\mathbf{02020}^\dagger\mathbf{10}^\dagger]$		$-1\otimes 0\otimes 11$		$0\hat{\otimes}0\otimes 0\otimes 1\hat{\otimes}0$	$1\otimes 1\otimes 1$	$-1\otimes 0\hat{\otimes}0\otimes 0\hat{\otimes}0$	$-0\hat{\otimes}0\otimes 1\otimes 0\hat{\otimes}0$	$0\hat{\otimes}0\otimes 0\hat{\otimes}0\otimes 1$	
$[\mathbf{020}^\dagger\mathbf{2010}]$		$-1\otimes 0\otimes 11$		$-0\hat{\otimes}0\otimes 0\otimes 1\hat{\otimes}0$	$1\otimes 1\otimes 1$	$-1\otimes 0\hat{\otimes}0\otimes 0\hat{\otimes}0$	$0\hat{\otimes}0\otimes 1\otimes 0\hat{\otimes}0$	$-0\hat{\otimes}0\otimes 0\hat{\otimes}0\otimes 1$	
$[\mathbf{020}^\dagger\mathbf{20}^\dagger\mathbf{10}^\dagger]$	$1\otimes 2\hat{\otimes}0$	$1\otimes 0\otimes 11$	$-0\hat{\otimes}0\otimes 21$	$-0\hat{\otimes}0\otimes 0\otimes 1\hat{\otimes}0$	$1\otimes 1\otimes 1$	$1\otimes 0\hat{\otimes}0\otimes 0\hat{\otimes}0$	$-0\hat{\otimes}0\otimes 1\otimes 0\hat{\otimes}0$	$-0\hat{\otimes}0\otimes 0\hat{\otimes}0\otimes 1$	
⋮									

$$
\begin{aligned}
& \partial U^{\mathrm{pKdV}} = \partial\big(PU^{\mathrm{mKdV}}\big) + \partial U^{\mathrm{mKdV}} \\
\Leftrightarrow\quad & \partial U^{\mathrm{pKdV}} = \partial\big(PU^{\mathrm{mKdV}}\big) + U^{\mathrm{mKdV}}\partial(P^2)U^{\mathrm{mKdV}} \\
\Rightarrow\quad & \partial[\![G^{\mathrm{pKdV}}]\!] = \partial[\![G^{\mathrm{mKdV}}]\!] + [\![G^{\mathrm{mKdV}}]\!]^2.
\end{aligned}
$$

In the last step we used the Pöppe product rule. This represents the Miura transformation giving the connection between the non-commutative potential, and modified, Korteweg–de Vries hierarchies. A natural question is what the translation (likely non-trivial) of this result is at even orders?

At the abstract algebra level, for the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}] \cong \mathbb{C}[C] \times \mathbb{R}\langle\mathbb{B}\rangle$, there are many open questions. For example, the skew-Pöppe algebra $\mathbb{C}[\mathbb{Z}_{\mathbf{0}}]$, endowed with the triple product in Lemma 12, constitutes a *triple system* or *ternary algebra*. See for example Meyberg [57, p. 21] or Ricciardo [58, p. 23]. Exploring this context is very much of interest.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix. Tables of coefficients

See Tables 2–5.

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