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Conjugacy languages in virtual graph products

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ABSTRACT

In this paper we study the behaviour of conjugacy languages in virtual graph products, extending results by Ciobanu et al. [13]. We focus primarily on virtual graph products in the form of a semi-direct product. First, we study the behaviour of twisted conjugacy representatives in right-angled Artin and Coxeter groups. We prove regularity of the conjugacy geodesic language for virtual graph products in certain cases, and highlight properties of the spherical conjugacy language, depending on the automorphism and ordering on the generating set. Finally, we give a criterion for when the spherical conjugacy language is not unambiguous context-free for virtual graph products. We can extend this further in the case of virtual RAAGs, to show the spherical conjugacy language is not context-free.

1. Introduction

Graph products are a well studied class of groups in geometric group theory, which include right-angled Artin and Coxeter groups (RAAGs/RACGs). Given a finite simple graph Γ, where each vertex has a group assigned to it, we define the associated graph product G_Γ as the group generated by the vertex groups, with the added relations that

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two elements commute if they lie in distinct vertex groups which are adjacent in the graph. Conjugacy in these groups has been studied in detail - for example the conjugacy problem in graph products was first solved by Green in the 90s [22], and a linear-time solution was found for RAAGs by Crisp, Godelle and Wiest [16]. In this paper, we focus on conjugacy in virtual graph products, which we define as groups which contain a finite index subgroup isomorphic to a graph product.

Virtual graph products are quasi-isometric to graph products by the Milnor-Schwarz Lemma [27]. While there exist many properties, such as hyperbolicity and standard growth, which are quasi-isometry invariant, there are also important properties which are not. A relevant example is given by Collins and Miller [14], who constructed groups $H \leq G$, where H is of finite index in G, such that H has solvable conjugacy problem but G does not. We therefore cannot immediately assume that the conjugacy problem is solvable in virtual graph products. One area of research which has been used to look at conjugacy classes in finitely generated groups is formal language theory. The language theoretic properties of representatives of conjugacy classes have been studied for various types of groups (see [13]). These languages are interlinked with conjugacy growth series, and in some cases we can establish the nature of these series through a language theoretic approach.

More formally, if G is a finitely generated group with generating set X, we define the *conjugacy geodesic language*, denoted $\mathsf{ConjGeo}(G, X)$, to be the set of words $w \in X^*$ such that the word length of w is smallest in its conjugacy class. The *spherical conjugacy language*, denoted $\mathsf{ConjSL}(G, X)$, consists of a unique representative word for each conjugacy class in G (see Section 2.5 for further details). By taking a formal power series over $\mathsf{ConjSL}(G, X)$, we obtain the conjugacy growth series of the group. It was shown by Ciobanu and Hermiller (Theorem 3.1, [12]) that for a graph product G_Γ, with respect to the standard generating set X, the language $\mathsf{ConjGeo}(G_\Gamma, X)$ is regular, provided each of the languages $\mathsf{ConjGeo}(G_i, X_i)$ of the vertex groups of G_Γ are regular. They also give an example of a RAAG, namely the free group on two generators, such that the language $\mathsf{ConjSL}(A_\Gamma, X)$ is not context-free. We note here that it remains open as to whether the algebraic type (i.e. rational/algebraic/transcendental) of the conjugacy growth series is a quasi-isometry invariant. In particular, the nature of formal languages has already been shown to not be quasi-isometry invariant (see Proposition 2.12), and so to establish how conjugacy growth may behave in virtual graph products, we need to study these groups directly.

We focus primarily on virtual graph products of the form $G_\phi = G_\Gamma \rtimes_\phi \langle t \rangle$ (see Equation (2)), where $\phi \in \mathrm{Aut}(G_\Gamma)$ and t are of finite order. Conjugacy in these types of group extensions is equivalent to twisted conjugacy in G_Γ. For a group G and automorphism $\phi \in \mathrm{Aut}(G)$, we say two elements $u, v \in G$ are *ϕ-conjugate* if there exists an element $x \in G$ such that $u = \phi(x)^{-1} v x$. Twisted conjugacy is an active topic of research, and relates to proving the R_∞ property in various types of groups, or solving the conjugacy problem in group extensions (see [9], [15], [18], [34]).

In a RAAG, which we denote A_Γ, it is well known that two cyclically reduced words are conjugate if and only if they are related by a finite sequence of cyclic permutations and commutation relations. To prove a twisted analogue of this result, we define ψ-cyclically reduced words in Section 3, where $\psi \in \mathrm{Aut}(A_\Gamma)$. Informally, these are words which do not decrease in length after performing any sequence of twisted cyclic permutations and commutation relations. These twisted cyclic permutations are similar to cyclic permutations, except we need to consider images of first and last letters of words under powers of ψ.

Automorphisms of RAAGs can be split into two categories: length preserving, where the length of any geodesic word is preserved under the automorphism, and non-length preserving otherwise. Our first result gives an analogue for length preserving automorphisms as to how twisted conjugate elements are related in RAAGs.

Theorem 3.13. *Let $A_\Gamma = \langle X \rangle$ be a RAAG, and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be length preserving. Let $u, v \in X^*$ be two ψ-cyclically reduced words. Then u and v are ψ-conjugate if and only if u and v are related by a finite sequence of ψ-cyclic permutations and commutation relations.*

We can extend Theorem 3.13 to all finite order automorphisms.

Corollary 3.14. *Let $A_\Gamma = \langle X \rangle$ be a RAAG, and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be of finite order. Let $u, v \in X^*$ be two ψ-cyclically reduced words. Then u and v are ψ-conjugate if and only if u and v are related by a finite sequence of ψ-cyclic permutations, commutation relations and free reduction.*

We then use Theorem 3.13 to prove the following for the conjugacy geodesic language in certain types of virtual RAAGs and RACGs.

Theorem 4.8. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for A_ϕ, and $\phi \in \mathrm{Aut}(A_\Gamma)$ is a composition of inversions. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular.*

It still remains open whether Theorem 4.8 holds for all length preserving automorphisms. We do, however, find an example which proves the negation of Theorem 4.8 for non-length preserving automorphisms.

Corollary 4.16. *There exists a virtual RAAG A_ϕ such that $\mathsf{ConjGeo}(A_\phi, X)$ is not context-free. In particular, let $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)) where A_Γ, ϕ are defined in Proposition 4.14. Let $\widehat{X}$ be the standard generating set for A_ϕ. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is not context-free.*

Next we study the spherical twisted conjugacy language for virtual graph products (see Definition 5.5). This language, denoted $\mathsf{ConjSL}_\phi(G, X)$, consists of a unique representative word for each ϕ-twisted conjugacy class in G. We first give an example of a RAAG such that the spherical twisted conjugacy language changes from regular to not context-free, depending on the automorphism or the ordering on the generating set.

Corollary 5.9. *There exists a RAAG $A_\Gamma = \langle X \rangle$, with an order on X, and automorphism $\psi \in \mathrm{Aut}(A_\Gamma)$, such that $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ is regular, but $\mathsf{ConjSL}_{\psi^2}(A_\Gamma, X)$ is not context-free. Moreover, there exists a different order on X, such that $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ is not context-free.*

For the language $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$, we prove the following result which does not require any restrictions on the automorphism, unlike our results for the conjugacy geodesic language.

Corollary 5.14. *Let G_ϕ be a virtual graph product of the form $G_\phi = G_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for G_ϕ, endowed with an order. For an induced subgraph $\Lambda \subseteq \Gamma$, let G_Λ be the induced graph product with respect to Λ, with standard generating set X_Λ, with an induced ordering from X. If $\mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ is not regular, then $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ is not regular. This result also holds when we replace regular with either unambiguous context-free or context-free.*

For RAAGs, we can extend this further. We denote $\tilde{\sigma}(G, X)$ to be the strict growth series of $\mathsf{ConjSL}(G, X)$, for a group G with respect to a generating set X. This is also known as the *spherical conjugacy growth series* (see Section 2).

Theorem 5.15. *Let A_Γ be a RAAG which is not free abelian. Let X be the standard generating set for A_Γ, endowed with an order. Then $\mathsf{ConjSL}(A_\Gamma, X)$ is not context-free.*

Corollary 5.16. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for A_ϕ, endowed with an order. If A_ϕ is not virtually abelian, then $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$ is not context-free. Otherwise, the spherical conjugacy growth series $\tilde{\sigma}\left(A_\phi, \widehat{X}\right)$ is rational.*

As an aside, we prove another result for more general virtual RAAGs and RACGs, where the finite index subgroup is acylindrically hyperbolic.

Corollary 5.2. *Let G be a group with a finite index normal subgroup $H \trianglelefteq G$, such that H is an acylindrically hyperbolic RAAG (or RACG). Then $\mathsf{ConjSL}(G, X)$ is not unambiguous context-free, with respect to any generating set X.*

The organisation of this paper is as follows. After providing necessary definitions in Section 2, we study the behaviour of twisted conjugate elements in RAAGs in Section 3. We then use these results to study the geodesic conjugacy language in Section 4. In Section 5, we study the spherical conjugacy language in virtual graph products.

2. Preliminaries

All groups in this paper are finitely generated with inverse-closed generating sets. We write $u \sim v$ when two group elements $u, v \in G$ are conjugate.

2.1. Group extensions

As is standard, an extension of a group H by N is a group G such that $N \trianglelefteq G$ and $G/N \cong H$. This can be encoded by a short exact sequence of groups

$$1 \to N \to G \to H \to 1.$$

A group extension is split if and only if G is a semi-direct product, that is $G \cong N \rtimes_\alpha H$ for some homomorphism $\alpha \colon H \to \mathrm{Aut}(N)$. For this paper we assume H is a finite cyclic group, and so N has finite index in G. It is sufficient to specify $\alpha(t)$ for a generator t of H. We let $\phi = \alpha(t)$, and so G has presentation

$$G \cong N \rtimes_\phi H = \langle s_1, \ldots, s_r, t \mid R(N),\ t^m = 1,\ t^{-1} s_i t = \phi(s_i) \rangle, \tag{1}$$

where $R(N)$ denotes the relations in N, and $\phi \in \mathrm{Aut}(N)$ is of finite order which divides m. Any element of G can be written in the form $g = t^l v$ where $v \in N$ and $0 \leq l \leq m-1$.

2.2. Twisted conjugacy

Let G be a group, let $u, v \in G$, and let $\phi \in \mathrm{Aut}(G)$. We say u and v are *ϕ-conjugate*, denoted $u \sim_\phi v$, if there exists an element $x \in G$ such that $u = \phi(x)^{-1} v x$. There is an immediate link between conjugacy in a split group extension $G \cong N \rtimes_\phi H$, and twisted conjugacy in the normal subgroup N.

Lemma 2.1. *([8], Page 4) Let G be a group extension of the form (1). Let $t^a u, t^b v \in G$ be two elements in G, with $0 \leq a, b \leq m-1$. Then $t^a u \sim t^b v$ if and only if $a = b$ and $v \sim_{\phi^a} \phi^k(u)$, for some integer k where $0 \leq k \leq m-1$.*

2.3. Graph products

Let Γ be an undirected finite simple graph, that is, with no loops or multiple edges. We let $V(\Gamma)$ denote the vertex set of Γ, and let $E(\Gamma)$ denote the edge set of Γ. For a collection of groups $\mathcal{G} = \{G_v \mid v \in V(\Gamma)\}$ which label the vertices of Γ, the associated *graph product*, denoted G_Γ, is the group defined as the quotient

$$\left(*_{v \in V(\Gamma)} G_v\right) / \langle\langle st = ts,\ s \in G_u,\ t \in G_v,\ \{u, v\} \in E(\Gamma) \rangle\rangle.$$

These groups include two well studied classes: right-angled Artin groups (RAAGs), denoted A_Γ, where each vertex group is isomorphic to $\mathbb{Z}$, and right-angled Coxeter groups (RACGs), denoted W_Γ, where each vertex group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. With this convention, elements commute if and only if an edge exists between the corresponding vertex groups. Some authors use the opposite convention, for example in [16].

We can create a virtual graph product G_ϕ from the following split short exact sequence

$$1 \to G_\Gamma \to G_\phi \to H \to 1,$$

where $G_\Gamma = \langle s_1, \ldots, s_r \rangle$ is a graph product such that G_Γ is a finite index subgroup in G_ϕ, and H is a finite cyclic group. In this case we have

$$G_\phi = G_\Gamma \rtimes_\phi \langle t \rangle = \langle s_1, \ldots s_r, t \mid R(G_\Gamma),\ t^m = 1,\ t^{-1} s_i t = \phi(s_i) \rangle, \tag{2}$$

where $\phi \in \mathrm{Aut}(G_\Gamma)$ is of finite order which divides m. Of course not all virtual graph products arise from such a short exact sequence, but in this paper we will focus primarily on virtual graph products of the form (2). The exception will be in Section 5, where we look at acylindrically hyperbolic cases.

For notation, if X is the standard generating set for G_Γ, then we define $\widehat{X} = \{X, t\}$ to be the standard generating set for the virtual graph product $G_\phi = G_\Gamma \rtimes_\phi \langle t \rangle$.

2.4. Automorphisms of graph products

For any word $w \in X^*$, we let $l(w)$ denote the word length of w over X.

Definition 2.2. Let G_Γ be a graph product. We say $\phi \in \mathrm{Aut}(G_\Gamma)$ is:

(i) *length preserving* if $l(w) = l(\phi(w))$ for any word $w \in X^*$, and
(ii) *non-length preserving* otherwise.

For length preserving cases, $l(\phi(x)) = 1$ if and only if $x \in X$.

For RAAGs and RACGs, generators of the automorphism groups were classified by Servatius and Laurence [32] [24], and consist of four types:

1. **Inversions**: For some $x \in V(\Gamma)$, send $x \mapsto x^{-1}$, and fix all other vertices.
2. **Graph automorphisms**: Induced by the defining graph Γ.
3. **Partial conjugations**: Let $x \in V(\Gamma)$, let $Q \subseteq \Gamma$ be the subgraph obtained by deleting x, all vertices adjacent to x and all incident edges. Let $P \subseteq Q$ be a union of connected components of Q. A partial conjugation maps $p \mapsto xpx^{-1}$ for all $p \in P$, and fixes all other vertices.

4. **Dominating transvections**: Let $x, y \in V(\Gamma)$, $x \neq y$, and assume y is adjacent to all vertices in Γ which are adjacent to x. A dominating transvection maps $x \mapsto xy^{\pm 1}$ or $x \mapsto y^{\pm 1}x$, and fixes all other vertices.

The following proposition can be shown using a straightforward induction argument.

Proposition 2.3. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be a length preserving automorphism. Then ψ is a finite composition of inversions and graph automorphisms, and hence has finite order.*

2.5. Languages and growth series

We recall some basic notions from formal language theory, and refer the reader to [19] and [23] for further details.

Definition 2.4. For a finite set X, we define a *language* L to be any subset of X^*, where X^* is the set of all words over X. A *finite state automaton* over X is a tuple $M = (Q, X, \mu, A, q_0)$, where Q is the finite set of states, $\mu\colon Q \times X \to Q$ the transition function, $A \subseteq Q$ the set of accept states, and $q_0 \in Q$ the start state. A word $w = x_1 \dots x_n \in X^*$, where $x_i \in X$ for all $1 \leq i \leq n$, is *recognised* by a finite state automaton if $\mu(\dots(\mu(\mu(q_0, x_1), x_2)\dots, x_n) \in A$. A language L is *regular* if and only if L is recognised by some finite state automaton.

Informally, a pushdown automaton is a finite state automaton combined with a 'stack': this stores a string of symbols, and acts as a way of remembering information. A language is *context-free* if and only if it is accepted by a pushdown automaton. Alternatively, a language is context-free if it is generated by a context-free grammar. If this grammar produces each word in a unique way, we say the language is *unambiguous context-free.*

We recall the following properties of regular and context-free languages.

Lemma 2.5. *Let L, L' be regular languages over a finite alphabet X. Then the languages L^* (Kleene closure), $X^* \setminus L$, LL', $L \cap L'$ and $L \cup L'$ are also regular, i.e. regularity is closed under Kleene closure, complement, concatenation, intersection and union. Similarly, context-free languages are closed under Kleene closure, concatenation and union.*

Note that context-free languages are not closed under complement.

Lemma 2.6. *([23], Theorem 7.24 and 7.25) Let $h\colon X^* \to X^*$ be a monoid homomorphism. Then if P is a context-free language over X, so is $h(P)$, i.e. context-free languages are closed under monoid homomorphisms.*

The following result can be shown using the fact that if L is context-free, then the language L^R, consisting of all words in L in reverse, is context-free.

Lemma 2.7. *If L is a context-free language, then so is*

$$\mathsf{suff}(L) = \{u \in X^* \mid vu \in L \text{ for some } v \in X^*\}.$$

Lemma 2.8. *([23], Theorem 7.27) Let L be a regular language over a finite alphabet X.*

1. *If N is an unambiguous context-free language over X, then $N \cap L$ is unambiguous context-free.*
2. *If P is a context-free language over X, then $P \cap L$ is context-free.*

Definition 2.9. A formal power series $f(z) \in \mathbb{Z}[[z]]$ is *rational* if there exist non-zero polynomials $p(z), q(z) \in \mathbb{Z}[[z]]$ such that $f(z) = \frac{p(z)}{q(z)}$. We say $f(z)$ is *algebraic* over $\mathbb{Q}(z)$ if there exists a non-trivial polynomial $p(z, u) \in \mathbb{Q}(z, u)$ such that $p(z, f(z)) = 0$. If $f(z)$ is not algebraic, then $f(z)$ is *transcendental*.

Any language $L \subseteq X^*$ gives rise to a *strict growth function*, defined as

$$\phi_L(n) := |\{w \in L \mid l(w) = n\}|.$$

The generating series associated to this function, defined as

$$f_L(z) := \sum_{i=0}^{\infty} \phi_L(i) z^i,$$

is known as the *strict growth series*. It is well known that if L is a regular language, then the series $f_L(z)$ is rational [28]. Another key link between formal languages and growth series comes from the following result:

Theorem 2.10. *(Chomsky-Schützenberger, [10]) Let $L \subset X^*$ be a language. If L is unambiguous context-free, then the series $f_L(z)$ is algebraic.*

By taking the contrapositive, we immediately see that if the series $f_L(z)$ is transcendental, then the language L is not unambiguous context-free.

2.6. Languages in groups

Let X be a finite set. For a group $G = \langle X \rangle$ and words $u, v \in X^*$, we use $u = v$ to denote equality of words, and $u =_G v$ to denote equality of the group elements represented by u and v. For a group element $g \in G$, we define the *length* of g, denoted $|g|_X$, to be the length of a shortest representative word for the element g over X (if X is fixed or clear from the context, we write $|g|$). A word $w \in X^*$ is called a *geodesic* if $l(w) = |\pi(w)|$, where $\pi\colon X^* \to G$ is the natural projection (recall $l(w)$ denotes the word length of w over X).

Definition 2.11. We define the language of geodesic words of G, with respect to X, as

$$\mathsf{Geo}(G,X) := \{w \in X^* \mid l(w) = |\pi(w)|\}.$$

We let $[g]_c$ denote the conjugacy class of $g \in G$. We define the *length up to conjugacy* of an element $g \in G$ by

$$|g|_c := \min\{|h| \mid h \in [g]_c\}.$$

We now consider languages associated to conjugacy classes. These will be adapted to twisted conjugacy in Section 4. We define the conjugacy geodesic and cyclic geodesic languages of G, with respect to X, as follows:

$$\mathsf{ConjGeo}(G,X) := \{w \in X^* \mid l(w) = |\pi(w)|_c\},$$
$$\mathsf{CycGeo}(G,X) := \{w \in X^* \mid w \text{ is a cyclic geodesic}\},$$

where a cyclic geodesic is a word such that all cyclic permutations are geodesic. Given an order on X, let $\leq_{sl}$ be the induced shortlex ordering of X^*. For each conjugacy class c, we define the *shortlex conjugacy normal form* of g to be the shortlex least word z_c over X representing an element of c. We define the shortlex conjugacy language, for G over X, as

$$\mathsf{ConjSL}(G,X) := \{z_c \mid c \in G/\sim\}.$$

These languages, with respect to a generating set X of G, satisfy

$$\mathsf{ConjSL}(G,X) \subseteq \mathsf{ConjGeo}(G,X) \subseteq \mathsf{CycGeo}(G,X) \subseteq \mathsf{Geo}(G,X).$$

We collect a few results from the literature.

Proposition 2.12. *The property of being a regular language is not a quasi-isometry invariant for*

(i) *([30], Page 268)* $\mathsf{Geo}(G,X)$*,*
(ii) *([13], Propositions 5.3 and 5.4)* $\mathsf{ConjGeo}(G,X)$*, and*
(iii) *([13], Propositions 5.1 and 5.2)* $\mathsf{ConjSL}(G,X)$*.*

Cannon [30] provided an example of a virtually abelian group $G = \mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}/2\mathbb{Z}$ action swaps the generators of $\mathbb{Z}^2$, such that the language $\mathsf{Geo}(G,X)$ could be either regular or not depending on the generating set. In Section 5 of [13], further examples of virtually abelian groups were studied, and in certain cases the languages $\mathsf{ConjGeo}(G,X)$ and $\mathsf{ConjSL}(G,X)$ could be either regular or not depending on the generating set.

Notice that $\mathsf{ConjSL}(G, X)$ counts the number of conjugacy classes of G whose smallest element, with respect to the shortlex ordering, is of length n, for any $n \geq 0$. We denote $\tilde{\sigma} = \tilde{\sigma}(G, X)$ to be the strict growth series of $\mathsf{ConjSL}(G, X)$. This is also known as the *spherical conjugacy growth series.* The nature of this series has been studied for many types of groups. Proposition 2.13 collects the only known results so far which hold for any generating set.

Proposition 2.13. *Let G be a finitely generated group.*

(1) *[20] If G is virtually abelian, then $\tilde{\sigma}$ is rational with respect to any generating set.*
(2) *[3] If G is hyperbolic and not virtually cyclic, then $\tilde{\sigma}$ is transcendental with respect to any generating set.*

It has been conjectured that the only groups with rational $\tilde{\sigma}$ are virtually abelian groups (Conjecture 7.2, [11]).

3. Twisted conjugacy in RAAGs

In this section we prove Theorem 3.13, which is a twisted analogue of the well known fact that two cyclically reduced elements in a RAAG are conjugate if and only if they are related by a finite sequence of cyclic permutations and commutation relations [16]. We recall some of this terminology here.

Definition 3.1. Let X be a finite set, and let $w = x_1 \dots x_n \in X^*$, where $x_i \in X$ for all $1 \leq i \leq n$. We say w is *freely reduced* if there does not exist any i, for $1 \leq i < n$, such that $x_i = x_{i+1}^{-1}$. We define a *cyclic permutation* or *cyclic shift* of w to be any word of the form

$$w' = x_{i+1} \dots x_n x_1 \dots x_i,$$

for some $i \in \{1, \dots, n-1\}$.

Unless otherwise stated, we let A_Γ be a RAAG with standard generating set $X = V(\Gamma)$, and remind the reader that the generating set is inverse closed.

Definition 3.2. Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be non-empty. We define $\mathrm{Geod}(v)$ to be the word obtained from v by free reductions after commutation relations. Hence $\mathrm{Geod}(v)$ is a geodesic over X.

Definition 3.3. Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic. We say v is *cyclically reduced* if there does not exist a sequence of cyclic permutations, commutation relations and free reductions to a geodesic word $w \in X^*$, such that $l(v) > l(w)$.

We now define twisted versions of these definitions to understand twisted conjugacy in RAAGs. We use similar techniques to [4], where the authors studied twisted conjugacy in free groups.

Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic, where $l(v) > 1$. Then we can write v in the form $v = xy$ for some non-empty geodesic words $x, y \in X^*$. We say x is a *proper prefix* of v, and y is a *proper suffix* of v. We let $\mathcal{P}(v)$ and $\mathcal{S}(v)$ be the set of all possible proper prefixes and suffixes of v respectively.

Definition 3.4. Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic, where $l(v) > 1$. Let $v = xy$ for some $x \in \mathcal{P}(v),\ y \in \mathcal{S}(v)$. Let $\psi \in \mathrm{Aut}(A_\Gamma)$. We define a *$\psi$-cyclic shift of a prefix* of v to be the following operation on the word v:

$$v = xy \xmapsto{\psi-\text{cyclic shift}} y \cdot \mathrm{Geod}\left(\psi^{-1}(x)\right).$$

Similarly we define a *ψ-cyclic shift of a suffix* of v as

$$v = xy \xmapsto{\psi-\text{cyclic shift}} \mathrm{Geod}\left(\psi(y)\right) \cdot x.$$

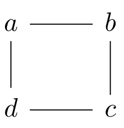
Remark 3.5. We note that the Geod notation is unnecessary for length preserving automorphisms, since $\mathrm{Geod}(\psi(v)) = \psi(v)$ for any geodesic $v \in X^*$. From now on, we will assume $\psi \in \mathrm{Aut}(A_\Gamma)$ is length preserving, and remove the Geod notation. This notation will be introduced again in Section 3.1, where we will generalise Theorem 3.13 to all finite order automorphisms.

For brevity, we will use $\overset{\psi}{\longleftrightarrow}$ to denote a ψ-cyclic shift of either a prefix or suffix. The double arrow is necessary here since any ψ-cyclic shift can be reversed, i.e.

$$xy \xmapsto{\psi-\text{cyclic shift}} y \cdot \left(\psi^{-1}(x)\right) \xmapsto{\psi-\text{cyclic shift}} \psi\left(\psi^{-1}(x)\right) \cdot y = xy.$$

Here is an example of how this operation works in practice.

Example 3.6. Let $A_\Gamma = F_2 \times F_2$, and label the vertices of the defining graph as follows:



Let $\psi\colon a \to b \to c \to d$ be a graph automorphism. Consider the geodesic word $w = ac^{-1}d$. We can compute the ψ-cyclic shift of the first and last letters as follows:

$$w = ac^{-1}d \overset{\psi}{\longleftrightarrow} \psi(d)ac^{-1} = a^2c^{-1}, \quad w = ac^{-1}d \overset{\psi}{\longleftrightarrow} c^{-1}d\psi^{-1}(a) = c^{-1}d^2.$$

Note that $w =_{A_\Gamma} adc^{-1}$, so we could also compute the following ψ-cyclic shift:

$$w =_{A_\Gamma} adc^{-1} \overset{\psi}{\longleftrightarrow} \psi(c^{-1})ad = d^{-1}ad.$$

Definition 3.7. Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic and $\psi \in \mathrm{Aut}(A_\Gamma)$. We define the set of all ψ-cyclic shifts of v to be the following set:

$$\psi[v] = \{y\psi^{-1}(x),\ \psi(y)x \mid v = xy \text{ for all possible } x \in \mathcal{P}(v), y \in \mathcal{S}(v)\}.$$

Recall that two words $u, v \in X^*$ representing groups elements of A_Γ are ψ-conjugate if there exists $w \in X^*$ such that $u =_{A_\Gamma} \psi(w)^{-1}vw$.

Lemma 3.8. *Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic. Then v is ψ-conjugate to all elements $w \in \psi[v]$.*

Proof. Write $v = xy$ for some $x \in \mathcal{P}(v)$, $y \in \mathcal{S}(v)$. Consider the words $y\psi^{-1}(x)$, $\psi(y)x \in \psi[v]$. The result follows by the following relations:

$$\psi\left(\psi^{-1}(x)\right) \cdot y\psi^{-1}(x) \cdot \psi^{-1}(x)^{-1} =_{A_\Gamma} x \cdot y\psi^{-1}(x) \cdot \psi^{-1}(x)^{-1} =_{A_\Gamma} xy = v.$$

$$\psi(y)^{-1} \cdot \psi(y)\, x \cdot y =_{A_\Gamma} xy = v. \quad \square$$

For ψ of order greater than two, we need to also consider further iterations of ψ-cyclic shifts. To do this, we define a ψ-cyclic permutation of a word.

Definition 3.9. Let $A_\Gamma = \langle X \rangle$, and let $w = x_1 \dots x_n \in X^*$ be a geodesic, where $x_i \in X$ for all $1 \le i \le n$. Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be of finite order m. We define a *ψ-cyclic permutation* of w to be any word of the form

$$w' = \psi^k(x_{i+1}) \dots \psi^k(x_n)\psi^{k-1}(x_1) \dots \psi^{k-1}(x_i),$$

for some $0 \le k \le m - 1$.

We note that if ψ is the trivial automorphism, a ψ-cyclic permutation is equivalent to a cyclic permutation (see Definition 3.1).

A ψ-cyclic permutation is equivalent to taking successive ψ-cyclic shifts of letters in w, including taking higher powers of ψ. By Lemma 3.8, we immediately have the following result.

Corollary 3.10. *Let $A_\Gamma = \langle X \rangle$, and let $w \in X^*$ be a geodesic. Let $w' \in X^*$ be a ψ-cyclic permutation of w. Then w is ψ-conjugate to w'.*

We now have the tools necessary to define a twisted version of cyclic reduction in RAAGs.

Definition 3.11. Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic. We say v is *ψ-cyclically reduced* (ψ-CR) if there does not exist a sequence of ψ-cyclic permutations, commutation relations and free reductions to a geodesic $w \in X^*$, such that $l(v) > l(w)$.

It is well known that a geodesic $v \in X^*$ is cyclically reduced if and only if v cannot be written in the form $v =_{A_\Gamma} u^{-1}wu$, where $l(w) < l(v)$ and $u^{-1}wu \in X^*$ is geodesic. We prove one direction of this idea with respect to twisted cyclic reduction.

Proposition 3.12. *Let $A_\Gamma = \langle X \rangle$, and let $v \in X^*$ be a geodesic. Let $\psi \in \mathrm{Aut}(A_\Gamma)$. Suppose v can be written in the form*

$$v =_{A_\Gamma} \psi(u)^{-1}wu,$$

where $l(w) < l(v)$ and $\psi(u)^{-1}wu \in X^*$ is geodesic. Then v is not ψ-CR.

Proof. We can perform a ψ-cyclic shift from v to w as follows:

$$v =_{A_\Gamma} \psi(u)^{-1}wu \overset{\psi}{\leftrightarrow} \psi(u) \cdot \psi(u)^{-1}w =_{A_\Gamma} w.$$

By assumption $l(w) < l(v)$, so by definition v cannot be ψ-CR. □

We are now able to prove the main theorem of this section.

Theorem 3.13. *Let $A_\Gamma = \langle X \rangle$ be a RAAG, and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be length preserving. Let $u, v \in X^*$ be two ψ-cyclically reduced words. Then u and v are ψ-conjugate if and only if u and v are related by a finite sequence of ψ-cyclic permutations and commutation relations.*

Proof. We start with the reverse implication. Suppose there exists a finite sequence

$$u = u_0 \leftrightarrow u_1 \leftrightarrow \cdots \leftrightarrow u_n = v$$

of ψ-cyclic permutations and commutation relations. Then by Corollary 3.10, $u_i \sim_\psi u_{i+1}$ for all $0 \leq i < n$, and so $u \sim_\psi v$.

For the forward direction, we can write u in the form $u =_{A_\Gamma} \psi(x)^{-1}vx$ for some $x \in X^*$. Since both u and v are ψ-CR, $l(u) = l(v)$, and so $\psi(x)^{-1}vx$ must contain cancellations by Proposition 3.12. We consider whether v cancels fully or not in $\psi(x)^{-1}vx$.

Case 1: v is not fully cancelling in $\psi(x)^{-1}vx$.

Suppose there are cancellations in $\psi(x)^{-1}v$ only. Since u is ψ-CR, we must have that $\psi(x)^{-1}$ is fully cancelled. In this case, write $v =_{A_\Gamma} \psi(x)v_1$. Then

$$u =_{A_\Gamma} \psi(x)^{-1}vx =_{A_\Gamma} \psi(x)^{-1}\psi(x)v_1x =_{A_\Gamma} v_1x.$$

Here $u =_{A_\Gamma} v_1x$ and $v =_{A_\Gamma} \psi(x)v_1$, so these words are related to each other by a ψ-cyclic shift of x.

Similarly suppose there are cancellations in vx only. Similar to before, we must have that x is fully cancelled. Let $v =_{A_\Gamma} v_2x^{-1}$. Then

$$u =_{A_\Gamma} \psi(x)^{-1}vx =_{A_\Gamma} \psi(x)^{-1}v_2.$$

Here $u =_{A_\Gamma} \psi(x)^{-1}v_2$ and $v =_{A_\Gamma} v_2x^{-1}$, so these words are related to each other by a ψ-cyclic shift of x^{-1}.

Finally, suppose there exist cancellations in both $\psi(x)^{-1}v$ and vx. We note that since v is ψ-CR, we can assume v is not of the form $v =_{A_\Gamma} \psi(x_s)^{-1}\tilde{v}x_s$ by Proposition 3.12. Hence there exists $\alpha, \beta \in X^*$, where $\alpha \neq_{A_\Gamma} \beta$, such that v can be written in the form $v =_{A_\Gamma} \psi(\alpha)^{-1}\tilde{v}\beta$, where $\tilde{v}$ is geodesic and non-empty. By assumption we can write $\psi(x)^{-1} =_{A_\Gamma} \psi(x_1)^{-1}\psi(\alpha)$ (cancellation in $\psi(x)^{-1}v$)) and $x =_{A_\Gamma} \beta^{-1}x_2$ (cancellation in vx). Now we have

$$u =_{A_\Gamma} \psi(x)^{-1}vx =_{A_\Gamma} \psi(x_1)^{-1}\psi(\alpha)\psi(\alpha)^{-1}\tilde{v}\beta\beta^{-1}x_2.$$

Since β^{-1} is a subword of x which can be moved via commutation relations to the left in x, $\psi(\beta)$ must be a subword of $\psi(x)^{-1}$ which can be moved via commutation relations to the right in $\psi(x)^{-1}$. Similarly α^{-1} must be a subword of x which can be moved via commutation relations to the left in x. Therefore, we can also write u in the form

$$u =_{A_\Gamma} \psi(x_r)^{-1}\psi(\beta)\psi(\alpha)\psi(\alpha)^{-1}\tilde{v}\beta\beta^{-1}\alpha^{-1}x_r.$$

Since u is ψ-CR, we can assume x_r is the empty word, and so $u =_{A_\Gamma} \psi(\beta)\tilde{v}\alpha^{-1}$. We can see u and v are related by the following sequence of ψ-cyclic shifts and commutation relations:

$$u =_{A_\Gamma} \psi(\beta)\tilde{v}\alpha^{-1} \overset{\psi}{\longleftrightarrow} \psi(\alpha)^{-1}\psi(\beta)\tilde{v} =_{A_\Gamma} \psi(\beta)\psi(\alpha)^{-1}\tilde{v} \overset{\psi}{\longleftrightarrow} \psi(\alpha)^{-1}\tilde{v}\beta =_{A_\Gamma} v.$$

We note this scenario also holds when $\tilde{v}$ is the empty word, which lies in the second case we're about to prove, when v is fully cancelling.

Case 2: v is fully cancelling in $\psi(x)^{-1}vx$.
First suppose v fully cancels in vx. We can assume there are cancellations of the remaining part of x with $\psi(x)^{-1}$, otherwise u would not be ψ-CR. We set

$$x =_{A_\Gamma} v^{-1}x_1 \quad \Rightarrow \psi(x)^{-1} =_{A_\Gamma} \psi(x_1)^{-1}\psi(v).$$

Then

$$u =_{A_\Gamma} \psi(x)^{-1}vx =_{A_\Gamma} \psi(x_1)^{-1}\psi(v)x_1.$$

We now consider the cancellations remaining in x. We first note that if no letters from x_1 cancel with letters in $\psi(v)$, then x_1 and $\psi(x_1)^{-1}$ must be inverses of each other and cancel after commutation relations, since u is ψ-CR. We therefore assume x_1 is of minimal length up to any cancellation with terms in $\psi(x_1)^{-1}$. Suppose x_1 fully cancels with $\psi(v)$. Then let $\psi(v) =_{A_\Gamma} v_2 x_1^{-1}$, and so $u =_{A_\Gamma} \psi(x_1)^{-1} v_2$. Now

$$\psi(v) =_{A_\Gamma} v_2 x_1^{-1} \quad \Rightarrow v =_{A_\Gamma} \psi^{-1}(v_2)\psi^{-1}(x_1^{-1}),$$

and so u and v are related by a sequence of ψ-cyclic shifts and commutation relations:

$$v =_{A_\Gamma} \psi^{-1}(v_2)\psi^{-1}(x_1^{-1}) \overset{\psi}{\longleftrightarrow} x_1^{-1}\psi^{-1}(v_2) \overset{\psi}{\longleftrightarrow} v_2 x_1^{-1} \overset{\psi}{\longleftrightarrow} \psi(x_1^{-1})v_2 =_{A_\Gamma} u.$$

If instead, $\psi(v)$ fully cancels in $\psi(v)x_1$, then we have

$$x_1 =_{A_\Gamma} \psi(v)^{-1} x_{11} \Rightarrow \psi(x_1)^{-1} =_{A_\Gamma} \psi(x_{11})^{-1}\psi^2(v),$$

and hence $u =_{A_\Gamma} \psi(x_{11})^{-1}\psi^2(v)x_{11}$. The result then follows by reverse induction on the length of x.

The case where v fully cancels in $\psi(x^{-1})v$, follows a similar proof. We already gave a proof in Case 1 for when v cancels in both $\psi(x)^{-1}v$ and vx, and so in all cases, u and v are related by a sequence of ψ-cyclic permutations and commutation relations. □

3.1. Non-length preserving cases

We mention briefly how the results from above can be extended to all finite order automorphisms. For $A_\Gamma = \langle X \rangle$, let $v \in X^*$ be ψ-cyclically reduced for some finite order $\psi \in \mathrm{Aut}(A_\Gamma)$.

1. If ψ is length preserving, then any word w which can be obtained from v via a sequence of ψ-cyclic permutations and commutation relations must also be ψ-cyclically reduced.
2. If ψ is non-length preserving, we can find words w from v via ψ-cyclic shifts and commutation relations such that $l(\mathrm{Geod}(w)) > l(v)$. In particular, w is not ψ-cyclically reduced.

To extend Theorem 3.13 for all finite order automorphisms, we need to also include free reduction in any sequence between twisted conjugate words, since any word in the sequence could increase in length after ψ-cyclic shifts. The definitions and results from Section 3 follow similarly as before, by adding Geod notation when required.

Corollary 3.14. *Let $A_\Gamma = \langle X \rangle$ be a RAAG, and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be of finite order. Let $u, v \in X^*$ be two ψ-cyclically reduced words. Then u and v are ψ-conjugate if and only if u and v are related by a finite sequence of ψ-cyclic permutations, commutation relations and free reduction.*

3.2. Twisted cyclic reduction: inversions

When considering standard cyclic reduction in a RAAG $A_\Gamma = \langle X \rangle$, we recall that any word $w \in X^*$ which is cyclically reduced cannot have the form $w = x_1 a_i^{\pm 1} x_2 a_i^{\mp 1} x_3$, where all letters of subwords x_1 and x_3 commute with a_i. For twisted cyclic reduction, however, we have a different situation. In particular, the reverse direction of Proposition 3.12 does not necessarily hold.

Example 3.15. Let A_Γ be the RAAG with defining graph

$$a \text{ —— } b \text{ —— } c \text{ —— } d.$$

Let $\psi\colon a \leftrightarrow d,\ b \leftrightarrow c$ be a reflection, and consider the geodesic word $w = c^{-1}aac$. Then $w \neq_{A_\Gamma} \psi(u)^{-1}vu$ for any shorter word $v \in X^*$, but w is not ψ-CR. This can be seen by the following sequence of ψ-cyclic shifts and commutation relations of w:

$$w = c^{-1}aac \overset{\psi}{\leftrightarrow} aac\cdot\psi^{-1}(c^{-1}) = aacb^{-1} =_{A_\Gamma} b^{-1}aac \overset{\psi}{\leftrightarrow} aac\cdot\psi^{-1}(b^{-1}) = aacc^{-1} =_{A_\Gamma} aa.$$

We can however show the reverse direction of Proposition 3.12 holds for inversions.

Definition 3.16. Let $A_\Gamma = \langle X \rangle$, let $\psi \in \mathrm{Aut}(A_\Gamma)$ and let $w \in X^*$ be a geodesic. We say w is a *ψ-cyclic geodesic* if all ψ-cyclic permutations of w are geodesic.

We now show that if ψ is an inversion, then all ψ-cyclic geodesics are necessarily ψ-CR.

Lemma 3.17. *Let $A_\Gamma = \langle X \rangle$ and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be a composition of inversions. Let $g \in A_\Gamma$ be arbitrary and let $w \in X^*$ be a geodesic representing g, such that w is a ψ-cyclic geodesic. Then any geodesic words which represent g are also ψ-cyclic geodesics.*

The proof is similar to Lemma 3.8 of [21], which considers the case where $\psi \in \mathrm{Aut}(A_\Gamma)$ is trivial.

Proof. Let $w = w_1 \dots w_n$, and suppose $[w_i, w_{i+1}] = 1$ for some $i \in \{1, \dots n-1\}$, where $w_i \neq w_{i+1}$. Consider the word $w' = w_1 \dots w_{i+1} w_i w_{i+2} \dots w_n =_{A_\Gamma} w$. Let w'' be a ψ-cyclic permutation of w'. We claim that w'' is geodesic. Since ψ has order two, we only have three cases to consider:

1. $w'' = \psi(w_{j+1}) \dots \psi(w_{i+1})\psi(w_i)\psi(w_{i+2}) \dots \psi(w_n) w_1 \dots w_j$, where $j < i$.
2. $w'' = \psi(w_{j+1}) \dots \psi(w_n) w_1 \dots w_{i+1} w_i w_{i+2} \dots w_j$, where $j > i$.
3. $w'' = \psi(w_i)\psi(w_{i+2}) \dots \psi(w_n) w_1 \dots w_{i-1} w_{i+1}$.

For the first case, we see that $w'' =_{A_\Gamma} \psi(w_{j+1}) \dots \psi(w_i)\psi(w_{i+1})\psi(w_{i+2}) \dots \psi(w_n)w_1 \dots w_j$, which is a ψ-cyclic permutation of w. This must be geodesic since w is a ψ-cyclic geodesic. The second case follows similarly. For the third case, we note that the subword $s = \psi(w_{i+2}) \dots \psi(w_n)w_1 \dots w_{i-1}$ must be geodesic, since it is a subword of $p = \psi(w_{i+1})\psi(w_{i+2}) \dots \psi(w_n)w_1 \dots w_{i-1}w_i$, which is a ψ-cyclic permutation of w. If w'' is not geodesic, then either $\psi(w_i)$ or w_{i+1} cancel with a letter in the subword s after applying commutation relations, or $\psi(w_i) = w_{i+1}^{-1}$, which cancel after applying commutation relations.

First suppose $\psi(w_i)$ cancels with $\psi(w_k)$, where $k \in \{i+2, \dots, n\}$, after applying commutation relations. Then w_i can cancel with w_k in w, and so w is not geodesic, giving a contradiction. Now suppose $\psi(w_i)$ cancels with w_l, where $l \in \{1, \dots, i-1\}$, after applying commutation relations. If we consider the ψ-cyclic permutation $q = \psi(w_i)\psi(w_{i+1}) \dots \psi(w_n)w_1 \dots w_{i-1}$ of w, then q is not geodesic, again giving a contradiction. The case for w_{i+1} cancelling with letters in s follows similarly.

Finally, suppose $\psi(w_i) = w_{i+1}^{-1}$ which cancel after applying commutation relations. Since ψ is a composition of inversions, we can assume that $\psi(w_i) = w_i^{\pm 1}$. If ψ acts as the identity on w_i, then our original word w would not be geodesic. Otherwise $\psi(w_i) = w_i^{-1} = w_{i+1}^{-1}$, which contradicts our original assumption that $w_i \neq w_{i+1}$. Indeed if this was the case, $w = w'$. This completes the proof. □

We note that the final case of this proof does not hold if ψ is a graph automorphism. This can be seen in Example 3.15, where the word $w = c^{-1}aac$ is a ψ-cyclic geodesic, but is not ψ-CR.

Corollary 3.18. *Let $A_\Gamma = \langle X \rangle$, and let $\psi \in \mathrm{Aut}(A_\Gamma)$ be a composition of inversions. Then any geodesic $v \in X^*$ is ψ-CR if and only if v cannot be written in the form*

$$v =_{A_\Gamma} \psi(u)^{-1}wu,$$

where $l(w) < l(v)$ and $\psi(u)^{-1}wu \in X^*$ is geodesic.

Proof. This follows immediately by Proposition 3.12 and Lemma 3.17. □

4. Conjugacy geodesics in virtual graph products

In this section we prove the following:

Theorem 4.8. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for A_ϕ, and $\phi \in \mathrm{Aut}(A_\Gamma)$ is a composition of inversions. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular.*

Our method follows a similar technique to [13], where the authors prove that $\mathsf{ConjGeo}(A_\Gamma, X)$ is regular for any RAAG (or RACG).

Definition 4.1. Let $G = \langle X \rangle$, and let $\psi \in \mathrm{Aut}(G)$. For any language $L \subset X^*$, let $\mathsf{Cyc}_\psi(L)$ denote the *ψ-cyclic closure* of L, which is the set of all ψ-cyclic permutations of words in L (recall Definition 3.9).

The following result is well known in the case where ψ is the identity map, for regular and other languages (see Lemma 2.1, [13]).

Proposition 4.2. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be length preserving. If L is regular, then $\mathsf{Cyc}_\psi(L)$ is regular.*

We acknowledge here a recent paper [26], where a similar statement is shown for twisting by antimorphic involutions in semigroups.

Proof. The idea of this proof is to take copies of the automaton accepting L, and adjust edge labels with respect to ψ, to account for ψ-cyclic shifts of letters.

Let M be a finite state automaton accepting L, with state set Q and initial state q_0. A word $w = x_1 x_2 \dots x_n$, where x_i are letters, is in $\mathsf{Cyc}_\psi(L)$ if and only if there exists a ψ-cyclic permutation

$$v = \psi^k(x_i) \dots \psi^k(x_n)\psi^{k-1}(x_1) \dots \psi^{k-1}(x_{i-1}) \in L$$

of w, for some $1 \leq i \leq n$, $0 \leq k \leq m-1$, where m is the order of ψ. In other words, w is in $\mathsf{Cyc}_\psi(L)$ if and only if there exist states $q, q' \in Q$, with q' accepting, such that M contains a path from q_0 to q labelled $\psi^k(x_i) \dots \psi^k(x_n)$, and a path from q to q' labelled $\psi^{k-1}(x_1) \dots \psi^{k-1}(x_{i-1})$. We first define an automaton for each value of i and k as follows.

If $i = 1$ and k is arbitrary, our original automaton M contains a path from q_0 to q' labelled $\psi^k(x_1) \dots \psi^k(x_n)$. We construct $M_{1,k}(q)$, for each k, by taking a copy of the states and transitions of M, and for each edge label e, applying a homomorphism $e \mapsto \psi^{-k}(e)$. The start state of $M_{1,k}(q)$ is the state q_0 in M, and the single accept state is the state q' in M. Therefore when $i = 1$, the word $w = x_1 \dots x_n$ is accepted by an automaton of the form $M_{1,k}(q)$, for some k.

If $i \neq 1$ and k is arbitrary, we take two copies of the states and transitions of M, which we denote by $\psi^{-k}(M)$ and $\psi^{-(k-1)}(M)$. For all edge labels $e \in \psi^{-k}(M)$ and $f \in \psi^{-(k-1)}(M)$, we apply a homomorphism

$$e \mapsto \psi^{-k}(e), \quad f \mapsto \psi^{-(k-1)}(f).$$

We define $M_{i,k}(q, q')$ to be the union of $\psi^{-k}(M)$ and $\psi^{-(k-1)}(M)$, where we add an ϵ-transition from the state q' in $\psi^{-(k-1)}(M)$ to the state q_0 in $\psi^{-k}(M)$. The start state of $M_{i,k}(q, q')$ is the state $q \in \psi^{-(k-1)}(M)$, and the single accept state is the state $q \in \psi^{-k}(M)$. With this construction, $M_{i,k}(q, q')$ will accept w (see Fig. 1).

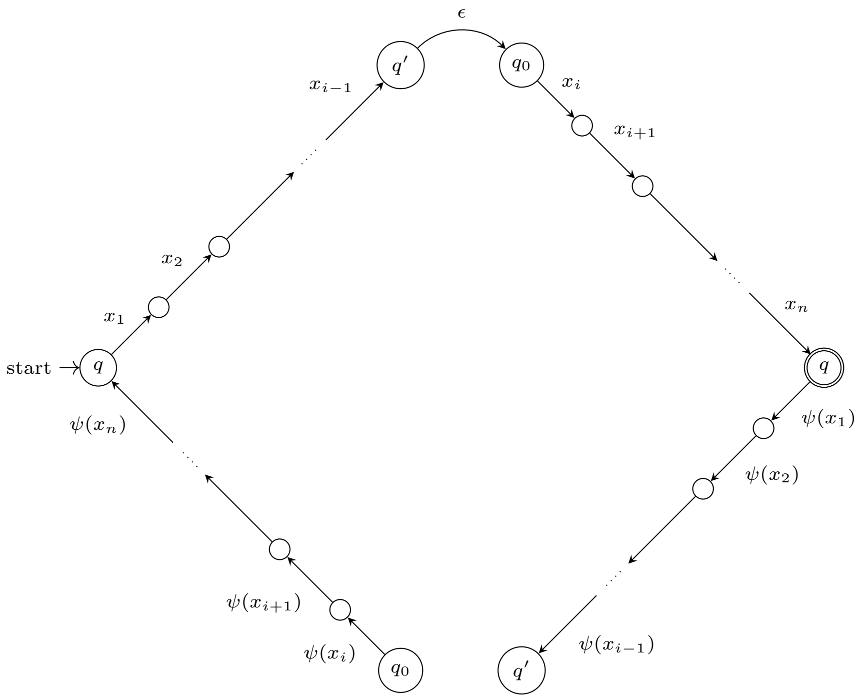


Fig. 1. Automata $M_{i,k}(q, q')$ for $i \neq 1$.

We now construct an automaton $\widehat{M}$ as follows. First construct all possible $M_{1,k}(q)$ for $0 \leq k \leq m-1$. Then construct all possible $M_{i,k}(q, q')$ for $2 \leq i \leq n$, $0 \leq k \leq m-1$. Add two new states S and E, which will become our start and end states for $\widehat{M}$. We add ϵ-transitions from S to each start state in $M_{1,k}(q)$ and $M_{i,k}(q, q')$, for all possible i, k, and similarly add ϵ-transitions from E to each end state in $M_{1,k}(q)$ and $M_{i,k}(q, q')$. Then

$$\widehat{M} = \{S, E\} \bigcup_k M_{1,k}(q) \bigcup_{i,k} M_{i,k}(q, q').$$

$\widehat{M}$ will accept $w = x_1 \dots x_n$ for all possible i, k, and so will accept w for all $w \in \mathsf{Cyc}_\psi(L)$. $\widehat{M}$ is a finite state automaton, since ψ is of finite order, hence $\mathsf{Cyc}_\psi(L)$ is regular. □

We now define twisted languages analogous to Definition 2.11, with respect to any automorphism $\psi \in \mathrm{Aut}(A_\Gamma)$.

Definition 4.3. Let G be a group. Let $[g]_\psi$ denote the twisted conjugacy class of a group element $g \in G$. We define the *length up to twisted conjugacy* of an element $g \in G$ as

$$|g|_\psi := \min\{|h| \mid h \in [g]_\psi\}.$$

Recall a word whose ψ-cyclic permutations are all geodesic is a ψ-cyclic geodesic word. We define the following languages for a group G, with respect to a generating set X.

$$\mathsf{ConjGeo}_{\psi}(G,X) := \{w \in X^* \mid l(w) = |\pi(w)|_{\psi}\},$$
$$\mathsf{CycGeo}_{\psi}(G,X) := \{w \in X^* \mid w \text{ is a } \psi - \text{cyclic geodesic}\}.$$

As before, we let $X = V(\Gamma)$ denote the standard generating set of a RAAG A_Γ.

Proposition 4.4. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be length preserving. Then the language $\mathsf{CycGeo}_{\psi}(A_\Gamma, X)$ is regular.*

Proof. We prove that

$$\mathsf{CycGeo}_{\psi} = X^* \setminus \mathsf{Cyc}_{\psi}(X^* \setminus \mathsf{Geo}).$$

($\supseteq$): Let $w \in X^* \setminus \mathsf{Cyc}_{\psi}(X^* \setminus \mathsf{Geo})$, and suppose $w \notin \mathsf{CycGeo}_{\psi}$. Then there exists a ψ-cyclic permutation w' of w such that w' is not geodesic. By definition this means $w \in \mathsf{Cyc}_{\psi}(X^* \setminus \mathsf{Geo})$, which is a contradiction.

($\subseteq$): Let $w \in \mathsf{CycGeo}_{\psi}$, and suppose $w \in \mathsf{Cyc}_{\psi}(X^* \setminus \mathsf{Geo})$. Then there exists a ψ-cyclic permutation w' of w such that w' is not geodesic. But since $w \in \mathsf{CycGeo}_{\psi}$, all ψ-cyclic permutations of w are geodesic, again giving a contradiction.

Since $\mathsf{Geo}(A_\Gamma, X)$ is regular [25], the result follows by Lemma 2.5 and Proposition 4.2. □

Corollary 4.5. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be length preserving. Then $\mathsf{ConjGeo}_{\psi}(A_\Gamma, X)$ consists precisely of all geodesic ψ-CR words.*

Proof. This follows immediately by Theorem 3.13. □

Proposition 4.6. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be a composition of inversions. Then*

$$\mathsf{ConjGeo}_{\psi}(A_\Gamma, X) = \mathsf{CycGeo}_{\psi}(A_\Gamma, X).$$

Proof. The $\subseteq$ direction is clear from the definitions, since ψ is length preserving.

Now suppose $v \in \mathsf{CycGeo}_{\psi}(A_\Gamma, X)$, but $v \notin \mathsf{ConjGeo}_{\psi}(A_\Gamma, X)$. By Corollary 4.5, v is not ψ-CR, and hence $v =_{A_\Gamma} \psi(u)^{-1}wu$ by Corollary 3.18. In particular, we can write

$$v = x_1 \psi\left(a_i^{\mp 1}\right) x_2 a_i^{\pm 1} x_3,$$

where $x_1, x_2, x_3 \in X^*$ are geodesic, all letters of x_1 commute with $\psi\left(a_i^{\mp 1}\right)$, and all letters of x_3 commute with $a_i^{\pm}$. Then there exists a ψ-cyclic permutation

$$v = x_1 \psi\left(a_i^{\mp 1}\right) x_2 a_i^{\pm 1} x_3 \overset{\psi}{\longleftrightarrow} \psi\left(a_i^{\pm 1}\right) \psi\left(x_3\right) x_1 \psi\left(a_i^{\mp 1}\right) x_2 =_{A_\Gamma} \psi(x_3) x_1 x_2.$$

This contradicts the fact that v is a ψ-cyclic geodesic, since the length of $\psi(x_3)x_1x_2$ is less than the length of v. □

The following is immediate by Propositions 4.4 and 4.6.

Corollary 4.7. *Let $\psi \in \mathrm{Aut}(A_\Gamma)$ be a composition of inversions. Then the language $\mathsf{ConjGeo}_\psi(A_\Gamma, X)$ is regular.*

Theorem 4.8. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t\rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for A_ϕ, and $\phi \in \mathrm{Aut}(A_\Gamma)$ is a composition of inversions. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular.*

Proof. By definition, we can write elements of this language in the form

$$\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right) = \bigcup_{l=1}^{m} \{t^l U \mid U \in \mathsf{ConjGeo}_{\phi^l}(A_\Gamma, X)\}.$$

For fixed l, the languages $\{t^l\}$ (singleton language) and $\mathsf{ConjGeo}_{\phi^l}(A_\Gamma, X)$ are both regular (by Corollary 4.7), and so the concatenation of these languages is regular by Lemma 2.5. Since t has finite order, the union of these regular languages across all possible l must also be regular. □

Remark 4.9. Theorem 4.8 also holds for RACGs with analogous proof.

One observation we make here is that the crucial step in the proof of Proposition 4.6 comes from Corollary 3.18. One might ask if Proposition 4.6 holds for other types of length preserving automorphisms, such as graph automorphisms. The next example illustrates how this is not the case.

Example 4.10. Recall Example 3.15 and consider the geodesic word $w = c^{-1}aac$. Then $w \in \mathsf{CycGeo}_\psi(A_\Gamma, X)$ since all ψ-cyclic permutations of w are geodesic:

$$c^{-1}aac \overset{\psi}{\longleftrightarrow} bc^{-1}aa \overset{\psi}{\longleftrightarrow} dbc^{-1}a \overset{\psi}{\longleftrightarrow} ddbc^{-1} \overset{\psi}{\longleftrightarrow} b^{-1}ddb \overset{\psi}{\longleftrightarrow} cb^{-1}dd \overset{\psi}{\longleftrightarrow} acb^{-1}d \overset{\psi}{\longleftrightarrow} aacb^{-1} \overset{\psi}{\longleftrightarrow} c^{-1}aac.$$

However $w \notin \mathsf{ConjGeo}_\psi(A_\Gamma, X)$, since there exists a sequence of ψ-cyclic shifts, commutation relations and free reduction to $w' = aa$, which is shorter than w.

4.1. Non-length preserving automorphisms and conjugacy geodesics

We now focus on non-length preserving automorphisms of RAAGs, in particular partial conjugations and dominating transvections (see Section 2.4). While these are infinite order automorphisms, they can be composed with inversions to give us order two automorphisms. We can then construct a virtual RAAG A_ϕ, with respect to this type of finite order non-length preserving automorphism, and study the language $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$.

Definition 4.11. For any vertex $v \in V(\Gamma)$, we define the *link* of a vertex $\mathrm{Lk}(v)$ as

$$\mathrm{Lk}(v) = \{x \in V(\Gamma) \mid \{v, x\} \in E(\Gamma)\}.$$

Similarly we define the *star* of a vertex $\mathrm{St}(v)$ as

$$\mathrm{St}(v) = \mathrm{Lk}(v) \cup \{v\}.$$

Our first result focuses on a composition of a partial conjugation and inversion. With this composition, we find that the finite extension A_ϕ is precisely a graph product, and hence $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular by Theorem 3.1 of [12].

Theorem 4.12. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t\rangle$ (as defined in (2)) where X is the standard generating set. For some $x \in X$ and a connected component $D \subseteq \Gamma \setminus \mathrm{St}(x)$, let $\phi \in \mathrm{Aut}(A_\Gamma)$ be the following order two automorphism:*

$$\begin{aligned} \phi \colon X &\to X \\ y &\mapsto xyx^{-1} \quad \forall\, y \in D, \\ x &\mapsto x^{-1}, \\ s &\mapsto s \quad \forall\, s \in X \setminus D \cup \{x\}. \end{aligned}$$

Then A_ϕ is a graph product, with vertex groups $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Proof. We start with the standard presentation for A_ϕ, and use Tietze transformations to rewrite the extension in the form of a graph product. We have

$$A_\phi = \langle X \cup \{t\} \mid R_1, R_2,\ t^2 = 1, (tx)^2 = 1, tyt = xyx^{-1} \text{ for all } y \in D\rangle,$$

where

$$R_1 = \{[a, b] = 1 : \{a, b\} \in E(\Gamma)\}, \quad R_2 = \{[t, s] = 1 : s \in X \setminus D \cup \{x\}\}.$$

Define a new generator $u = tx$, and remove x using this relation. Our presentation becomes

$$A_\phi = \langle X \setminus \{x\} \cup \{t, u\} \mid R_1, R_2, t^2 = 1, u^2 = 1, tyt = tuyut \text{ for all } y \in D\rangle,$$

where the only other relations that change are of the form $[x, a]$ in R_1. These change to $[tu, a]$ for all $a \in St(x), a \neq x$. For each of these relations we have

$$1 = [tu, a] = uta^{-1}tua = ua^{-1}ua = [u, a],$$

using the relation $[t, a] = 1$ from R_2. So each relation of the form $[tu, a]$ can be replaced by $[u, a]$. Also note

$$tyt = tuyut \Leftrightarrow y = uyu \Leftrightarrow uy^{-1}uy = 1 \Leftrightarrow [u, y] = 1,$$

so we can replace the relation $tyt = tuyut$ by $[u, y] = 1$. This leaves us with

$$A_\phi = \langle X \setminus \{x\} \cup \{t, u\} \mid R_1, R_2, t^2 = 1, u^2 = 1, [u, y] = 1 \text{ for all } y \in D\rangle,$$

where

$$R_1 = \begin{cases} [a, b] = 1, & \{a, b\} \in E(\Gamma),\ a, b \neq x, \\ [u, a] = 1, & a \in St(x),\ a \neq x, \end{cases}$$

and

$$R_2 = \{[t, s] = 1,\ s \in X \setminus D \cup \{x\}\}.$$

Our presentation is now in the form of a graph product with vertex groups $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. □

The following is immediate by Theorem 3.1 of [12], because $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ have regular conjugacy geodesics.

Corollary 4.13. *Let A_ϕ be a virtual RAAG as defined in Theorem 4.12, and let $\widehat{X}$ be the standard generating set for A_ϕ. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular.*

Regularity of the language $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is surprisingly not universal across all non-length preserving cases. This can be seen by the following example.

Proposition 4.14. *Let $A_\Gamma \cong F_2 \times \mathbb{Z}$ with defining graph Γ labelled as*

$$x \;\text{——}\; z \;\text{——}\; y,$$

Let $\phi \in \mathrm{Aut}(A_\Gamma)$ be the following composition of a dominating transvection and inversion:

$$\phi\colon x \mapsto xy, \quad y \mapsto y^{-1}, \quad z \mapsto z.$$

Let X be the standard generating set. Then the language $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$ is not context-free.

The proof of this proposition uses similar techniques to [12], where the authors show that the spherical conjugacy language on the free group on two generators is not context-free.

Proof. Suppose $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$ is context-free, and consider the intersection $I = \mathsf{ConjGeo}_\phi(A_\Gamma, X) \cap L$, where $L = x^+(xy)^+x^+$. Since L is regular, I must be context-free by Lemma 2.8. We claim that $I = \{x^p(xy)^q x^r \mid p \geq q, r \geq q\}$. By Corollary 3.14 we need to check all possible sequences of ϕ-cyclic permutations, commutation relations and free cancellations of $w = x^p(xy)^q x^r \in I$.

Any word $v \in [w]_\phi$ must contain only the letters x and y, and so we can assume no commutation relations exist in v. Also, $l(v) \geq l(w) = p + r + 2q$, since w is ϕ-cyclically reduced. This leaves us with the following possible forms for v, by taking ϕ-cyclic permutations of w:

1. $v = (xy)^{r_1} x^p (xy)^q x^{r_2}$, where $r = r_1 + r_2$. Here $l(v) > l(w)$.
2. $v = y^{-1}(xy)^r x^p (xy)^{q-1} x$. Again $l(v) > l(w)$.
3. $v = x(xy)^r x^p (xy)^{q-1}$. Here we need $p + 2q + 2r - 1 \geq p + r + 2q$, and so $r \geq 1$. If we keep repeating Cases 2 and 3, by moving letters from $(xy)^q$ to the front of the word, we find that $r \geq q$. This gives our first condition on I.
4. $v = (xy)^{p_1} x^q (xy)^r x^{p_2}$, where $p = p_1 + p_2$. Here we require $q \leq r + p_1$, which always holds since $q \leq r$ and $p_1 \geq 0$.
5. $v = x^s (xy)^p x^q (xy)^{r-s}$ where $1 \leq s \leq r$. Similar to Case 3, we can repeat this pattern of moving letters from $(xy)^r$ to find that $q \leq p$. This gives our second condition on I.
6. $v = (xy)^{q_1} x^r (xy)^p x^{q_2}$, where $q = q_1 + q_2$. Here we need $q \leq p + q_1$, which holds since $q \leq p$ and $q_1 \geq 0$.
7. $v = x^s (xy)^q x^r (xy)^{p-s}$ where $1 \leq s \leq p$. This case always works, since if $p - s > 0$, then $l(v) > l(w)$.

This proves the claim, and it remains to prove that the language I is not context-free. Let k be the constant given by the Pumping Lemma for context-free languages (see [23], Theorem 7.18, Page 281). Consider the word $W = x^n(xy)^n x^n \in I$ where $n > k$. Here W contains 3 blocks, namely $x^n, (xy)^n$ and x^n. Hence by the Pumping Lemma, W can be written as $W = stuvw$ where $l(tv) \geq 1, l(tuv) \leq k$ and $st^i uv^i w \in L$ for all $i \geq 0$. Since $l(tuv) \leq k < n$, tuv cannot be part of more than 2 consecutive blocks.

Case 1: tuv part of one block only.
a) tuv lies in first block. Then

$$s \cdot tuv \cdot w = x^{n-j_1-j_2} \cdot x^{j_1} \cdot x^{j_2}(xy)^n x^n.$$

When $i = 0$,

$$suw = x^{n-j_1-j_2} \cdot x^k \cdot x^{j_2}(xy)^n x^n = x^{n+k-j_1}(xy)^n x^n,$$

for some $k < j_1$. Here $p < n$ and $q = n$, and so $suw \notin I$, which is a contradiction.
b) tuv lies in 2nd block. Then

$$s \cdot tuv \cdot w = x^n (xy)^{i_1} \cdot (xy)^{i_2} \cdot (xy)^{i_3} x^n,$$

where $i_1 + i_2 + i_3 = n$. If we let $i \geq 2$, then for $st^i uv^i w$ we have $q > n$ and $p = r = n$, and so $st^i uv^i w \notin I$.
c) tuv lies in third block. Similar to a), we have that for $i = 0$, $r < q$.

Case 2: tuv is part of more than one block.
If either t or v contains both x and xy letters, then for $i \geq 2$, the word $st^i uv^i w$ contains at least 4 blocks alternating between powers of x and powers of xy, and so can't lie in I. Hence either

$$t = x^a,\ v = (xy)^b \quad \text{or} \quad t = (xy)^a,\ v = x^b.$$

i) t in first block, v in second block. Then

$$stuvw = x^{i_1} \cdot x^{i_2} \cdot x^{i_3}(xy)^{j_1} \cdot (xy)^{j_2} \cdot (xy)^{j_3} x^n,$$

where $s = x^{i_1}, t = x^{i_2}, u = x^{i_3}(xy)^{j_1}, v = (xy)^{j_2}, w = (xy)^{j_3} x^n$, and $i_1 + i_2 + i_3 = j_1 + j_2 + j_3 = n$. When $i \geq 2$, then for $st^i uv^i w$ we have $q > n$ and $r = n$, so $st^i uv^i w \notin I$.
ii) t in second block, v in third block. Again for $i \geq 2$, $q > p$.

All cases give a contradiction, and so I is not context-free. Hence the language $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$ is not context-free. □

Now we consider the extension $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ where $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)) and $\widehat{X} = \{X, t\}$ for this example. We first prove the following:

Proposition 4.15. *Let $G_\phi = G_\Gamma \rtimes \langle t \rangle$ (as defined in (2)). Then if $\mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)$ is not context-free (or not unambiguous context-free)* ***for some*** *l, then $\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right)$ is not context-free (or not unambiguous context-free), where $\widehat{X}$ is the standard generating set for G_ϕ.*

In other words, we have two different methods for showing whether languages in the extension are regular or not.

1. If the twisted conjugacy classes in the base group are regular **for all** powers of the automorphism, then the conjugacy language of the extension is regular (see for example Theorem 4.8).

2. If **at least one** twisted conjugacy class is not context-free (not unambiguous context-free), then the conjugacy language of the extension is not context-free (not unambiguous context-free).

Therefore in order to show an extension is not context-free, we just need to find one twisted class which is not context-free in the base group.

Proof. (of Proposition 4.15). Recall by definition that

$$\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right) = \bigcup_{l=1}^{m} \{t^l U \mid U \in \mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)\}.$$

Choose l such that $\mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)$ is not context-free (this exists by assumption). Suppose the set $\{t^l U \mid U \in \mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)\}$ is context-free. Then by Lemma 2.7, $\mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)$ is context-free. This contradicts our assumption and so $\{t^l U \mid U \in \mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)\}$ is not context-free.

Now suppose $\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right)$ is context-free. Then consider

$$\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right) \cap t^l X^* = t^l \cdot \mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X),$$

for l such that $\mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)$ is not context-free. By Lemma 2.8, it must be that $t^l \cdot \mathsf{ConjGeo}_{\phi^l}(G_\Gamma, X)$ is context-free. But this contradicts our previous claim, and so $\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right)$ is not context-free. □

Corollary 4.16. *There exists a virtual RAAG A_ϕ such that $\mathsf{ConjGeo}(A_\phi, X)$ is not context-free. In particular, let $A_\phi = A_\Gamma \rtimes \langle t \rangle$ (as defined in (2)) where A_Γ, ϕ are defined in Proposition 4.14. Let $\widehat{X}$ be the standard generating set for A_ϕ. Then $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is not context-free.*

4.2. Graph products

We mention a more general result for graph products, with some restrictions on vertex groups, as a twisted adaptation of Theorem 2.4 in [13]. Further details can be found in the preprint [17].

Recall a graph product G_Γ has generating set $X = \cup_i X_i$ where each X_i are the generators of the vertex groups assigned to the defining graph.

Definition 4.17. For $\psi \in \mathrm{Aut}(G_\Gamma)$, we say ψ *fixes the vertex groups* if for all $x_i \in X_i$, $\psi(x_i) \in X_i$ for every vertex group $\langle X_i \rangle$.

Corollary 4.18. *Let G_Γ be a graph product with vertex groups $\langle X_i \rangle$. Let G_ϕ be a virtual graph product of the form $G_\phi = G_\Gamma \rtimes \langle t \rangle$ (as defined in (2)) with standard generating set $\widehat{X}$ where:*

(i) $\mathsf{Geo}(G_i, X_i)$ *is regular for each vertex group,*
(ii) $\phi \in \mathrm{Aut}(G_\Gamma)$ *is length preserving and fixes the vertex groups, and*
(iii) $\mathsf{CycGeo}_{\phi^l}(G_i, X_i) = \mathsf{ConjGeo}_{\phi^l}(G_i, X_i)$ *for each vertex group, where* $1 \leq l \leq m-1$.

Then $\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right)$ *is regular.*

5. The conjugacy shortlex language in virtual graph products

We now turn our attention to the language ConjSL in virtual graph products. As a slight detour, we first study acylindrically hyperbolic RAAGs and RACGs (see [31] for background on acylindrically hyperbolic groups).

5.1. $\mathcal{AH}$-accessability

We begin with the following result by Antolín and Ciobanu.

Theorem 5.1. *([3], Theorem 1.5) If G is non-elementary acylindrically hyperbolic, then* $\mathsf{ConjSL}(G, X)$ *is not unambiguous context-free with respect to any generating set X.*

We note that the algebraic type of the conjugacy growth series, for acylindrically hyperbolic groups, is still open.

Theorem 5.1 will allow us to prove the following.

Corollary 5.2. *Let G be a group with a finite index normal subgroup $H \trianglelefteq G$, such that H is an acylindrically hyperbolic RAAG (or RACG). Then* $\mathsf{ConjSL}(G, X)$ *is not unambiguous context-free, with respect to any generating set X.*

We will show in this case that G must necessarily be acylindrically hyperbolic, using a property of groups known as *acylindrically hyperbolic accessibility* (see [1] for further details). In a correction note of [29], Minasyan and Osin proved a criterion for when extensions of acylindrically hyperbolic groups are acylindrically hyperbolic. We note that it is still an open question in general as to whether acylindrical hyperbolicity is a quasi-isometry invariant.

Theorem 5.3. *([29], Lemma 6). If $H \trianglelefteq G$ of finite index, where H is*

(i) *acylindrically hyperbolic, and*
(ii) *$\mathcal{AH}$-accessible,*

then G is acylindrically hyperbolic.

RAAGs are acylindrically hyperbolic if they are not cyclic or a direct product (see [31]). To study the $\mathcal{AH}$-accessible property in RAAGs and RACGs, we need to look at *hierarchically hyperbolic groups* (HHGs). These were first defined by Behrstock, Hagen and Sisto in 2017 (see [6], Definition 1.21 for the definition of a HHG, and [33] for a survey on these groups). By Proposition B and Theorem G from [5], RAAGs and RACGs are HHGs.

Proposition 5.4. *Every HHG is $\mathcal{AH}$-accessible.*

Proof. By definition (see [1]), a group is $\mathcal{AH}$-accessible if the poset $\mathcal{AH}(G)$, consisting of hyperbolic structures corresponding to acylindrical actions, contains the largest element. Theorem A from [2] states that every HHG admits a largest acylindrical action, and so is $\mathcal{AH}$-accessible. □

Proof. (of Corollary 5.2) Let $H = A_\Gamma$ be acylindrically hyperbolic, where A_Γ is a RAAG. By Proposition 5.4, H is $\mathcal{AH}$-accessible, and so G is acylindrically hyperbolic by Theorem 5.3. Hence by Theorem 5.1, $\mathsf{ConjSL}(G, X)$ is not unambiguous context-free for any generating set X. The proof is analogous for RACGs. □

5.2. Virtual graph products

We now return our attention to extensions of graph products $G_\phi = G_\Gamma \rtimes \langle t\rangle$ as defined in (2). To study $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$, we again focus on twisted conjugacy in the base group.

Definition 5.5. Let $G = \langle X\rangle$ be a finitely generated group, and let $\psi \in \mathrm{Aut}(G)$. Given an order on X, we define $\mathsf{ConjSL}_\psi(G, X)$ to be the set of all shortlex least words of each twisted conjugacy class (with respect to ψ):

$$\mathsf{ConjSL}_\psi(G, X) := \{z_c \mid c \in G/\sim_\psi\}.$$

By definition, in the extension G_ϕ, where ϕ has order m, we can write

$$\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right) = \bigcup_{l=1}^{m} \{t^l U \mid U \in \mathsf{ConjSL}_{\phi^l}(G_\Gamma, X)\},$$

where $t <_{SL} x$ for all $x \in X$. Similar to Theorem 4.8 and Proposition 4.15 for $\mathsf{ConjGeo}\left(G_\phi, \widehat{X}\right)$, one can show the following.

Proposition 5.6. *Let $G_\phi = G_\Gamma \rtimes \langle t\rangle$ be a virtual graph product as defined in (2).*

1. *If $\mathsf{ConjSL}_{\phi^l}(G_\Gamma, X)$ is regular **for all** l, then $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ is regular.*

2. *If* $\mathsf{ConjSL}_{\phi^l}(G_\Gamma, X)$ *is not context-free (or unambiguous context-free)* ***for some*** l*, then* $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ *is not context-free (or unambiguous context-free).*

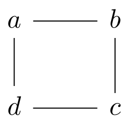
The nature of the language $\mathsf{ConjSL}_\psi(G_\Gamma, X)$ can vary depending on the type of automorphism we focus on. This language can change from regular to not context-free simply by taking different powers of the automorphism, or relabelling the vertices of the defining graph.

Definition 5.7. Let $G_\Gamma = \langle X \rangle$ be a graph product, with vertex groups G_v for every $v \in V(\Gamma)$. Let $g = g_1 \dots g_n \in X^*$ be a geodesic, where each $g_i \in G_v$ for some $v \in V(\Gamma)$. We define the *support* of g as

$$\mathrm{supp}(g) = \{v \in V(\Gamma) \mid \text{there exists } i \in \{1, \dots, n\} \text{ such that } g_i \in G_v \setminus \{1\}\}.$$

We note that Corollary 5.2 does not apply to the following example, since direct products of infinite groups are not acylindrically hyperbolic (Corollary 7.2, [31]).

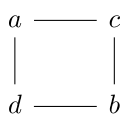
Proposition 5.8. *Let* $A_\Gamma = F_2 \times F_2$*, and label the defining graph as follows:*



Let $\psi \colon a \to b \to c \to d$ *be a rotation, and let* X *be the standard generating set.*

(1) *If vertices are ordered as* $a < b < c < d$*, then* $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ *is regular.*
(2) *If we consider* $\psi^2 \colon a \leftrightarrow c,\ b \leftrightarrow d$*, and vertices ordered as* $a < b < c < d$*, then* $\mathsf{ConjSL}_{\psi^2}(A_\Gamma, X)$ *is not context-free.*
(3) *If vertices are ordered as* $a < c < b < d$*, then* $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ *is not context-free.*

Note that Case (3) in Proposition 5.8 is equivalent to swapping the vertices b and c and relabelling the graph as



Proof. (1): We claim that $\mathsf{ConjSL}_\psi(A_\Gamma, X) = \{a^n \mid n \in \mathbb{Z}\}$, which is regular.
First note that any word in $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ must start with an a letter - otherwise we could ψ-cyclically permute the entire word in each of the cases:

$$b^{\pm 1} w \overset{\psi}{\leftrightarrow} \psi^3(b^{\pm 1})\psi^3(w) = a^{\pm 1}\psi^3(w),$$

$$c^{\pm 1}w \overset{\psi}{\longleftrightarrow} \psi^2(c^{\pm 1})\psi^2(w) = a^{\pm 1}\psi^2(w),$$

$$d^{\pm 1}w \overset{\psi}{\longleftrightarrow} \psi(d^{\pm 1})\psi(w) = a^{\pm 1}\psi(w).$$

We now rule out some cases. No words over $\{a, b\}$ can belong to $\mathsf{ConjSL}_\psi(A_\Gamma, X)$, since these would be of the form $a^n b^m =_{A_\Gamma} b^m a^n$, and we could ψ-cyclic shift all the b letters from $b^m a^n$ to a^{n+m} as follows:

$$b^m a^n \overset{\psi}{\longleftrightarrow} a^n \psi^{-1}(b^m) = a^{n+m},$$

which is a smaller word lexicographically. Similarly we can rule out all words over $\{a, d\}$.

Consider words over $\{a, c\}$. We can first rule out words of the form $a^{-1}w_1 \dots w_n c$ or $aw_1 \dots w_n c^{-1}$, where each $w_i \in \{a^{\pm 1}, c^{\pm 1}\}$, since we can apply ψ-cyclic shifts to obtain a shorter word. For example:

$$\begin{aligned} a^{-1}w_1 \dots w_n c &\overset{\psi}{\longleftrightarrow} \psi(c)a^{-1}w_1 \dots w_n \\ &= da^{-1}w_1 \dots w_n \\ &=_{A_\Gamma} a^{-1}w_1 \dots w_n d \\ &\overset{\psi}{\longleftrightarrow} \psi(d)a^{-1}w_1 \dots w_n \\ &= aa^{-1}w_1 \dots w_n, \end{aligned}$$

which after cancellation gives a word of shorter length. We also can't have words of the form $aw_1 \dots w_n c$, where each $w_i \in \{a^{\pm 1}, c^{\pm 1}\}$ belong to $\mathsf{ConjSL}_\psi(A_\Gamma, X)$, since we can apply ψ-cyclic shifts to get

$$\begin{aligned} aw_1 \dots w_n c &\overset{\psi}{\longleftrightarrow} \psi(c)aw_1 \dots w_n = daw_1 \dots w_n =_{A_\Gamma} aw_1 \dots w_n d \\ &\overset{\psi}{\longleftrightarrow} \psi(d)aw_1 \dots w_n = aaw_1 \dots w_n. \end{aligned}$$

In particular, $w_1 = a$ must hold for the word to be in $\mathsf{ConjSL}_\psi(A_\Gamma, X)$. By induction, each $w_i = a$, but then we can ψ-cyclic shift to

$$a \dots ac \overset{\psi}{\longleftrightarrow} a^{n+2}.$$

A similar argument holds for words of the form $a^{-1}w_1 \dots w_n c^{-1}$. We can now rule out all words over $\{a, c\}$. Indeed, any word would have to be of the form $a^{\pm 1}w_1 \dots w_n a^{\pm 1}$ where $w_i \in \{a^{\pm 1}, c^{\pm 1}\}$, which can be ψ-cyclic shifted to

$$a^{\pm 1}w_1 \dots w_n a^{\pm 1} \overset{\psi}{\longmapsto} \psi(a^{\pm 1})a^{\pm 1}w_1 \dots w_n = b^{\pm 1}a^{\pm 1}w_1 \dots w_n =_{A_\Gamma} a^{\pm 1}b^{\pm 1}w_1 \dots w_n.$$

Then $w_1 = a$ must hold, and again by induction, $w_i = a$ for all i. Hence we can rule out all words w such that $|\mathrm{supp}(w)| = 2$.

Now consider 3 letters. If $w \in \{a,b,c\}^*$, then it can be written in the form $w = b^n x$ where $x \in \{a,c\}^*$. Then we can ψ-cyclic shift $b^n x$ to xa^n:

$$b^n x \overset{\psi}{\leftrightarrow} x\psi^{-1}(b^n) = xa^n,$$

which can then be ψ-cyclic shifted to a^k. Similar arguments show that no words w such that $|\mathrm{supp}(w)| = 3$ belong to $\mathsf{ConjSL}_\psi(A_\Gamma, X)$. Finally suppose $|\mathrm{supp}(w)| = 4$. Since A_Γ is a product, we can write w in the form w_1w_2, where $w_1 \in \{a,c\}^*, w_2 \in \{b,d\}^*$. This can be ψ-cyclic shifted to $\psi(w_2)w_1$, a word over $\{a,c\}$, which can be ψ-cyclic shifted to a^k.

We've now ruled out all cases except when a word is over one letter. The smallest lexicographically of these is a^n, which proves the claim.

(2): Throughout we let $\gamma = \psi^2$. Suppose $\mathsf{ConjSL}_\gamma(A_\Gamma, X)$ is context-free and consider the intersection $I = \mathsf{ConjSL}_\gamma(A_\Gamma, X) \cap L$ where $L = a^+c^+a^+$. Since L is a regular language, I must be context-free by Lemma 2.8. We first prove that

$$I = \{a^pc^qa^r \mid p,q,r \geq 1\} = I_1 \sqcup I_2,$$

where

$$I_1 = \{a^pc^qa^r \mid p > q, p > r\}, \quad I_2 = \{a^pc^qa^p \mid p \geq q\}.$$

Let $w = a^pc^qa^r \in \mathsf{ConjSL}_\gamma(A_\Gamma, X)$. Since $a^pc^qa^r$ is γ-CR and γ is length preserving, the γ-conjugacy class consists of all possible γ-cyclic permutations of $a^pc^qa^r$. The images of any letters by γ must lie in $\{a,c\}$, which do not commute in A_Γ, and so the only possible γ-cyclic permutations which could be lexicographically less than w are of the form

$$\gamma(c^q)\gamma(a^r)a^p = a^qc^ra^p \quad \text{or} \quad a^r\gamma^{-1}(a^p)\gamma^{-1}(c^q) = a^rc^pa^q.$$

Therefore

$$w = a^pc^qa^r \leq a^qc^ra^p \Rightarrow \begin{cases} p > q, \\ p = q, q \leq r, \end{cases}$$

and

$$w = a^pc^qa^r \leq a^rc^pa^q \Rightarrow \begin{cases} p > r, \\ p = r, p \geq q. \end{cases}$$

Comparing possible cases for both options gives the disjoint sets I_1 and I_2.

Now let k be the constant given by the Pumping Lemma for context-free languages applied to the set I (see [23], Theorem 7.18, Page 281) and consider the word $W = a^nc^na^n$, where $n > k$. Note that W is composed of 3 blocks, namely a^n, c^n and a^n. By the Pumping Lemma, W can be written as $W = uvwxy$ where $l(vx) \geq 1, l(vwx) \leq k$ and $uv^iwx^iy \in I$ for all $i \geq 0$. Since $l(vwx) \leq k < n$, vwx cannot be part of more than two consecutive blocks.

Case 1: vwx is part of one block only.
a) vwx lies in first block, i.e.

$$u \cdot vwx \cdot y = a^{n-s-t} \cdot a^s \cdot a^t c^n a^n.$$

For $i = 0$,

$$uwy = a^{n-s-t} \cdot a^r \cdot a^t c^n a^n,$$

where $r < s$ (cannot have $r = s$ since $l(vx) \geq 1$). If $uwy \in I$, then $p = n + r - s$ and $q = n$. Hence $p < q$, which is a contradiction.
b) vwx is in second block, i.e.

$$u \cdot vwx \cdot y = a^n c^{s_1} \cdot c^{s_2} \cdot c^{n-s_1-s_2} a^n.$$

For $i \geq 2$, we have that $p < q$, so again $uv^iwx^iy \notin I$.
c) vwx is in third block, i.e.

$$u \cdot vwx \cdot y = a^n c^n \cdot a^s \cdot a^{n-s}.$$

For $i = 0$, we have that $r < n$ and $p = q = n$, so again $uwy \notin I$.

Case 2: vwx is part of more than one block.
Suppose vwx contains both a and c letters. If one of v or x contains both a and c, then for $i \geq 2$ the word uv^iwx^iy contains at least four blocks alternating between powers of a and c, and so can't lie in $a^+c^+a^+$. Therefore v must be a power of one letter only, and x must be a power of the other letter.
a) v in first block, x in second, i.e.

$$uvwxy = a^{i_1} \cdot a^{i_2} \cdot a^{i_3} c^{j_1} \cdot c^{j_2} \cdot c^{j_3} a^n,$$

where $i_1 + i_2 + i_3 = j_1 + j_2 + j_3 = n$. For $i = 0$,

$$uwy = a^{i_1} a^{i_3} c^{j_1} c^{j_3} a^n.$$

Here $p < n, q < n$ and $r = n$, so $uwy \notin I$.
b) v in second block, x in third. Here for $i \geq 2$, the word uv^iwx^iy is of the form where $p = n, q > n$ and $r > n$, so $uv^iwx^iy \notin I$. This covers all cases and so I is not context-free. Hence $\mathsf{ConjSL}_\gamma(A_\Gamma, X)$ is context-free.

The proof for Case (3) is identical to Case (2), with some additional justification needed for the set of words I. $\square$

Corollary 5.9. *There exists a RAAG $A_\Gamma = \langle X \rangle$, with an order on X, and automorphism $\psi \in \mathrm{Aut}(A_\Gamma)$, such that $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ is regular, but $\mathsf{ConjSL}_{\psi^2}(A_\Gamma, X)$ is not context-free. Moreover, there exists a different order on X, such that $\mathsf{ConjSL}_\psi(A_\Gamma, X)$ is not context-free.*

Recall the *spherical conjugacy growth series*, denoted $\tilde{\sigma}(G, X)$, to be the strict growth series of $\mathsf{ConjSL}(G, X)$, for a group G with respect to generating set X.

We now consider more general properties of $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$. We first consider irreducible graph products, that is, graph products which cannot be written as a direct product, using the following dichotomy by Minasyan and Osin.

Corollary 5.10. *([29], Corollary 2.13) Let G_Γ be an irreducible graph product, where $V(\Gamma) \geq 2$. Then G_Γ is either virtually cyclic or acylindrically hyperbolic.*

Lemma 5.11. *Let G_Γ be an irreducible graph product, where $V(\Gamma) \geq 2$. Let G_ϕ be a virtual graph product of the form $G_\phi = G_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for G_ϕ, endowed with an order. If G_ϕ is not virtually abelian, the language $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ is not unambiguous context-free. Otherwise, the spherical conjugacy growth series $\tilde{\sigma}\left(G_\phi, \widehat{X}\right)$ is rational.*

We recall that even if the series $\tilde{\sigma}\left(G_\phi, \widehat{X}\right)$ is rational, this does not necessarily imply regularity of the language $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$. This can be seen by Proposition 2.12 for virtually abelian groups.

Proof. By Corollary 5.10, we need to check two cases. If G_Γ is virtually cyclic, it is necessarily virtually abelian. Therefore G_ϕ is virtually abelian, and so $\tilde{\sigma}\left(G_\phi, \widehat{X}\right)$ is rational [20].
If G_Γ is acylindrically hyperbolic but not virtually cyclic, then $\mathsf{ConjSL}(G_\Gamma, X)$ is not unambiguous context-free by Theorem 5.1. Hence $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ is not unambiguous context-free by Proposition 5.6, by considering $l = 0$. □

When proving results for graph products, a common technique is to study whether certain properties are closed under taking graph products. For example, it was shown in Theorem 3.1 [12], that regularity of the conjugacy geodesic language is preserved by graph products.

This method is less clear when we consider the language $\mathsf{ConjSL}(G_\Gamma, X)$. For example, consider the graph product $G_\Gamma = F_2 = \mathbb{Z} * \mathbb{Z}$. Here $\mathsf{ConjSL}(\mathbb{Z}, X)$ is regular over any generating set X, however $\mathsf{ConjSL}(F_2, \{a, b\})$ is not context-free by Proposition 2.2 of [12] (where $\{a, b\}$ is a free basis). We generalise this result further by considering subsets of the vertex groups in a graph product.

Definition 5.12. Let Γ be a graph, and let $S \subseteq V(\Gamma)$. We say $\Lambda \subseteq \Gamma$ is an *induced subgraph* of Γ if Λ is the graph with vertices S, such that each pair of vertices in Λ are connected by an edge in Λ if and only if they are connected by an edge in Γ.

Let G_Γ be a graph product with vertex groups $G_i = \langle X_i \rangle$. Let $\Lambda \subseteq \Gamma$ be an induced subgraph of Γ. We define G_Λ to be the *induced graph product*, with respect to Λ. In particular, $G_\Lambda = \bigcup_{v \in V(\Lambda)} G_v$. For notation, we denote X_Λ to be the standard generating set for G_Λ. In particular, $X_\Lambda = \cup_{v \in V(\Lambda)} X_v$.

It was shown in Lemma 3.12 of [21] that if two cyclically reduced elements $x, y \in G_\Gamma$ are conjugate, then $\mathrm{supp}(x) = \mathrm{supp}(y)$. We use this result to prove the following.

Proposition 5.13. *Let G_Γ be a graph product with standard generating set X, endowed with an order. For an induced subgraph $\Lambda \subseteq \Gamma$, let G_Λ be the induced graph product with respect to Λ, with standard generating set X_Λ, with an induced ordering from X. If $\mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ is not regular, then $\mathsf{ConjSL}(G_\Gamma, X)$ is not regular. This result also holds when we replace regular with either unambiguous context-free or context-free.*

Proof. We claim that

$$\mathsf{ConjSL}(G_\Gamma, X) \cap X_\Lambda^* = \mathsf{ConjSL}(G_\Lambda, X_\Lambda).$$

The implication $\subseteq$ is clear. For the other direction, suppose there exists a word $x \in \mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ such that $x \notin \mathsf{ConjSL}(G_\Gamma, X)$. Then there exists a word $y \in \mathsf{ConjSL}(G_\Gamma, X)$ such that $x \sim y$ and $y <_{SL} x$. By definition, both x and y are cyclically reduced in the graph product, and hence $\mathrm{supp}(x) = \mathrm{supp}(y)$. Since $\mathrm{supp}(x)$ contains only vertices from the vertex groups which generate G_Λ, so must $\mathrm{supp}(y)$. Hence y is precisely a word over X_Λ, and so $y \in \mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ by the $\subseteq$ implication. Now we have a contradiction since $x \in \mathsf{ConjSL}(G_\Lambda, X_\Lambda)$.

Now suppose $\mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ is not regular, but $\mathsf{ConjSL}(G_\Gamma, X)$ is regular. Then we reach a contradiction by Lemma 2.5, which completes the proof. This argument also holds for unambiguous context-free and context-free using Lemma 2.8. □

Corollary 5.14. *Let G_ϕ be a virtual graph product of the form $G_\phi = G_\Gamma \rtimes \langle t \rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for G_ϕ, endowed with an order. For an induced subgraph $\Lambda \subseteq \Gamma$, let G_Λ be the induced graph product with respect to Λ, with standard generating set X_Λ, with an induced ordering from X. If $\mathsf{ConjSL}(G_\Lambda, X_\Lambda)$ is not regular, then $\mathsf{ConjSL}\left(G_\phi, \widehat{X}\right)$ is not regular. This result also holds when we replace regular with either unambiguous context-free or context-free.*

Proof. This result follows from Proposition 5.13 and Proposition 5.6, by considering $l = 0$. □

For RAAGs, we can prove a stronger result by considering the trivial automorphism. By showing $\mathsf{ConjSL}(A_\Gamma, X)$ is not context-free for any RAAG, we find that the equivalent language in the extension, $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$, is also not context-free (recall Proposition 5.6). This may surprise readers considering Proposition 5.8, where we found two twisted components of the language $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$, one of which was regular and the other was not context-free. In this scenario, $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$ is also not context-free, again by Proposition 5.6, even though this language contains a regular twisted component.

One therefore might ask if there exists a RAAG such that the twisted components, $\mathsf{ConjSL}_{\phi^l}(A_\Gamma, X)$ are regular for all l, and hence the extension language, $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$, is also regular. The following theorem proves the negative of this question.

Theorem 5.15. *Let A_Γ be a RAAG which is not free abelian. Let X be the standard generating set for A_Γ, endowed with an order. Then $\mathsf{ConjSL}(A_\Gamma, X)$ is not context-free.*

Proof. By assumption, there exists two generators $x, y \in X$ such that $\langle x, y\rangle \cong F_2$. Let ϕ^l be trivial, i.e. $\mathsf{ConjSL}_{\phi^l}(A_\Gamma, X) \cong \mathsf{ConjSL}(A_\Gamma, X)$. We claim that

$$\mathsf{ConjSL}(A_\Gamma, X) \cap \{x, y\}^* = \mathsf{ConjSL}(F_2, \{x, y\}).$$

The proof follows similarly to the proof of Proposition 5.13, recalling that in RAAGs, cyclically reduced conjugate elements are related by cyclic permutations and commutation relations.

Suppose $\mathsf{ConjSL}(A_\Gamma, X)$ is context-free. By Lemma 2.8, this would imply $\mathsf{ConjSL}(F_2, \{x, y\})$ is also context-free. This is a contradiction since $\mathsf{ConjSL}(F_2, \{x, y\})$ is not context-free by Proposition 2.2 of [12]. Therefore $\mathsf{ConjSL}(A_\Gamma, X)$ is not context-free. □

Corollary 5.16. *Let A_ϕ be a virtual RAAG of the form $A_\phi = A_\Gamma \rtimes \langle t\rangle$ (as defined in (2)), where $\widehat{X}$ is the standard generating set for A_ϕ, endowed with an order. If A_ϕ is not virtually abelian, then $\mathsf{ConjSL}\left(A_\phi, \widehat{X}\right)$ is not context-free. Otherwise, the spherical conjugacy growth series $\tilde{\sigma}\left(A_\phi, \widehat{X}\right)$ is rational.*

Proof. If A_ϕ is not virtually abelian, the result follows by Theorem 5.15 and Proposition 5.6. □

6. Open questions

We finish with some unanswered problems arising from this paper.

For conjugacy geodesics, we have shown both positive and negative examples of virtual RAAGs with respect to having regular $\mathsf{ConjGeo}$, both in length and non-length preserving cases. It is still unclear however, whether this language is regular or not in the case of graph automorphisms.

Question 6.1. *Let A_ϕ be a virtual RAAG (as defined in (2)) such that $\phi \in \mathrm{Aut}(A_\Gamma)$ is a graph automorphism. Is* $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ *regular?*

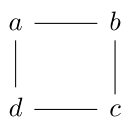
We know in general that for graph automorphisms, $\mathsf{ConjGeo}_\psi \subsetneq \mathsf{CycGeo}_\psi$ (see Example 4.10), and while CycGeo_ψ is regular, a subset of a regular language may not be regular. We can however give a partial answer to Question 6.1, when considering direct products of free groups.

For $\psi \in \mathrm{Aut}(A_\Gamma)$, where $A_\Gamma = A_{\Gamma_1} \times A_{\Gamma_2}$, we say ψ *fixes elements within vertex groups* if for all $v \in V(\Gamma_i)$, $\psi(v) \in V(\Gamma_i)$. In this scenario, we can immediately deduce that

$$\mathsf{ConjGeo}_\psi(A_\Gamma, X) = \mathsf{ConjGeo}_\psi(A_{\Gamma_1}, X_1) \cdot \mathsf{ConjGeo}_\psi(A_{\Gamma_2}, X_2),$$

where $X_i = V(\Gamma_i)$ for $i \in \{1, 2\}$. By Lemma 2.5, $\mathsf{ConjGeo}_\psi(A_\Gamma, X)$ is regular if both $\mathsf{ConjGeo}_\psi(A_{\Gamma_1}, X_1)$ and $\mathsf{ConjGeo}_\psi(A_{\Gamma_2}, X_2)$ are regular. Using this result, we can find an example of a graph automorphism ϕ such that $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular.

Example 6.2. Let $A_\Gamma = F_2 \times F_2$ and label the defining graph as follows:



Let $\phi\colon a \leftrightarrow c,\ b \leftrightarrow d$ be a reflection, and let X be the standard generating set. To determine whether $\mathsf{ConjGeo}\left(A_\phi, \widehat{X}\right)$ is regular, we need to check two twisted classes, namely $\mathsf{ConjGeo}(A_\Gamma, X)$ and $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$. The first language is regular by Corollary 2.5, [13]. Since ϕ fixes elements within the vertex groups, showing $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$ is regular reduces to showing that $\mathsf{ConjGeo}_\phi(F_2, X_1)$ is regular. Suppose for a contradiction that this language is not regular, and consider the language $\mathsf{ConjGeo}\left(F_\phi, \widehat{X}\right)$, where F_ϕ is a finite extension as defined in Equation (2). By assumption this language cannot be regular, however the group F_ϕ is virtually free, and hence has regular $\mathsf{ConjGeo}$ over any generating set. Hence the language $\mathsf{ConjGeo}_\phi(F_2, X_1)$ must be regular, and so $\mathsf{ConjGeo}_\phi(A_\Gamma, X)$ is regular. Since both twisted classes are regular, $\mathsf{ConjGeo}_\phi\left(A_\phi, \widehat{X}\right)$ must also be regular.

We note that if we consider the same RAAG with the rotation $\phi\colon a \to b \to c \to d$, then this question becomes more difficult. Here we would need to check four twisted classes. Two of these, namely $\mathsf{ConjGeo}$ and $\mathsf{ConjGeo}_{\phi^2}$, are regular by Example 6.2, but it is still unclear whether $\mathsf{ConjGeo}_\phi$ or $\mathsf{ConjGeo}_{\phi^3}$ are regular or not.

For the spherical conjugacy language, it would be interesting to see if we can strengthen our results in the case of extensions of RACGs.

Question 6.3. *Does there exist a virtual RACG* $W_\phi = W_\Gamma \rtimes \langle t \rangle$*, such that* $\mathsf{ConjSL}\left(W_\phi, \widehat{X}\right)$ *is not context-free?*

We cannot take the same approach as Theorem 5.15, since a similar intersection would give

$$\mathsf{ConjSL}(W_\Gamma, X) \cap \{x, y\}^* = \mathsf{ConjSL}(D_\infty, \{x, y\}).$$

This is inconclusive since $\mathsf{ConjSL}(D_\infty, \{x, y\})$ is regular (Theorem 2.4, [12]), so a new approach will be needed.

For more general graph products, it may be possible to extend Corollary 5.2, if the following two questions can be answered.

Question 6.4. *When are graph products* $\mathcal{AH}$*-accessible?*

Some example of groups which are $\mathcal{AH}$-accessible can be found in Theorem 2.18 of [1], which includes RAAGs.

Question 6.5. *Are all graph products hierarchically hyperbolic groups?*

We mention a recent result, Theorem C of [7], which states that if each of the vertex groups of a graph product are hierarchically hyperbolic, then the graph product itself is a hierarchically hyperbolic group.

Data availability

No data was used for the research described in the article.

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References

[1] Carolyn Abbott, Sahana Balasubramanya, Denis Osin, Hyperbolic structures on groups, Algebraic Geom. Topol. 19 (4) (2019) 1747–1835.

[2] Carolyn Abbott, Jason Behrstock, Matthew Durham, Largest acylindrical actions and Stability in hierarchically hyperbolic groups, Trans. Amer. Math. Soc. Ser. B 8 (3) (2021) 66–104.

[3] Yago Antolín, Laura Ciobanu, Formal conjugacy growth in acylindrically hyperbolic groups, Int. Math. Res. Not. 1 (2017) 121–157.

[4] Valerij Bardakov, Leonid Bokut, Andrei Vesnin, On the conjugacy problem for cyclic extensions of free groups, Commun. Algebra 33 (6) (2005) 1979–1996.

[5] Jason Behrstock, Mark Hagen, Alessandro Sisto, Hierarchically hyperbolic spaces, I: curve complexes for cubical groups, Geom. Topol. 21 (3) (2017) 1731–1804.
[6] Jason Behrstock, Mark Hagen, Alessandro Sisto, Hierarchically hyperbolic spaces II: combination theorems and the distance formula, Pac. J. Math. 299 (2) (2019) 257–338.
[7] Daniel Berlyne, Jacob Russell, Hierarchical hyperbolicity of graph products, Groups Geom. Dyn. 16 (2022) 523–580.
[8] O. Bogopolski, A. Martino, O. Maslakova, E. Ventura, The conjugacy problem is solvable in free-by-cyclic groups, Bull. Lond. Math. Soc. 38 (2006) 787–794.
[9] José Burillo, Francesco Matucci, Enric Ventura, The conjugacy problem in extensions of Thompson's group F, Isr. J. Math. 216 (1) (2016) 15–59.
[10] N. Chomsky, M.P. Schützenberger, The Algebraic Theory of Context-Free Languages, 1963, pp. 118–161.
[11] Laura Ciobanu, Alex Evetts, Meng-Che "Turbo" Ho, The conjugacy growth of the soluble Baumslag-Solitar groups, N.Y. J. Math. 26 (2020) 473–495.
[12] Laura Ciobanu, Susan Hermiller, Conjugacy growth series and languages in groups, Trans. Am. Math. Soc. 366 (5) (2013) 2803–2825.
[13] Laura Ciobanu, Susan Hermiller, Derek Holt, Sarah Rees, Conjugacy languages in groups, Isr. J. Math. 211 (1) (2016) 311–347.
[14] Donald J. Collins, Charles F. Miller, The conjugacy problem and subgroups of finite index, Proc. Lond. Math. Soc. s3–34 (3) (1977) 535–556.
[15] Charles Garnet Cox, Twisted conjugacy in Houghton's groups, J. Algebra 490 (2017) 390–436.
[16] John Crisp, Eddy Godelle, Bert Wiest, The conjugacy problem in subgroups of right-angled Artin groups, J. Topol. 2 (3) (2009) 442–460.
[17] Gemma Crowe, Conjugacy languages in virtual graph products, https://arxiv.org/abs/2212.07111v1, 2022.
[18] Karel Dekimpe, Pieter Senden, The R_∞-property for right-angled Artin groups, Topol. Appl. 293 (2021).
[19] David B.A. Epstein, Word Processing in Groups, A K Peters/CRC Press, 1992.
[20] Alex Evetts, Rational growth in virtually abelian groups, Ill. J. Math. 63 (4) (2019).
[21] Michal Ferov, On conjugacy separability of graph products of groups, J. Algebra 447 (2016) 135–182.
[22] E.R. Green, Graph Products of Groups, PhD thesis, The University of Leeds, 1990.
[23] J. Hopcroft, R. Motwani, J.D. Ullman, Introduction to Automata Theory, Languages, and Computation, 3rd edition, Pearson Addison Wesley, 2007.
[24] Michael R. Laurence, A generating set for the automorphism group of a graph group, J. Lond. Math. Soc. 52 (2) (1995) 318–334.
[25] Joseph Loeffler, John Meier, James Worthington, Graph products and cannon pairs, Int. J. Algebra Comput. 12 (2002) 747–754.
[26] Kalpana Mahalingam, Anuran Maity, Watson-Crick conjugates of words and languages, https://arxiv.org/abs/2208.03123v1, 2022.
[27] Margalit Dan, Matt Clay (Eds.), Office Hours with a Geometric Group Theorist, Princeton University Press, 2017.
[28] John Meier, Groups, Graphs and Trees, Cambridge University Press, 2008.
[29] Ashot Minasyan, Denis Osin, Acylindrical hyperbolicity of groups acting on trees, Math. Ann. 362 (3–4) (2015) 1055–1105.
[30] Walter D. Neumann, Michael Shapiro, Automatic structures, rational growth, and geometrically finite hyperbolic groups, Invent. Math. 120 (1) (1995) 259–287.
[31] D. Osin, Acylindrically hyperbolic groups, Trans. Am. Math. Soc. 368 (2) (2015) 851–888.
[32] Herman Servatius, Automorphisms of graph groups, J. Algebra 126 (1) (1989) 34–60.
[33] Alessandro Sisto, What is a hierarchically hyperbolic space?, http://arxiv.org/abs/1707.00053, 2017.
[34] Melanie Stein, Jennifer Taback, Peter Wong, Automorphisms of higher rank lamplighter groups, Int. J. Algebra Comput. 25 (2015) 1275–1299.