Controlling investment risk in a pooled annuity fund to improve income withdrawals

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Abstract

Pooled annuity fund offer a third way of decumulation, between life annuities and income drawdown (called programmed withdrawals). In such funds, participants receive an income while they are alive. Their account values within the fund earn both investment returns and an additional return due to the pooling of longevity risk.

A question of how to invest in such funds is studied here. In the absence of systematic longevity risk and no re-investment risk, a guaranteed minimum income can be secured by investing entirely in the risk-free asset. How should a participant invest if they wish to have a higher income?

An optimal strategy is derived, based on a quadratic loss function. The strategy shows increasing risk aversion with time, as well as responding to gains and losses above a target value. Analysis of numerical simulations shows that the desired income to be withdrawn must be chosen carefully, so as to enable investment risk-taking and hence the income to be achieved with a reasonable chance.

Keywords: tontine; longevity risk; mortality; retirement; decumulation.

1. Introduction

Pooled annuity funds provide their participants with a retirement income for life. They do this by pooling directly the longevity risk of their participants. Just like in life annuities and defined benefit pension schemes, deaths among the shorter-lived subsiside payments to the longer-lived. A

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key difference with the former structures is that there is no life insurance or sponsoring employer between or behind the participants. Instead, the participants are exposed to the volatility of deaths in the fund, which manifests itself through volatility in the income paid to the participants.

The consequence on the income paid out of using a quadratic loss function to set investment risk is studied here. The optimizing investment strategy exhibits both decreasing risk aversion over time – just like life-styling strategies – as well as a dynamic response to over- and under-funding relative to a target value. It is found that the level of desired income must be carefully chosen in this setting, to get a reasonable income distribution.

Structures in which longevity risk is pooled directly among participants have increasingly gained industry and academic interest in recent years. Various ways of doing longevity risk-sharing have been proposed. Some of these work for single-cohort pools in which every member is an independent and identical copy of each other (for example, Milevsky and Salisbury 2015; Stamos 2008) and some are intended for multi-cohort pools (for example, Piggott et al. 2005; Stamos 2008; Sabin 2010; Qiao and Sherris 2013; Donnelly et al. 2014; Milevsky and Salisbury 2016). There is heterogeneity among the participants in the latter funds since successive cohorts age and have different amounts of money in the fund, according to what has occurred.

Denuit and Robert (2021) present various fair linear risk-sharing rules, and a conditional mean risk-sharing rule and study their convergence. Weinert and Gründl (2021) derive a distribution to model the longevity credits paid from the pooled annuity fund, rather than modelling directly the mortality experience of the pool of participants.

Milevsky and Salisbury (2015) calculate the payout from a pooled annuity fund which maximizes the expected discounted value of lifetime consumption. The optimal payout to participants varies according to the utility-maximization problem considered, which can also be observed in Chen et al. (2020, 2021) who use the approach of Milevsky and Salisbury (2015).

Some authors calculate what proportion of the funds of those who have died should be received by each participant – whether explicitly (Stamos, 2008; Sabin, 2010; Donnelly et al., 2014) or implicitly (Piggott et al., 2005; Qiao and Sherris, 2013). Both of these explicit and implicit schemes then use an annuity value to calculate the income paid out. Here, an explicit rule is used to share out the funds of those who have died.

Due to a pooled annuity fund being akin to a life annuity without the implicit financial and mortality guarantees, a study of the demand for pooled annuity funds compared to life annuities is a natural one. It has been studied by authors such as Piggott et al. 2005; Valdez et al. 2006; Donnelly et al. 2013; Hanewald et al. 2013; Milevsky and Salisbury 2016; Chen et al. 2021. The results show that pooled annuity funds become increasingly preferred to life annuities as the loadings on life annuities increase. Additionally, the attractiveness of pooled annuity funds increases as the risk aversion of the retiree reduces. Retirees who are less risk averse are happier to bear the volatility of pooled annuity funds in exchange for their higher expected return.

The income paid from a pooled annuity fund is volatile whereas the income paid under a conventional life annuity contract is not. Some investors may prefer to switch their pensions savings to a life annuity at some future age, as they prefer the income stability of an annuity contract at higher ages. Chen et al. (2018) study this problem, considering an individual receiving income from a pooled annuity fund up to some fixed age, followed by income from a deferred life annuity. They determine the optimal age at which to switch between these contracts, with optimality determined using a CRRA utility function-based criterion.

Bequests are another important area of research, since many individuals value a death benefit which can be used by their dependents. This is studied by various authors in different settings, such as Bernhardt and Donnelly (2019); Dagpunar (2021); Zhou et al. (2021); Chen and Rach

(2022).

Turning to the literature most relevant to the study of this paper, Bernhardt and Donnelly (2020) calculate how income stability is affected by the number of homogeneous members – who share the same characteristics – in a single cohort fund. They propose a measure of income stability which is applied here. They derive an analytical expression that can be used to determine for how many years is the income stable in a pooled annuity fund, for a given number of members. A numerical study of the same results, seen in Bernhardt and Donnelly (2021, Figure 2), shows that at least 2 000 participants are needed to eliminate most of the idiosyncratic longevity risk.

Bernhardt and Qu (2022) extend the study to a heterogeneous fund, in which some of the members are wealthier than the others. In all other aspects, members are identical and the setting is again a single cohort fund. They again derive an analytical expression that can be used to determine for how many years is the income stable. In the analytical expression, a measure of heterogeneity is observed. Their study then focuses on determining criteria for when it is beneficial for a particular collection of members to stay in a large fund rather than form smaller but homogeneous funds.

Qiao and Sherris (2013) study systematic longevity risk in a pooled annuity fund. They find that allowing a pooled annuity fund to remain open improves income stability. This illustrates again the general rule of pooled annuity funds: the more members, the better. While their charts show income falling gradually over time, this is likely due to the life annuity value used to calculate the income paid out, being discordant with the calculation of the longevity credit rather than a feature of pooled annuity funds.

Donnelly (2022) also studies systematic longevity risk, studying the question of how many members are needed to join an open fund each year, to have sufficient longevity pooling. She finds that if around 100 homogeneous members, most of the idiosyncratic longevity risk is eliminated. In a study of the distribution of income paid to each cohort, she finds that cohorts experience a similar outcome, in terms of the distribution of income, with the exception of the last cohorts. The last cohorts to join the fund have a similar experience up until the time when they are old. At that point, there are no new cohorts to join to continue to diversify idiosyncratic longevity risk. The last cohorts are, in effect, in a version of single cohort fund when they are old.

The motivation for the particular pooled annuity fund model is detailed in Section 2, followed by a mathematical specification of the model in Section 3. The stochastic optimal control problem and its solution are given in Section 4. The numerical study is in Section 5 and the conclusion is in Section 6.

2. Motivation for the pooled annuity fund model

The purpose of a pooled annuity fund is to pay a lifetime income to its members by pooling their longevity risk. Longevity risk can be decomposed into two components: idiosyncratic longevity risk and systematic longevity risk. In the model used here, it is assumed that idiosyncratic longevity risk is totally diversified away and that systematic longevity risk has been passed to an insurer. The justification for these assumptions is given next.

Idiosyncratic longevity risk can be eliminated by having enough participants in the fund. In other words, it is completely diversifiable. Various authors have studied the income streams paid from pooled annuity funds when it has not been diversified away.

Donnelly et al. (2013) find that the cost of removing residual idiosyncratic longevity risk is very low. They study a pooled annuity fund in which all participants have the same future distribution of deaths. As participants experience the same probabilities of death as assumed in the mortality distribution, there is no systematic longevity risk present. They compare it to a fund in which participants earn the expected longevity credit but less a fee, with the fee paying for the removal of any residual idiosyncratic longevity risk. They consider an instantaneous 'snapshot' of an open fund, in which there is a range of ages and fund values present in the fund. They find that the fee charged for the removal of residual idiosyncratic longevity risk is very low. For a 75-year-old, it would imply investing at most an additional 0.60% of the participant's account value in the risky asset in order to get the same expected return as in a pooled annuity fund with 1 000 participants.

Bernhardt and Donnelly (2020) study the stability of the income paid out in a single-cohort pooled annuity fund, with no systematic longevity risk present. All participants are the same age, have the same account value and face the same distribution of deaths. As the figure in Bernhardt and Donnelly (2021) shows for a typical UK mortality distribution, idiosyncratic longevity risk is mostly removed with around 2 000 participants in the fund.

Bernhardt and Qu (2022) continue the study of a single-cohort pooled annuity fund, again with no systematic longevity risk, when participants have different amounts of wealth. Their work is further explored in a numerical setting by Donnelly (2023). The numerical study looks at various levels of heterogeneity that, To have the same level of idiosyncratic longevity risk, roughly, the number of members in the fund should be increased by about half to two-thirds, in the presence of moderate heterogeneity compared to the situation in which members are independent copies of each other, are the same age and have the same chance of dying at each future age.

Systematic longevity risk cannot be eliminated through pooling more and more participants' longevity risk. It is the risk that, for example, participants in the fund live longer than was estimated. In such a scenario, too much money would be paid out. The longer-lived participants and later-joining participants would bear the consequences of this risk, through having less money paid out to them. This would manifest itself through lower longevity credits since the account values of participants would have been depleted at a higher rate than expected, due to the earlier too-high income payments.

Systematic longevity risk is cumulative over time. If future lifetimes are sometimes over-estimated and at other times under-estimated, then its net effect may be zero at some future time. However, it is also noted that the rate of improvements in life expectancy have often been under-estimated.

Additionally, the financial impact of systematic longevity risk is borne collectively by all participants in the fund at each point in time. Changes in the number of members will affect the amount of systematic longevity risk borne by each member. For example, if the number of members suddenly fell then the amount of systematic longevity risk borne by each of the remaining participants would increase. This will certainly happen if the fund closes to new members.

The impact of systematic longevity risk may not be as large as the effect of investment returns, depending on the investment strategy chosen. Nonetheless, it should be managed appropriately. For example, at least some of it could be transferred to a third-party provider in exchange for a fee.

The assumption in this paper is that systematic longevity risk has been removed from the participants. Their account values would be lower as a consequence. There is no attempt here to cost the removal of systematic longevity risk; this is simply not the focus of the current paper.

3. The pooled annuity fund structure

3.1. The financial market

In the pooled annuity fund, the account value of each participant is invested in a frictionless financial market. The financial market consists of two traded assets: a risky asset and a risk-free asset. The risk-free asset has price $B(t)$ at time t with dynamics

$$dB(t) = rB(t)dt, \qquad B(0) = B_0 > 0 \text{ constant},$$

with constant risk-free rate of return r.

The price process S of the risky asset is driven by a one-dimensional standard Brownian motion $\mathcal{Z}$, so that at time t it has dynamics

$$dS(t) = S(t)\left(\mu dt + \sigma d\mathcal{Z}(t)\right), \qquad S(0) = S_0 > 0 \text{ constant},$$

with constant $\mu > r$ and constant $\sigma > 0$.

A Poisson process could be introduced for each participant, to model their time of death. It would be assumed that all of these Poisson processes are independent of each other and the Brownian motion. However, as the notation would only appear here, it is not written down.

Instead, with $\mathcal{N}(\mathbb{P}) := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$, the information $\mathcal{F}_t$ available to individuals at time t is represented by the natural filtration generated by these Poisson processes and the Brownian motion, augmented by the null sets $\mathcal{N}(\mathbb{P})$. In practical terms, the information $\mathcal{F}_t$ allows the surviving participants to know which participants have died (via knowledge of the Poisson processes) and the price of the risky asset (via knowledge of the Brownian motion which generates the price of the risky asset) at all times up to and including at time t.

3.2. Participants in the fund

Consider a prototype member who joins the fund upon their retirement at time 0 when they are age 65 years. They bring an account value $W(0) = w_0 > 0$ with them. Their account value earns both investment returns and longevity credits. Since it is assumed that there is no longevity risk in the fund, the longevity credits are earned continuously at the same rate as their force of mortality.

Denote by $\lambda(t)$ the deterministic force of mortality of the member at time t, i.e. $\lambda : [0, T] \to [0, \infty)$. Participants with different forces of mortality will earn longevity credits at a rate equal to their own force of mortality. It is assumed that each participant dies on or before some constant limiting time $\omega \in (0, \infty]$. The time would vary by participant but it is only necessary to specify one value in this setting, since only one participant is considered.

Upon the death of a member, their account value is surrendered to the surviving participants. This is a key difference to, for example, income drawdown, also known as programmed withdrawals. Losing the account value upon death is a typical feature of life annuities and defined benefit annuity benefits since it is the foundation of how longevity risk is pooled.

The prototype member invests an amount $\pi(t)$ of their account value in the risky asset. The remainder of their account value is invested in the risk-free asset. In addition to earning investment returns, their account earns longevity credits at a rate equal to their force of mortality, $\lambda(t)$. To live on, they withdraw income at the constant rate $c_0 \geq 0$ from their account value.

Then, as long as the member survives, their account value evolves with dynamics

$$dX(t) = \left((r + \lambda(t))X(t) + \sigma\beta\pi(t) - c_0\right)dt + \sigma\pi(t)dW(t), \tag{1}$$

subject to $X(0) = x_0$ a.s, in which $x_0 > 0$ is a constant and $\beta := (\mu - r)/\sigma$ is the Sharpe ratio of the risky asset. It is assumed that $\pi := \{\pi(t) : t \in [0, T]\}$ is an $\mathcal{F}_t$-progressively measurable process.

3.3. Risk appetite of the participant

In the theoretical setting presented here, the participant could gain a guaranteed annual lifetime income starting from time 0 at the rate

$$c_0^{\text{no risk}} := \frac{x_0}{\int_0^\omega e^{-\int_0^s (r+\lambda(u))du} ds}. \tag{2}$$

This value is obtained by setting $\pi(t) := 0$ for all t. The resultant dynamics of the account value are

$$dX(t) = ((r + \lambda(t))X(t) - c_0)\, dt,$$

which has solution

$$X(t) = e^{\int_0^t (r+\lambda(s))ds} \left(x_0 - c_0 \int_0^t e^{-\int_0^s (r+\lambda(u))du} ds \right).$$

Letting t go to the limiting time ω, the income rate $c_0 := c_0^{\text{no risk}}$ is obtained. Thus with no longevity risk (which is implied by perfect pooling and the absence of systematic longevity risk) and no investment risk, the annual income rate $c_0^{\text{no risk}}$ can be withdrawn by the participant without any chance of variations in that income.

To have a higher income rate than $c_0^{\text{no risk}}$, either all the time or for some of the time, it is necessary to take some investment risk. This brings in the possibility of losses, and consequently the risk of a lower income than $c_0^{\text{no risk}}$.

Suppose that the participant is willing to take more risk in order to gain a higher income. For example, they withdraw income at a rate $c_0 \geq c_0^{\text{no risk}}$ per annum and aim to have an account value of F at a fixed future time $T > 0$. F is called the *terminal target*. What F represents can vary according to the investor.

The goal assumed here is that F is sufficient to pay the participant an income of c_T^{target} per annum, starting from a fixed time $T > 0$. For this reason, time T is called the *annuitization* time. The desired income paid from time T, c_T^{target}, does not have to be the same value as the income paid before time T, c_0. Additionally, c_T^{target} is only a target income and the actual income paid from time T is likely to be different to it.

Further suppose that the participant wants to minimise the quadratic loss about the terminal target, via the function

$$L(T, X(T)) := (F - X(T))^2,$$

in which

$$F := c_T^{\text{target}} \int_T^\omega e^{-\int_T^s (r+\lambda(u))du} ds$$

is the value which, if invested entirely in the risk-free asset at the annuitisation time T, is sufficient to provide income at the rate $c_T^{\text{target}} > 0$ per annum for the rest of the participant's life.

It is also assumed that the participant cares about their funding position during the time period $[0, T]$. Regulators of the fund may also care about the funding position of the fund and its ability to meet the terminal target. Thus interim targets of amount $F(t)$ at time $t \in [0, T)$ are introduced.

The participant wishes also to minimise the the quadratic loss about the interim targets over the time period $[0, T)$. This is measured by the loss function

$$L(t, X(t)) := (F(t) - X(t))^2. \tag{3}$$

The goal of keeping their account value close to the ultimate target. The amount $F(t)$ is selected so that, if invested in the risk-free asset, it will provide the desired income profile. Thus define

$$F(t) := Fe^{-\int_t^T (r+\lambda(s))ds} + c_0 \int_t^T e^{-\int_t^s (r+\lambda(u))du} ds, \tag{4}$$

at each time $t \in [0, T]$. Furthermore, define $F(T) := F$. Note that, if the interim target were to be achieved and invested in the risk-free asset, then the loss function $L(t, X(t))$ would be zero from that time onwards. The minimisation is done by determining which of all possible investment strategies will minimise a function of these quadratic loss function.

These targets are the ones studied in Gerrard et al. (2004) in an income drawdown setting. There is no pooling of longevity risk in their model. That is different to the setting here. The setting here in this paper is for a retiree who joins a pooled annuity fund, whereas their setting is for a retiree who chooses to bear their own idiosyncratic longevity risk (again, without systematic longevity risk) and use income drawdown to provide an income in retirement.

Di Giacinto et al. (2014) further extend Gerrard et al. (2004) to include constraints. Di Giacinto et al. (2014, Remark 3.5.2) shows that the same optimal strategy is obtained over the set $\{(t, x) : t \in [0, T], x \leq F(t)\}$ if the expression for the loss function in (3) is replaced by

$$\tilde{L}(t, x) := \begin{cases} (F(t) - x)^2, & x \leq F(t) \\ 0, & x > F(t). \end{cases}$$

In other words, as long as the participant's account value at time t lies at or below the interim target $F(t)$ – which constitute the events of interest – the optimal investment strategy is the same as if no loss value were assigned when the account value is higher than the interim target.

The constraints included by Di Giacinto et al. (2014) are a guaranteed minimum value of the account value at the annuitisation time T and a no-short-selling constraint on the investment in the risky asset. It is very challenging to obtain an analytical solution with these type of constraints, to which the complexity of the solution in Di Giacinto et al. (2014) is testament.

Introducing a guaranteed minimum constraint adds another layer of risk aversion from the participant, on top of that implied by the loss function. It may give different outcomes for members, comparing the version of the problem with and without the guaranteed minimum constraint. This appears to particularly the case for participants with higher levels of risk aversion. For example, there is a 60% chance that the members who are dubbed to be of high risk aversion in Di Giacinto et al. (2014) end up with an account value at the guaranteed minimum value. That is a 60% chance that they have an income past time T of $2/3c_0$. Thus they have only a 40% chance of having more than c_0 per annum. In contrast, removing the minimum constraint means that participants have a much better chance of getting a higher income. From Gerrard et al. (2004), a participant with similar targets (labelled as having medium risk aversion therein) has a 93% chance of ending up with an income of more than c_0 per annum beyond the annuitisation time T. While they have a 3% chance of the participant being ruined, arguably this is a risk worth taking.

Guaranteed minimum constraints and no-short-selling constraints are not studied here due to the additional complexities which they bring, both in terms of mathematics and the understanding of what risk aversion means when a minimum constraint is imposed and the consequent, intricate implications for the participant.

To specify the problem and solve it, the standard machinery of stochastic optimal control theory is applied, as detailed next.

4. The stochastic optimal control problem

The pooled annuity fund participant takes account of the chance of surviving to each age, in their optimization problem and, hence, in their investment strategy. This is justified since the participant's account value gains longevity credits due to the pooling of longevity risk, in addition to investment returns. In contrast, an income drawdown investor has to choose an age to which they may survive and then invest as if they were certain to survive to that age. An income drawdown investor's account value earns only investment returns. They have to optimize over a fixed time.

The ability to take into account the distribution of the future lifetime of the participant has two consequences.

First, given identical investment strategies, the pooled annuity fund participant can withdraw money at a significantly higher rate compared to an income drawdown investor. For example, assume a constant return of 5% per annum effective and a starting account value of £100 000. Then a 65-year-old income drawdown investor can withdraw an income of £6, 757 per annum paid at the start of each year, until age 90 years. In contrast, a 65-year-old participant in the pooled annuity fund can withdraw an income of £8 463 per annum until their time of death (assuming the mortality table S1PMA).

The pooled annuity participant has a 25% higher annual income due to pooling longevity risk. If the income drawdown investor wants to withdraw the same income as the pooled annuity participant, then they would run out of money by age 82 years. The numbers will vary according to the underlying mortality model. However, the main point is that pooled annuity participant's income will be higher.

Second, the pooled annuity fund investor could take less investment risk than the income drawdown investor, and still have a higher income. This is because an income drawdown investor needs to invest in riskier assets in order to get an income comparable to that of the pooled annuity participant. Continuing the example above, the income drawdown investor would need an annual return of close to 8% per annum to be confident of withdrawing £8 463 per annum until age 90. A higher return-seeking investment strategy would bring the risk that the income drawdown investor suffers investment losses and is unable to sustain the desired, higher income.

4.1. The value function

Consider again a participant in the pooled annuity fund who has account value $X(t)$ at time t. Denote their time of death by the random variable $\tau > 0$. Suppose the participant has a constant time preference rate ρ. The goal of the participant is to choose an investment strategy to minimize the expected value

$$\mathbb{E}\left(\int_0^{\tau\wedge T} e^{-\rho t}\, L(t, X(t))dt + \epsilon\, \chi_{[\tau>T]}\, e^{-\rho T}\, L(T, X(T)) \,\middle|\, X(0) = x_0\right).$$

The constant weighting factor $\epsilon > 0$ allows the participant to choose the relative importance of the terminal loss function. The function $\chi_{[\tau>T]}$ is the zero-one penalty function which takes value one if the event $[\tau > T]$ holds and value zero otherwise.

Integrating over the stochastic time of death τ, the last expression becomes

$$\mathbb{E}\left(\int_0^T e^{-\int_0^t \lambda(u)du} \int_0^{t\wedge T} e^{-\rho s} L(s,X(s))ds\,dt + \epsilon \int_0^T e^{-\int_0^t \lambda(u)du} \chi_{[t>T]}\, e^{-\rho T}\, L(T,X(T))\,dt \,\Big|\, X(0)=x_0\right)$$
$$=\mathbb{E}\left(\int_0^T e^{-\int_0^s(\rho+\lambda(u))du}\, L(s,X(s))ds + \epsilon\, e^{-\int_0^T(\rho+\lambda(u))du}\, L(T,X(T)) \,\Big|\, X(0)=x_0\right). \tag{5}$$

In order for the problem to be well-defined, some technical conditions are required. For example, the investment strategies over which the problem is optimised must be admissible, where an investment strategy process π is said to be admissible if it is $\mathcal{F}_t$-progressively measurable and it satisfies $\int_0^T \pi^2(s)ds < \infty$.

It is shown in the rest of this section that, for the account value dynamics given by (1), an optimal investment strategy that minimises the objective function (5) is

$$\pi^\star(t,X(t)) = \frac{\beta}{\sigma}\left(-\frac{1}{2}\frac{B(t)}{A(t)} - X(t)\right),$$

in which the deterministic functions $A(t)$ and $B(t)$ are given by (11) and (13).

4.2. The Hamilton–Jacobi–Bellman equation

In order to determine a minimising investment strategy, the Hamilton–Jacobi–Bellman (HJB) approach is applied. A clear exposition of this methodology can be found in Bjork (2009, Chapter 19), with the same methodology followed here.

Define the value function

$$V(t,x) := \inf_{\{(\pi(s))_{s=t}^T\}} \mathbb{E}\left(\int_t^T e^{-\int_t^s(\rho+\lambda(u))du}\, L(s,X(s))ds + \epsilon e^{-\int_t^T(\rho+\lambda(u))du}\, L(T,X(T)) \,\Big|\, X(t)=x\right), \tag{6}$$

in which the dynamics of X given by (1) show the dependence on the investment strategy π.

For convenience, define the infinitesimal operator $\mathcal{A}_t^\pi$ on a suitably differentiable function $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ as

$$\mathcal{A}_t^\pi f(t,x) := \frac{\partial f}{\partial t}(t,x) + \left[(r+\lambda(t))\,x + (\mu-r)\,\pi - c\right]\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\pi^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x).$$

Assume that from time t to time $t+h$, the participant follows an arbitrary investment strategy $\{\pi(s)\}_{s=t}^{t+h}$. After time $t+h$, the participant follows an optimal investment strategy. Then, denoting by $\mathbb{E}^{t,x}$ the expectation conditional on $X(t)=x$,

$$V(t,x) \le \mathbb{E}^{t,x}\left(\int_t^{t+h} e^{-\int_t^s(\rho+\lambda(u))du}\, L(s,X(s))ds\right) + \mathbb{E}^{t,x}\left(e^{-\int_t^{t+h}(\rho+\lambda(u))du}\, V(t+h,X(t+h))\right). \tag{7}$$

Assuming sufficient differentiability, apply Ito's lemma to the product $e^{-\int_t^{t+h}(\rho+\lambda(u))du}V(t+h,X(t+h))$, and substitute for the wealth dynamics in (1) to find

$$e^{-\int_t^{t+h}(\rho+\lambda(u))du}V(t+h,X(t+h)) = V(t,X(t)) + \int_t^{t+h} \mathcal{A}_s^\pi e^{-\int_t^s(\rho+\lambda(u))du}\, V(s,X(s))\,ds$$
$$+ \int_t^{t+h} e^{-\int_t^s(\rho+\lambda(u))du}\, \sigma\pi(s)\,\frac{\partial V}{\partial x}(s,X(s))\,dW(s).$$

Substituting this last expression into the last term of equation (7), the expectation of the stochastic integral vanishes and the $V(t,x)$ terms cancel, leaving

$$0 \leq \mathbb{E}^{t,x}\left(\int_t^{t+h} e^{-\int_t^s(\rho+\lambda(u))du}\, L(s,X(s))ds\right) + \mathbb{E}^{t,x}\left(\int_t^{t+h} \mathcal{A}_s^{\pi} e^{-\int_t^s(\rho+\lambda(u))du}\, V(s,X(s))\, ds\right).$$

Divide by $h > 0$, let h go to zero and assume enough regularity so that we can take the limit within the expectation to obtain

$$0 \leq L(t,x) - (\rho+\lambda(t))V(t,x) + V_t(t,x) + \left[(r+\lambda(t))\,x + \sigma\beta\pi - c_0\right] V_x(t,x) + \frac{1}{2}\pi^2\sigma^2 V_{xx}(t,x).$$

If the optimal investment strategy is chosen then, upon substitution for $L(t,x)$ from equation (3), the HJB equation is obtained:

$$\begin{aligned}&(\rho+\lambda(t))V(t,x)\\ =&\inf_{\pi}\left\{(F(t)-x)^2 + V_t(t,x) + \left[(r+\lambda(t))\,x + \sigma\beta\pi - c_0\right] V_x(t,x) + \frac{1}{2}\pi^2\sigma^2 V_{xx}(t,x)\right\},\end{aligned}$$

with boundary condition $V(T,x) = \epsilon\ (F(T)-x)^2$.

4.3. A candidate optimal investment strategy

Having derived the HLB equation, the next step is to determine an expression for the optimal investment strategy in terms of V and its derivatives. Set

$$f(\pi) = \sigma\beta\pi V_x(t,x) + \frac{1}{2}\pi^2\sigma^2 V_{xx}(t,x).$$

Then $f_x(t,x) = 0$ at $\pi = \pi^\star$, i.e.

$$\pi^\star = -\frac{\beta}{\sigma}\frac{V_x(t,x)}{V_{xx}(t,x)}.$$

Substituting for $\pi^\star$ into the HJB equation gives

$$\begin{aligned}&(\rho+\lambda(t))V(t,x)\\ =&(F(t)-x)^2 + V_t(t,x) + \left[(r+\lambda(t))\,x - c_0\right] V_x(t,x) - \frac{1}{2}\beta^2\frac{(V_x(t,x))^2}{V_{xx}(t,x)}.\end{aligned} \tag{8}$$

4.4. A candidate solution for the value function

A candidate solution for $V(t,x)$ is

$$H(t,x) := A(t)x^2 + B(t)x + C(t), \tag{9}$$

in which $A(t)$, $B(t)$ and $C(t)$ are deterministic functions. Since $V(T,x) = L(T,x) = (F-x)^2$, for all $x > 0$, it follows that

$$A(T) = \epsilon, \qquad B(T) = -2\epsilon F, \qquad C(T) = \epsilon F^2. \tag{10}$$

From the HJB equation, the first-order linear ordinary differential equations satisfied by these functions are derived next. For convenience, define the constant

$$\alpha := \beta^2 + \rho - 2r.$$

Substitution of the quadratic trial solution for $V(t,x)$ into equation (8) gives the ODEs

$$\begin{aligned} A'(t) &= (\alpha - \lambda(t))\, A(t) - 1. \\ B'(t) &= (\alpha + r)\, B(t) + 2F(t) + 2c_0 A(t). \\ C'(t) &= (\rho + \lambda(t))\, C(t) - F^2(t) + c_0 B(t) + \frac{\beta^2 B^2(t)}{4A(t)}. \end{aligned}$$

As their boundary conditions are given in (10), the standard 'integrating factor' approach to solving first-order linear ODEs yields the solutions

$$A(t) = \epsilon e^{-\int_t^T (\alpha-\lambda(s))ds} + \int_t^T e^{-\int_t^s (\alpha-\lambda(u))du} ds, \tag{11}$$

$$-\frac{1}{2}B(t) = \epsilon F e^{-(\alpha+r)(T-t)} + \int_t^T (F(s) + c_0 A(s))\, e^{-(\alpha+r)(s-t)} ds \tag{12}$$

and

$$C(t) = \epsilon F^2 e^{-\int_t^T (\rho+\lambda(s))ds} + \int_t^T \left(F^2(s) - c_0 B(s) - \frac{\beta B^2(s)}{4A(s)} \right) e^{-\int_t^s (\rho+\lambda(u))du} ds, \tag{13}$$

in which $F(t)$ and $F = F(T)$ are given by expression (4). With these solutions, a candidate optimal investment strategy is

$$\pi^\star(t,x) = -\frac{\mu - r}{\sigma^2} \frac{H_x(t,x)}{H_{xx}(t,x)} = \frac{\beta}{\sigma}\left(G(t) - x\right), \tag{14}$$

in which

$$G(t) := -\frac{1}{2}\frac{B(t)}{A(t)}.$$

A suitable verification theorem, such as Bjork (2009, Theorem 19.6) verifies that $V(t,x) = H(t,x)$ and that $\pi^\star(t,x)$ is an optimal investment strategy that minimizes the last expression in (5).

With the above optimal investment strategy, the solution to the wealth equation (1) is a.s. for $t \in [0,T]$,

$$X(t) = \xi^{-1}(t)\left(x_0 + \int_0^t \xi(s)\left(2\beta^2 G(s) - c_0\right) ds + \int_0^t \xi(s)\beta G(s) dW(s) \right),$$

in which

$$\xi(t) = \exp\left(\int_0^t \left(\frac{3}{2}\beta^2 - r - \lambda(s) \right) ds + \beta W(t) \right).$$

Remark 4.1. *From expression (11), $A(t)$ is composed of exponential terms, with one of them weighted by $\epsilon > 0$. Hence $A(t) > 0$ for all $t \geq 0$. A similar consideration applies to the expression (12) for $-B(t)/2$, since $A(t) > 0$ and, from (4), $F(t) > 0$. Hence $-B(t)/2 > 0$ for all $t \geq 0$. Thus $G(t) > 0$.*

Letting $t \to T$, from the terminal conditions in (10) and the continuity of $A(t)$ and $B(t)$, it is observed that $G(t) \to F$. Thus as the account value becomes closer to its target value F at time T, the optimal amount in the risky asset would get closer to zero at time T. It would be zero if the account value was able to achieve its target value. However, in this continuous time setting, the target value is never achieved (due to the difference between the target and account value being a Brownian motion, as discussed in Gerrard et al. 2004).

Remark 4.2. *Upon setting $\lambda(t) \equiv 0$ for all $t \geq 0$, the solutions for $A(t)$, $B(t)$ and $C(t)$ are identical to those in Gerrard et al. (2004, equation (30)), aside from the difference in how investment in the risky asset is expressed (here, as an amount while, in Gerrard et al. 2004, as a fraction of the account value).*

Remark 4.3. *From the expression (14), the optimal amount invested in the risky asset is positive if*

$$G(t) = -\frac{1}{2}\frac{B(t)}{A(t)} > X(t),$$

in which $X(t)$ is the account value at time t assuming the investor invests in line with the optimal strategy. There is no guarantee that the account value stays below $G(t)$, meaning that the latter inequality holds. Indeed, Gerrard et al. (2004, Figure 2) show that negative holdings in the risky asset occurs in their 'highly risk averse' investor scenario. This latter constraint is included in the set-up of Di Giacinto et al. (2014), albeit the latter is in the income drawdown setting and so does not include a longevity credit.

Furthermore, the plots in Gerrard et al. (2004, Figure 2) imply borrowing in the risk-free asset in some future states of the world, corresponding to more than 100% investment of the account value in the risky stock. Including a 'no borrowing' (i.e. no short selling) constraint on both the risky asset and risk-free asset makes the problem extremely challenging to solve and is not addressed here. That aside, these constraints are non-binding in the numerical simulations shown here.

4.5. A discussion of the optimal investment strategy

The optimal investment strategy (14) exhibits two features:

- Decreasing risk aversion with time. In other words, less money is invested in the risky asset as the terminal time T is approached. This feature is seen in the numerical simulations below, through observation of the development of the deterministic function $G(t)$ over time t. It is a feature also shared by life-styling strategies, which are model-free investment strategies often used in defined contribution pension plans in the lead up to a member's retirement date. Broadly speaking, life-styling strategies invest less in risky assets as the pension saver approaches retirement.

- Dynamically responding to how far the account value is from the target. If the account value equals $G(t)$ at time t, the optimal investment strategy is to invest all of the account value in the risk-free asset. If the account value falls below $G(t)$, more and more is invested in the risky asset. The idea is to take investment risk in order to attain $G(t)$ and thus reach the target value.

  Conversely, if the account value were to rise above $G(t)$, the risky asset would be short-sold and the proceeds invested in the risk-free asset. Considering the optimal proportion - instead of the amount - invested in the risky asset, it is observed from expression (14) that

  $$\lim_{x\uparrow\infty}\left\{\frac{\pi^\star(t,x)}{x}\right\} = -\frac{\beta}{\sigma}.$$

  Thus as the account value gets larger and larger, the proportion invested in the risky asset tends to a constant, $-\frac{\beta}{\sigma}$. If the mean return, μ, on the risky stock is higher than the risk-free interest rate, r, then $\beta > 0$ and the limiting proportion of the account value invested in the risky asset is negative.

  In the continuous-time setting of the problem, as long as the account value is insufficient to secure the desired income stream through investment in the risk-free asset alone, this point is moot. However, in the discretised version of the optimal investment strategy, it is possible to have the account value be more than enough to secure the desired income. In that case, the dynamic strategy involves trying to lose money to get as close to the target value as possible.

The above comments imply note that the proportion of the account value that is short-sold in the risky asset is bounded by $-\frac{\beta}{\sigma}$.

To avoid short-selling the two assets, a non-negativity constraint would have to be imposed on the amounts invested in them. It is very challenging technically to solve the consequent optimization problem, to which Di Giacinto et al. (2014) is testament. However, it is seen in the numerical solutions here that, for the chosen model and its parameters, these constraints are rarely binding.

5. Numerical study

5.1. Setting and parameters

Consider a man who has joined the pooled annuity fund at age 65 with pension savings of amount £100. He is happy to take investment risk for $T = 10$ years. However, he does not think that he will be capable of working after age 75 and so is averse to taking any risk from that age. Thus he wants certainty about his income from age 75.

Whatever remains of his pension savings at age 75, he will use it to secure a certain income. In the setting of this paper, that is achieved by the man staying in the pooled annuity fund and investing entirely in the risk-free asset at age 75. In the absence of systematic longevity risk, it is equivalent to buying a life annuity in this simplified setting.

Before age 75, he will withdraw continuously income at a rate c_0 per annum. From age 75, when his residual pension savings are the amount $X(T)$ and are then invested entirely in the risk-free asset, the income withdrawn from his pension savings becomes the annual rate

$$c_T^{\text{actual}} := \frac{X(T)}{\int_T^{\omega} e^{-\int_T^s (r+\lambda(u))du} ds}. \tag{15}$$

The annual rate c_T^{actual} will, in general, not be known until time T. The investor's ambition for what the random variable c_T^{actual} should be, is represented by c_T, which is a constant fixed at time 0.

5.1.1. Income decisions

At time 0, the investor must choose two income streams, one known and one desired:

- How much constant income c_0 to withdraw each year, from age 65 to age 75. In the numerical simulations, it is assumed that $c_0/12$ is withdrawn at the start of each month. It is possible for the investor to be ruined before reaching age 75, through withdrawing the constant annual income c_0 up to age 75 and not achieving sufficient investment returns.

  It seems reasonable that c_0 should be at least equal to $c_0^{\text{no risk}}$. Otherwise, the investor would be tempted to avoid the investment risk and simply secure the lifetime income $c_0^{\text{no risk}}$ at age 65. The lifetime income $c_0^{\text{no risk}}$ is a baseline income against which the merits of taking investment risk to secure a higher income can be measured.

- How much constant annual income c_T^{target} to target for income withdrawn from age 75 until his death. The actual income secured at age 75 is calculated via equation (15) at time T, and is likely to be different to c_T^{target}. Again, this may be due to poor investment returns up to age 75.

  The desired income to be paid from age 75, c_T^{target}, should be at least as large as c_0. It is unlikely the investor would be happy with a drop in income from age 75.

5.1.2. Investment strategies

Furthermore, at time 0, the investor must choose how to invest his pension savings from age 65 to age 75. He will invest his residual savings entirely in the risk-free asset starting from age 75. Three different investment strategies are studied numerically.

- A discretization of the optimal investment strategy for the problem of minimizing the last expression in (5). At the start of each month, the amount $\frac{\beta}{\sigma}\left(G(t) - X(t)\right)$ is invested in the risky asset.
- A constant proportion investment strategy, in which a constant proportion of the account value is invested in the risky asset. In the simulations, the proportion is re-balanced to the desired constant at the start of each month. The constant proportion is chosen to be consistent with the optimal strategy, namely as

$$p := \frac{\beta}{\sigma}\left(G(0) - x_0\right).$$

- A life-styling, or decreasing proportion, investment strategy. Initially, a proportion of the account value is invested in the risky asset. The proportion is gradually reduced over time until, at age 75, nothing is invested in the risky asset. The initial proportion is chosen to be consistent with the optimal and constant proportion strategies, namely as p. At time $t \in \{0, 1/12, 2/12, \dots, T\}$, the proportion of the account value invested in the risky asset is

$$\left(1 - \frac{t}{T}\right)p.$$

5.1.3. Parameterization

Finally, it remains to specify the parameters of the problem, the financial market model and the investor's characteristics. In terms of the problem specification, it is assumed that the participant is very keen to have the desired income at age 75. If necessary, the participant could seek work before age 75 in order to save in their account value. For this reason, the time preference rate up to age 75 is chosen to be negative. The implication is that cashflows further into the future are given a greater weight in the optimization problem than earlier ones.

Similarly, a greater weight is put on the terminal loss function, $L(10, X(10))$, compared to the earlier loss functions $\{L(t, X(t)); t \in [0, 10)\}$, by choosing $\epsilon > 1$. Again, this relates to a strong desire to get as near as possible to the desired post-age 75 income stream.

In summary, the baseline choices of these parameters are:

$$T = 10, \quad \epsilon = 10 \quad \text{and} \qquad \rho = -0.05.$$

The baseline financial market parameters are chosen to be

$$r = 0.05, \quad \mu = 0.1, \quad \sigma = 0.2 \quad \Rightarrow \quad \beta = 0.25.$$

The investor, who is age 65 years at time 0, is assumed to have a mortality that follows the UK pensioner male mortality table S1PMA. With these choices, the annual income that the investor could secure at age 65 by investing all of his pension savings in the risk-free bond is £8.90.

5.1.4. The deterministic function $G(t)$

The function $G(t)$ is a key determiner of the amount invested in the risky asset. The numerical simulations (Figures A.1a-A.1d) show that $G(t)$ decreases over time, demonstrating the decreasing risk aversion of the investor as he approaches age 75. How sensitive is $G(t)$ to some of the choices of parameters?

Figure A.1a shows how $G(t)$ changes as ϵ varies, assuming that the investor seeks a 10% higher income than can be done through risk-free investment only, i.e. $c_{10} = c_0 = 1.1c_0^{\text{no risk}}$. The function $G(0)$ gets smaller as ϵ increases.

For very small values of ϵ, such as $\epsilon = 0.1$ or smaller, the time T account value has virtually no weight. However, there is an in-built need to have a certain account value at time T through the interim targets $\{F(t)\}$. The targets counter-balance the very small values of ϵ, rendering them somewhat meaningless. Once ϵ climbs above the value 1, the time T loss function, $L(T, X(T))$ starts being noticed in the optimization.

As ϵ climbs towards 10, there is less and less invested in the risky asset. As the time T loss function carries more and more weight, less investment risk is taken in order to secure it. However, once ϵ gets above 10, the reduction in investment risk slows up. Notice that the time T target value $F = F(T)$ is independent of ϵ and all the strategies have the objective of minimising the expected squared distance that the time T account value is from $F = F(T)$. This requirement acts to lessen the speed of reduction in investment risk.

Increasing the income to 20% above the risk-free income, i.e. $c_{10} = c_0 = 1.2c_0^{\text{no risk}}$, it is observed that $G(t)$ shifts upwards (Figure A.1b). More investment risk needs to be taken, in order to secure the desired income.

The stock volatility parameter, σ, drives the relative attractiveness of the risky asset over the risk-free one. Varying it while keeping the income aspiration to a constant 10% over the baseline, risk-free income, shows that $G(t)$ is very sensitive to the value of σ (Figures A.1c-A.1d). Reducing the stock volatility leads to a substantial increase in investment in the risky asset. With a fall in stock volatility, the risky asset becomes a more attractive asset as its Sharpe Ratio doubles.

Increasing the stock volatility causes $G(t)$ to become relatively insensitive to the choice of ϵ (Figure A.1d). This is due to a combination of the quadratic loss functions, with the loss measured about the target amounts $\{F(t)\}$ which are independent of σ, and the reduction in risk-adjusted returns for the risky asset.

Allowing the time at which all of the residual pension savings are invested in the risk-free asset to vary (Figure A.2), it is seen that more investment risk is taken, the longer the investment time horizon. Concretely, the value of $G(t)$ is higher for each time t, as T increases.

5.1.5. The income secured at time T

The investor could follow one of three possible investment strategies, which are detailed in Section 5.1.2. The income secured at time T under these strategies is considered next. A total of 10 000 simulations were run.

While the investor would like to have c_T paid from time T, how does it compare to the actual income c_T^{actual} secured at time T? Under the optimal strategy, the distribution of c_T^{actual} is significantly higher than under the model-free strategies (Table 5.1.1).

Consider the setting under $T = 10$ when the desired income was $c_T = 9.79$. In that case, the mean and median income secured at age 75 under the optimal strategy is 11% below the desired income. It is even lower than the baseline, risk-free income of £8.90. The comparable values under the constant proportion and decreasing proportion strategies are significantly lower (around 33% lower) than the desired income. Although the optimal strategy gives a higher income, on a percentile comparison, than the other strategies. Nonetheless, it has a higher standard deviation of income at age 75 under the optimal strategy, compared to the other strategies.

A reason for this disappointing performance may be that, by withdrawing 10% more than the baseline, risk-free income in the first T years, the account value is depleted too much. Indeed, this suggestion is given strength by the even worse performance when both the income withdrawn in the first 10 years and the desired income is set at 20% above the baseline, risk-free income (Table 5.1.2). The problem is that, under the quadratic loss function, having an account value far from the interim targets will be heavily penalised. However, if investment risk is taken because the account value is not close enough to the interim targets, then the risk is that, in some of the future states of the world, the account value becomes even further from the interim targets. To avoid this situation occurring, less investment risk is taken.

Going further, enabling a higher account value in the first 10 years should improve the outcomes at time T. Effectively, 'spare' assets are needed in order to take investment risk. This is done through reducing the income withdrawn up to time T, to the baseline, risk-free income (i.e. $c_0 = c_0^{\text{actual}}$) and then setting the desired income at time T to 10% above that income rate ($c_T = 1.1c_0^{\text{actual}}$). The results in Table 5.1.3 show a more acceptable range of outcomes, with the median income being very close to the desired income of £9.79, and only 5% of incomes being 5% below £9.79.

However, looking at the distribution of the investment in the risky asset (Figure A.3c), the proportion turns negative for many simulations near time T. This means that the strategy has been very successful in many future states of the world, and the optimal strategy tries to get rid of this excess money as it aims to be close to the target amounts $\{F(t)\}$. It also suggests that there is scope to take more investment risk.

Keeping the income withdrawn up to time T at the baseline, risk-free income (i.e. $c_0 = c_0^{\text{actual}}$) but choosing a higher desired income of 20% above the baseline ($c_T = 1.2c_0^{\text{actual}}$), it is observed that the proportion of the account value in the risky asset always stays positive (Figure A.3d). The distribution shifts mostly upward, albeit with a higher standard deviation (Table 5.1.4). While the investor is less likely to meet their desired income at time T, it is generally higher than when a lower desired income is targeted.

In summary, withdrawing too much income before time T results in the desired income target at time T not being met. In order to have a good chance of meeting it, the income withdrawn in the first T years should be around the value of the baseline, lifetime income. Otherwise, the account value will be too depleted to allow sufficient investment risk to be taken.

The results above have been done within a pooled annuity fund. For an investor who prefers income drawdown, his results are shown in the last two columns of Table 5.1.3. They demonstrate the poorer outcomes in the absence of longevity risk pooling.

6. Conclusion

Pooled annuity funds offer investors both an investment return and returns due to the pooling of longevity risk. These latter returns are non-negative and represent the share of the funds of the fund members who have died.

Within such funds and in the simplified setting of this paper, a guaranteed income can be achieved by investing pension savings entirely in the risk-free asset. However, what about investors who would like a higher income? To enable them to control the level of investment risk, an objective function is specified.

The idea is that the investor will withdraw a known income from age 65 to age 75. They would like to have at least as high an income from age 75, which they will secure by investing all of their

Statistic	Optimal strategy $\pi^\star(t, X(t))$	Constant proportion	Decreasing proportion
Mean	8.74	6.08	6.60
Standard deviation	0.45	0.24	0.18
5% percentile	7.97	5.72	6.33
25% percentile	8.44	5.91	6.48
50% percentile	8.75	6.07	6.59
75% percentile	9.05	6.23	6.71
95% percentile	9.45	6.49	6.90

Table 5.1.1: Statistics of the random income c_T^{actual} secured from age 75, for $T = 10$, under the three considered investment strategies when the stock volatility parameter is $\sigma = 0.2$ and $c_{10} = c_0 = 1.1c_0^{\text{no risk}} = 9.79$.

Statistic	Optimal strategy $\pi^\star(t, X(t))$	Constant proportion	Decreasing proportion
Mean	8.12	3.71	4.54
Standard deviation	0.75	0.23	0.21
5% percentile	6.83	3.35	4.22
25% percentile	7.63	3.54	4.39
50% percentile	8.15	3.70	4.52
75% percentile	8.65	3.86	4.66
95% percentile	9.28	4.12	4.90

Table 5.1.2: Statistics of the random income c_T^{actual} secured from age 75, for $T = 10$, under the three considered investment strategies when the stock volatility parameter is $\sigma = 0.2$ and $c_{10} = c_0 = 1.2c_0^{\text{no risk}} = 10.68$.

Statistic	PAF Optimal strategy $\pi^\star(t, X(t))$	PAF Constant proportion	PAF Decreasing proportion	INCD Constant proportion	INCD Decreasing proportion
Mean	9.68	8.21	8.52	3.87	4.27
Standard deviation	0.25	0.20	0.13	0.18	0.14
5% percentile	9.26	7.90	8.32	3.59	4.05
25% percentile	9.51	8.07	8.42	3.74	4.17
50% percentile	9.68	8.20	8.51	3.86	4.26
75% percentile	10.00	8.35	8.60	3.98	4.36
95% percentile	10.10	8.56	8.75	4.18	4.52

Table 5.1.3: Statistics of the random income c_T^{actual} secured from age 75, for $T = 10$, under the three considered investment strategies when the stock volatility parameter is $\sigma = 0.2$ and $c_{10} = 1.1c_0^{\text{no risk}} = 9.79$ and $c_0 = c_0^{\text{no risk}} = 8.90$. The right-most two columns show the results under income drawdown (INCD), with the other columns being for investment within a pooled annuity fund (PAF).

Statistic	Optimal strategy $\pi^\star(t, X(t))$	Constant proportion	Decreasing proportion
Mean	10.00	8.06	8.45
Standard deviation	0.33	0.22	0.15
5% percentile	9.44	7.72	8.22
25% percentile	9.78	7.91	8.34
50% percentile	10.01	8.05	8.44
75% percentile	10.23	8.36	8.54
95% percentile	10.53	8.45	8.71

Table 5.1.4: Statistics of the random income c_T^{actual} secured from age 75, for $T = 10$, under the three considered investment strategies when the stock volatility parameter is $\sigma = 0.2$ and $c_{10} = 1.2c_0^{\text{no risk}} = 10.68$ and $c_0 = c_0^{\text{no risk}} = 8.90$.

remaining pension savings in the risk-free asset. The objective function encapsulates this desire, using a quadratic loss function.

The results show that target setting must be carefully considered. There must be sufficient assets to enable investment risk taking, since the latter is required to have a good chance of delivering on the targets. For example, withdrawing a 10% higher income than the baseline, risk-free income, does not give sufficient scope for investment risk-taking. Instead, withdrawing the baseline, risk-free income for the first 10 years can lead to significantly better outcomes from age 75.

This is due to the nature of the objective function, which is based upon a quadratic loss functions. For example, taking a lot of investment risk implies an increased risk of loss, and these losses would be penalised heavily. This leads to lower investment risk-sharing in order to minimise the quadratic loss function.

Note that the baseline, risk-free income in a pooled annuity fund is higher than in the alternative pensions vehicles of income drawdown. This is due to the pooling of longevity risk in the pooled annuity fund, which provides an annual return roughly equal to the annual chance of dying of the investor. While the comparison in the paper has been within pooled annuity funds, the income would be significantly lower under income drawdown. An investor may be satisfied with 'only' withdrawing this baseline income if it is higher than that attainable under income drawdown.

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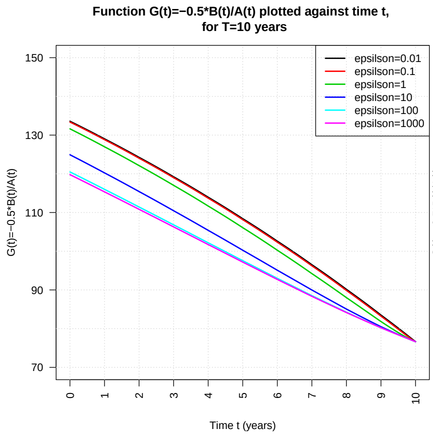
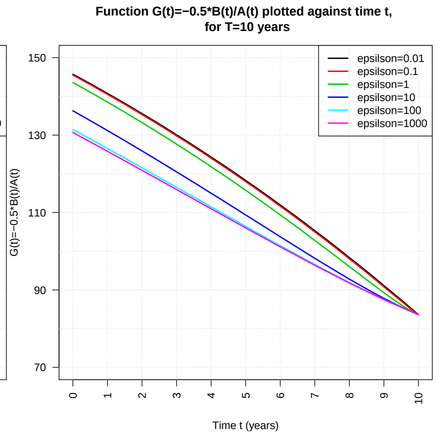
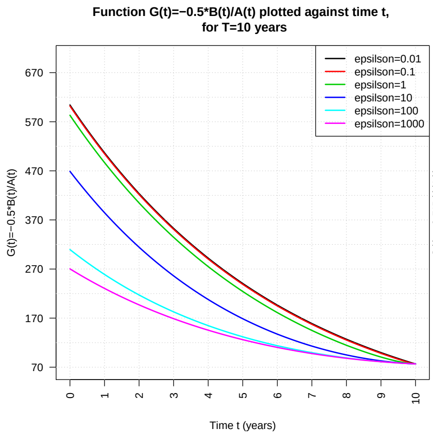
A. Figures



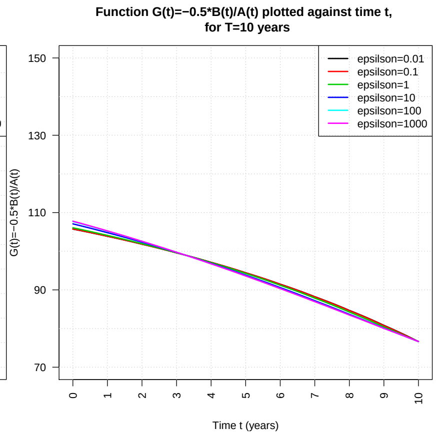
(a) $c_{10} = c_0 = 1.1 c_0^{\text{no risk}}$, $\sigma = 0.2$.



(b) $c_{10} = c_0 = 1.2 c_0^{\text{no risk}}$, $\sigma = 0.2$.

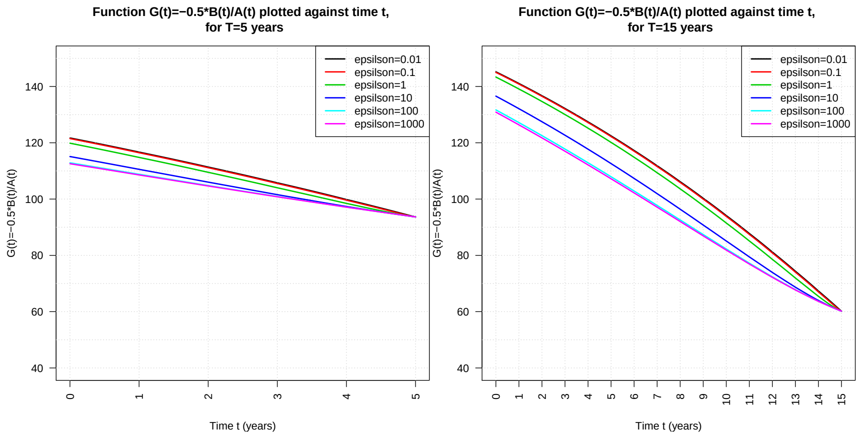


(c) $c_{10} = c_0 = 1.1 c_0^{\text{no risk}}$, $\sigma = 0.1$.



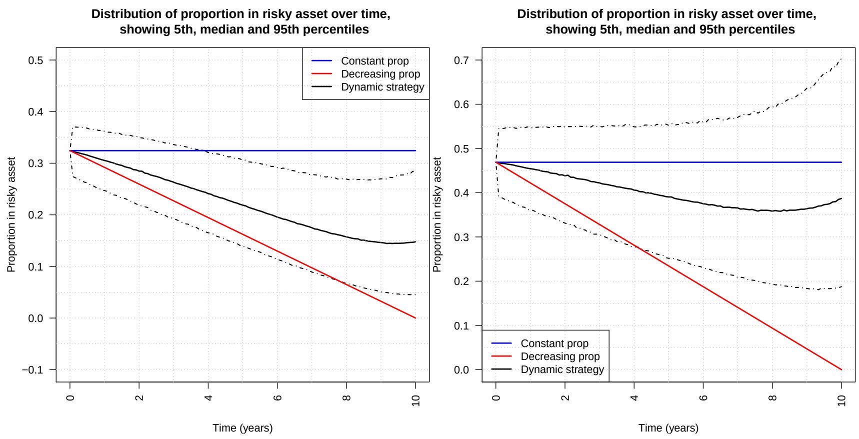
(d) $c_{10} = c_0 = 1.1 c_0^{\text{no risk}}$, $\sigma = 0.3$.

Figure A.1: The deterministic function $G(t)$ plotted against $t \in [0, 10]$ for $T = 10$ and $c_{10} = c_0$. The top two plots use that the stock volatility is 20% per annum. The bottom two plots vary the stock volatility parameter value.

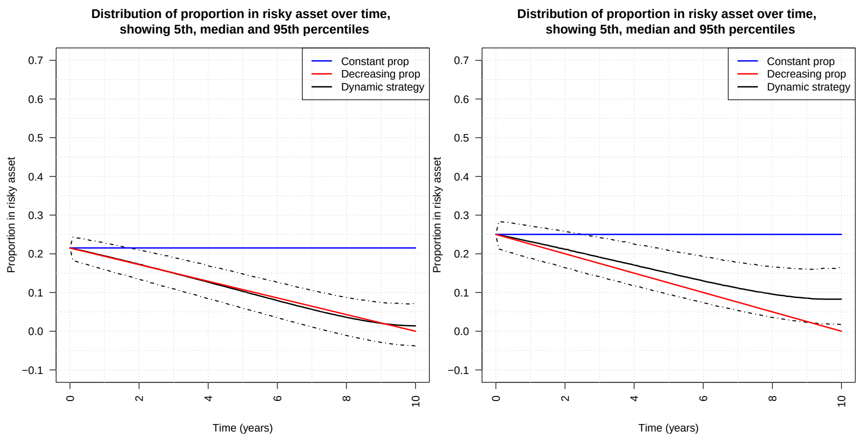


(a) $T = 5$. (b) $T = 15$.

Figure A.2: The deterministic function $G(t)$ plotted against $t \in [0, T]$ for $c_T = c_0 = 1.1c_0^{\text{no risk}}$ and $\sigma = 0.2$, as T varies.



(a) $c_{10} = c_0 = 1.1c_0^{\text{no risk}}$. (b) $c_{10} = c_0 = 1.2c_0^{\text{no risk}}$.



(c) $c_{10} = 1.1c_0^{\text{no risk}}$, $c_0 = c_0^{\text{no risk}}$. (d) $c_{10} = 1.2c_0^{\text{no risk}}$, $c_0 = c_0^{\text{no risk}}$.

Figure A.3: The proportion of the account value invested in the risky asset, plotted against $t \in [0, 10]$ for $T = 10$ and $\sigma = 0.2$.