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Abstract

Model risk in a collective defined contribution (CDC) pension scheme is analysed. It is seen that the scheme manager's view of the future affects the balance of payments between generations in the scheme. This is also an important risk because the scheme manager chooses the model but the members bear the consequences of the model risk.

The surprising result is that it is the entropy of the probabilistic model chosen by the manager to predict the future, which affects the payments. The higher the entropy, the greater the difference in the payments between the generations in the scheme.

The conclusion is that the manager of a CDC scheme should optimise the member's outcomes by minimising the model risk. This may be a more effective tool to optimise outcomes than controlling the investment risk, since the *raison d'être* of CDC pension schemes is to take investment risk.

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1 Introduction

Collective defined contribution (CDC) pension schemes provide a means of balancing poorer investment returns against better returns, over the generations of members in the scheme, in order to provide their members with higher benefit payments in retirement than standard alternatives. By joining a CDC scheme, a member can take investment risk for longer than they would in, for example, an individual defined contribution (IDC) scheme as they are more protected from the volatility of investment returns (Bonenkamp and Westerhout, 2014).

Model risk has not been studied explicitly in the academic literature. In other words: what if the model used by the scheme manager to predict the future is wrong. It is generally assumed that model which models the true evolution of investment returns and the scheme members' longevity is known to the scheme managers. In this paper, it is assumed that the manager does not use the true model. It is found that the benefit payments to generations of members in the scheme will be different, in accordance with the manager's view. Their view is expressed as a probabilistic model of future investment returns. The model which results in the higher entropy, gives the greatest imbalance in the payments between the generations.

The academic literature on CDC pension schemes tend, on the whole, to show that CDC pension schemes provide a higher expected lifetime utility of consumption in retirement compared to alternatives (Gollier, 2008; Cui et al., 2011; Owadally et al., 2022). Broadly, the academic literature views CDC schemes positively and more desirable than standard alternatives such as IDC schemes.

There are more nuanced results on the attractiveness of CDC schemes. For example, inter-generational risk-sharing is advantageous when market prices are volatile (Chen et al., 2021). The inter-generational risk-sharing mechanism of the CDC pension scheme studied by Chen et al. (2021) smooths out the effect of volatile market prices on the income paid to pensioners. Moreover, they show that members must be suitably risk-averse before a CDC pension scheme is more attractive than an IDC plan.

Another example is on the the attractiveness of CDC pension schemes when there is an existing deficit (Cui et al., 2011). They study the outcome for a particular generation who can join one of three different CDC pension schemes. In one of their analyses, they consider the outcomes for the generation if the CDC pension scheme has a pre-existing deficit. They show that a member of that generation is worse off joining any of their studied CDC schemes compared to the alternative of investing in an IDC scheme followed by income drawdown. Barajas-Paz and Donnelly (2022) recommend reviewing contributions regularly in a CDC scheme, to avoid the build up of deficits, since deficits reduce the attractiveness of CDC pension schemes. Indeed, Beetsma et al. (2012) suggest that mandatory participation is required for CDC pension schemes, to avoid a failure to participate by new generations when deficits arise.

That there will be a deficit in a CDC scheme at some point in time is highly likely. In a particular CDC scheme structure, Donnelly (2022a) sets out different types of investment risk-sharing, one of which is found in every inter-generational risk-sharing scheme. This aside, it may not be apparent to potential joiners of a CDC scheme that there is a deficit. Moreover, even with a deficit in the scheme, it may still be in their financial interests to join as they may lose the employer contribution if they do not join a specified pension scheme.

The management of a CDC pension scheme requires predictions of the future: what investment returns will the scheme assets earn in the future? What will be the mortality distribution of the members? These predictions are used to value the benefits accrued by the current membership. The valuation of the accrued benefits is compared to the scheme asset value, which is itself a reflection of both the historical experience of the scheme and the historical predictions of that and future experience. From the comparison, a suitable adjustment is made to the accrued benefits of the scheme.

One of the purported advantages of inter-generational risk-sharing schemes is that they diversify the highs and lows of asset returns over time. The income paid out to retired scheme members is not as low or as high as it would be, if it fully reflected investment returns immediately. What is a 'high' and 'low' asset return is done by reference to either a model of the asset prices or historical returns.

What if the model of investment returns used by the CDC scheme manager is wrong? How does it affect the benefit payments to scheme members? What is shown in this paper is that the entropy of the model is the important factor. The more uncertainty in the distribution of the investment model used by the manager, the larger is the difference in the mean benefit payments between generations. Surprisingly, it is not an overly- or underly-optimistic view which matters. It might be thought that a manager who thinks returns are more likely to be higher in the future, will pay out more to earlier generations than later generations when this sunnier outlook does not materialise. However, due to the delicate calculations in a CDC scheme, this is not the case.

The CDC scheme and the mathematical representation of model risk in the context of the scheme is described in Section 2. The comparator IDC scheme is given in Section 3. The analysis of model risk via a numerical study is in Section 4. Finally, the paper concludes with a wider discussion in Section 5.

2 The CDC pension scheme

A CDC pension scheme in which members pay a single contribution upon entry and receive a benefit payment two periods later is considered. A total of three generations join the scheme. The initial amount of benefits accrued by each generation is calculated as the expected accumulated value of the contribution paid, under the manager's model of the future and using the correct future investment returns. Thus the scheme is actuarially fair on an individual contribution basis.

There is no surplus or deficit in the scheme when the first generation joins. The second generation is in the scheme with the other two generations; they are the middle generation. The last generation have no future generation with whom to share the accumulation of risk in the scheme and thus the risk crystallises with them.

In the considered scheme, surpluses and deficits against the manager's expectations of discounted future benefit payment are shared immediately among the members. Chen et al. (2017) suggest that fewer people exit a CDC scheme when the period over which surpluses and deficits are recovered is shortened.

It is assumed that there is no longevity risk in the scheme, as the focus is on model risk with respect to investment risk. Idiosyncratic longevity risk can be diversified away with a sufficient number of members; the results of Donnelly (2022b) suggest that the order of 100 members joining the scheme each year is sufficient. Systematic longevity cannot be diversified away, but introducing it will obscure the results and message of the paper.

The benefits paid out to each generation, and their calculation, are shown in Section 2.4. These quickly become very complicated, particularly as not assumption of independence of investment returns from year-to-year is made. It is difficult to see the effect of the financial market model on the benefits paid out, which is why a numerical study of model risk is done in Section 4.

2.1 Mathematical representation of model risk

Let $R = \{R(n) \in (-1, \infty); n = 0, 1, 2, \ldots\}$ be the stochastic process representing the investment returns to be achieved on the scheme's assets. At each time n, the value of $R(n)$ is observed. It represents the return achieved on the pension scheme assets over the time period $(n-1, n]$. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be the natural filtration generated by R.

Suppose the joint probability density function (p.d.f.) $f(R)$ is used by the CDC pension scheme manager to model future investment returns. The manager believes that the p.d.f. f will help them best determine the stochastic evolution of the returns $R(1), R(2), \ldots$.

Expectations with respect to the p.d.f. f are denoted $\mathbb{E}$ and conditional expectations with respect to $\mathcal{F}_n$ are denoted by $\mathbb{E}_n$. For example, the manager's expectation of the time n return is, with f_m representing the marginal p.d.f. of $R(m)$ under the manager's model,

$$\mathbb{E}\left(R(m)\right) = \int_{-1}^{\infty} r f_m(r) dr$$

and the manager's prediction at time n of the investment return over $(m-1, m]$ is

$$\mathbb{E}_n\left(R(m)\right) := \mathbb{E}\left(R(m) \,|\, \mathcal{F}_n\right), \quad \text{for } m > n.$$

When discounting or accumulating future cashflows, the manager uses the p.d.f. f.

Meanwhile, the true model from which the observed returns are generated is represented by the joint p.d.f. $\tilde{f}(R)$. Expectations with respect to the p.d.f. $\tilde{f}$ are denoted $\tilde{\mathbb{E}}$ and conditional expectations with respect to $\mathcal{F}_n$ are denoted by $\tilde{\mathbb{E}}_n$. For example, the true expectation of the time n return is, with $\tilde{f}_m$ representing the marginal p.d.f. of $R(m)$ under the true model,

$$\tilde{\mathbb{E}}\left(R(m)\right) = \int_{-1}^{\infty} r \tilde{f}_m(r) dr$$

and the true prediction at time n of the investment return over $(m-1, m]$ is

$$\tilde{\mathbb{E}}_n\left(R(m)\right) := \tilde{\mathbb{E}}\left(R(m) \,|\, \mathcal{F}_n\right), \quad \text{for } m > n.$$

If $f \neq \tilde{f}$ then model risk exists in respect of the financial market model. That model risk exists is almost certainly the case.

Two questions around model risk are studied: does the manager's view of the future have an impact on the benefits paid out to members, and does it affect the balance of payments between generations.

2.2 Scheme membership

There are three generations in the CDC scheme, Generations 0, 1 and 2, who join at time 0, 1 and 2, respectively. Since no longevity risk is pooled and therefore there is no advantage to having lots of members in each generation, it is assumed that there is exactly one member in each generation.

Each member pays a single contribution $C > 0$ upon joining the scheme. In return, they receive a lump-sum benefit paid exactly two time periods after joining.

2.3 Contribution-benefit structure

The benefits are calculated on a age-related basis. This means that the benefits are actuarially fair on an individual contribution basis: the expected value of the benefits equals the contribution so that, if invested individually, the contribution is expected to provide the benefits in full without any additional funding required. Actuarial fairness means that each member does not enter with the expectation of requiring additional funds from the existing or future membership to pay their benefits. A lack of actuarial fairness can result in inter-generational cross-subsidies which favour the first generations in the scheme at the expense of the later ones (Donnelly, 2022a).

Over the first time period, the first generation has no-one with whom to share their investment risk. Due to the actuarial fairness imposed on the anticipated benefits, this means that the CDC scheme is identical to a DC pension scheme for the first year, from time 0 to time 1. From time 1, investment risk is shared albeit the first generation shares it for only one time period. The second generation is in the scheme with another generation over their scheme membership. The third generation shares risk for one time period only. In their last time period in the scheme, just before they retire, they bear the residual risk in the scheme. From this viewpoint, it may be expected that the last generation has the highest risk since they bear the cumulative effects of the investment risk-sharing between generations.

When a member of Generation g joins at time g, the expected accumulation of their fixed contribution C is calculated. The expected value takes account of what has happened up to time g. Mathematically, let $R(n) > -1$, a.s. represent the actual investment return achieved over time $(n-1, n]$. All returns are expressed as annual effective rates. The amount of lump-sum benefit predicted at time g to be paid out to them at time $g+2$ is calculated as

$$B^{(g)}(g) = C\,\mathbb{E}_g\left(\prod_{k=g+1}^{g+2}(1+R(k))\right),$$

for $g \in \{0, 1, 2\}$.

At the end of each of the next two time periods, the benefit predictions are adjusted to take account of the scheme experience. This is done via the benefit increase factors $\{\gamma(k); k = 1, 2, 3, 4\}$. Specifically, the factors reflect the experience of the scheme against the predictions of that experience. The factor $\gamma(k)$ is calculated by setting the asset value at time k equal to the value of the discounted future benefits at time k, for the benefits which were accrued before time k.

At time $g+1$ and $g+2$, the benefit predictions are updated to new values

$$B^{(g)}(g+1) = (1+\gamma(g+1))B^{(g)}(g) \qquad \text{and} \qquad B^{(g)}(g+2) = (1+\gamma(g+2))B^{(g)}(g+1),$$

with the final amount, $B^{(g)}(g+2)$, paid out to each member of Generation g at time $g+2$.

2.4 Evolution of the scheme benefits

It is convenient to define the notation

$$\Lambda(0) := \frac{\mathbb{E}_1\left(\prod_{k=2}^{3}(1+R(k))\right)}{1+R(1)}$$

and, for $n = 1, 2, 3$,

$$\Lambda(n) := \Lambda(n-1)\,\mathbb{E}_n\left(\frac{1}{1+R(n+1)}\right).$$

Let $A(n)$ be the asset value of the scheme at time n, with $A(0_-) = 0$ and $A_0 = C$. Thus the scheme has no assets before the first generation joins at time 0.

2.4.1 Evolution to time 1

At time 1, just before Generation 1 joins the scheme, the asset value is

$$A(1_-) = A(0)(1+R(1)) = C(1+R(1)).$$

The scheme is due to pay a lump-sum benefit at time 2 to each member of Generation 0. The discounted value at time 1 of the prediction at time 1 of the lump-sum benefit payable at time 2 to each member of Generation 0, $B^{(0)}(1) = B^{(0)}(0)\,(1+\gamma(1))$, is

$$(1+\gamma(1))\,B^{(0)}(0)\,\mathbb{E}_1\left(\frac{1}{1+R(2)}\right) = (1+\gamma(1))\,B^{(0)}(0)\,\frac{\Lambda(1)}{\Lambda(0)}.$$

The value of $\gamma(1)$ is determined by equating the asset value $A(1)$ to this latter discounted value. The result is that

$$1+\gamma(1) = \frac{\Lambda(0)}{\Lambda(1)}\frac{1+R(1)}{\mathbb{E}_0\left(\prod_{k=1}^{2}(1+R(k))\right)}$$

and hence the benefit predicted at time 1 to be paid to Generation 0 at time 2 is

$$B^{(0)}(1) = C\,\frac{\Lambda(0)}{\Lambda(1)}\,(1+R(1))\,.$$

The amount, $B^{(0)}(1)$, contains nothing of the prediction made at time 0 of future returns. The reason is that, until the next generation joins, Generation 0 is in a scheme of its own. The scheme operates like an individuals defined contribution pension scheme from time 0 until time 1. All that matters at time 1 is what investment return was actually achieved and the future predictions of the returns over the next time period.

At time 1, Generation 1 joins the scheme. They pay the same amount C to join as the previous generation and the asset value just after the payment of the contribution is $A(1) = A(1_{-}) + C$. Since

$$B^{(1)}(1) = C\,\mathbb{E}_1\left(\prod_{k=2}^{3}(1+R(k))\right) = C\,\Lambda(0)\,(1+R(1))\,.$$

2.4.2 Evolution to time 2

At time 2, Generation 0 is to retire with benefit payment $B^{(0)}(2)$ per member. The benefit adjustment $\gamma(2)$ must be calculated in order to know how much to pay out.

Equating the discounted value at time 2 of the two generation's benefits to the asset value at time 2 gives that

$$1+\gamma(2) = \frac{\prod_{k=1}^{2}(1+R(k)) + (1+R(2))}{(1+R(1))\,\frac{\Lambda(0)}{\Lambda(1)}\,(1+\Lambda(2))}.$$

Thus the amount

$$B^{(0)}(2) = C\,\frac{1}{1+\Lambda(2)}\left(\prod_{k=1}^{2}(1+R(k)) + 1 + R(2)\right)$$

is paid out to each member of Generation 0 at time 2. In the expression for $B^{(0)}(2)$, there is a reliance on the future return $R(3)$, via the term $\Lambda(2)$, which is an investment return experienced after Generation 0 has exited the scheme. The act of paying out an amount $B^{(0)}(2)$, crystallises the model risk implied by the predictions made at time 1 and time 2.

The predicted value at time 2 of the benefit payable to each member of Generation 1 at time 3 is

$$B^{(1)}(2) = C\,\frac{\Lambda(1)}{1+\Lambda(2)}\left(\prod_{k=1}^{2}(1+R(k)) + 1 + R(2)\right).$$

The predicted value of the benefits to be paid to Generation 1, depends on what happened before Generation 1 joined the scheme, namely the return $R(1)$. It is not only the observed value of $R(1)$ but also the past predictions of that value, which affect $B^{(1)}(2)$.

2.4.3 Evolution to time 3

Setting

$$\Theta := \frac{\Lambda(3)}{\Lambda(2)} \frac{1+\Lambda(2)}{\Lambda(1)} \frac{\mathbb{E}_2\left(\prod_{k=3}^{4}(1+R(k))\right)}{\prod_{k=1}^{2}(1+R(k))+1+R(2)},$$

and continuing with the algebra, the benefit paid out to the second generation, Generation 1, at time 3 is

$$B^{(1)}(3) = C\,\frac{1+R(3)+\frac{\Lambda(2)}{1+\Lambda(2)}\left(\prod_{k=1}^{3}(1+R(k))+\prod_{k=2}^{3}(1+R(k))\right)}{1+\Theta}.$$

The benefit predicted at time 3 to be paid out to Generation 2 at time 4 is

$$B^{(2)}(3) = C\,\frac{\Lambda(2)}{\Lambda(3)}\,\frac{1+R(3)+\frac{\Lambda(2)}{1+\Lambda(2)}\left(\prod_{k=1}^{3}(1+R(k))+\prod_{k=2}^{3}(1+R(k))\right)}{1+\Theta^{-1}}.$$

After benefits are paid to Generation 1, the asset value at time 3 is

$$\begin{aligned} A(3) =& A(3_-) - B^{(1)}(3) \\ =& C(1+R(3))\left(\frac{\Lambda(2)}{1+\Lambda(2)}\left(\prod_{k=1}^{2}(1+R(k))+1+R(2)\right)+1\right)\frac{\Theta}{1+\Theta}. \end{aligned}$$

2.4.4 Evolution to time 4

The benefits paid out to Generation 3 at time 4 will be the asset value remaining at time 4. As $A(4_-) = A(3)(1+R(4))$,

$$\begin{aligned} B^{(2)}(4) =& A(4_-) \\ =& C\left(\prod_{k=3}^{4}(1+R(k))\right)\left(\frac{\Lambda(2)}{1+\Lambda(2)}\left(\prod_{k=1}^{2}(1+R(k))+1+R(2)\right)+1\right)\frac{\Theta}{1+\Theta} \end{aligned}$$

and the asset value at the end of the scheme's life falls to zero, i.e. $A(4) = A(4_-) - B^{(2)}(4) = 0$.

3 Individual Defined Contribution (IDC) pension scheme

As a baseline comparator, the results of the CDC pension scheme are compared to the benefits paid out in an Individual Defined Contribution (IDC) pension scheme.

It is assumed that in the IDC scheme, the fixed contributions C are invested directly in the financial market. Their accumulated value is paid out at retirement to the IDC members. For example, the members of Generation g would, in the IDC scheme, receive

$$C \prod_{k=g+1}^{g+2} (1 + R(k)),$$

at retirement, for $g \in \{0, 1, 2\}$.

4 Model risk in the CDC scheme

For our analysis, consider a market model in which the investment returns $R(1), \ldots, R(4)$ are independent and identically distributed random variables. It is not essential to make either of these assumptions; however, they make the calculations easier.

4.1 The manager's beliefs about the future

Suppose that each investment return random variable can take one of two possible values, namely, for real numbers $r_1, r_2 \in (-1, \infty)$ with $r_1 \neq r_2$, it is assumed that, under the manager's model

$$R(k) = \begin{cases} r_1 & \text{with probability } p \\ r_2 & \text{with probability } 1 - p, \end{cases}$$

for $k = 1, 2, 3, 4$. For the discrete random variables $\{R(k)\}$, the probability p is akin to the p.d.f. f: it represents the manager's belief about the future.

Suppose the investment return values are fixed at $r_1 = +0.2$ and $r_2 = -0.2$. Without loss of generality, set the contribution paid by each member upon joining the scheme to $C = 1$. Denote the manager's model by the set $\{p, 1 - p\}$.

Three managers imply three different models

Consider three possible managers of the scheme, each with a different viewpoint of the future. Each of them believes a specified model of the future: one is a pessimistic view, the second is neutral and the third is optimistic.

Pessimistic manager

Under the pessimistic manager's model, returns of -20% are most likely and the investment return has annual mean -12% and annual standard deviation 16%. Table 4.1.1 shows statistics of the benefits paid at retirement to each of the three generations in the scheme. The results are calculated entirely under the manager's model; in other words, there is no model risk.

It is observed that the mean benefit increases over time under the pessimistic model. The calculations of the benefit under-state the expected returns, meaning that the first generations get less. The last generations gains what the first two generations did not get, albeit at the cost of a higher standard deviation of benefit.

The standard deviation of the benefit paid out increases over the generations. These results are consistent with those in Donnelly (2022a) and Døskeland and Nordahl (2008), where the same general effect is seen. The first generations – here represented by Generation 0 – have the lowest mean and standard deviation of benefits paid out at retirement. The middle generations, exemplified by Generation 1, have a mean and standard deviation of benefits which increases at a slower rate as the generation number increases. The mean and standard deviation of benefits increase sharply for the last generations who are in the scheme when it closes to new members; here, this is Generation 2. In summary, the later that a generation joins a scheme, the higher the mean and the higher the standard deviation borne by that generation.

The risk-return trade-off can be analysed by the ratio of the mean benefit paid at retirement to the standard deviation of that benefit. It is highest for the last generation. This means that the average return per unit of risk increases. However, since the amount of risk taken by each generation increases, this may mean that the later generations are taking an uncomfortably high level of risk for their risk appetite.

The results are compared to the benefits paid out in the IDC pension scheme, described in Section 3, in which contributions are invested exactly the same way as in the CDC scheme. The first two generations in the CDC scheme have a slightly lower average benefit paid out, and a slightly lower standard deviation, compared to the IDC scheme. These lower average payments work to the expected advantage of the last generation, who gets a higher average pension than in the IDC scheme with a standard deviation that is the same in both schemes for the last generation.

Neutral manager

Under the neutral manager, returns of $+20\%$ are as likely as returns of -20% and the investment return has annual mean 0% and annual standard deviation 20%. The results in Table 4.1.1 show that the relative rankings of the mean and standard deviation of the retirement benefit are the same as in the pessimistic manager's model. Again, the results are calculated under the neutral manager's model so that there is not model risk. The comparison with the IDC scheme gives similar conclusions too.

The outcomes between the first and last generations are the most extreme under

Manager's model $\{p, 1-p\}$	Generation	CDC Mean	CDC Std Dev	CDC Mean /Std Dev	IDC Mean	IDC Std Dev	IDC Mean /Std Dev
Pessimistic	0	0.753	0.197	3.829	0.774	0.201	3.857
$\{0.2, 0.8\}$	1	0.772	0.202	3.829	0.774	0.201	3.857
	2	0.794	0.206	3.857	0.774	0.201	3.857
Neutral	0	0.960	0.276	3.466	1.000	0.286	3.500
$\{0.5, 0.5\}$	1	0.999	0.288	3.465	1.000	0.286	3.500
	2	1.041	0.298	3.500	1.000	0.286	3.500
Optimistic	0	1.223	0.250	4.896	1.254	0.255	4.925
$\{0.8, 0.2\}$	1	1.258	0.257	4.896	1.254	0.255	4.925
	2	1.290	0.262	4.924	1.254	0.255	4.925

Table 4.1.1: Statistics of the benefits, $\{B^{(g)}(g+2); g = 0, 1, 2\}$, paid out under each of the manager's models for the CDC scheme, when the true model is the manager's model. Similar statistics are shown for an individual defined contribution (IDC) pension scheme.

the neutral manager's model. The last generation's mean benefit is over 8% of the first generation's mean benefit payment. This ratio is about 5.5% under both the pessimistic and optimistic manager. A similar magnitude of difference can be calculated for the standard deviation. When the results are re-run under model risk, it is observed that the largest differences in the mean and standard deviation between the first and last generations continues to be under the neutral manager's model.

Optimistic manager

Under the optimistic manager, returns of +20% are most likely and the investment return has annual mean +12% and annual standard deviation 16%. In this case, the results are almost the same as for the previous managers, but with Generation 1 – the middle generation – getting a slightly higher mean benefit and higher standard deviation of that benefit, than in an IDC scheme.

4.2 Results under model risk

Now introduce model risk into the operation of the CDC scheme. Fix a probability $\tilde{p} \in [0, 1]$. Suppose that the true model of the evolution of future returns is

$$R(k) = \begin{cases} r_1 & \text{with probability } \tilde{p} \\ r_2 & \text{with probability } 1 - \tilde{p}, \end{cases}$$

for $k = 1, 2, 3, 4$. The probabilities are the representation of the true model of the future, which was described earlier via a p.d.f. $\tilde{f}$. Returns are sampled from the true model $\{\tilde{p}, 1 - \tilde{p}\}$.

Each manager is unaware of the true model and uses their own model, as represented by the specification of the value of p, to value future benefits. For example, each manager would calculate at time 2 the return expected from time 2 to time 3 on the basis of their model, i.e. $\mathbb{E}_2(1 + R(3))$, rather than the true model $\tilde{\mathbb{E}}_2(1 + R(3))$. This means that the expected outcomes, as the manager expects them to occur, are calculated using p.

However, the evolution of the benefit outcomes as they truly occur, are calculated using $\tilde{p}$. The true evolution will still be affected by the manager's beliefs about the future, since the manager's beliefs shape how much is paid out to each generation.

4.3 Outcomes under model risk

Suppose that the three managers - the pessimistic, neutral and optimistic ones - run identical schemes. It has been shown above that, in the absence of model risk, the outcomes are different for the schemes' members. However, this is not surprising - with no model risk, each scheme experiences a different future to the other ones. Their future was individually correctly predicted by the model; at least, in terms of the probabilistic model of the future.

Assume that each scheme experiences the same future. Suppose that the future is represented by the pessimistic model: this is the true model. The neutral and optimistic managers will continue to run their schemes as if their own probabilities of the future returns holds true. Meanwhile, the actual returns are drawn from the pessimistic model. What happens to the members of each scheme? They experience exactly the same investment returns, but the managers' viewpoints of the future will change the level of benefits paid out. Once different amounts of benefits are paid out, the outcomes from each scheme will differ.

When the pessimistic model is the true model, both the pessimistic manager and optimistic manager's views give identical outcomes (Table 4.7.1). This is due to the symmetry in the model assumed by the managers.

In contrast, the neutral manager's view skew the outcomes: the first generation has the lowest mean and standard deviation of the retirement benefit over all the managers' schemes. The last generation in the neutral manager's scheme is the reverse: it has the highest mean and standard deviation over all the managers' schemes.

Varying the true model, this observation about the neutral manager's scheme giving the most extreme outcomes continues to hold. It is observed that the schemes run by the pessimistic manager and optimistic manager have a more balanced outcome for the generations: there is a smaller difference in the mean and standard deviation between the three generations in their schemes (Tables 4.7.2-4.7.3).

4.4 Entropy as an explanation

Continuing the assumption that the scheme manager does not change their assessment of the future investment returns over time, define for notational simplicity

$$\nu := \mathbb{E}_n\left(1 + R(m)\right) = p(1 + r_1) + (1 - p)(1 + r_2)$$

and

$$\beta := \mathbb{E}_n\left(\frac{1}{1 + R(m)}\right) = p\frac{1}{1 + r_1} + (1 - p)\frac{1}{1 + r_2}.$$

4.5 First CDC generation gets less than in an IDC scheme

After some algebra, the benefit paid to the first generation can be expressed as

$$B^{(0)}(2) = \prod_{k=1}^{2}(1 + R(k))\frac{1 + R(1) + 1}{1 + R(1) + \nu^2\beta^2}.$$

Note the following.

- If $p = 0$ or $p = 1$ then the first generation gets the same as in an IDC scheme, regardless of the interest rate used in the manager's model. For these two values of p, the manager's model uses a single rate for discounting and accumulating future cashflows. In that case, the product of the annual discount and accumulation factors, $\nu\beta = 1$, and $B^{(0)}(2) = \prod_{k=1}^{2}(1 + R(k))$, i.e. the same benefit amount as accumulating their unit contribution at the actual investment return.

- More generally, the first generation is paid less than in an IDC scheme, in all future states of the world. The product of the annual discount and accumulation factors can be expressed as

$$\nu\beta = 1 + p(1 - p)\frac{(r_1 - r_2)^2}{(1 + r_1)(1 + r_2)}$$

Since $r_1, r_2 \in (-1, \infty)$ and $r_1 \neq r_2$, the coefficient of $p(1-p)$ is positive. Thus

$$\nu\beta \geq 1,$$

with equality occurring only at either $p = 0$ or $p = 1$. Thus, restricting to $p \in (0, 1)$,

$$B^{(0)}(2) = \prod_{k=1}^{2}(1 + R(k))\frac{1 + R(1) + 1}{1 + R(1) + \nu^2\beta^2} < \prod_{k=1}^{2}(1 + R(k)).$$

Thus the first generation is paid less than in an IDC scheme, in all future states of the world, when the manager uses a stochastic model for future investment returns.

- The first generation has the highest payment at retirement when the manager's model has the highest entropy, i.e. at $p = 0.5$. By differentiation, it is shown that the product $\nu\beta$ is maximised at $p = 0.5$. It is minimised when either $p = 0$ or $p = 1$.

4.6 Middle CDC generation gets close to an IDC scheme

Defining

$$\lambda := \frac{1 + R(1) + 1}{1 + R(1) + \nu^2\beta^2}, \quad \text{it is found that} \quad B^{(1)}(3) = \prod_{k=2}^{3}(1+R(k))\,\lambda\,\frac{1 + (1 + R(2))\lambda\nu^2\beta^2}{1 + (1 + R(2))\lambda},$$

The benefit paid to the middle (second) generation may or may not be less than the benefit paid from an IDC scheme. The latter amount is $\prod_{k=2}^{3}(1 + R(k))$. This can be observed from a comparison of the benefits paid in the CDC scheme versus the IDC scheme in Tables **??**-

4.7 Last CDC generation gets more than in an IDC scheme

$$\eta := (1+R(2))\frac{1 + R(1) + 1}{1 + R(1) + \nu^2\beta^2}, \quad \text{it is found that} \quad B^{(2)}(4) = \prod_{k=3}^{4}(1+R(k))\,\lambda\,\frac{1 + \eta\nu^2\beta^2}{1 + \eta}.$$

Now

The neutral manager's model is the maximum entropy probability distribution in the considered class of models for the investment returns, since

$$0.5 = \underset{p\in[0,1]}{\arg\max}\left\{-p\ln(p) - (1-p)\ln(1-p)\right\}.$$

Manager's model $\{p, 1-p\}$	Generation	CDC Mean	CDC Std Dev	CDC Mean /Std Dev	IDC Mean	IDC Std Dev	IDC Mean /Std Dev
Pessimistic	0	0.753	0.197	3.829	0.774	0.201	3.857
$\{\mathbf{0.2}, \mathbf{0.8}\}$	1	0.772	0.202	3.829	0.774	0.201	3.857
	2	0.794	0.206	3.857	0.774	0.201	3.857
Neutral	0	0.741	0.194	3.813	0.774	0.201	3.857
$\{0.5, 0.5\}$	1	0.770	0.202	3.813	0.774	0.201	3.857
	2	0.804	0.209	3.857	0.774	0.201	3.857
Optimistic	0	0.753	0.197	3.829	0.774	0.201	3.857
$\{0.8, 0.2\}$	1	0.772	0.202	3.829	0.774	0.201	3.857
	2	0.794	0.206	3.857	0.774	0.201	3.857

Table 4.7.1: Statistics of the benefits, $\{B^{(g)}(g+2); g = 0, 1, 2\}$, paid out under each of the manager's models, when the true model is the pessimistic model. Similar statistics are shown for an IDC pension scheme.

The numerical results suggest that the more uncertainty in the manager's model, the greater is difference in the outcomes between the first and last generations. This reason comes from the balance of expected accumulation and discounting factors, with the value of their product maximised when $p_1 = 0.5$.

This is contrary to what might be alternative hypothesis: that the optimistic manager would pay out the most to the earlier generations, under the belief that investment returns are more likely to be +20% per annum. They believe that these higher returns would allow the later generations to receive a similar level of payment to the first generations.

However, this misses the complexity of the CDC scheme evolution. There is a trade-off between accumulation factors (arising when the accrued benefit is first calculated) and discount factors (arising when the assets are compared to the discounted benefits). The product of the annual expected accumulation factor and the annual discount factor is maximised when $p_1 = 0.5$. It is this product which affects the benefits paid out rather than only the expected accumulation factor

In summary, the scheme manager's view of the future will change the balance of payments between each generation. The higher the entropy in the manager's model, the larger the difference in the payments between generations.

Manager's model $\{p, 1-p\}$	Generation	CDC Mean	CDC Std Dev	CDC Mean /Std Dev	IDC Mean	IDC Std Dev	IDC Mean /Std Dev
Pessimistic	0	0.974	0.280	3.478	1.000	0.286	3.500
$\{0.2, 0.8\}$	1	1.000	0.287	3.478	1.000	0.286	3.500
	2	1.026	0.293	3.501	1.000	0.286	3.500
Neutral	0	0.960	0.276	3.466	1.000	0.286	3.500
$\{\mathbf{0.5}, \mathbf{0.5}\}$	1	0.999	0.288	3.465	1.000	0.286	3.500
	2	1.041	0.298	3.500	1.000	0.286	3.500
Optimistic	0	0.974	0.280	3.478	1.000	0.286	3.500
$\{0.8, 0.2\}$	1	1.000	0.287	3.478	1.000	0.286	3.500
	2	1.026	0.293	3.500	1.000	0.286	3.500

Table 4.7.2: Statistics of the benefits, $\{B^{(g)}(g+2); g = 0, 1, 2\}$, paid out under each of the manager's models, when the true model is the neutral model. Similar statistics are shown for an IDC pension scheme.

Manager's model $\{p, 1-p\}$	Generation	CDC Mean	CDC Std Dev	CDC Mean /Std Dev	IDC Mean	IDC Std Dev	IDC Mean /Std Dev
Pessimistic	0	1.223	0.250	4.896	1.254	0.255	4.925
$\{0.2, 0.8\}$	1	1.258	0.257	4.896	1.254	0.255	4.925
	2	1.290	0.262	4.924	1.254	0.255	4.925
Neutral	0	1.206	0.247	4.881	1.254	0.255	4.925
$\{0.5, 0.5\}$	1	1.259	0.258	4.880	1.254	0.255	4.925
	2	1.309	0.266	4.921	1.254	0.255	4.925
Optimistic	0	1.223	0.250	4.896	1.254	0.255	4.925
$\{\mathbf{0.8}, \mathbf{0.2}\}$	1	1.258	0.257	4.896	1.254	0.255	4.925
	2	1.290	0.262	4.924	1.254	0.255	4.925

Table 4.7.3: Statistics of the benefits, $\{B^{(g)}(g+2); g = 0, 1, 2\}$, paid out under each of the manager's models, when the true model is the optimistic model. Similar statistics are shown for an IDC pension scheme.

5 Discussion and conclusion

The manager's view of the future affects the outcomes for members of a CDC scheme. Three different managers were studied, and each resulted in different benefit payments to scheme members. This is in spite of the scheme having an identical structure under all three managers and the scheme's assets earning the same (percentage) investment returns. Moreover, the manager's view affects the balance of payments between generations. This is an important result.

Of the three types of investment risk-sharing between generations which were identified in Donnelly (2022a), one applies here. The two other investment risk-sharing types do not apply here, since they relied on actuarially-unfair initial benefits and a different approach to sharing investment gains across the generations.

The type of investment risk-sharing in the CDC scheme in this paper is the ubiquitous variety, found in all investment risk-sharing schemes. In this variety, the cumulative notional gains or losses of earlier generations, which arise from the difference between what is expected to happen and what actually happens, are borne by later generations. This is seen clearly in our results. The lower the payment to earlier generations, the higher the payment to later generations, and vice versa. The final generation, who has no future generation with whom to share their bear the ultimate consequences of the distribution of the payments between generations.

The study in this paper shows that the manager's model adds a further dimension to this type of risk-sharing. A different choice of model by the manager, i.e. a different view of what might happen in the future, changes the balance of payments between generations. The results suggest in the model considered here, that the higher the entropy in the manager's model, the greater the difference between the payments to different generations.

The conclusion is that the choice of model has important consequences. It changes which generation gets what. A possible outcome may be for the scheme to set a specific objective. The scheme manager should choose a model of the future which optimises the objective. In more mathematical terminology, the model of the future becomes a control variable itself.

To manipulate the risk-return trade-off across generations, this paper has shown that the manager's model is one tool. Another possible tool is the investment strategy. However, to increase the risk of earlier generations will only increase the risk of later generations.

First consider the return-risk trade-off for each generation, where the return is the mean benefit paid out at retirement and the risk is the standard deviation. In the models presented here, the IDC scheme offers outcomes with a risk-return trade-off across all generations. The CDC scheme changes the balance of payments between generations, and hence the generations in a scheme have a different risk-return trade-off compared to each other. To have a risk-return trade-off for each CDC scheme member which is constant, would arise only when the CDC scheme is no longer a collective scheme but an individual one.

It is observed in studies of other investment risk-sharing schemes, that the risk-return trade-off increases over time (Døskeland and Nordahl, 2008; Donnelly, 2022a). This is an unavoidable consequence of investment risk-sharing across generations. The fact that the amount of payments made to earlier generations affects the payments to later generations will increase the risk for those later generations, as it increases the range of possible payments made to later generations.

There are three distinct phases in the development of the risk-return trade-off across generations (Døskeland and Nordahl, 2008; Donnelly, 2022a). In the starting phase, before the scheme membership is growing, the risk and return increase. The risk and return for the first generations is lower than in a comparable IDC scheme. In the middle phase, when the scheme membership is stable with new entrants balancing exits, the risk and return increase but at a slower pace. In the end phase, the scheme membership shrinks as there are no new entrants, and the risk and return increase sharply. Risk-sharing schemes tend to hold back money from the first generations, as is also seen in our paper, and the amount held back is, on average, paid out to the last generations.

In all the risk-sharing schemes considered here and in (Døskeland and Nordahl, 2008; Donnelly, 2022a), the investment risk borne by each generation is the same; they follow the same investment strategy. However, as discussed above, the volatility of their benefit payments are not the same, and they increase over the generations.

Suppose it is desired to increase the risk borne by the first generations by increasing their investment risk. Then later generations would have a higher volatility of their benefit payments which is attributed to investment risk. Consequently, to reduce the volatility of later generation's benefit payments would mean that the later generations take less investment risk. The downside of a lower risk investment strategy followed by the later generations is that, if the returns experienced by the earlier generations are less than expected, the later generations will be unlikely to compensate for those earlier losses. The upside is that if the returns experienced by the earlier generations are better than expected, the later generations are less likely to lose these earlier due to experiencing severe losses themselves. The later generations would experience a lower risk but at the cost of a lower return.

At a high level, what does taking less investment risk mean for the later generations? It means that the later generations are getting a greater exposure to past investment returns than to current investment returns. This means that the later geneations' benefit payments will reflect past returns to a greater extent than current returns.

One way to mitigate the risk for not taking enough risk, is to follow a dynamic investment strategy. This means tailoring the investment strategy so that its assessed risk is at the desired level. However, a dynamic strategy requires an investment model, which means that model risk materialises in the investment strategy. As is shown in this paper, model risk affects the balance of payments between generations, independently of the investment strategy. The conclusion that, if it is desired to control the risk of each generation, both the investment strategy and the model should be considered as control variables. That aside, it is seems unlikely that controlling the investment risk can result in both a control of the risk experienced by each generation and a high enough return to make the CDC scheme an attractive scheme to join under all market conditions. The point of the CDC scheme is to take investment risk and try to balance the poor investment returns at one point in time with better investment returns at other times.

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