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Citation for published version:
Lacey, AA, Kavallaris, N, Nikolopoulos, C & Tzanetis, D 2015, 'On the Quenching Behaviour of a Semilinear Wave Equation Modelling MEMS Technology', *Discrete and Continuous Dynamical Systems-Series A*, vol. 35, no. 3, 13, pp. 1009-1037. https://doi.org/10.3934/dcds.2015.35.1009

Digital Object Identifier (DOI):
10.3934/dcds.2015.35.1009

Published In:
Discrete and Continuous Dynamical Systems-Series A

Submitted to
Discrete and Continuous Dynamical Systems-Series A (DCDS-A)

ON THE QUENCHING BEHAVIOUR OF A SEMILINEAR WAVE EQUATION MODELLING MEMS TECHNOLOGY

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Abstract. In this work we study the semilinear wave equation of the form

$$u_{tt} = u_{xx} + \lambda/(1-u)^2,$$

with homogeneous Dirichlet boundary conditions and suitable initial conditions, which, under appropriate circumstances, serves as a model of an idealized electrostatically actuated MEMS device. First we establish local existence of the solutions of the problem for any $\lambda > 0$. Then we focus on the singular behaviour of the solution, which occurs through finite-time quenching, i.e. when $||u(\cdot,t)||_\infty \to 1$ as $t \to t^*- < \infty$, investigating both conditions for quenching and the quenching profile of u. To this end, the non-existence of a regular similarity solution near a quenching point is first shown and then a formal asymptotic expansion is used to determine the local form of the solution. Finally, using a finite difference scheme, we solve the problem numerically, illustrating the preceding results.

1. Introduction

The main purpose of this work is to study the singular behaviour of the hyperbolic problem

$$u_{tt} = u_{xx} + \frac{\lambda}{(1-u)^2}, \quad 0 < x < 1, \quad t > 0, \tag{1.1a}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x) < 1, \quad u_t(x,0) = u_1(x), \quad 0 < x < 1\,, \tag{1.1c}$$

where λ is a positive parameter. Problem (1.1) can model the deformation of an elastic membrane inside an idealized electrostatically actuated *MEMS.*

AMS subject classifications: Primary 35K55, 35J60; Secondary 74H35, 74G55, 74K15.
Keywords: Electrostatic MEMS, Quenching of Solution, Hyperbolic Problems.

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"MEMS" stands for micro electro-mechanical systems, and refers to precision devices which combine mechanical processes with electrical circuits. MEMS devices range in size from millimetres down to microns, and involve precision mechanical components that can be constructed using semiconductor manufacturing technologies. The devices are widely applied as sensors and have fluid mechanical, optical, radio frequency (RF), data storage, and biotechnology applications. In particular, examples of microdevices of this kind include microphones, temperature sensors, RF switches, resonators, accelerometers, data-storage devices etc., [7, 37, 41].

The key part of such a MEMS device usually consists of an elastic plate suspended above a rigid ground plate. In the simplest geometry, the elastic plate (or membrane) is rectangular and held fixed at two ends while the other two edges remain free to move. An alternative configuration could entail the plate or membrane (no longer necessarily rectangular) being held fixed around its entire edge. When a potential difference V_d is applied between the membrane and the plate, the membrane deflects towards the ground plate. Under the realistic assumption that the width of the gap, between the membrane and the bottom plate, is small compared to the device length, then the deformation of the elastic membrane is described by a dimensionless equation of the form

$$\epsilon^2 u_{tt} + u_t - \Delta u = \frac{\lambda\, f_d(x,t)}{(1-u)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega, \tag{1.2a}$$

$$u(x,0) = u_0(x) < 1, \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2b}$$

where $u = u(x,t)$ stands for the (dimensionless) deflection of the membrane,

$$\epsilon^2 = \frac{\text{inertial terms}}{\text{damping terms}} \quad \text{and} \quad \lambda = \frac{V_d^2 L_c^2 \varepsilon_0}{2T_m l_c^2} \propto V_d^2.$$

Here $u_0(x)$ and $u_1(x)$ represent the initial deflection and velocity, respectively, of the elastic membrane. The function $f_d(x,t)$ describes the varying dielectric properties of the membrane; for simplicity we assume here that $f_d(x,t) \equiv 1$. Furthermore, T_m stands for the tension in the membrane, L_c is the width of the parallel plates, each of them denoted by Ω, l_c is the unperturbed width of the gap between the membrane and the ground electrode, and ε_0 is the permittivity of free space. The boundary condition represents the membrane being kept in its unperturbed position along its edge.

When the damping terms dominate, i.e. when $\epsilon^2 \ll 1$, then (1.2) reduces to the parabolic problem

$$u_t - \Delta u = \frac{\lambda}{(1-u)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega,$$
$$u(x,0) = u_0(x) < 1, \quad x \in \Omega,$$

which has been extensively studied in [7, 10, 13, 21].

On the other hand, when the contribution of the inertial terms dominates, i.e. $\epsilon^2 \gg 1$, we derive, after rescaling, the model

$$u_{tt} - \Delta u = \frac{\lambda}{(1-u)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega, \tag{1.3a}$$

$$u(x,0) = u_0(x) < 1, \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.3b}$$

In general, the two parallel plates are of arbitrary shape. However, if the parallel plates are thin and narrow homogeneous strips of fixed width L_c, see [1, 34], then, with suitable scaling, (1.3) can be reduced to the one-dimensional model (1.1). The one-dimensional problem can also be used as a simple model to get better insight into the operation of devices with more general geometries, and especially for the two-dimensional radially symmetric case, i.e. when Ω is a disk, which will be investigated in a forthcoming paper. For certain MEMS-type devices, e.g. resonators and some devices with applications in data storage and optical engineering, [14, 39, 40, 41], the rectangular geometry is practical and it is in fact used. The investigation of the one-dimensional model (1.1) is thus of importance in its own right.

When the MEMS device is connected in series with a voltage and a fixed capacitor one can derive a non-local model of the form

$$\epsilon^2 u_{tt} + u_t - \Delta u = \frac{\lambda}{(1-u)^2\left(1+\gamma\int_0^1 \frac{1}{1-u}dx\right)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega, \tag{1.4a}$$

$$u(x,0) = u_0(x) < 1, \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.4b}$$

where the parameter γ represents the ratio of a fixed capacitance to a reference capacitance, see [10]. Model (1.4), depending on the contribution of the inertial and damping terms, gives rise to the non-local parabolic problem

$$u_t - \Delta u = \frac{\lambda}{(1-u)^2\left(1+\gamma\int_0^1 \frac{1}{1-u}dx\right)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega,$$
$$u(x,0) = u_0(x) < 1, \quad x \in \Omega,$$

which has been studied in [16, 17, 19, 35], or to the hyperbolic non-local model

$$u_{tt} - \Delta u = \frac{\lambda}{(1-u)^2\left(1+\gamma\int_0^1 \frac{1}{1-u}dx\right)^2}, \quad x \in \Omega \subset \mathbb{R}^2, \quad u = 0, \quad x \in \partial\Omega,$$
$$u(x,0) = u_0(x) < 1, \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

whose behaviour for the one-dimensional case was investigated in [22].

Recently some authors initiated the investigation of fourth-order models, using the bi-laplacian operator which models the moving part of a MEMS device as an elastic plate (with non-zero thickness), rather than as a simple, thin, membrane, [18, 25]. There are also papers investigating the quenching behaviour for fourth order parabolic equations, [24, 32]. Some recent works investigate the wave equation with damping, [9, 30, 31].

For a more detailed account of the modelling of MEMS devices, see the books [7, 37, 41].

From the above, it is clear that the applied voltage V_d controls the operation of the MEMS device. Indeed, when V_d takes values above a critical threshold V_{cr}, called the *pull-in voltage*, this can lead to the phenomenon of *touch-down* (or *pull-in instability* as it is also known in MEMS literature) when the elastic membrane touches the rigid ground plate, possibly causing destruction of the device in some applications. (The designers of such MEMS devices consequently need to tune the voltage load so that stays away from the pull-in voltage.) Equivalently, this means that there should be some critical value λ_{cr}, depending upon the initial data, of the parameter λ above which singular behaviour should be expected for the solution of problem (1.1). Looking at the nonlinear term of problem (1.1), one can notice that singular behaviour is possible only when u takes the value 1, a phenomenon known in the literature as *quenching*, see also Section 4. From the point of view of applications it is important to determine whether quenching occurs and, if it does, to clarify when, how and where it might happen. We address two of these questions in this manuscript.

Many authors have investigated the occurrence of quenching for the hyperbolic problem (1.1), [5, 6, 20, 29, 38]. However, to the best of our knowledge, the behaviour close to quenching, i.e. the quenching profile, has not been studied previously. In the current work, we first prove some quenching results for problem (1.1) which improve some of the results in [5] for the one-dimensional case and in [29, 38] for higher dimensions. Although we mainly focus on the one-dimensional case, our quenching results can be easily extended to higher dimensions as we note in the text.

The outline of the current work is as follows. In Section 2 the local and global existence of solutions to problem (1.1) are studied, while the steady problem is briefly looked at in Section 3. Then, in Section 4, we establish some conditions under which the solution u of (1.1) quenches in finite time, see Theorems 4.2, 4.4 and 4.6. Section 5 is devoted to the investigation of the question of existence of a regular self-similar quenching solution and we finally give a negative answer, see Theorem 5.1. This result is rather surprising since, in standard semilinear wave equations with nonlinearities leading to blow-up, the local behaviour close to blow-up is usually of self-similar type, see [2, 3, 11, 33]. The result is also in contrast to the existence of a self-similar quenching profile for the corresponding parabolic problem, [10]. In Section 6 we go on to use an asymptotic expansion to obtain the local form of the quenching profile as const.$\times(x-\text{quenching point})^{\frac{4}{3}}$. Finally, in Section 7, a moving mesh adaptive method is used to obtain a numerical solution of the problem, corroborating the results regarding the quenching profile. We close the paper with a short discussion of our results.

2. Local and Global Existence

In this section we establish local existence of problem (1.1) where $u_0, u_1 \in C^1([0,1])$ and satisfy the compatibility conditions $u_0(0) = u_0(1) = 0$.

Definition 2.1. *We say that u is a weak solution of (1.1) in $Q_T \equiv (0,1)\times(0,T)$, for some $T>0$, if:*

- *(i) u is continuous in $\bar{Q}_T$ and satisfies the initial and boundary conditions there,*
- *(ii) $u \le 1-\delta$ in $\bar{Q}_T$, for some $\delta > 0$ and for $x \in (0,1)$,*
- *(iii) u has weak derivatives, $u_x, u_t \in L^2(\bar{Q}_T)$ and for all $t \in (0,T)$, $u_x, u_t \in L^2([0,1])$,*
- *(iv) for any function $\zeta(x,t) \in C^2(\bar{Q}_T)$ satisfying the boundary conditions (1.1b) and for $0 \le t \le T$, the following equality holds:*

$$\begin{aligned}\int_0^1 \zeta(x,t)u_t(x,t)dx &= \int_0^t\int_0^1 [\zeta_\tau(x,\tau)u_\tau(x,\tau) - \zeta_x(x,\tau)u_x(x,\tau)]dxd\tau \\ &+ \lambda\int_0^t\int_0^1 \frac{\zeta(x,\tau)dxd\tau}{(1-u(x,\tau))^2},\end{aligned} \tag{2.1}$$

  where $\zeta(x,0)=0$.

By Sobolev's and Poincaré's inequalities,

$$||u||_\infty \le C\,||u_x||_2, \quad C>0, \tag{2.2}$$

where C depends only on the interval $(0,1)$, and we get that a weak solution of (1.1) is actually a $C^{1,1}_{x,t}-$solution. Under the assumptions $u_0 \in C^2((0,1))$ and $u_1 \in C^1([0,1])$ we obtain, via D'Alembert's formula, that $u(x,t)$ is a regular $C^{2,2}_{x,t}-$solution to (1.1) except (possibly) on the set

$$\{(x,t) \in (0,1) \times [0,T] \,|\, x-t \text{ or } x+t \text{ is an integer}\},$$

see also [5].

Moreover, the total energy of any weak solution of (1.1) is preserved, i.e.

$$E_T(t) = \frac{1}{2}\int_0^1 (u_x^2 + u_t^2)\,dx + \int_0^1 \frac{\lambda}{1-u}\,dx = E_T(0) := E_0, \quad 0<t<T. \tag{2.3}$$

Regarding local existence of problem (1.1) we have:

Theorem 2.2. [5] *: For any $\lambda > 0$, if the initial data $u_0(x)$ and $u_1(x) \in C^1([0,1])$ satisfy the condition*

$$||u_0||_\infty + T||u_1||_\infty < 1 - 2\delta, \quad \textit{for some positive } \delta > 0, \tag{2.4}$$

with T sufficiently small, then problem (1.1) has a unique weak $C^{1,1}_{x,t}-$solution on Q_T. Furthermore, the solution can be extended to any interval of the form $[0, T+\tau]$ for τ sufficiently small and positive as long as $|u| < 1$ on $\bar{Q}_T$.

We notice that Theorem 2.2 implies that the solution of problem (1.1) ceases to exist by quenching. For local existence results in the higher dimensions $N = 2,3$, see [29, 38]. For a different existence proof see [30].

For small initial data and for $0 < \lambda < \lambda^*_{R_0} \le \lambda^*$, the following global existence result is available.

Theorem 2.3. [5, 29] *: If the initial data $u_0(x)$ and $u_1(x)$ satisfy the condition*

$$||u_0||^2_{H^1_0((0,1))} + ||u_1||^2_{L^2((0,1))} < R_0, \quad \textit{for some small enough positive } R_0 > 0,$$

then there exists $\lambda^*_-(R_0) \le \lambda^*$, where λ^* is the critical parameter for the steady problem (3.2), see below, so that for $0 < \lambda < \lambda^*_-(R_0)$ problem (1.1) has a global-in-time solution, i.e. there exists a constant $K = K(\lambda, R_0) < 1$ such that $||u(\cdot,t)||_\infty \le K$ for any $t \ge 0$.

Remark 2.4. *Athough it is conjectured that $\sup_{R_0} \lambda^*_-(R_0) = \lambda^*$ there is still no proof of this equality.*

Remark 2.5. *By the definition of λ_{cr} in the Introduction, if $||u_0||^2_{H^1_0((0,1))} + ||u_1||^2_{L^2((0,1))} < R_0$, then $\lambda_{cr} \ge \lambda^*_-(R_0)$.*

3. The Steady-State Problem

The steady-state problem of (1.1) is

$$w'' + \frac{\lambda}{(1-w)^2} = 0\,, \quad 0<x<1\,, \qquad w(0) = w(1) = 0\,, \;\; 0<w<1. \tag{3.1}$$

For the steady problem it is known that there exists a critical value λ^* such that problem (3.1) has exactly two solutions (the minimal solution $\underline{w}$ and the maximal one $\overline{w}$) for any $\lambda < \lambda^*$, moreover, there is a unique solution $0 < w^* < 1$ for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$ (see [12, 27]).

We can actually calculate the critical value λ^*. If we set $W = 1 - w$ then (3.1) becomes

$$W'' = \lambda/W^2\,, \quad 0<x<1\,, \qquad W(0) = W(1) = 1\,. \tag{3.2}$$

Multiplying both sides of (3.1) by W' and integrating from $m=\min\{W(x),\ x\in[0,1]\}=W(1/2)$ to $W(x)$, we derive

$$\int_0^{W'} W'dW' = \int_{\frac12}^{x} W''W'dx = \lambda\int_{\frac12}^{x}\frac{W'}{W^2}\,dx = \lambda\int_m^W \frac{dW}{W^2}\,.$$

Hence

$$\frac12 (W')^2 = \lambda\left(\frac1m - \frac1W\right)\,.$$

This gives, equivalently,

$$\frac{dx}{dW} = \sqrt{\frac{m}{2\lambda}}\sqrt{\frac{W}{W-m}}\,,$$

which implies

$$x-\tfrac12 = \sqrt{\frac{m}{2\lambda}}\left[\sqrt{W(W-m)} - \frac12 m\ln(m) + m\ln\left(\sqrt{W}+\sqrt{W-m}\right)\right]\,.$$

The latter yields, on setting $x=1$ so that $W=1$,

$$\lambda = 2m\left[\sqrt{1-m} - \frac12 m\ln(m) + m\ln\left(1+\sqrt{1-m}\right)\right]^2\,.$$

By the above relation we conclude that the maximum of $m=m(\lambda)$ is attained for $\lambda=\lambda^*\approx 1.4$, see Figure 1.
The computation of the value λ^*, with a different way, is also given in [10].

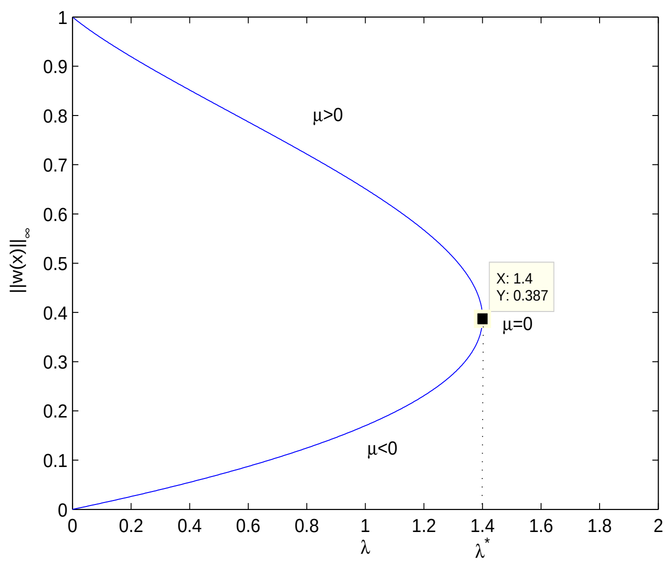

FIGURE 1. *Bifurcation diagram for problem (3.1). Here μ is the parameter of the linearized problem (4.2).*

4. Finite-Time Quenching

By Theorem 2.2 we derive that the solution of (1.1) ceases to exist only when u reaches the value 1 at some point $(x,t) \in [0,1] \times (0,\infty]$, i.e. for finite or infinity quenching time. This phenomenon is usually called quenching or touch-down (or pull-in instability) since it corresponds to the situation where the elastic membrane touches down on the rigid plate in the MEMS device. Rather more mathematical discussions of this phenomenon can be found in Sections 5 and 6.

Definition 4.1. *The solution $u(x,t)$ of problem (1.1) quenches at some point $x^* \in [0,1]$ in finite time $0 < t^* < \infty$ if there exists sequences $\{x_n\}_{n=1}^{\infty} \in (0,1)$ and $\{t_n\}_{n=1}^{\infty} \in (0,\infty)$ with $x_n \to x^*$ and $t_n \to t^*$ as $n \to \infty$ such that $u(x_n, t_n) \to 1-$ as $n \to \infty$. In the case where $t^* = \infty$ we say that $u(x,t)$ quenches in infinite time at x^*.*

We now present two results regarding the finite-time quenching of solution $u(x,t)$ of (1.1). The first one proves that finite-time quenching occurs when the parameter λ is too big for steady-state solutions to exist. This resembles a result valid for the corresponding parabolic problem, [13, 21]. The occurrence of quenching for $\lambda > \lambda^*$ resembles also the results obtained in [18, 25] for the fourth order wave equation.

Theorem 4.2. *If $\lambda > \lambda^*$ the solution $u(x,t)$ of problem* (1.1) *quenches in finite time.*

Proof. For the proof we use the spectral method developed in [23]. We assume that for $\lambda > \lambda^*$ problem (1.1) has a solution for $0 < t < T_{max} \leq \infty$, i.e.

$$u(x,t) < 1 \quad \text{almost everywhere in} \quad (0,1) \quad \text{for any } 0 < t < T_{max}. \tag{4.1}$$

We first provide some results for the associated linearized eigenvalue problem

$$\phi'' + \frac{2\,\lambda}{(1-w)^3}\phi = \mu\phi, \quad 0 < x < 1, \;\; \phi(0) = \phi(1) = 0. \tag{4.2}$$

Set $\mu_1 = \mu_1(\lambda; w)$, the principal eigenvalue of problem (4.2), then $\mu_1(\lambda; \overline{w}) > 0$ and $\mu_1(\lambda; \underline{w}) < 0$ for any $0 < \lambda < \lambda^*$, [12], which indicates the stability of $\underline{w}$ and the instability of $\overline{w}$, see also Figure 1. Moreover, $\mu_1 \to \mu_1^* = \mu_1(\lambda^*; w^*) = 0$ as $\lambda \to \lambda^*-$, as stated in Theorem 1.3 of [12]. Let ϕ^* be the eigenfunction corresponding to the eigenvalue $\mu_1^* = 0$, taken to be strictly positive, [12], and normalized so that

$$\int_0^1 \phi^* dx = 1, \tag{4.3}$$

i.e. ϕ^* satisfies

$$\phi^{*\prime\prime} + \frac{2\,\lambda^*}{(1-w^*)^3}\,\phi^* = 0, \quad 0 < x < 1, \quad \phi^*(0) = \phi^*(1) = 0. \tag{4.4}$$

For $\lambda > \lambda^*$, set $u(x,t,;\lambda) = w^*(x) + z(x,t;\lambda)$, then z satisfies

$$u_{tt} = z_{tt} = w^{*\prime\prime} + z_{xx} + \frac{\lambda}{(1-u)^2}. \tag{4.5}$$

Now define the functional

$$A(t) = \int_0^1 z(x,t)\phi^*(x)dx.$$

Multiplying both sides of equation (4.5) with the eigenfunction ϕ^*, integrating over the interval $[0,1]$, using Green's identity and equation (4.4), we then obtain

$$\begin{aligned}
A''(t) &= \int_0^1 w^{*''}\phi^* dx + \int_0^1 z_{xx}\phi^* dx + \lambda \int_0^1 \frac{\phi^*}{(1-u)^2} dx \\
&= -\lambda^* \int_0^1 \frac{\phi^*}{(1-w^*)^2}\, dx + \int_0^1 z\phi^{*''} dx + \lambda \int_0^1 \frac{\phi^*}{(1-u)^2} dx, \\
&= -\lambda^* \int_0^1 \frac{\phi^*}{(1-w^*)^2}\, dx - \lambda^* \int_0^1 \frac{2\,\phi^*}{(1-w^*)^3}\, z dx + (\lambda-\lambda^*) \int_0^1 \frac{\phi^*}{(1-u)^2}\, dx \\
&\quad +\lambda^* \int_0^1 \frac{\phi^*}{(1-u)^2}\, dx \\
&= (\lambda-\lambda^*) \int_0^1 \frac{\phi^*}{(1-u)^2}\, dx + \lambda^* \int_0^1 \left[\frac{1}{(1-u)^2} - \frac{1}{(1-w^*)^2} - \frac{2\,z}{(1-w^*)^3}\right] \phi^* dx, \quad (4.6)
\end{aligned}$$

for $\lambda > \lambda^*$.

From conservation of energy, (2.3),

$$||u_x||_2^2 \leq 2E_0 - 2\,\lambda \int_0^1 \frac{1}{1-u}\, dx < 2E_0,$$

and we then derive, on combining Sobolev's and Poincaré's inequalities (2.2), that

$$u(x,t) > -C_0 \quad \text{for any} \quad x \in [0,1], \;\; 0 < t < T_{max}, \tag{4.7}$$

where C_0 is a positive constant depending only upon λ and the initial data.

Note that due to (4.7) the first term of the right-hand side of (4.6) is estimated from below by

$$(\lambda-\lambda^*) \int_0^1 \frac{\phi^*}{(1-u)^2}\, dx \geq (\lambda-\lambda^*) \inf_{t\in(0,T_{max})} \int_0^1 \frac{\phi^*}{(1-u)^2}\, dx \geq \frac{(\lambda-\lambda^*)}{(1+C_0)^2},$$

since also (4.3) holds.

On the other hand, the integrand of the second term of the right-hand side of (4.6) is non-negative since

$$\frac{1}{(1-u)^2} - \frac{1}{(1-w^*)^2} - \frac{2\,z}{(1-w^*)^3} \geq \frac{3\,z^2}{(1-\xi)^4} > 0, \tag{4.8}$$

for some $\xi \in < w^*, u >$, where $< w^*, u >= \{\xi : \xi = \theta w^* + (1-\theta)u,\ \theta \in [0,1]\}$.

Thus, we obtain the differential inequality

$$A''(t) \geq \frac{(\lambda-\lambda^*)}{(1+C_0)^2} = K > 0, \quad \text{for} \quad \lambda > \lambda^*,$$

which integrated twice yields

$$A(t) \geq K\frac{t^2}{2} + A_1 t + A_0 = G(t), \quad \text{for any} \quad 0 < t < T_{max}, \tag{4.9}$$

where

$$A_0 = \int_0^1 z(x,0)\phi^*(x)dx = \int_0^1 \left(u_0(x) - w^*(x)\right)\phi^*(x)dx < 1,$$

and

$$A_1 = \int_0^1 z_t(x,0)\phi^*(x)dx = \int_0^1 u_1(x)\phi^*(x)dx.$$

It is readily seen that the positive root of the equation $G(t) = 1$ is

$$0 < t_+ = \frac{-A_1 + \sqrt{A_1^2 - 2K(A_0 - 1)}}{K} < \infty, \tag{4.10}$$

thus $\lim_{t \to t_1} A(t) = 1-$ for some $t_1 \le t_+$ by (4.9). However, the latter, since

$$A(t) = \int_0^1 z(x,t)\phi^*(x)\,dx \le \|u - w^*\|_\infty \le \|u\|_\infty, \quad \text{for} \quad w^* > 0\,,$$

implies that $\lim_{t \to t^*} ||u(\cdot,t)||_\infty = 1$ for some $t^* \le t_+$. $\square$

Theorem 4.2 improves the result of Theorem 3.2 of [5] where quenching was proved only for $\lambda > \lambda_+^*$, for some $\lambda_+^* > \lambda^*$, and left a gap for the range $(\lambda^*, \lambda_+^*]$.

Moreover, the result of Theorem 4.2 can be easily extended to the practically important two-dimensional case and indeed to three dimensions. The proof follows exactly the same steps. The existence of $\lambda^* < \infty$ for the higher-dimensional steady-state problem

$$\Delta w + \frac{\lambda}{(1-w)^2} = 0\,, \quad x \in \Omega, \quad w(x) = 0,\ x \in \partial\Omega,\ \ 0 < w < 1,$$

for Ω being a bounded domain of $\mathbb{R}^N$, $N = 2, 3$, is guaranteed by the results in [7, 12], where the C^2-regularity of the extremal solution $w^*(x) = w(x;\lambda^*)$ is also proved. In [7, 12], it is additionally proved that the principal eigenvalue of the linearized problem

$$\Delta\phi^* + \frac{2\,\lambda^*}{(1-w^*)^3}\,\phi^* = \mu^*\phi^*,\ \ x \in \Omega, \quad \phi^*(x) = 0, \quad x \in \partial\Omega,$$

is $\mu^* = 0$. Moreover, a lower estimate of the form (4.7) still holds due to Sobolev's inequality holding for $N = 2, 3$. Therefore the quenching result in [38] can also be improved. In addition, the estimates of the quenching time presented in the next remark are also applicable for $N = 2, 3$.

Remark 4.3. *The upper bound of the quenching time obtained in Theorem 4.2 can be used to estimate the quenching time, from above, in the asymptotic limit of $\lambda \to \lambda^*+$.*

For $u_1 \ge 0$ and not identically zero, so that $A_1 > 0$, taking $\lambda \to \lambda^*+$ so that $K \to 0$, (4.10) gives $t_+ \to (1 - A_0)/A_1$. For λ close to the critical value, any upward perturbation leads to quenching in an order-one time, or less. With u_1 identically zero, so that $A_1 = 0$, $t_+ = (2(1 - A_0)/K)^{1/2}$ and thus $t^* \le t_+ = O((\lambda - \lambda^*)^{-1/2})$ for $\lambda \to \lambda^*+$. With $u_1 \le 0$ and not identically zero, so that $A_1 < 0$, taking $\lambda \to \lambda^*+$, so that $K \to 0$ in (4.10) now gives $t_+ \sim -2A_1/K$. Then $t^* \le t_+ = O((\lambda - \lambda^*)^{-1})$.

The middle estimate of the quenching time agrees with one which holds for the corresponding parabolic problem, see [13, 23].

We cannot easily get a good bound in the same way in the opposite limit of $\lambda \to \infty$. This is due to C_0 being potentially unbounded.

We should note, however, that in this one-dimensional case we can proceed slightly differently.

Writing $F(x,t) = \lambda/(1 - u(x,t))^2$ and $\tilde{F}(x,t) = F(x,t)$ for $0 < x < 1$, $\tilde{F}(x,t) = -F(-x,t)$ for $-1 < x < 0$ and $\tilde{F}(x,t) = -F(2-x,t)$ for $1 < x < 2$, the D'Alembert *solution for (1.1), applying for $t \geq 0$, gives*

$$\begin{aligned} u(x,t+1) &= \frac{1}{2}(\tilde{u}(x-1,t)+\tilde{u}(x+1,t)) + \frac{1}{2}\int_{x-1}^{x+1} \tilde{u}_t(y,t)\,dy \\ &+ \frac{1}{2}\int_0^1 \int_{x-1+s}^{x+1-s} \tilde{F}(y,t+s)\,dy\,ds \\ &= -\frac{1}{2}(u(1-x,t)+u(1-x,t)) + \frac{1}{2}\int_0^1 \int_{\max\{1-s-x,x-1+s\}}^{\min\{x+1-s,1-x+s\}} F(y,t+s)\,dy\,ds \\ &> -u(1-x,t)\,, \end{aligned}$$

where $\tilde{u}$ is the (odd) extension of u defined by $\tilde{u}(x,t) = u(x,t)$ for $0 < x < 1$, $\tilde{u}(x,t) = -u(-x,t)$ for $-1 < x < 0$, and $\tilde{u}(x,t) = -u(2-x,t)$ for $1 < x < 2$. We deduce that if u falls to -1 at some time t_1, quenching must then occur before $t_1 + 1$.

Combining this result with the estimate got from (4.10) (based on assuming that u remains greater than -1) gives the quenching time estimate $t^* \lesssim 1$ for $\lambda \to \infty$.

Since $u(x,t)$ represents the deflection of the elastic membrane inside MEMS device, one expects touch-down to occur when the initial deformation $u_0(x)$ of the elastic membrane is big enough and/or there is movement towards the rigid plate, meaning that $u_1(x)$ is positive. This expectation is verified by the following.

Theorem 4.4. *Let $0 < \lambda \leq \lambda^*$, then the solution of problem* (1.1) *quenches in finite time provided that the initial data $u_0(x)$ is greater than or equal to the maximal steady-state solution $\overline{w}(x;\lambda)$ and $u_1(x)$ is non-negative, with $u_0(x) > \overline{w}(x;\lambda)$ or $u_1(x) > 0$ for some x.*

Proof. Again we proceed as in [23]. Let us assume that the maximum existence time of problem (1.1) is $0 < T_{max} \leq \infty$. For any $0 < \lambda \leq \lambda^*$ set $u(x,t;\lambda) = \overline{w}(x;\lambda) + z(x,t;\lambda)$, (note that $\overline{w} = w^*$ for $\lambda = \lambda^*$). Then z satisfies

$$\begin{aligned} u_{tt} = z_{tt} &= \overline{w}'' + z_{xx} + \frac{\lambda}{(1-u)^2} \\ &= \lambda\left[\frac{1}{(1-u)^2} - \frac{1}{(1-\overline{w})^2} - \frac{2\,z}{(1-\overline{w})^3}\right] \\ &+ z_{xx} + \frac{2\lambda\,z}{(1-\overline{w})^2} - \mu_1\,z + \mu_1\,z. \end{aligned} \tag{4.11}$$

Let (μ_1, ϕ_1) be the principal eigenpair of the problem:

$$\phi'' + \frac{2\,\lambda}{(1-\overline{w})^3}\,\phi = \mu\phi, \quad 0 < x < 1, \quad \phi(0) = \phi(1) = 0, \tag{4.12}$$

where ϕ_1 is considered to be positive and normalized according to (4.3) and μ_1 is known to be non-negative, since $\overline{w}$ is unstable, see [7, 12].

We now define $A(t)$ by

$$A(t) = \int_0^1 z(x,t)\phi_1(x)\,dx. \tag{4.13}$$

Differentiating (4.13) twice and using equation (4.11) combined with Green's identity, we obtain

$$\begin{aligned} A''(t) &= \int_0^1 z_{tt}(x,t)\phi_1(x)\,dx \\ &= \int_0^1 \lambda\left[\frac{1}{(1-u)^2}-\frac{1}{(1-\overline{w})^2}-\frac{2\,z}{(1-\overline{w})^3}\right]\phi_1\,dx \\ &+ \int_0^1\left[\phi_1''+\frac{2\,\lambda\,\phi_1}{(1-\overline{w})^3}-\mu_1\,\phi_1\right]z\,dx+\mu_1\int_0^1 z\,\phi_1\,dx. \end{aligned} \tag{4.14}$$

Since ϕ_1 satisfies (4.12) with $\mu=\mu_1$, the second term on the right-hand side of (4.14) vanishes, hence

$$A''(t)\geq\lambda\int_0^1\frac{3\,z^2\,\phi_1}{(1+C_0)^4}\,dx+\mu_1\,A(t) \tag{4.15}$$

taking also into account

$$\frac{1}{(1-u)^2}-\frac{1}{(1-\overline{w})^2}-\frac{2\,z}{(1-\overline{w})^3}\geq\frac{3\,z^2}{(1-\xi)^4}>0,\quad\text{for some } \xi\in<u,\overline{w}>,$$

as well as the fact that (4.7) is still valid. By virtue of Jensen's inequality, (4.15) yields

$$A''(t)\geq\Lambda A^2(t)+\mu_1 A(t),\quad\text{for any } 0<t<T_{max}, \tag{4.16}$$

where $\Lambda=\frac{3\,\lambda}{(1+C_0)^4}$. Now the differential inequality (4.16) under the initial conditions

$$A(0)=A_0=\int_0^1 z(x,0)\,\phi_1(x)\,dx=\int_0^1(u_0(x)-\overline{w}(x))\,\phi_1(x)\,dx\geq 0, \tag{4.17}$$

and

$$A'(0)=A_1=\int_0^1 z_t(x,0)\,\phi_1(x)\,dx=\int_0^1 u_1(x)\,\phi_1(x)\,dx\geq 0, \tag{4.18}$$

with $A(0)>0$ or $A'(0)>0$ implies that $A(t)>0$ for any $0<t<T_{max}$. Therefore the right-hand side of (4.16) is positive, since $\mu_1\geq 0$, so

$$A(t)>A_1t+A_0,\quad\text{for } 0<t<T_{max}\,.$$

Substituting back into (4.16), and integrating, gives

$$A(t)>A_1t+A_0+\Lambda\left(A_0^2t+A_0A_1t^2+\frac{1}{3}A_1^2t^3\right),\quad\text{for } 0<t<T_{max}\,.$$

This yields that $\lim_{t\to t_1-}A(t)=1$ for some finite positive t_1. Since

$$A(t)=\int_0^1 z(x,t)\phi_1(x)\,dx\leq\|u-\overline{w}\|_\infty\leq\|u\|_\infty,\quad\text{for}\quad\overline{w}>0,$$

we also have that $||u(\cdot,t)||_\infty\to 1-$ as $t\to t^*\leq t_1<\infty$, which also implies, [6, 22], that $||u_{tt}(\cdot,t)||_\infty\to\infty$ as $t\to t^*$. □

Remark 4.5. *Theorem 4.4 can be easily extended to the higher dimensions* $N=2,3$ *since the linearized problem for* $0<\lambda\leq\lambda^*$,

$$\Delta\phi+\frac{2\,\lambda}{(1-\overline{w})^3}\,\phi=\mu\phi,\quad x\in\Omega,\qquad\phi(x)=0,\ \ x\in\partial\Omega,$$

has non-negative principal eigenvalue as well as a lower estimate of the form (4.7) *still valid.*

We close this section with a quenching result applying for higher dimensions $N > 3$, where the estimate (4.7) obtained via Sobolev's inequality is no longer valid. For a similar result see also [10, 18].

Theorem 4.6. *If $\lambda > \lambda_+^* = 4\nu_1/27 \geq \lambda^*$, where $\nu_1 > 0$ is the principal eigenvalue of the problem*

$$-\Delta\psi = \nu\psi,\ x \in \Omega,\quad \psi = 0,\quad x \in \partial\Omega, \tag{4.19}$$

then the solution of problem

$$u_{tt} - \Delta u = \frac{\lambda}{(1-u)^2},\quad x \in \Omega \subset \mathbb{R}^N,\quad u = 0,\quad x \in \partial\Omega, \tag{4.20a}$$

$$u(x,0) = u_0(x) < 1,\quad u_t(x,0) = u_1(x)\,,\quad x \in \Omega, \tag{4.20b}$$

quenches in finite time.

Proof. We define the functional

$$F(t) = \int_\Omega u(x,t)\,\psi_1(x)\,dx \leq ||u(\cdot,t)||_\infty\,, \tag{4.21}$$

where $\psi_1 > 0$ is the eigenfunction of (4.19) normalized so that $\int_\Omega \psi_1\,dx = 1$. Differentiating $F(t)$ twice and using integration by parts together with equation (4.20a) and Jensen's inequality, yield the differential inequality

$$\begin{aligned} F''(t) &= \int_\Omega \left(\Delta u + \frac{\lambda}{(1-u)^2}\right)\psi_1\,dx = \int_\Omega u\Delta\psi_1\,dx + \int_\Omega \frac{\lambda\,\psi_1}{(1-u)^2}\,dx, \\ &\geq -\nu_1\,F(t) + \frac{\lambda}{(1-F(t))^2}, \end{aligned} \tag{4.22}$$

with associated initial conditions

$$F(0) = F_0 < 1 \quad \text{and} \quad F'(0) = F_1\,. \tag{4.23}$$

It can be easily seen that for $\lambda > \lambda_+^* = 4\nu_1/27$,

$$\frac{\lambda}{(1-s)^2} - \nu_1 s > 0\,,\quad \text{for any} \quad 0 \leq s < 1\,,$$

which, by virtue of (4.22) and (4.23), guarantees that

$$F''(t) > C_2 > 0 \quad \text{for any} \quad 0 < t < T_{max}\,,$$

thus

$$F(t) > C_2 t^2 + F_1 t + F_0 \quad \text{for any} \quad 0 < t < T_{max}\,. \tag{4.24}$$

But relation (4.24) implies that $\lim_{t\to t_2} F(t) = 1$ for some $t_2 \leq t_+$ where

$$0 < t_+ = \frac{-F_1 + \sqrt{F_1^2 + 4C_2(1-F_0)}}{2\,C_2} < \infty\,. \tag{4.25}$$

Finally, (4.21) implies that $||u(\cdot,t)||_\infty \to 1$ as $t \to t^* \leq t_+$.

The statement that $\lambda_+^* \geq \lambda^*$ is clear by contradiction (on assuming that $\lambda^* > \lambda_+^*$ and then taking $\lambda^* > \lambda > \lambda_+^*$ with initial conditions $u_0 = \underline{w}$ (or $\overline{w}$) and $u_1 = 0$).

□

Remark 4.7. *An estimate on the quenching time for large λ can easily be got from (4.25) (c.f. Remark 4.3).*

Remark 4.8. *For convenience the proofs of all the quenching results given in the current section concern smooth solutions. However, the same results can be proved for the weak solutions defined by Definition 2.1 under the assumption that $u_0(x), u_1(x) \in L^2([0,1])$.*

5. Non-Existence of Regular Similarity Solutions

In many cases the existence of a similarity solution can provide us with a description of the solution profile during quenching. However, as we will see in this section, we do not have such a similarity solution for our problem. Our analysis will also be a guide towards obtaining an asymptotic expansion describing the quenching profile in the following section. For simplicity, we take the quenching time to be $t = 0$ and position to be $x = \frac{1}{2}$ both in this section and in Section 6, provided we consider initial data symmetric with respect to $x = \frac{1}{2}$. Also for simplicity we may consider $\lambda = 1$.

We take an alternative form of the local hyperbolic problem by setting $U = 1 - u$. Thus we have

$$U_{tt} = U_{xx} - 1/U^2, \quad 0 < x < 1, \quad t > 0, \tag{5.1a}$$

$$U(0,t) = 1, \quad U(1,t) = 1, \quad t > 0, \tag{5.1b}$$

$$U(x,0) = U_0(x), \quad U_t(x,0) = U_1(x), \quad 0 < x < 1\,, \tag{5.1c}$$

with $0 < U < 1$ and quenching occurring when $U = 0$.

We set $U = (-t)^{\alpha} v(\eta)$, for $\eta = (x - \frac{1}{2})/(-t)$ and we have $\partial\eta/\partial t = (x - \frac{1}{2})/t^2$. Then the terms in equation (5.1a) become:

$$\begin{aligned}
\frac{\partial^2 U}{\partial t^2} &= (-t)^{\alpha-2}\left[\alpha(\alpha-1)v - (2\alpha\eta v' - 2\eta v') + \eta^2 v''\right], \\
\frac{\partial^2 U}{\partial x^2} &= (-t)^{\alpha-2} v'', \\
\frac{1}{U^2} &= (-t)^{-2\alpha} v^{-2}.
\end{aligned}$$

To eliminate time we must take $\alpha - 2 = -2\alpha$ or that $\alpha = \frac{2}{3}$ and we obtain the relevant equation for v,

$$\eta^2 v''(\eta) + \frac{2}{3}\eta v'(\eta) - \frac{2}{9} v(\eta) = v''(\eta) - \frac{1}{v^2(\eta)},$$

or

$$(1-\eta^2) v''(\eta) - \frac{2}{3}\eta v'(\eta) + \frac{2}{9} v(\eta) = \frac{1}{v^2(\eta)}. \tag{5.2}$$

A constant and regular solution of this equation is $v = a$ with $a = \left(\frac{9}{2}\right)^{\frac{1}{3}}$.

We want to show that $v = a$ is the only symmetric regular positive solution of the equation and thus there is no non-trivial similarity solution of the problem. This means that not all of the conditions $v(0) > 0$, $v'(0) = 0$, $v(\eta) \to +\infty$ for $\eta \to \infty$ and v being smooth in its domain can be satisfied simultaneously. Indeed we have the following:

Theorem 5.1. *The solution $v = a$ is the only symmetric regular positive solution of equation* (5.2).

Proof. We set $v = a + V$ and then for $V = V(\eta)$ we get

$$(1-\eta^2)V''(\eta) - \frac{2}{3}\eta V'(\eta) + \frac{2}{9}a + \frac{2}{9}V(\eta) - \frac{1}{a^2} + \frac{2V(\eta)}{a^3} - g(\eta) = 0,$$

or

$$(1-\eta^2)V''(\eta) - \frac{2}{3}\eta V'(\eta) + \frac{2}{3}V(\eta) - g(\eta) = 0, \tag{5.3}$$

for

$$g(\eta) = \frac{1}{(a+V(\eta))^2} - \frac{1}{a^2} + \frac{2V(\eta)}{a^3} = \frac{(3a+2V(\eta))\,V^2(\eta)}{a^3\,(a+V(\eta))^2} \geq 0\,.$$

Noting that $V = \eta$ is a solution of (5.3) if the g term is neglected, we set $V = \eta q$ and obtain

$$(1-\eta^2)\,(\eta q'' + 2q') - \frac{2}{3}\,(\eta q' + q) + \frac{2}{3}\eta q = g(\eta)\,,$$

or

$$\eta(1-\eta^2)q'' + 2\left(1 - \frac{4}{3}\eta^2\right)q' = g(\eta),$$

and

$$q'' + \frac{2\left(1-\frac{4}{3}\eta^2\right)}{\eta(1-\eta^2)}q' = \frac{g(\eta)}{\eta(1-\eta^2)}.$$

Using the integating factor $\eta^2(1-\eta^2)^{\frac{1}{3}}$ we obtain

$$\left(\eta^2(1-\eta^2)^{\frac{1}{3}}q'\right)' = \eta(1-\eta^2)^{\frac{2}{3}}g(\eta).$$

This gives

$$\eta^2(1-\eta^2)^{\frac{1}{3}}q' = A_c + G(\eta),$$

where

$$G(\eta) = \int_0^\eta s(1-s^2)^{\frac{2}{3}}g(s)ds\,.$$

Then we get

$$\begin{aligned} q(\eta) &= B_c - \int_\eta^1 \frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}}ds = B_c - \int_\eta^1 \left(\frac{A_c+G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} + \frac{A_c}{s^2}\right)ds \\ &= B_c - \frac{A_c}{\eta} + A_c - \int_\eta^1 \left(\frac{A_c+G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2}\right)ds. \end{aligned}$$

Thus due to the fact that $V = \eta q$ we have

$$V(\eta) = \eta\left[B_c + A_c - \int_\eta^1 \left(\frac{A_c+G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2}\right)ds\right] - A_c\,.$$

In order to obtain regularity at $\eta = 0$, with $v'(0) = V'(0) = 0$, (i.e. demand the symmetry condition) we have

$$B_c = \int_0^1 \left(\frac{A_c+G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2}\right)ds - A_c\,,$$

and thus

$$V(\eta) = \eta \int_0^\eta \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds - A_c \,.$$

For $0 < c = v(0) < a$, we have $-a < V(0) = -A_c < 0$ and $0 < A_c < a$. Also for $V > -a$ we have $g(V) \geq 0$ which additionally implies that $G(s) \geq 0$. Now for $0 < \eta < 1$ we have

$$\begin{aligned} \frac{dV}{d\eta} &= \int_0^\eta \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds + \eta \left(\frac{A_c + G(\eta)}{\eta^2(1-\eta^2)^{\frac{1}{3}}} - \frac{A_c}{\eta^2} \right) \\ &> \eta \left(\frac{A_c + G(\eta)}{\eta^2(1-\eta^2)^{\frac{1}{3}}} - \frac{A_c}{\eta^2} \right) > \frac{A_c}{\eta} \left(\frac{1}{(1-\eta^2)^{\frac{1}{3}}} - 1 \right) \to \infty \text{ as } \eta \to 1- \,, \end{aligned}$$

which implies that the solution, V, develops a singularity at $\eta = 1$.

Hence any regular symmetric solution must have $v(0) > a$, i.e. $V(0) > 0$. From (5.2), it is clear that v is then decreasing for η small and positive. Since v must remain positive, either it must reach a positive local minimum, say v^*, at some point η^* in $(0, 1)$, or v remains decreasing throughout $[0, 1]$, taking some positive value v_0 at $\eta = 1$. We examine the former case first.

As before,

$$V(\eta) = \eta \int_0^\eta \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds - A_c$$

and

$$V'(\eta) = \int_0^\eta \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds + \frac{A_c + G(\eta)}{\eta(1-\eta^2)^{\frac{1}{3}}} - \frac{A_c}{\eta} \,. \tag{5.4}$$

Given that V has a minimum at $\eta = \eta^*$, $V'(\eta^*) = 0$ so

$$\int_0^{\eta^*} \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds = \frac{A_c}{\eta^*} - \frac{A_c + G(\eta^*)}{\eta^*(1-\eta^{*2})^{\frac{1}{3}}} \tag{5.5}$$

and hence

$$V(\eta) = \eta \left[\int_{\eta^*}^\eta \left(\frac{A_c + G(s)}{s^2(1-s^2)^{\frac{1}{3}}} - \frac{A_c}{s^2} \right) ds + \frac{A_c}{\eta^*} - \frac{A_c + G(\eta^*)}{\eta^*(1-\eta^{*2})^{\frac{1}{3}}} \right] - A_c \,. \tag{5.6}$$

In particular,

$$V(\eta^*) = -\frac{A_c + G(\eta^*)}{(1-\eta^{*2})^{\frac{1}{3}}} \,. \tag{5.7}$$

Turning again to (5.2), it is seen that for v and V to have local minima at $\eta = \eta^*$, $v < a$ and hence V is negative at that point. It follows that

$$A_c + G(\eta^*) = -(1-\eta^{*2})^{\frac{1}{3}} V(\eta^*) > 0 \,. \tag{5.8}$$

Now $A_c + G(\eta) = A_c + G(\eta^*) + \int_{\eta^*}^\eta s(1-s^2)^{\frac{2}{3}} g(s) ds \geq A_c + G(\eta^*) > 0$ for $\eta > \eta^*$ and from (5.4) we again see that $V' \to \infty$ as $\eta \to 1-$ so that V develops a singularity at $\eta = 1$.

For the other case, if v is to be regular, it will have a first derivative, say v_1, at $\eta = 1$. Then (5.2) gives

$$v_1 = \frac{1}{3}v_0 - \frac{3}{2}\frac{1}{v_0^2} = \frac{1}{3}v_0^{-2}(v_0^3 - a^3)\,.$$

We see that for $v_0 > a$, $v_1 > 0$ so that v is locally increasing, contradicting the assumption of v decreasing in $[0, 1]$. Taking $v_0 = a$, $v_1 = 0$, and we regain the trivial solution $v \equiv a$ (contradicting $v(0) > a$). We are left with $0 < v_0 < a$ and $v_1 < 0$, so that now v decreases in a neighbourhood of $\eta = 1$.

If v is to be smooth, we can differentiate (5.2) to get

$$(\eta^2 - 1)v''' + \frac{8}{3}\eta v'' = \frac{2v'}{v^3} - \frac{4v'}{9} = \frac{4v'}{9v^3}\left(\frac{9}{2} - v^3\right)\,. \tag{5.9}$$

At $\eta = 1$, $v = v_0 < a$ and $v' = v_1 < 0$ so $v'' < 0$. As long as $v'' \le 0$, $v' < 0$ for $\eta \ge 1$. Supposing that there is a first point $\eta_* > 1$ where $v'' = 0$, so that $v''' \ge 0$ at $\eta = \eta_*$, $v > 0$ (for the solution to still exist) and $v' < 0$ at that point. Then (5.9) gives $v'''(\eta_*) < 0$, another contradiction. This means that $v'' < 0$ for $\eta \ge 1$, so that v must fall to 0, and the solution ceases to exist, at a finite value of η. □

Remark 5.2. *The local behaviour of solutions which are singular at $\eta = 1$ can be determined formally. We write $\eta = 1 + \sigma$ so that for σ small we are close to $\eta = 1$ and the equation has the form*

$$(2\sigma + \sigma^2)\frac{d^2v}{d\sigma^2} + \frac{2}{3}(1 + \sigma)\frac{dv}{d\sigma} - \frac{2}{9}v + v^{-2} = 0\,.$$

We assume that v has the form of a power-series expansion $v \sim v_0 + v_1\sigma^\alpha + \dots$, for some constants v_0, v_1 and α. Then the equation becomes

$$2\alpha(\alpha - 1)v_1\sigma^{\alpha-1} + \dots + \frac{2}{3}\alpha v_1\sigma^{\alpha-1} + \dots - \frac{2}{9}v_0 + \dots + v_0^{-2} + \dots = 0\,. \tag{5.10}$$

The leading-order terms, which must balance, are either the first and second, for $\alpha \le 1$, or the third and fourth, for $\alpha \ge 1$.

Taking the first two terms to be small gives $v_0 = a$. This is the special case of $v \equiv a$. With all the terms of the same size, $\alpha = 1$ and we obtain, to first-order, $v \sim v_0 + v_1\sigma$ with $v_1 = \frac{1}{3}v_0 - \frac{3}{2}v_0^{-2}$. These are the locally regular, but non-trivial, solutions noted in the above theorem.

With the first two terms dominating, so $v_1 \neq 0$, $2\alpha(\alpha - 1) + \frac{2}{3}\alpha = 2\alpha(\alpha - \frac{2}{3}) = 0$. Clearly we want α to be non-zero to get a locally varying solution so $\alpha = \frac{2}{3}$. We then have a two-parameter family of locally singular solutions, $v \sim v_0 + v_1\sigma^{\frac{2}{3}}$ (as indicated by the earlier estimates on the first derivative of v).

Remark 5.3. *Asymmetric regular solutions can also be eliminated. Taking $0 < c = v(0) < a$, we are no longer able to fix B_c since we do not know that $V'(0) = v'(0)$ vanishes. However, $A_c = -V(0) = a - v(0)$ is again positive, so the key steps for this case still apply and it is still clear that the solution to (5.2) is singular at $\eta = 1$.*

With $v(0) \ge a$, we may assume, without loss of generality, that $v'(0) < 0$. We again have the two possible cases to consider: (i) v attains some positive minimum (which must be less than a) at some η^* in $(0, 1)$; or (ii) v is decreasing in $[0, 1]$, taking a positive value v_0 at $\eta = 1$. Case (ii) is ruled out as before. Case (i) is likewise eliminated since, although

B_c is not fixed, (5.5) again applies so that (5.6) - (5.8) all follow and a singularity occurs at $\eta = 1$.

6. Formal Asymptotics for the Quenching Profile

For simplicity we again consider initial data $u_0(x)$ and $u_1(x)$ symmetric with respect to $x = \frac{1}{2}$ then quenching is expected to take place at $x = \frac{1}{2}$.

Now we rescale time as $\tau = -\ln(-t)$ and set $\eta = (x - \frac{1}{2})/(-t)$. We then have $d\tau/dt = 1/(-t)$, $\partial\eta/\partial t = (-t)^{-1}\eta$ and we set $U = U(x,t) = (-t)^{\frac{2}{3}}v$, with $v = v(\eta, \tau)$. Thus

$$\begin{aligned} U_t &= -\frac{2}{3}(-t)^{-\frac{1}{3}}v + (-t)^{-\frac{1}{3}}v_\tau + (-t)^{-\frac{1}{3}}\eta v_\eta, \\ U_{tt} &= (-t)^{-\frac{4}{3}}\left(v_{\tau\tau} + 2\eta v_{\eta\tau} - \frac{1}{3}v_\tau + \eta^2 v_{\eta\eta} + \frac{2}{3}\eta v_\eta - \frac{2}{9}v\right) \\ U_{xx} &= (-t)^{-\frac{4}{3}}v_{\eta\eta}, \quad U^{-2} = (-t)^{-\frac{4}{3}}v^{-2}. \end{aligned}$$

Therefore equation (5.1a) becomes

$$v_{\tau\tau} + 2\eta v_{\eta\tau} - \frac{1}{3}v_\tau + \eta^2 v_{\eta\eta} + \frac{2}{3}\eta v_\eta - \frac{2}{9}v = v_{\eta\eta} - v^{-2},$$

or

$$v_{\tau\tau} + 2\eta v_{\eta\tau} - \frac{1}{3}v_\tau = (1-\eta^2)v_{\eta\eta} - \frac{2}{3}\eta v_\eta + \frac{2}{9}v - v^{-2}. \tag{6.1}$$

We initially investigate the form of the solution near the quenching point.

Inner Solution. We expect that v tends to a, with $a = \left(\frac{9}{2}\right)^{\frac{1}{3}}$, near the quenching point and therefore we assume that v has an expansion of the form $v \sim a + v_1 + v_2 + \dots$. Thus equation (6.1) gives

$$v_{1\tau\tau} + 2\eta v_{1\eta\tau} - \frac{1}{3}v_{1\tau} = (1-\eta^2)v_{1\eta\eta} - \frac{2}{3}\eta v_{1\eta} + \frac{2}{9}v_1 + \frac{2}{9}a - a^{-2} + 2a^{-3}v_1 + \dots,$$

on neglecting terms in v_1^2, v_2, etc. Given that $2a^{-3} = \frac{4}{9}$,

$$v_{1\tau\tau} + 2\eta v_{1\eta\tau} - \frac{1}{3}v_{1\tau} = (1-\eta^2)v_{1\eta\eta} - \frac{2}{3}\eta v_{1\eta} + \frac{2}{3}v_1\,. \tag{6.2}$$

Supposing that v_1 decays algebraically in τ, (6.2) reduces to

$$(1-\eta^2)v_{1\eta\eta} - \frac{2}{3}\eta v_1\eta + \frac{2}{3}v_1 = 0\,,$$

which is (5.3) without the $g(\eta)$ term, has no regular non-trivial solution, and is therefore not of interest.

Therefore the next reasonable choice is to assume that v_1 has a τ dependence of the form $v_1 = e^{-\alpha\tau}p(\eta)$. In such a case we obtain the equation for p

$$(1-\eta^2)p'' + 2\left(\alpha - \frac{1}{3}\right)\eta p' + \left(\frac{2}{3} - \frac{\alpha}{3} - \alpha^2\right)p = 0\,. \tag{6.3}$$

From what we have seen before, it might be expected that for general α, (6.3) has no non-trivial regular solution. We now look for values of α for which there is a non-trivial solution for all η.

We seek an even (symmetric) solution of (6.3) as a power series, $p(\eta) = \sum_{n=0}^{\infty} a_n \eta^{2n}$ with $a_0 \neq 0$ and get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} \left(\left(\frac{2}{3} - \frac{\alpha}{3} - \alpha^2 \right) + 4n \left(\alpha - \frac{1}{3} \right) - 2n(2n-1) \right) a_n \eta^{2n} + \sum_{n=1}^{\infty} 2n(2n-1) a_n \eta^{2n-2} \\ &= -\sum_{n=0}^{\infty} (\alpha - (2n-1)) \left(\alpha - \left(2n + \frac{2}{3} \right) \right) \eta^{2n} + \sum_{n=0}^{\infty} 2(n+1)(2n+1) a_{n+1} \eta^{2n}, \end{aligned}$$

which gives

$$\frac{a_{n+1}}{a_n} = \frac{(2n-1-\alpha)(2n+\frac{2}{3}-\alpha)}{2(n+1)(2n+1)} \,. \tag{6.4}$$

We see from (6.4) that the radius of convergence of the power series for $p(\eta)$ is (in general) 1, consistent with p having a singularity at $\eta = 1$. However, the series terminates, with p being a polynomial, $p_n^{\pm}(\eta)$, and hence smooth for all η, for $\alpha = \alpha_n^{\pm}$ with $\alpha_n^- = 2n-1$ for $n = 1, 2, 3, \ldots$ and $\alpha_n^+ = 2n + \frac{2}{3}$ for $n = 0, 1, 2, \ldots$.

For the dominant, slowest decaying, behaviour, with spatial variation, i.e. dependence upon η, we need $\alpha = \alpha_1^- = 1$ and then $a_1 = a_0/3$ and $v_1^- = ce^{-\tau}(\eta^2+3)$ on writing $a_0 = 3c$.

On the other hand, a slower shrinking solution still, but without η dependence, is given by $\alpha = \alpha_0^+ = \frac{2}{3}$, so that we have $v_1^+ = de^{-\frac{2}{3}\tau}$, on writing $a_0 = d$.

Therefore we have that the solution to the v problem near the quenching time has the form

$$v \sim a + ce^{-\tau}\left(\eta^2 + b_0\right) + de^{-\frac{2}{3}\tau}$$

and therefore for U we get approximately

$$U \sim e^{-\frac{2}{3}\tau}\left[a + ce^{-\tau}(\eta^2+3) + de^{-\frac{2}{3}\tau}\right] \sim e^{-\frac{2}{3}\tau}\left[a + ce^{-\tau}\eta^2 + de^{-\frac{2}{3}\tau}\right] ,$$

for large η as well as large τ, or, in terms of t and x,

$$U \sim (-t)^{\frac{2}{3}}\left[a + c\frac{(x-\frac{1}{2})^2}{-t} + d(-t)^{\frac{2}{3}}\right]. \tag{6.5}$$

Outer Solution. We want an approximation for the solution of equation (6.1) valid for η large. For $\eta \gg 1$ the equation becomes

$$v_{\tau\tau} + 2\eta v_{\eta\tau} - \frac{1}{3}v_\tau = -\eta^2 v_{\eta\eta} - \frac{2}{3}\eta v_\eta + \frac{2}{9}v - v^{-2},$$

i.e. $v_{\eta\eta}$ is negligible compared with $\eta^2 v_{\eta\eta}$. The neglected term corresponds to the diffusion term, U_{xx} of the original equation $U_{tt} = U_{xx} - 1/U^2$. Therefore to determine the outer solution we must solve the equation

$$\frac{d^2U}{dt^2} = -\frac{1}{U^2}. \tag{6.6}$$

Multiplying both sides of (6.6) with dU/dt and integrating results in

$$\left(\frac{dU}{dt}\right)^2 = \frac{2}{U} + b = \frac{2+bU}{U},$$

where b is a constant of integration. Therefore we have

$$\frac{dt}{dU} = -\left(\frac{U}{2+bU}\right)^{\frac{1}{2}},$$

and

$$t_0 - t = \int \left(\frac{U}{2+bU}\right)^{\frac{1}{2}} dU.$$

We set $U = \frac{2}{b}\tan^2\theta$ with $dU = \frac{4}{b}\tan(\theta)\sec^2(\theta)d\theta$ and we obtain

$$\begin{aligned} t_0 - t &= \frac{4}{b^{\frac{3}{2}}}\int \left(\frac{\tan^2(\theta)}{\sec^2(\theta)}\right)^{\frac{1}{2}} \tan(\theta)\sec^2(\theta)d\theta, \\ &= \frac{4}{b^{\frac{3}{2}}}\int \tan^2(\theta)\sec(\theta)d\theta. \end{aligned}$$

In addition we have

$$\begin{aligned} \int \tan^2(\theta)\sec(\theta)d\theta &= \int \left(\sec^2(\theta) - \sec(\theta)\right) d\theta \\ &= \tan(\theta)\sec(\theta) - \int \left(\sec(\theta) + \sec(\theta)\tan^2(\theta)\right) d\theta, \\ \int \sec(\theta)\tan^2(\theta)d\theta &= \frac{1}{2}\tan(\theta)\sec(\theta) - \frac{1}{2}\int \sec(\theta)d\theta, \\ \int \sec(\theta)d\theta &= \ln\left(\tan(\theta) + \cos(\theta)\right). \end{aligned}$$

Thus

$$\begin{aligned} t_0 - t &= \frac{2}{b^{\frac{3}{2}}}\left[\tan(\theta)\sec(\theta) - \ln\left(\tan(\theta) + \sec(\theta)\right)\right] \\ &= \frac{2}{c^{\frac{3}{2}}}\left[2^{-\frac{1}{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}\left(1+\frac{bU}{2}\right)^{\frac{1}{2}} - \ln\left(2^{-\frac{1}{2}}b^{\frac{1}{2}}U^{\frac{1}{2}} + \left(1+\frac{bU}{2}\right)^{\frac{1}{2}}\right)\right]. \end{aligned}$$

Finally we have that

$$t_0 - t = \frac{U^{\frac{1}{2}}}{b}(2+bU)^{\frac{1}{2}} - 2b^{-\frac{3}{2}}\ln\left[\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}} + \left(1+\frac{bU}{2}\right)^{\frac{1}{2}}\right]. \tag{6.7}$$

The quantity inside the logarithm can be written in the following way

$$\left[\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}} + \left(1+\frac{bU}{2}\right)^{\frac{1}{2}}\right] \sim 1 + \frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}} + \frac{bU}{4} - \frac{b^2U^2}{32} + \dots .$$

Therefore we have that

$$\begin{aligned} t_0 - t \sim\ & \frac{2^{\frac{1}{2}}U^{\frac{1}{2}}}{b}\left(1+\frac{bU}{4}-\frac{b^2U^2}{32}\right) - 2b^{-\frac{3}{2}}\left[\left(1+\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}+\frac{bU}{4}-\frac{b^2U^2}{32}+\dots\right)\right. \\ & -\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}+\frac{bU}{4}-\frac{b^2U^2}{32}+\dots\right)^2 + \frac{1}{3}\left(1+\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}+\frac{bU}{4}-\frac{b^2U^2}{32}+\dots\right)^3 \\ & -\frac{1}{3}\left(1+\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}+\frac{bU}{4}-\frac{b^2U^2}{32}+\dots\right)^4 \\ & \left.+\frac{1}{5}\left(1+\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}}+\frac{bU}{4}-\frac{b^2U^2}{32}+\dots\right)^5 + \dots\right]. \end{aligned}$$

Expanding the quantities in the brackets we obtain

$$\begin{aligned} t_0 - t \quad \sim \quad & \frac{2^{\frac{1}{2}}U^{\frac{1}{2}}}{b} + 2^{-\frac{3}{2}}U^{\frac{3}{2}} - b2^{-\frac{9}{2}}U^{\frac{5}{2}} \\ & -2b^{-\frac{3}{2}}\left[\frac{1}{\sqrt{2}}b^{\frac{1}{2}}U^{\frac{1}{2}} + \frac{bU}{4} - \frac{bU}{4} - \frac{1}{4\sqrt{2}}b^{\frac{3}{2}}U^{\frac{3}{2}} + \frac{1}{6\sqrt{2}}b^{\frac{3}{2}}U^{\frac{3}{2}} - \frac{b^2U^2}{32} \right. \\ & \left. -\frac{b^2U^2}{32} + \frac{1}{8}b^2U^2 - \frac{1}{16}b^2U^2 + 2^{-\frac{11}{2}}b^{\frac{5}{2}}U^{\frac{5}{2}} + 2^{-\frac{9}{2}}b^{\frac{5}{2}}U^{\frac{5}{2}} - 2^{-\frac{5}{2}}b^{\frac{5}{2}}U^{\frac{5}{2}} + \frac{1}{5}2^{-\frac{5}{2}}b^{\frac{5}{2}}U^{\frac{5}{2}} + \dots\right]. \end{aligned}$$

After doing the appropriate eliminations we get, to leading order, that

$$t_0 - t \sim \frac{\sqrt{2}}{3}U^{\frac{3}{2}} + \frac{3}{5}bU^{\frac{5}{2}} + \dots .$$

This implies that

$$U^{\frac{3}{2}} \sim \frac{3}{\sqrt{2}}(-t)\left(1 + t_0(-t)^{-1} - \frac{3}{5}b(-t)^{-1}U^{\frac{5}{2}} + \dots\right)$$

or alternatively

$$\begin{aligned} U \quad & \sim \quad a(-t)^{\frac{2}{3}}\left(1 + t_0(-t)^{-1} - \frac{3}{5}b(-t)^{-1}a^{\frac{5}{2}}(-t)^{\frac{5}{3}} + \dots\right)^{\frac{2}{3}} \\ & \sim \quad a(-t)^{\frac{2}{3}}\left(1 + \frac{2}{3}t_0(-t)^{-1} - \frac{2}{5}ba^{\frac{5}{2}}(-t)^{\frac{2}{3}} + \dots\right), \end{aligned}$$

and we obtain an expression for the outer approximation.

An alternative way for solving the equation for the outer solution is the following: We have the equation $\frac{dt}{dU} = -\left(\frac{U}{2-bU}\right)^{\frac{1}{2}}$, or that $t_0 - t = \int\left(\frac{U}{2-bU}\right)^{\frac{1}{2}} dU$ and we set $U = \frac{2}{b}\sin^2(\theta)$ with $dU = \frac{4}{b}\sin(\theta)\cos(\theta)d\theta$. Then $t_0 - t = 4b^{-\frac{3}{2}}\int \sin^2(\theta)d\theta = 2b^{-\frac{3}{2}}\int(1-\cos(2\theta))\,d\theta = 2b^{-\frac{3}{2}}\left(\theta - \frac{1}{2}\sin(2\theta)\right) = 2b^{-\frac{3}{2}}\left(\theta - \frac{1}{2}\cos(\theta)\sin(\theta)\right)$.
Thus again

$$t_0 - t = 2b^{-\frac{3}{2}}\left[\sin^{-1}\left(\frac{bU}{2}\right)^{\frac{1}{2}} - \left(\frac{bU}{2}\right)^{\frac{1}{2}}\left(1 - \frac{bU}{2}\right)^{\frac{1}{2}}\right].$$

The same leading-order approximation results.
Matching. We have the approximation for the inner region being in the form, for

$$v \sim a + ce^{-\tau}\left(\eta^2 + b_0\right) + de^{-\frac{2}{3}\tau},$$

and $U = (-t^{\frac{3}{2}})v(\eta)$,

$$U \sim (-t)^{\frac{2}{3}}\left[a + c\frac{(x-\frac{1}{2})^2}{-t} + d(-t)^{\frac{2}{3}}\right].$$

In addition the approximation for the outer region has the form

$$U \sim a(-t)^{\frac{2}{3}}\left(1 + \frac{2}{3}t_0(-t)^{-1} - \frac{2}{5}ba^{\frac{5}{2}}(-t)^{\frac{2}{3}} + \dots\right),$$

and in an intermediate region these expressions should be the same and therefore we must have $\frac{2}{3}at_0 = c(x-\frac{1}{2})^2$ or $t_0 = \frac{3c(x-\frac{1}{2})^2}{2a}$. Similarly $-\frac{2}{5}ba^{\frac{7}{2}} = d$ or $b = -\frac{2}{5}a^{-\frac{7}{2}}d$ and we get t_0 and b from c and d which are determined by the initial and boundary conditions of the problem.

Finally for $b > 0$ and by equation (6.7), we have

$$\frac{U^{\frac{1}{2}}}{b}(2+bU)^{\frac{1}{2}} - 2b^{-\frac{3}{2}} \ln\left[\frac{1}{\sqrt{2}} b^{\frac{1}{2}} U^{\frac{1}{2}} + \left(1+\frac{bU}{2}\right)^{\frac{1}{2}}\right] \sim \frac{3c\left(x-\frac{1}{2}\right)^2}{2a} - t$$

and as $t \to 0-$

$$\frac{U^{\frac{1}{2}}}{b}(2+bU)^{\frac{1}{2}} - 2b^{-\frac{3}{2}} \ln\left[\frac{1}{\sqrt{2}} b^{\frac{1}{2}} U^{\frac{1}{2}} + \left(1+\frac{bU}{2}\right)^{\frac{1}{2}}\right] \sim \frac{3c\left(x-\frac{1}{2}\right)^2}{2a},$$

while for $x \to \frac{1}{2}$ and $U \to 0$ we obtain

$$\frac{3c\left(x-\frac{1}{2}\right)^2}{2a} \sim \frac{\sqrt{2}}{3} U^{\frac{3}{2}},$$

or that the profile of the solution at the quenching point is

$$U \sim \left(\frac{9c}{2\sqrt{2}a}\right)^{\frac{2}{3}} \left(x - \frac{1}{2}\right)^{\frac{4}{3}}.$$

This gives us an $(x-\frac{1}{2})^{\frac{4}{3}}$ dependence of the solution profile near the quenching point $x = \frac{1}{2}$.

Note also that if we rescale to put the factor λ back into the equation, so that (5.1a) is replaced by $U_{tt} = U_{xx} - \lambda/U^2$, then, according to the above analysis, we have that $U(\frac{1}{2}, t) = 1 - u(\frac{1}{2}, t) \sim a\lambda^{\frac{1}{3}}(t^* - t)^{\frac{2}{3}}$ for $t \to t^*-$.

We note that this asymptotic behaviour for the semilinear hyperbolic problem differs substantially from that for the corresponding parabolic problem, see [15] for results specifically for MEMS devices and [8] for general results on the monotonic quenching of solutions of problems which can be written in the form $u_t - u_{xx} = \lambda(1-u)^{-\beta}$. For the parabolic problem, centre manifold techniques have been used in showing (i) that the spatially uniform quenching solution is unstable and (ii) that the quenching profile differs from that suggested by the apparently obvious similarity solution (const.$\times|x - x^*|^{2/3}$ for $\beta = 2$ and x^* the blow-up point) by a factor of $|\ln|x - x^*||^{-1/3}$. For the hyperbolic PDE, although the simple-minded guess of a self-similar solution again suggests an $|x - x^*|^{2/3}$ profile, our formal asymptotics now give local behaviour like $|x - x^*|^{4/3}$, a quite different power of distance, apparently without logarithmic dependence.

These formal asymptotic results are not yet proved. It is possible that a centre manifold-type approach might be useful in trying to do so.

7. Numerical Solution

We now carry out a brief numerical study of problem (1.1), with a variety of initial conditions. A moving mesh adaptive method, based on the techniques suggested in [4], is used. This captures the behaviour of the solution near a singularity.

More specifically we take initially a partition of $M+1$ points in $[0,1]$, $0 = \xi_0, \xi_0 + \delta\xi = \xi_1, \cdots, \xi_M = 1$. For the solution $u = u(x,t)$, we introduce a computational coordinate ξ in the interval $[0,1]$ and we consider the mesh points X_i to be the images of the points ξ_i (uniform mesh) under the map $x(\xi, t)$ so that $X_i(t) = x(i\delta\xi, t)$. Given this transformation, we have, for the approximation of the solution $u_i(t) \simeq u(X_i(t), t)$, that $\frac{du(X_i(t),t)}{dt} = u_t(X_i, t) + u_x\dot{X}_i$ or $u_t = \frac{du}{dt} - u_x x_t$.

The way that the map, $x(\xi,t)$, is determined is controlled by the monitor function $\mathcal{M}(u)$ which, in a sense, follows the evolution of the singularity. This function is determined by the scale invariants of the problem ([4]). In our case for the semilinear wave equation of the form $U_{tt} = U_{xx} - 1/U^p$ for $U = 1-u$, $p=2$ an appropriate monitor function should be $\mathcal{M}(U) = |U|^{-(p+1)/2}$.

At the same time we need also a rescaling of time of the form $\frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt}$ for $\frac{dt}{d\tau} = g(u)$, where $g(u)$ is a function determining the way that the time scale changes as the solution approaches the singularity, and is given by $g(u) = \frac{1}{\|\mathcal{M}(u)\|_\infty}$ (see [4]).

In addition the evolution of $X_i(t)$ is given by a moving mesh PDE (see [4]) which has the form $-x_{\tau\xi\xi} = \frac{g(u)}{\varepsilon}\left(\mathcal{M}(u)x_\xi\right)_\xi$. Here ε is a small parameter accounting for the relaxation time scale.

Thus finally we obtain a system of ODE's for X_i and u_i. We set $\frac{du}{dt} = v$ and the ODE system takes the form

$$\begin{aligned}
\frac{dt}{d\tau} &= g(u),\\
u_\tau - x_\tau u_x &= g(u)v,\\
v_\tau - x_\tau v_x &= g(u)\left(u_{xx} - \lambda\frac{1}{(1-u)^2}\right),\\
-x_{\tau\xi\xi} &= \frac{g(u)}{\epsilon}\left(\mathcal{M}(u)x_\xi\right)_\xi.
\end{aligned}$$

We may apply now a discretization in space and we have

$$\begin{aligned}
u_x(X_i,\tau) &\simeq \delta_x u_i(\tau) := \frac{u_{i+1}(\tau)-u_{i-1}(\tau)}{X_{i+1}(\tau)-X_{i-1}(\tau)},\\
u_{xx}(X_i,\tau) &\simeq \delta_x^2 u_i(\tau) := \left(\frac{u_{i+1}(\tau)-u_i(\tau)}{X_{i+1}(\tau)-X_i(\tau)} - \frac{u_i(\tau)-u_{i-1}(\tau)}{X_i(\tau)-X_{i-1}(\tau)}\right)\frac{2}{X_{i+1}(\tau)-X_{i-1}(\tau)},\\
x_{\xi\xi}(\xi_i,\tau) &\simeq \delta_\xi^2 x_i(\tau) := \frac{X_{i+1}(\tau)-2X_i(\tau)+X_{i-1}(\tau)}{\delta\xi^2},\\
\left(\mathcal{M}(u)x_\xi\right)_\xi &\simeq \delta_\xi(\mathcal{M}\delta_\xi x) := \left(\frac{\mathcal{M}_{i+1}-\mathcal{M}_i}{2}\frac{x_{i+1}-x_i}{\delta\xi} - \frac{\mathcal{M}_i-\mathcal{M}_{i-1}}{2}\frac{x_i-x_{i-1}}{\delta\xi}\right)\frac{1}{\delta\xi}.
\end{aligned}$$

Therefore the resulting ODE system to be solved, for

$$\begin{aligned}
y &= \left(t(\tau), v_1(\tau), v_2(\tau), \dots v_M(\tau), u_1(\tau), u_2(\tau), \dots u_M(\tau), X_1(\tau), X_2(\tau), \dots X_M(\tau)\right),\\
&= \left(t(\tau), \mathbf{v}, \mathbf{u}, \mathbf{X}\right), \qquad \mathbf{v},\, \mathbf{u},\, \mathbf{X} \in \mathbb{R}^M,
\end{aligned}$$

will have the form

$$A(\tau,y)\frac{dy}{d\tau} = b(\tau,y),$$

where the matrix $A \in \mathbb{R}^{3n+1}$ has the block form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & I & 0 & -\delta_x u\\ 0 & 0 & I & -\delta_x v\\ 0 & 0 & 0 & -\delta_\xi^2 \end{bmatrix}, \quad y = \begin{bmatrix} t(\tau)\\ \mathbf{u}\\ \mathbf{v}\\ \mathbf{X}\end{bmatrix}, \quad b = g(u)\begin{bmatrix} 1\\ \mathbf{v}\\ \delta^2\mathbf{u} - \lambda\frac{1}{(1-\mathbf{u})^2}\\ \delta_\xi(\mathcal{M}\delta x_\xi)\end{bmatrix}.$$

For the solution of the above system a standard ODE solver can be used such as the matlab function “ode15i”, see [4].

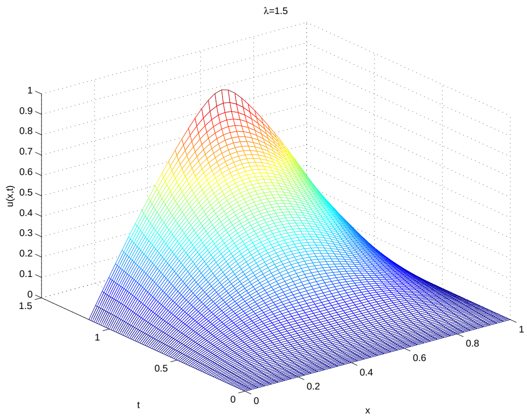

Figure 2. *The numerical solution of problem against space and time for* $\lambda = 1.5$.

In the figures of this section, the results of various numerical simulations are presented. In the numerical method we took $M = 161$.

The parameter used in the differential equation was $\lambda = 1.5$, this value being chosen so as to be slightly greater than the approximate value 1.4 found in Section 3 for λ^*, above which quenching should occur by Theorem 4.2.

In Figure 2, $u(x,t)$ is plotted against $x \in [0,1]$ and $t \in [0,T]$ for $T = 1.1547$. In the next plot, Figure 3, $u(x,t)$ is plotted against x for various times. The uppermost line corresponds to the solution of the problem near quenching. The initial conditions were $u_0 = 0$ and $u_1 = 0$.

Similar numerical simulations were carried out for smaller values of λ, still with zero initial data. Taking λ approaching 1.4 from above, the same quenching behaviour was observed. This indicates that, for $u_0 = u_1 = 0$, $\lambda_{cr} \approx 1.4$. The same was seen to happen for non-zero initial data u_0 lower than $\overline{w}$ still with $u_1 = 0$.

From the numerical solution of problem (1.1) we have that near quenching time t^*, $\ln\left(1 - u(\frac{1}{2}, t)\right) \propto \ln\left(t^* - t\right)$ with constant of proportionality $\frac{2}{3}$. This is demonstrated in Figure 4, where the fluctuations for t beyond $t^* - e^{-30}$ are apparently due to numerical errors.

A similar plot of $\ln u(x,t^*)$ against $\ln(x - \frac{1}{2})$, in Figure 5, shows that $u(x,t^*)$ behaves like $C(x - \frac{1}{2})^{\frac{4}{3}}$ near quenching. The agreement is also illustrated in Figure 6, where the solid line shows the numerical solution of problem (1.1) at the quenching time t^*, while the dotted curve displays $1 - \left(\frac{9c}{2\sqrt{2}a}\right)^{\frac{2}{3}} (x - \frac{1}{2})^{\frac{4}{3}}$. The constant $c = 2.1$ is chosen in such a way so that there is agreement of the plots at the boundaries, $x = 0, 1$.

Similarity Solution. It is also of some interest to investigate numerically the behaviour of possible similarity solutions, even though we have seen that they do not give local behaviour near quenching. We recall that we have taken $1 - u = U = (-t)^\alpha v(\eta), \quad \eta = (x - \frac{1}{2})/(-t)$, with U the solution of the equation $U_{tt} = U_{xx} - 1/U^2$. In this case the equation for v

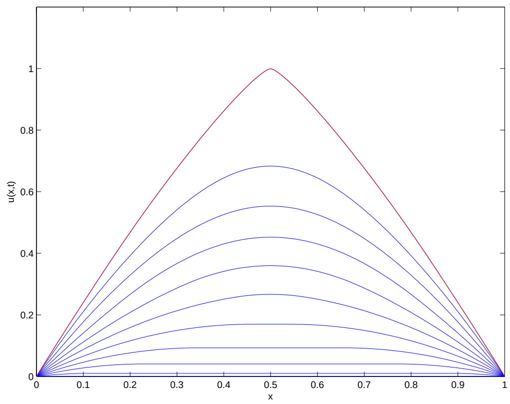

FIGURE 3. *Profile of the numerical solution of problem (1.1) for various times, taking* $\lambda = 1.5$.

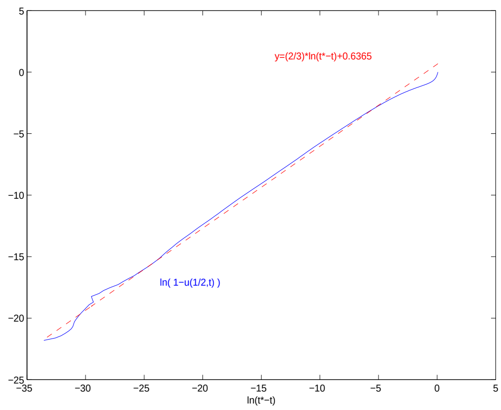

FIGURE 4. *Plot of* $y = \ln\big(1 - u(\frac{1}{2}, t)\big)$ *(solid curve) against* $\ln(t^* - t)$ *for* $\lambda = 1.5$. *The straight line (dashed) has slope* $\frac{2}{3}$ *and indicates good agreement between* $1 - u(\frac{1}{2}, t)$ *and const.*$\times (t^* - t)^{\frac{2}{3}}$. *The straight line, based on the analysis of Section 6 from which we have* $1 - u(\frac{1}{2}, t) \sim a\lambda^{\frac{1}{3}}(t^* - t)^{\frac{2}{3}}$ *for* $t \to t^*-$, *shows* $y = \ln\left[a\lambda^{\frac{1}{3}}(t^* - t)^{\frac{2}{3}}\right] = 0.6365 + \frac{2}{3}\ln(t^* - t)$ *for* $\lambda = 1.5$ *with* $a = (9/2)^{\frac{1}{3}}$.

becomes

$$(\eta^2 - 1)v'' - \frac{2}{3}\eta v' - \frac{2}{9}v = -\frac{1}{v^2}. \tag{7.1}$$

For equation (7.1) with initial conditions $v(0) = c$, a positive constant, and $v'(0) = 0$, we consider a uniform partition of an interval $[0, L]$ of M points with $\delta\eta = \frac{L}{M-1}$ and

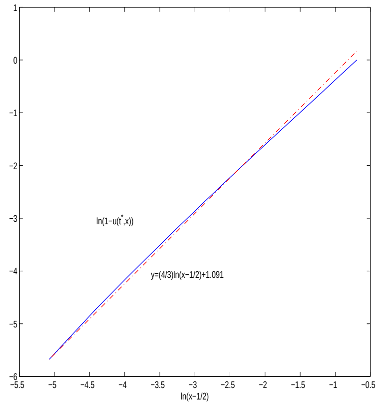

FIGURE 5. *Plot of* $\ln(1-u(x,t^*))$ *(solid curve) against* $\ln(x-\frac{1}{2})$*, for* $\lambda = 1.5$*. The straight line (dotted) has slope* $4/3$ *and the constant* 1.091 *is chosen so that it passes through the point* $(-5.075, -5.675)$ *on the curve* $\ln(1-u(x,t^*))$*.*

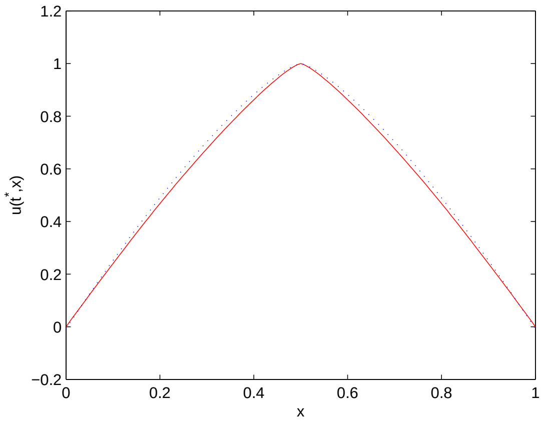

FIGURE 6. *Plot of* $u(x,t^*)$ *against* x *(solid curve) together with the curve* $\left(\frac{9c}{2\sqrt{2}a}\right)^{\frac{2}{3}}(x-\frac{1}{2})^{\frac{4}{3}}$ *for* $c = 2.1$*, (dotted).*

$\eta_j = (j-1)\delta\eta$. Using a simple finite difference scheme and writing $v_j = v(\eta_j)$, $j = 1 \ldots M$, we have

$$(-1)\frac{2v_2 - 2v(0)}{\delta\eta^2} - \frac{2}{9}v(0) = -\frac{1}{v(0)^2} \tag{7.2}$$

for $j = 1$,

$$(\eta_2^2 - 1)\frac{v_3 - 2v_2 + v(0)}{\delta\eta^2} - \frac{2}{3}\eta_2\frac{v_3 - v_1}{2\delta\eta} - \frac{2}{9}v_2 = -\frac{1}{v_2^2} \tag{7.3}$$

for $j = 2$ and

$$(\eta_j^2 - 1)\frac{v_{j+1} - 2v_j + v_{j-1}}{\delta\eta^2} - \frac{2}{3}\eta_j \frac{v_{j+1} - v_{j-1}}{2\delta\eta} - \frac{2}{9}v_j = -\frac{1}{v_j^2}, \tag{7.4}$$

for $j = 3, \ldots, M$. From equation (7.2) we can determine v_2, from equation (7.3) we can determine v_3 and then using equation (7.4) we can obtain a recursive relation giving us successively v_{j+1} for $j = 3, \ldots, M-1$.

In Figure 7 the numerical solutions of problem (7.1) are shown for $v(0) = a-1,\ a,\ a+1$, where $a = \sqrt[3]{\frac{9}{2}}$ and $M = 750$ for $\eta \in [0, L]$, $L = 3.2$. We notice that for $v(0) = a - 1$ the

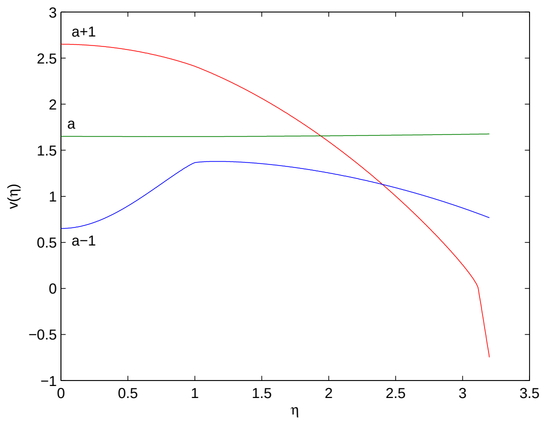

Figure 7. *Plots of the solutions* $v(\eta)$ *of equation* (7.1) *against* η *for* $v(0) = a-1,\ a,\ a+1$.

solution attains a singularity at $\eta = 1$, for $v(0) = 1$ we get the constant solution $v \equiv a$, while for $v(0) = a+1$ the solution falls to zero at some $\eta < \infty$ – as indicated by the analysis of Section 5.

Discussion

In the current work we have investigated the quenching behaviour of a one-dimensional semilinear wave equation modelling the operation of an electrostatic MEMS device. After establishing local existence, we have proved that the solution u of the equation quenches in finite time, i.e. $||u(\cdot, t)||_\infty \to 1$ as $t \to t^*- < \infty$ whenever the parameter of the problem $\lambda > \lambda^*$, where λ^* is the supremum of the spectrum of the associated stationary problem. Although this type of result is fairly standard for related parabolic problems, it is (as far as we are aware) new for nonlinear hyperbolic equations. We also showed that quenching occurs in the parameter range $0 < \lambda < \lambda^*$ if the initial conditions are large enough. Similar results were also found for the practically important two-dimensional case, and indeed for the three-dimensional case. Furthermore, in the second part of this work the quenching profile of the solution was studied. In particular, the existence of self-similar solutions was investigated and our main result in this direction was the surprising one that no non-constant regular self-similar solutions occur. With the aid of this result we studied the profile of the solution near a quenching point and, by use of formal asymptotics, we got that the solution resembles a curve of the form $(x - \text{quenching point})^{4/3}$. Finally, numerical solutions of the problem confirmed the results on the quenching profile.

Acknowledgements

This work was partially funded by PEVE (basic research programme) 2010, National Technical University of Athens (NTUA).

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