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Citation for published version:

Digital Object Identifier (DOI):
10.1364/JOSAA.36.000277

Link:
Link to publication record in Heriot-Watt Research Portal

Document Version:
Peer reviewed version

Published In:
Journal of the Optical Society of America A

Publisher Rights Statement:
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The development of polarization speckle based on random polarization phasor sum

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Received XX Month XXXX; revised XX Month, XXXX; accepted XX Month XXXX; posted XX Month XXXX (Doc. ID XXXXX); published XX Month XXXX

The random walk approach has been extended and applied to study the development of polarization speckle by taking the vector nature into account for stochastic electric fields. Based on the random polarization phasor sum, the first and second moments of the Stokes parameters of the resultant polarization speckle have been examined. Under certain assumptions about the statistics of the component polarization phasors that make up the sum, we present some of the details of the spatial derivation that leads to the expressions for the degree of polarization and the newly proposed Stokes contrast which are suitable for describing the polarization speckle development. This vectorial extension of the random walk will provide an intuitive explanation for the development of the polarization speckle.

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OCIS codes: (000.5490) Probability theory, stochastic processes, and statistics; (030.6140) Speckle; (030.6600) Statistical optics; (110.5405) Polarimetric imaging; (260.5430) Polarization

http://dx.doi.org/10.1364/AO.99.099999

1. INTRODUCTION

As a stochastic process, a random walk describes a path consisting of a succession of random steps on some mathematical space; explains the observed behaviors of many processes; and serves as a fundamental model for the recorded stochastic activity. Due to its theoretical importance and practical interest, random walk has been applied to many scientific fields including physics, chemistry, biology, psychology, ecology, computer sciences, as well as economics [1,2]. In optics, speckle phenomena are such examples, which is composed of a multitude of independently phased additive complex components with both random lengths (amplitudes) and directions (phases) in the complex plane. When these complex components are added together, they constitute what is known as a “random walk.” The resultant of the sum may be large or small, depending on whether constructive or destructive interference dominates the sum, and the squared length of the resultant is what we usually observed intensity with a typical granularity distribution [3-5].

On the other hand, we often deal with vectorial signals with two orthogonal components being the complex-valued signals in many areas of physics, and particularly in polarization optics [6]. Up until now, most of the investigation of the random walk problem in speckle has treated the optical fields as scalar fields without considering their polarization properties. In this paper, we will extend the random walk theory by taking the vector nature into account for stochastic electromagnetic fields referred to as polarization speckle [7-25]. This vectorial extension of the random walk will provide an intuitive explanation for the development of the polarization speckle. Based on the random polarization sum, we will calculate the mean and the second moment of the Stokes parameters where the average is replaced with the ensemble average. for the polarization speckle. Following these results, we examine the relations between the degree of polarization and the newly proposed Stokes contrast with the phase standard deviations and the random walk number.

2. FIRST AND SECOND MOMENTS OF THE STOKES PARAMETERS OF THE RESULTANT POLARIZATION PHASOR

We begin with explicit expression for the random polarization phasor sum of interests. Polarization speckle, known as stochastic electric fields, is a vectorial signal which could be understood as a sum of many contributions from polarization phasors with random vector directions (random polarization states) and random complex amplitudes (random amplitudes and random phases) representing a monochromatic or nearly monochromatic vectorial electric field disturbance. For each component of polarization speckle a complex addition of many small independent contributions from the scalar components of the polarization phasors constitutes a random walk in the complex plane and the resultant phasor of the sum has its total complex amplitude. Figure 1 illustrates an example of polarization phasor sum based on the vector random walks for \( \vec{E}_x \) and \( \vec{E}_y \), respectively.
Given that the complex electric field of the random walk is expressible as

\[
\vec{E} = \vec{E}_x \hat{x} + \vec{E}_y \hat{y} = a_x e^{i\phi_x} \hat{x} + a_y e^{i\phi_y} \hat{y} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \vec{E}_n
\]

\[
= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \left( a_{xn} e^{i\phi_{xn}} \hat{x} + a_{yn} e^{i\phi_{yn}} \hat{y} \right),
\]

where \( \vec{E}_x \) represents the resultant polarization phasor (a complex electric field vector), \( \vec{E}_n \) and \( \vec{E}_y \) are two Cartesian components of \( \vec{E} \) along \( \hat{x} \) and \( \hat{y} \) directions, with them being unit vectors. \( a_x \) and \( a_y \) are the lengths and \( \phi_x \) and \( \phi_y \) are the phases of the resultant polarization components, respectively. \( N \) represents the number of polarization phasors in the random walk. \( \vec{E}_n \) represents the \( n \)th component of polarization phasor in the sum (a complex vector), \( a_{xn} \) and \( \phi_{xn} \) are the length and phase for \( k = x \) or \( y \) components of \( \vec{E}_n \) respectively. The scaling factor \( 1/\sqrt{N} \) is introduced here and in what follows in order to preserve finite energy, i.e. \( < \vec{E}^2 > \) even when the number of component polarization phasors approach infinity.

Because of the great importance of the Stokes parameters in polarization optics, we present our derivations based on the random polarization phasor sums. From their definitions, the Stokes parameters can be given by

\[
S_0 = \vec{E}_x \cdot \vec{E}_y^* + \vec{E}_y \cdot \vec{E}_x^* = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xm} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right],
\]

\[
S_1 = \vec{E}_x \cdot \vec{E}_y^* - \vec{E}_y \cdot \vec{E}_x^* = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xm} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right],
\]

\[
S_2 = \vec{E}_x \cdot \vec{E}_y^* + \vec{E}_y \cdot \vec{E}_x^* = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xm} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right],
\]

\[
S_3 = i(\vec{E}_x \cdot \vec{E}_y^* - \vec{E}_y \cdot \vec{E}_x^*) = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xm} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right],
\]

where * indicates a complex conjugate. Allowing for the moment, arbitrary statistics for both the amplitudes \( a_{yn} \) and \( a_{yn} \) and the phases \( \phi_{xn} \) and \( \phi_{yn} \), but assuming that amplitude and phase of the \( n \)th elementary polarization phasor are statistically independent of each other and of the amplitudes and phases of all other elementary polarization phasors. We can express the mean stokes parameters as

\[
<S_0> = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] + <a_y^2 + a_x^2 + (N-1) a_x^2 M_x^2(-1) M_x^2(1) + (N-1) a_y^2 M_y^2(-1) M_y^2(1) ,
\]

\[
<S_1> = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] + <a_y^2 - a_x^2 + (N-1) a_x^2 M_x^2(-1) M_x^2(1) - (N-1) a_y^2 M_y^2(-1) M_y^2(1) ,
\]

\[
<S_2> = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] = N <a_y^2 > a_x > [M_x^2(1) M_y^2(-1) + M_x^2(-1) M_y^2(1) ,
\]

\[
<S_3> = \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] = -iN <a_y^2 > a_x > [M_x^2(1) M_y^2(-1) - M_x^2(-1) M_y^2(1) ,
\]

When Eqs. (6)~(9) have been derived, we have assumed that all \( a_{yn} (k = x \ or \ y) \) are identically distributed, with mean \( <a_q> \) and second moment \( <a^2_q> \). We have also assumed that all \( \phi_{xn} \) are identically distributed and therefore have common characteristic function \( M_q^2(\omega) \) given by \( M_q^2(\omega) = \exp(iaQ_\omega) \).

We now turn attention to the more difficult problem of calculating the second moment of the stokes parameters.

\[
<S_0^2 > = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] \times \left[ a_{xp} a_{yq} e^{i(\phi_{xn} - \phi_{yn})} + a_{yp} a_{xq} e^{i(\phi_{ym} - \phi_{xm})} \right] ,
\]

\[
<S_1^2 > = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] \times \left[ a_{xp} a_{yq} e^{i(\phi_{xn} - \phi_{yn})} - a_{yp} a_{xq} e^{i(\phi_{ym} - \phi_{xm})} \right] ,
\]

\[
<S_2^2 > = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} + a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] \times \left[ a_{xp} a_{yq} e^{i(\phi_{xn} - \phi_{yn})} + a_{yp} a_{xq} e^{i(\phi_{ym} - \phi_{xm})} \right] ,
\]

\[
<S_3^2 > = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \left[ a_{xm} a_{yn} e^{i(\phi_{xn} - \phi_{yn})} - a_{yn} a_{xm} e^{i(\phi_{ym} - \phi_{xm})} \right] \times \left[ a_{xp} a_{yq} e^{i(\phi_{xn} - \phi_{yn})} - a_{yp} a_{xq} e^{i(\phi_{ym} - \phi_{xm})} \right] ,
\]
where we have assumed again that the amplitudes and phases of the component polarization phasors are statistically independent. For these summations corresponding to each Stokes parameter, there are fifteen different cases that must be considered, as follows [4,5].

\[
\begin{align*}
(1) & \ n = m = p = q & N \text{ terms} \\
(2) & \ n = m, p = q, n \neq p & N(N-1)(N-2) \text{ terms} \\
(3) & \ n = m, p \neq q \neq n & N(N-1)(N-2) \text{ terms} \\
(4) & \ n = p, m = q, n \neq m & N(N-1) \text{ terms} \\
(5) & \ n = p, m \neq q \neq n & N(N-1)(N-2) \text{ terms} \\
(6) & \ n = q, m = p, n \neq m & N(N-1) \text{ terms} \\
(7) & \ n = q, m \neq p \neq n & N(N-1)(N-2) \text{ terms} \\
(8) & \ n = m, p, n \neq q & N(N-1) \text{ terms} \\
(9) & \ n = m, q, n \neq p & N(N-1) \text{ terms} \\
(10) & \ n = p = q, n \neq m & N(N-1) \text{ terms} \\
(11) & \ n = p, q, m = n \neq m & N(N-1) \text{ terms} \\
(12) & \ n \neq m \neq p \neq q & N(N-1)(N-2)(N-3) \text{ terms} \\
(13) & \ p = q, n \neq m \neq p & N(N-1)(N-2) \text{ terms} \\
(14) & \ m = q, n \neq m \neq p & N(N-1)(N-2) \text{ terms} \\
(15) & \ m = p, n \neq m \neq q & N(N-1)(N-2) \text{ terms}
\end{align*}
\]

After summation of all fifteen terms, we have our desired second moments for each individual Stokes parameter. These results are

\[
\begin{align*}
<S_i^2> & = \frac{1}{N} \langle a_i^4 \rangle + 2(N-1) \langle a_i^2 \rangle^2 + \langle a_i^2 \rangle^2 \rangle \\
& + 2N \langle a_i^2 \rangle \langle a_i \rangle^2 + (N^2 - 6N^2 + 11N - 6) \langle a_i \rangle^4 \\
& + (M_i^2(1))^2 + \langle a_i \rangle^4 + (M_i^2(1))^4 + (M_i^2(1))^4 \{4(N-1) \\
& - \langle a_i \rangle^2 + 2(N^2 - 3N + 2) \langle a_i \rangle^2 - \langle a_i \rangle^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + 2(4(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (N-1) \langle a_i > ^2 - a_i ^2 \rangle \\
& + 2(N-1)^2 \langle a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(1))^2 + (M_i^2(1))^4 + 2 \langle a_i \rangle^2 (M_i^2(2) + 2) \langle a_i \rangle^2 \rangle
\end{align*}
\]

(13)

After summation of all fifteen terms, we have our desired second moments for each individual Stokes parameter. These results are

\[
\begin{align*}
&S_i^2 : = \frac{1}{N} \langle a_i^4 \rangle + 2(N-1) \langle a_i^2 \rangle^2 + \langle a_i^2 \rangle^2 \rangle \\
& + 2N \langle a_i^2 \rangle \langle a_i \rangle^2 + (N^2 - 6N^2 + 11N - 6) \langle a_i \rangle^4 \\
& + (M_i^2(1))^2 + \langle a_i \rangle^4 + (M_i^2(1))^4 + (M_i^2(1))^4 \{4(N-1) \\
& - \langle a_i \rangle^2 + 2(N^2 - 3N + 2) \langle a_i \rangle^2 - \langle a_i \rangle^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + 2(4(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (N-1) \langle a_i > ^2 - a_i ^2 \rangle \\
& + 2(N-1)^2 \langle a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(1))^2 + (M_i^2(1))^4 + 2 \langle a_i \rangle^2 (M_i^2(2) + 2) \langle a_i \rangle^2 \rangle
\end{align*}
\]

(14)

After summation of all fifteen terms, we have our desired second moments for each individual Stokes parameter. These results are

\[
\begin{align*}
<S_i^2> & = \frac{1}{N} \langle a_i^4 \rangle + 2(N-1) \langle a_i^2 \rangle^2 + \langle a_i^2 \rangle^2 \rangle \\
& + 2N \langle a_i^2 \rangle \langle a_i \rangle^2 + (N^2 - 6N^2 + 11N - 6) \langle a_i \rangle^4 \\
& + (M_i^2(1))^2 + \langle a_i \rangle^4 + (M_i^2(1))^4 + (M_i^2(1))^4 \{4(N-1) \\
& - \langle a_i \rangle^2 + 2(N^2 - 3N + 2) \langle a_i \rangle^2 - \langle a_i \rangle^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(2) + 2) - 2N(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + 2(4(N-1) < a_i > ^2 - a_i ^2 \rangle \\
& + (N-1) \langle a_i > ^2 - a_i ^2 \rangle \\
& + 2(N-1)^2 \langle a_i > ^2 - a_i ^2 \rangle \\
& + (M_i^2(1))^2 + (M_i^2(1))^4 + 2 \langle a_i \rangle^2 (M_i^2(2) + 2) \langle a_i \rangle^2 \rangle
\end{align*}
\]

(15)

When these results have been derived, we make use of the Hermitian symmetry, $M_k^P(-\omega) = [M_k^P(\omega)]^*$ due to the fact that the probability density functions of $\varphi_k$ is real-valued. All these resulting expressions in Eqs. (14)-(17) are sufficiently complex, we prefer to make some simplifying assumptions at this point. Let the lengths of all the elementary polarization phasors be unity, i.e. $a_k = 1$, for all $n$ and $k = x$ or $y$, and hence

\[
\begin{align*}
< a_i^4 > & = \langle a_i^4 \rangle \langle a_i \rangle^2 \langle a_i \rangle^2 \rangle \\
& = \langle a_i \rangle^2 \langle a_i \rangle^2 \rangle \\
& = \langle a_i \rangle^2 \langle a_i \rangle^2 \rangle \\
& = \langle a_i \rangle^2 \langle a_i \rangle^2 \rangle \\
& = \langle a_i \rangle^2 \langle a_i \rangle^2 \rangle
\end{align*}
\]

(18)

In addition, we assume that the phases $\varphi_k$ are zero mean Gaussian random variables with standard deviations $\sigma_k^P[4]$

\[
P_\varphi(\varphi_k) = \exp[-\varphi_k^2 / (2\sigma_k^P)^2] / (\sqrt{2\pi}\sigma_k^P) ,
\]

(19)

in which case the characteristic functions are given by

\[
M_k^P(\omega) = \exp[-\omega^2 (\sigma_k^P)^2 / 2] .
\]

(20)

To describe the statistical properties of the polarization speckle, two quantities are of great importance: the spatial degree of polarization and the Stokes contrast. The formula for the spatial degree of polarization is

\[
P_S = \sqrt{1 - [4\text{Det}(J)]/[\text{Tr}(J)]^2} ,
\]

(21)

with
\[
J = \frac{1}{2} \left( <S_0> + <S_i> - i <S_i> + i <S_i> - <S_i> > - <S_i> > - <S_i> > + <S_i> > + <S_i> > - <S_i> > \right),
\]

(22)

where \(\text{Tr} \) and \(\text{Det} \) signify the trace operation and the determinant of the coherency matrix \(J\), respectively.

Note the fact that the well-known contrast for the conventional laser speckle has been defined as the standard deviation of intensity divided by the average intensity, i.e. \(C = \sigma_I/I\) [4]. Since the Stokes parameter \(S_0\) is equivalent to the intensity \(I\) and \(S_0^2 = S_i^2 + S_j^2 + S_k^2\) because the stochastic electric field discussed in this paper is fully coherent, the Stokes contrast can be introduced by

\[
C_s = \sqrt{\frac{<S_i^2 > + < S_j^2 > + < S_k^2 > - < S_0 >^2}{< S_0 >^2}},
\]

(23)

which is a measure of how strong the fluctuations of Stokes vectors are on a Poincaré sphere surface as compared with the average value of spherical radius for \(S_0\) (intensity).

After considerable simplification, by using these assumptions in Eqs. (18) and (20) we have the spatial degree of polarization as

\[
P_s = [(N-1)^2 e^{-2(\sigma_p^2)} + (N-1)^2 e^{-2(\sigma_y^2)} + 2(N^2 + 2N - 1) \\
\times e^{-2(\sigma_x^2) + (\sigma_y^2)}]^{1/2} \times (2 + (N-1)[e^{-2(\sigma_x^2)} + e^{-2(\sigma_y^2)}])^{-1}.
\]

(24)

The resulting expression for the Stokes contrast of the polarization speckle is

\[
C_s = \{8(1-1/N)e^{-2(\sigma_x^2)}[N-1 + \cosh((\sigma_x^2)/2)] \\
\times \sinh^2((\sigma_x^2)/2) + 8(1-1/N)e^{-2(\sigma_x^2)}[N-1] \\
+ \cosh((\sigma_x^2)/2)]\sinh^2((\sigma_y^2)/2)^{1/2} \times (2 + (N-1) \\
\times (e^{-2(\sigma_x^2)} + e^{-2(\sigma_y^2)}))^{-1}.
\]

(25)

It’s interesting to note that for \(\vec{E}_x = 0\) (or \(\vec{E}_y = 0\)) with the corresponding standard deviation equal to zero, the expression above reduces to

\[
C = \{8(N-1)[N-1 + \cosh((\sigma_x^2))\sinh^2((\sigma_y^2)/2)^{1/2} \\
\times [N][N-1 + e^{-(\sigma_x^2)/2}]^{3/2} \\
\times \sinh^2((\sigma_y^2)/2)^{1/2}\}
\]

(26)

which is the same as the conventional contrast used widely in the laser speckle, as of course it should be [3].

3. STATISTICAL PROPERTIES OF PARTIALLY DEVELOPED POLARIZATION SPECKLE

Figure 2 shows the relationship between the degree of polarization of partially developed polarization speckles, \(P_s\), and the phase standard deviation of \(\sigma_p\) and \(\sigma_y\) for various values of \(N\), i.e. \(N = 2, 10\) and \(100\). There is no significant change in these graphs when the number of random walks increased from 10 to 100, indicating that the number of random walks does not play an important role when the value is chosen to be larger than 10. It is clear from the figure that the spatial degree of polarization starts at one for smaller values of \(\sigma_p\) and \(\sigma_y\), and then begins to fall to zero. It can also be noted that the point and rate at which the spatial degree of polarization falls to zero is dependent on the value of the number of random walks. For increasing values of random walks, it can be seen that the spatial degree of polarization remains at one for higher values of \(\sigma_p\) and \(\sigma_y\), however, the rate at which it falls to zero increases rapidly.
Fig 2. Spatial degree of polarization $P_3$ of partially developed polarization speckles vs. phase standard deviation of $\sigma_x^P$ and $\sigma_y^P$ for various values of $N$.

Figure 3 plots the relationship between the spatial degree of polarization of partially developed polarization speckles $P_3$ and random walk number $N$ for various values of $\sigma_x^P/\sigma_y^P$. It can be seen that for smaller values of $N$, increasing $\sigma_x^P/2\pi$ will cause the general trend of the decreased values for the spatial degree of polarization. However, as $N$ increases, all these values for $P_3$ will eventually converge towards one. The rate at which these values converge towards one can be seen to decrease with increasing values of $\sigma_x^P/\sigma_y^P$. It can also be seen that for larger spatial degrees of polarization, $\sigma_x^P/2\pi$, must be kept low, as this will keep the spatial degree of polarization closer to one for all values of $N$.

Fig 3. Spatial degree of polarization $P_3$ of partially developed polarization speckles vs. random walk number $N$ for various values of $\sigma_x^P/\sigma_y^P$.

The relationship between the Stokes contrast of partially developed polarization speckle $C_S$ and the phase standard deviations $\sigma_x^P$ and $\sigma_y^P$ for various values of $N$ is illustrated in Fig 4. As both $\sigma_x^P$ and $\sigma_y^P$ increase, the Stokes contrast $C_S$ becomes saturated and tends towards a constant $\sqrt{2}/2$ [7]. Note that when both $\sigma_x^P$ and $\sigma_y^P$ are equal to zero for birefringent material with a flat surface, such as a wave plates, the Stokes contrast vanishes as expected.

Fig 4. Stokes contrast $C_S$ of partially developed polarization speckle vs. phase standard deviation $\sigma_x^P$ and $\sigma_y^P$ for various values of $N$.

Figure 5 plots the Stokes contrast $C_S$ vs. $N$ for values of $\sigma_x^P$ and $\sigma_y^P$. For a fixed ratio $\sigma_x^P/\sigma_y^P$, the behavior of the Stokes contrast shows a pronounced maximum, first rising with $N$ and then falling towards zero. Similar to the contrast $C$ for the laser speckle, the
surprising fact, not visible in the figure, is that all the curves for $C_S$ eventually fall towards zero, regardless of the value of $\sigma_x^p$ and $\sigma_y^p$, but for large standard deviations, the value of $N$ where this begins to happen is extremely large. An explanation for this phenomenon rests on the different dependencies of the average intensity $< S_0 >$ and deviations of the Stokes vectors on $N$. As shown in the expression for the Stokes contrast in equation (25), the dependence of average intensity on $N$ is $< S_0 >$ for large $N$, while the dependence of the numerator of $C_S$ indicating the expected fluctuation of the Stokes vector, is proportional to $\sqrt{N}$. As a stochastic process, random walk describes a path consisting of a succession of random steps. Due to its large applications to many scientific fields including ecology, psychology, computer science, physics, chemistry, biology as well as economics, random walks serve as a fundamental model for the recorded stochastic activity.

Thus, for any $\sigma_x^p$ and $\sigma_y^p$, eventually the numerator of the Stokes contrast will begin to determine and will cause $C_S$ to fall for a large enough $N$, although for large phase standard deviation, the value of $N$ required to see this effect may be extremely large.

Fig 5. Stokes contrast $C_S$ of partially developed polarization speckles vs. random walk number $N$ for various values of $\sigma_x^p/\sigma_y^p$.

4. CONCLUSION

Due to its great importance in statistical optics, we present in this paper an extension of the random walk approach by considering the polarization properties for stochastic electric fields. Based on some assumptions for the random polarization phases, we have derived the results of the first and second moments of the Stokes parameters for the polarization speckle. The vectorial extension of the random walk provides an intuitive explanation for the development of the polarization speckle. To characterize the development of the polarization speckle, the newly introduced Stokes contrast and the spatial degree of polarization over various values of the number of random walk and the phase standard deviations have been investigated. Further a systematic analysis of the random walk of the complex vector and the associated random polarization phasor sum will facilitate the understanding of the origin and development of polarization speckle phenomena from and for various applications and will open up new opportunities in the study of the partial polarization in statistical optics.

Fundings
SUPA START AWARD (SSG040); EPSRC (EP/K03643/1); Natural Sciences and Engineering Research Council of Canada (NSERC 2017-04932) and Canadian Dermatology Foundation.

References